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MULTIVARIATE RISKS AND DEPTH-TRIMMED REGIONS

Ignacio Cascos and Ilya Molchanov*

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It is shown that the concept of depth-trimmed (or central) regions from the multivariate statistics is closely related to the definition of risk measures. In particular, the halfspace trimming corresponds to the Value-at-Risk, while the zonoid trimming yields the expected shortfall. In the abstract framework, it is shown how to establish a both-ways correspondence between risk measures and depth-trimmed regions. It is also demonstrated how the lattice structure of the space of risk values influences this relationship.

Keywords: Acceptance set; cone; depth-trimmed region; multivariate risk; risk measure.

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Multivariate risks and depth-trimmed regions

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1 Introduction

Risk measures are widely used in financial engineering to assess the risk of investments and to optimise the capital allocation. The modern theory of coherent risk measures [2, 8] aims to derive properties of risk measures from several basic axioms: translation-invariance, monotonicity, homogeneity and convexity. The risk measures are mostly considered in the univariate case, i.e. it is assumed that all assets have been transferred to their monetary values. The quantile-based risk measures gain a particular importance in the form of so-called spectral risk measures that are weighted integrals of the quantile function, see [1].

When assessing risks of multivariate portfolios, the situation becomes more complicated. The quantile function is not a numerical function any more, and it is not possible to represent all portfolios as functions of a uniform random variable. The multivariate analogue of the Value-at-Risk discussed in [9] is based on set-valued quantiles of the multivariate cumulative distribution function. Multivariate coherent risk measures have been studied in [14] following the techniques from [8] based on the duality representations. The risk measures considered in [14] are actually set-valued and the preference order corresponds to the ordering of sets by inclusion. It is interesting to note that this order has the same meaning for risk, but formally is the exactly opposite to the ordering of univariate risks from [2].

Because of this reason and in order to unify several existing definitions we decided to consider risk measures as maps that have values in a certain partially ordered cone, which may be, e.g., the real line or the Euclidean space or the family of convex sets in the Euclidean space. We single out the main properties of so defined risk measures and then describe the main technical constructions that make it possible to produce new risk measures from the existing ones while respecting their properties, e.g. the homogeneity or coherence. It is not always assumed that the risk measures are coherent. It should be noted that risk measures with values in a partially ordered cone have been considered in [13], where however it was assumed that this cone is embeddable into a linear space. This is not the case for set-valued risk measures which are also covered by the current work.

We show that the multivariate setting for the risk measures has a number of common features with the concept of central (or depth-trimmed) regions well known in multivariate statistics [21, 22]. They associate a random vector with a set that consists of the points in space which are located near to the "central value" of this random vector. The risk measure is generated by considering all translations of a random vector that bring its central region to the positive (acceptable) part of the space. Correspondingly we identify two concepts: the acceptance cone is a subset of the space of risk values, and the acceptance set is the family of random vectors with risks belonging to the acceptance cone.

Despite the fact that the definition of central regions (and indeed the name also) treats all directions in the same way, it is possible to use them in order to construct risk measures by imposing a partial order relation on the Euclidean space. This establishes a two-way link between depth-trimmed regions and risk measures.

The paper is organised as follows. Section 2 introduces the main concept of a risk measure with values in an abstract cone. As special cases one obtains the classical risk measures [2], set-valued risk measure of [14] and vector-valued risk measures. Two important concepts here are the function that assigns risks to deterministic outcomes and another function that controls changes of the risk if a deterministic amount is being added to a portfolio.

The acceptance cone constitutes a subset of acceptable values for the risk measure, while the acceptance set is the family of random vectors whose risks belong to the acceptance cone. The partial order relation on the space of risks makes it possible to consider it as a lattice. Section 3 treats the acceptance cone and the acceptance set and adapts some concepts from Heijmans [11] (concerning mathematical morphology in abstract lattices) to the framework of risk measures.

Section 4 describes several ways to construct new risk measures: re-centring, homogenisation, worst conditioning and transformations of risks. In particular, the worst conditioning is a generic construction that yields the expected shortfall if applied to the expectation. It is shown that by transforming risks it is possible to produce vector-valued risk measures from set-valued risk measures. This construction can be applied, for instance, to the set-valued risk measures from [14].

The definition of depth-trimmed regions and their essential properties in view of relationships to risk measures are given in Section 5. In particular, the well-known halfspace trimmed regions [17, 20] correspond to the Value-at-Risk and the zonoid trimming [19] produces the expected shortfall. This analogy goes much further and leads to a systematic definition of a risk measure from a family of depth-trimmed regions in Section 6. The main idea here is to map the depth-trimmed region of a random vector into the risk space using the function that assigns risks to deterministic outcomes and then consider all translations of the image (of the depth-trimmed region) that place it inside the acceptance cone. Examples of basic risk measures obtained this way are described in Section 7. It is shown in Section 8 that the correspondence between risk measures and depth-trimmed regions goes both ways, i.e. it is possible to construct a family of depth-trimmed regions from a risk measure, so that, under some conditions, the initial risk measure is recoverable from the obtained family of depth-trimmed regions.

Finally, Section 9 deals with dual representation of coherent risk measures and depth-trimmed regions using families of measures, in a way similar to the well-known approach [8] for real-valued coherent risk measures.

2 Risk measures in abstract cones

A risky portfolio is modelled as an essentially bounded random vector X that represents a financial gain. Let $\mathcal{L}_d^\infty$ denote the set of all essentially bounded d-dimensional random vectors on the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. In order to combine several definitions of risk measures, it is sensible to regard them as functionals on $\mathcal{L}_d^\infty$ with values in a certain *convex cone* $\mathbb{C}$ (i.e. a topological abelian semigroup equipped with multiplication by positive numbers). By $\mathbf{e}$ we denote the neutral element of $\mathbb{C}$ with respect to the addition, which is also assumed to be the multiplicative zero, i.e. the only element of $\mathbb{C}$ such that $t\mathbf{e} = \mathbf{e}$ for all $t > 0$. Furthermore, assume that $\mathbb{C}$ is endowed with a partial order $\preceq$ that is compatible with the (commutative) addition operation and multiplication by scalars, i.e. $x \preceq y$ implies that $x+z \preceq y+z$ for all z and $tx \preceq ty$ for all $t>0$. The partial order on the cone generates the maximum and minimum operations $\vee$ and $\wedge$ respectively. Since this partial order may differ from the conventional order for real numbers, we retain the notation supremum and infimum for the conventional order on the real line. For $x, y \in \mathbb{R}^d$, the order relation $x \leq y$ is always understood coordinatewisely. For convenience, letters x, y, z with or without subscripts stand for points in $\mathbb{R}^d$, letters t, s represent real numbers, whereas letters a, b denote elements of $\mathbb{C}$.

Note that the cone $\mathbb{C}$ is not necessarily embeddable in a linear space, since the addition operation does not necessarily obey the cancellation law. Accordingly, it is not possible to view $\mathbb{C}$ as a Fréchet space, i.e. partially ordered linear space.

The risk measure is a functional on $\mathcal{L}_d^\infty$ with values in $\mathbb{C}$. As the first step of its proper definition, one should specify how this functional acts on degenerate random variables. This action is defined by a function $f : \mathbb{R}^d \mapsto \mathbb{C}$, which is interpreted as the risk associated with the degenerate random variable $X = x$ a.s.

Definition 2.1. *A functional* $\mathfrak{r} : \mathcal{L}_d^\infty \mapsto \mathbb{C}$ *is called a* risk measure *associated with* f *if* $\mathfrak{r}(X) = f(x)$ *in case* $X = x$ *a.s. and the following conditions hold*

R1 $g(y) + \mathfrak{r}(X - y) = \mathfrak{r}(X)$ *for all* $y \in \mathbb{R}^d$, *where* $g : \mathbb{R}^d \mapsto \mathbb{C}$ *such that* $g(y) + g(-y) = \mathbf{e}$ *for all* y*;*

R2 $\mathfrak{r}(Y) \preceq \mathfrak{r}(X)$ *whenever* $Y \leq X$ *a.s.;*

is called a homogeneous *risk measure if also*

R3 $\mathfrak{r}(tX) = t\mathfrak{r}(X)$ *for all* $t > 0$*;*

and a coherent risk measure *if additionally*

R4 $\mathfrak{r}(X) + \mathfrak{r}(Y) \preceq \mathfrak{r}(X + Y)$

holds for all $X, Y \in \mathcal{L}_d^\infty$. *A functional* $\mathfrak{r}$ *that satisfies* ***R4*** *alone, is called* superadditive.

It is also possible to consider not necessarily homogeneous risk measures that satisfy the assumption

$$\mathfrak{r}(tX) + \mathfrak{r}((1-t)Y) \preceq \mathfrak{r}(tX + (1-t)Y) \tag{2.1}$$

for all $t \in [0, 1]$, which are traditionally called *convex* [10] (despite the fact that the inequality in our setting actually means that $\mathfrak{r}$ is concave).

Note that the multiplication by numbers in $\mathbb{C}$ is not needed if the property **R3** is not considered. In this case one can only require that $\mathbb{C}$ is a partially ordered abelian semigroup.

Condition **R1** can be formulated also as

$$g(y) + \mathfrak{r}(X) = \mathfrak{r}(X + y)\,, \tag{2.2}$$

which also implies

$$g(y) + f(x) = f(x + y)\,, \quad x, y \in \mathbb{R}^d\,. \tag{2.3}$$

Note that **R1** implies that

$$g(x) + f(0) = f(x) \tag{2.4}$$

and $g(x)+f(-x) = f(0)$ for all $x \in \mathbb{R}^d$. If $\mathbb{F}$ and $\mathbb{G}$ denote the sets of values for the functions f and g respectively, then $\mathbb{G} + f(0) = \mathbb{F}$. If $f(0)$ is the neutral element $\mathbf{e}$, then $g(x) = f(x)$ and $f(x) + f(-x) = f(0)$ for all x. This is however not always the case. Note also that instead of the function g it is possible to begin with linear operators G_y indexed by $y \in \mathbb{R}^d$, so that $G_y a = a + g(y)$.

Lemma 2.2. *If $\mathfrak{r}$ is a risk measure associated with f, then f is necessarily non-decreasing (i.e. $f(y) \preceq f(x)$ if $y \leq x$) and*

$$f(x) + f(y) = f(x + y) + f(0) \tag{2.5}$$

for all $x, y \in \mathbb{R}^d$. If $\mathfrak{r}$ is superadditive, then $f(x) + f(y) \preceq f(x + y)$ for all x, y; if $\mathfrak{r}$ is homogeneous, then f is homogeneous, i.e. $f(tx) = tf(x)$ for all x and $t > 0$ and $f(x)+f(0) \preceq f(x)$ for all x.

Proof. The monotonicity of f follows from **R1**. The superadditivity and homogeneity properties directly follow from **R4** and **R3** applied to deterministic random vectors.

Take $X = x$ a.s. in **R1** to show that

$$g(-y) + f(x + y) = f(x)\,.$$

If we add $f(y)$ to the both sides and use the fact that $g(-y) + f(y) = f(0)$, we arrive at (2.5). □

Example 2.3 (Multivariate set-valued risk measures). Let $\mathbb{C}$ be the family of closed sets in $\mathbb{R}^d$ partially ordered by inclusion with the addition defined as the closure of Minkowski (elementwise) sum. Note that the Minkowski sum of two non-compact sets is not necessarily closed. Define $f(x) = \{y \in \mathbb{R}^d : \; -x \leq y\}$ and $g(x) = \{-x\}$. Here we see the main difference between the functions f and g, the former can be set-valued, while the latter is a point in $\mathbb{R}^d$ that controls translations of the risk values. In particular, the fact that $\mathfrak{r}(X) \supset [0, \infty)^d$ means that X has a negative risk. In this case Definition 2.1 turns into [14, Def. 2.1].

Example 2.4 (Univariate risk measures). The classical definition of coherent risk measures from Artzner *et al.* [2] can be recovered from the setting of Example 2.3 for $d = 1$ and $\mathfrak{r}(X) = [\rho(X), \infty)$, where $\rho(X)$ is the risk measure of X as in [2]. An alternative approach is to let $\mathbb{C}$ be the real line $\mathbb{R}$ with the reversed order and conventional addition and multiplication operations. In this case $g(x) = -x$ and $f(x) = -x$. We will briefly recall three univariate risk measures: the value at risk, which is the most widely used risk measure, and two coherent risk measures, the expected shortfall and the expected minimum.

The *value at risk* is defined as the amount of extra capital that a firm needs in order to reduce the probability of going bankrupt to a fixed threshold α. It is the opposite of the α-quantile of a random variable X, i.e.

$$\mathrm{V@R}_\alpha(X) = -\inf\{x : \; \mathbf{P}\{X \leq x\} > \alpha\} = -F_X^{-1}(\alpha)\,.$$

It can be shown that the value at risk is a homogeneous risk measure, but not a coherent one. It satisfies properties **R1**, **R2** and **R3**, but not necessarily **R4**.

The *expected shortfall* is a coherent risk measure defined as

$$\mathrm{ES}_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha F_X^{-1}(t)\mathrm{d}t\,,$$

where $\alpha \in (0,1]$.

The *expected minimum* is another coherent risk measure defined as

$$\mathrm{EM}_{1/n}(X) = -\mathbf{E}\min\{X_1, X_2, \dots, X_n\},$$

where $X_1, X_2, \dots, X_n$ are independent copies of X. The expected minimum belongs to the family of weighted $\mathrm{V@R}_\alpha$ and is called Alpha $\mathrm{V@R}_\alpha$ in [7].

Example 2.5 (Marginalised multivariate vector-valued risk measures). Let $\mathbb{C}$ be $\mathbb{R}^d$ with the usual addition and the reversed lexicographical order, i.e. $a \preceq b$ if $b \le a$. Given a d-dimensional random vector $X = (X_1, X_2, \dots, X_d)$, any of the aforementioned univariate risk measures $\mathfrak{r}$ yields a risk measure $\mathfrak{r}(X) = (\mathfrak{r}(X_1), \mathfrak{r}(X_2), \dots, \mathfrak{r}(X_n))$ which takes values in $\mathbb{R}^d$. In this case $g(x) = f(x) = -x$.

3 Acceptance cones and acceptance sets

The concept of an acceptance set is the dual one to the risk measure, see [2, 10, 14]. The main idea is that a portfolio X is acceptable if $\mathfrak{r}(X)$ belongs to a certain subcone $\mathbb{A} \subset \mathbb{C}$ called the *acceptance cone*. In the classical setting (see Example 2.4) $\mathbb{C} = \mathbb{R}$ with the reversed order and $\mathbb{A} = (-\infty, 0]$. Furthermore, $\mathbb{A}$ is upper with respect to $\preceq$, i.e. if $a \preceq b$ and $a \in \mathbb{A}$, then $b \in \mathbb{A}$.

We always assume that

$$f(x) \in \mathbb{A}\,, \quad x \ge 0\,, \tag{3.1}$$

i.e. all non-negative portfolios are acceptable. This is always the case if $\mathbb{A}$ is chosen to be the set of all elements from $\mathbb{C}$, which are greater than or equal to $f(0)$. If $f(0) = \mathbf{e}$, $\mathbb{A}$ consists of all $a \in \mathbb{C}$ such that $\mathbf{e} \preceq a$. In this case $b \in \mathbb{A} + a$ if and only if $a \preceq b$.

Recall that $\mathbb{G}$ denotes the family of values of the function g from Definition 2.1.

Definition 3.1. *The acceptance cone* $\mathbb{A}$ *is* compatible with the partial order *in* $\mathbb{C}$ *if*

$$f(0) + \bigvee\{a \in \mathbb{G} :\ b \in \mathbb{A} + a\} = b$$

for all $b \in \mathbb{C}$*.*

Definition 3.2 (see [11])**.** *If* $\mathbb{G} \subset \mathbb{C}$*, we say that* $\mathbb{G}$ *is a* sup-generating family *if*

$$b = \bigvee\{a \in \mathbb{G} :\ a \preceq b\}, \quad \textit{for all}\, b \in \mathbb{C}\,.$$

Theorem 3.3. *If $f(0) = \mathbf{e}$, then the acceptance cone $\mathbb{A} = \{a \in \mathbb{C} : \ \mathbf{e} \preceq a\}$ is compatible with the partial order if the family $\mathbb{G}$ is sup-generating.*

Proof. Given $b \in \mathbb{C}$, by the definition of $\mathbb{A}$,

$$f(0) + \bigvee\{a \in \mathbb{G} : \ b \in \mathbb{A} + a\} = \bigvee\{a \in \mathbb{G} : \ a \preceq b\} = b\,.$$

□

Example 3.4 (Set-valued risk measures). Let A be a convex cone which is a strict subset of $\mathbb{R}^d$ such that $\mathbb{R}^d_+ \subset A$. Define the cone $\mathbb{C}$ to be the family of convex closed sets F ordered by inclusion and such that the closure of the Minkowski sum $F + A$ equals F. Note that $A \in \mathbb{C}$ and also A is the neutral element of $\mathbb{C}$ for the closed Minkowski addition operation. If $g(y) = -\{y\}$ and $f(y) = -y + A$ for $y \in \mathbb{R}^d$, then $\mathbb{G}$ is a sup-generating family for $\mathbb{C}$ and the acceptance cone $\mathbb{A} = \{F \in \mathbb{C} : \ F \supset A\}$ is compatible with the partial order. The same holds also if $g(y) = f(y)$ for all y. In the univariate case ($d = 1$) we inevitably choose $A = [0, \infty)$.

Given the risk measure $\mathfrak{r}$, the set $\mathcal{A} \subset \mathcal{L}^\infty_d$ of acceptable portfolios (called the *acceptance set*) is given by

$$\mathcal{A} = \{X \in \mathcal{L}^\infty_d : \ \mathfrak{r}(X) \in \mathbb{A}\}\,.$$

It follows from (2.2) that

$$\{y : \ X - y \in \mathcal{A}\} = \{y : \ \mathfrak{r}(X - y) \in \mathbb{A}\} = \{y : \ \mathfrak{r}(X) + g(-y) \in \mathbb{A}\}\,.$$

The f-image of the set in the right-hand side is

$$\begin{aligned}
\mathfrak{r}_{\mathcal{A}}(X) &= \{f(y) : \ y \in \mathbb{R}^d, \ \mathfrak{r}(X) + g(-y) \in \mathbb{A}\} \\
&= \{f(y) : \ y \in \mathbb{R}^d, \ \mathfrak{r}(X) \in \mathbb{A} + g(y)\} \\
&= \{f(0) + g(y) : \ y \in \mathbb{R}^d, \ \mathfrak{r}(X) \in \mathbb{A} + g(y)\} \\
&= f(0) + \{a \in \mathbb{G} : \ \mathfrak{r}(X) \in \mathbb{A} + a\}\,.
\end{aligned}$$

Note that $\mathfrak{r}_{\mathcal{A}}(X)$ is not necessarily an element of $\mathbb{C}$. For instance, in Example 2.4 (with $\mathbb{C} = \mathbb{R}$), $\mathfrak{r}_{\mathcal{A}}(X)$ is the set $\{a \in \mathbb{R} : \ \mathfrak{r}(X) \in (-\infty, a]\}$. Note that in this case, one can retrieve $\mathfrak{r}(X)$ by taking the infimum of all members of $\mathfrak{r}_{\mathcal{A}}(X)$. This minimum corresponds to the maximum operation in the cone $\mathbb{C} = \mathbb{R}$ with the reversed order.

In particular, if the acceptance cone $\mathbb{A}$ is compatible with the partial order, then

$$\mathfrak{r}(X) = \bigvee \mathfrak{r}_{\mathcal{A}}(X)\,.$$

This fact corresponds to a well-known relationship between the risk measures and acceptance sets [2, 8].

4 Constructions of risk measures

4.1 Re-centring

All random vectors from $\mathcal{L}_d^\infty$ can be naturally centred by subtracting their expected values. This makes it possible to define a risk measure on centred random vectors and then use the translation invariance to extend it onto the whole $\mathcal{L}_d^\infty$.

If $\mathfrak{r}$ is defined on the family $\mathcal{L}_{d,0}^\infty$ of essentially bounded random vectors with mean zero, then the re-centred $\mathfrak{r}$ is defined by

$$\mathfrak{r}(X) = \mathfrak{r}(X - \mathbf{E}X) + g(\mathbf{E}X)\,, \quad X \in \mathcal{L}_d^\infty\,.$$

For this we usually rely on the canonical choice of the translation function g from Definition 2.1 by setting $g(y) = -y$ for all $y \in \mathbb{R}^d$.

The condition **R2** does not hold if $\mathcal{L}_d^\infty$ is equipped with the coordinatewise stochastic order. However it holds if the order $Y \leq X$ means that $\mathbf{E}X \leq \mathbf{E}Y$ coordinatewisely and $(Y - \mathbf{E}Y) \leq (X - \mathbf{E}X)$ with respect to a certain stochastic order on $\mathcal{L}_{d,0}^\infty$ that is respected by $\mathfrak{r}$ on $\mathcal{L}_{d,0}^\infty$.

The re-centring construction may be also used to extend the risk measure defined for random vectors with all non-negative coordinates to all essentially bounded random vectors by setting

$$\mathfrak{r}(X) = \mathfrak{r}(X - \operatorname{essinf} X) - \operatorname{essinf} X\,,$$

where $\operatorname{essinf} X$ is the vector composed of essential minima of the coordinates from X. In order to ensure the compatibility of this definition, the function $\mathfrak{r}$ should satisfy $\mathfrak{r}(X+z) = \mathfrak{r}(X) - z$ for all random vector X in $[0,\infty)^d$ and $z \in [0,\infty)^d$.

4.2 Homogenisation

If $\mathfrak{r}$ satisfies **R1** and **R2**, it is possible to construct a homogeneous risk measure from it by setting

$$\mathfrak{r}_{\mathrm{h}}(X) = \bigwedge_{t>0} \frac{1}{t}\,\mathfrak{r}(tX)\,, \tag{4.1}$$

where $\bigwedge$ is the infimum operation in $\mathbb{C}$. It is easy to see that $\mathfrak{r}_{\mathrm{h}}$ satisfies **R3**. Furthermore, it satisfies **R2** and also remains translation invariant (i.e. satisfies **R1**) if the function g is homogeneous. The latter is clearly the case if $g(y) = -y$ for all y.

A similar construction produces a translation-invariant risk measure from a general one by

$$\mathfrak{r}_{\mathrm{t}}(X) = \bigwedge_{z \in \mathbb{R}^d} (\mathfrak{r}(X+z) + z)\,. \tag{4.2}$$

Both (4.1) and (4.2) applied together to a function $\mathfrak{r}$ that satisfies **R2** and **R4** yield a coherent risk measure.

Example 4.1. If $\mathbb{C}$ is the real line with the reversed order and (4.1) results in a non-trivial function, then $\mathfrak{r}(tX) \to 0$ as $t \to 0$. Similarly, a non-trivial result of (4.2) yields that $\mathfrak{r}(X+z) \to -\infty$ as $z \to \infty$. For instance, these constructions produce trivial results if applied to the risk measure $\mathbf{E}(k-X)_+$ studied in [12].

If $d=1$ and $\mathfrak{r}(X) = e^{-\mathbf{E}X} - 1$, then $\mathfrak{r}_{\mathrm{h}} = 0$ if $\mathbf{E}X \geq 0$ and $\mathfrak{r}_{\mathrm{h}} = \infty$ otherwise.

4.3 Worst conditioning

A single risk measure $\mathfrak{r}$ can be used to produce a family of risk measures by taking the infimum of the risks associated to the random vectors obtained after certain rearrangements of the probability measure. For each $\alpha \in (0,1]$ define

$$\mathfrak{r}_\alpha(X) = \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}(X_\phi)\,,$$

where Φ_α is the family of measurable mappings $\phi : \Omega \mapsto \Omega$ such that $\mathbf{P}(\phi^{-1}(A)) \leq \alpha^{-1}\mathbf{P}(A)$ for all $A \in \mathfrak{F}$ and $X_\phi = X \circ \phi$. If X is an essentially bounded random vector, then for any $\alpha \in (0,1]$ and any $\phi \in \Phi_\alpha$, X_ϕ is an essentially bounded random vector.

It is possible to define the worst conditioning $\mathfrak{r}_\alpha(X)$ alternatively as

$$\mathfrak{r}_\alpha(X) = \bigwedge_{Y \in \mathcal{P}_\alpha(X)} \mathfrak{r}(Y)\,,$$

where $\mathcal{P}_\alpha(X)$ is the family of all random vectors Y such that $\mathbf{P}\{Y \in B\} \leq \alpha^{-1}\mathbf{P}\{X \in B\}$ for all Borel $B \subset \mathbb{R}^d$.

It is simple to show that $\mathfrak{r}_\alpha$ preserves any property that $\mathfrak{r}$ satisfies from **R1**–**R4**. For instance, if $Y \leq X$ a.s., then $Y_\phi \leq X_\phi$ a.s. for any $\phi \in \Phi_\alpha$ and thus

$$\mathfrak{r}_\alpha(Y) = \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}(Y_\phi) \preceq \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}(X_\phi) = \mathfrak{r}_\alpha(X)\,,$$

whenever $\mathfrak{r}$ satisfies **R2**. If $X, Y \in \mathcal{L}^\infty_d$ and $\mathfrak{r}$ satisfies **R4**, then

$$\begin{aligned}\mathfrak{r}_\alpha(X+Y) &= \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}\big((X+Y)_\phi\big) = \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}(X_\phi + Y_\phi) \\ &\succeq \bigwedge_{\phi \in \Phi_\alpha} \big(\mathfrak{r}(X_\phi) + \mathfrak{r}(Y_\phi)\big) \succeq \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}(X_\phi) + \bigwedge_{\phi \in \Phi_\alpha} \mathfrak{r}(Y_\phi) = \mathfrak{r}_\alpha(X) + \mathfrak{r}_\alpha(Y)\,.\end{aligned}$$

Consider now the setting of univariate risk measures from Example 2.4, that is, X is a random variable from $\mathcal{L}^\infty_1$ and the cone $\mathbb{C}$ is the real line with the reversed order. The simplest coherent risk measure is the opposite of the expectation of a random variable. In fact, this risk measure appears from the expected shortfall when $\alpha = 1$, i.e. $\mathrm{ES}_1(X) = -\mathbf{E}X$. Let us apply the worst conditioning to the opposite of the expectation,

$$\begin{aligned}(-\mathbf{E})_\alpha(X) &= \sup_{\phi \in \Phi_\alpha} \{-\mathbf{E}(X_\phi)\} = -\inf_{\phi \in \Phi_\alpha} \mathbf{E}(X_\phi) = -\inf_{\phi \in \Phi_\alpha} \int X(\phi(\omega))\mathbf{P}(\mathrm{d}\omega) \\ &= -\inf_{\phi \in \Phi_\alpha} \int X(\omega)\mathbf{P}\phi^{-1}(\mathrm{d}\omega) = -\inf_{\phi \in \Phi_\alpha} \mathbf{E}_{\mathbf{P}\phi^{-1}} X\,,\end{aligned}$$

where $\mathbf{E}_{\mathbf{P}\phi^{-1}}$ denotes the expectation with respect to the probability measure $\mathbf{P}\phi^{-1}$. Note that the infimum above depends on the underlying probability space.

Example 4.2. Let $\Omega = \{\omega_1, \omega_2\}$ with the both atoms of probability $1/2$ and let X be given by $X(\omega_1) = 0$, $X(\omega_2) = 1$. Since the only element in $\Phi_{2/3}$ is the identity, $(-\mathbf{E})_{2/3}(X) = -1/2$. Nevertheless, if Y induces the same probability distribution as X, but it is defined on a non-atomic probability space, then $(-\mathbf{E})_{2/3}(Y) = -1/4$.

In general $-\inf_{\phi\in\Phi_\alpha} \mathbf{E}_{\mathbf{P}\phi^{-1}} X \leq \mathrm{ES}_\alpha(X)$. If $(\Omega, \mathfrak{F}, \mathbf{P})$ is non-atomic, then the expected shortfall appears from the opposite of the expectation under the worst conditioning. Without loss of generality assume that $\Omega = [0,1]$, $\mathbf{P}$ is the Lebesgue measure restricted to $[0,1]$ and X is increasing mapping from $[0,1]$ into $\mathbb{R}$, which implies that $X(\omega) = F_X^{-1}(\omega)$ for all $\omega \in [0,1]$, where F_X is the cumulative distribution function of X. The infimum of $\mathbf{E}_{\mathbf{P}\phi^{-1}} X$ over all $\phi \in \Phi_\alpha$ is achieved when $X \circ \phi$ takes the smallest possible values with the highest possible probabilities, and thus it is attained at $\phi'(\omega) = \alpha\omega$. We conclude

$$(-\mathbf{E})_\alpha(X) = -\int X(\alpha\omega)\mathrm{d}\omega = -\frac{1}{\alpha}\int_0^\alpha F_X^{-1}(t)\mathrm{d}t = \mathrm{ES}_\alpha(X)\,.$$

Example 4.3 (Worst conditioning of the expected shortfall). Let us now apply the worst conditioning to the expected shortfall at level β,

$$\big(\mathrm{ES}_\beta\big)_\alpha(X) = \sup_{\phi_1\in\Phi_\alpha} \mathrm{ES}_\beta(X_{\phi_1}) = \sup_{\phi_1\in\Phi_\alpha}\Big(-\inf_{\phi_2\in\Phi_\beta} \mathbf{E}_{\mathbf{P}\phi_1^{-1}} X_{\phi_2}\Big) = -\inf_{\phi_1\in\Phi_\alpha,\,\phi_2\in\Phi_\beta} \mathbf{E}_{\mathbf{P}\phi_1^{-1}\phi_2^{-1}} X\,.$$

Clearly $\phi_2 \circ \phi_1 \in \Phi_{\alpha\beta}$ and thus $\big(\mathrm{ES}_\beta\big)_\alpha(X) \leq \mathrm{ES}_{\alpha\beta}(X)$. Nevertheless, if the probability space is non-atomic, all mappings from $\Phi_{\alpha\beta}$ can be written as a composition of a mapping from Φ_α and a mapping from Φ_β and thus $\big(\mathrm{ES}_\beta\big)_\alpha(X) = \mathrm{ES}_{\alpha\beta}(X)$. One can say that the expected shortfall risk measures are stable under the worst conditioning.

Example 4.4 (Worst conditioned $\mathrm{V@R}_\alpha$). Let us finally apply the worst conditioning construction to the value at risk at level β considered on a non-atomic probability space $\Omega = [0,1]$ with $\mathbf{P}$ being the Lebesgue measure. If X is increasing, then $X(\omega) = F_X^{-1}(\omega)$. The infimum below is attained at $\phi'(\omega) = \alpha\omega$ and since $X_{\phi'}$ is also increasing, we have $X_{\phi'}(\omega) = F_{X_{\phi'}}^{-1}(\omega)$. Thus

$$\big(\mathrm{V@R}_\beta\big)_\alpha(X) = -\inf_{\phi\in\Phi_\alpha} F_{X_\phi}^{-1}(\beta) = -X_{\phi'}(\beta) = -X(\alpha\beta) = -F_X^{-1}(\alpha\beta) = \mathrm{V@R}_{\alpha\beta}(X)\,.$$

4.4 Transformations of risks

Risk measures with values in a cone $\mathbb{C}$ may be further transformed by mapping $\mathbb{C}$ into another cone $\mathbb{C}'$ using a map h. The aim may be to change the dimensionality (cf [14]) or produce a vector-valued risk measure from a set-valued one.

The mapping $h : \mathbb{C} \mapsto \mathbb{C}'$ that transforms any $\mathbb{C}$-valued risk measure $\mathfrak{r}$, into the $\mathbb{C}'$-valued risk measure $h \circ \mathfrak{r}$, will be called a *risk transformation.* If h respects the coherence property of risk measures, it will be called a *coherent map.*

Let us denote by $\preccurlyeq$ the partial order in $\mathbb{C}'$ which we assume to be compatible with the (commutative) addition operation and multiplication by scalars. In the following result, we list the properties that a coherent map should possess.

Proposition 4.5. *A map $h : \mathbb{C} \mapsto \mathbb{C}'$ is a risk transformation if it is*

(i) non-decreasing, i.e. $h(a) \preccurlyeq h(b)$ if $a \preceq b$;

(ii) linear on $\mathbb{G}$, i.e. $h(a+b) = h(a) + h(b)$ for all $b \in \mathbb{C}$ and $a \in \mathbb{G}$.

Further, it is a coherent map if it is homogeneous, i.e. $h(ta) = th(a)$ for all $t > 0$ and $a \in \mathbb{C}$ and also satisfies

$$h(a) + h(b) \preccurlyeq h(a+b) \tag{4.3}$$

for all $a, b \in \mathbb{C}$.

Proof. Since $\mathfrak{r}$ satisfies **R1** and $g(y) \in \mathbb{G}$, we have for all $y \in \mathbb{R}^d$

$$h\big(\mathfrak{r}(X)\big) = h\big(g(y) + \mathfrak{r}(X-x)\big) = h\big(g(y)\big) + h\big(\mathfrak{r}(X-x)\big)\,,$$

i.e. **R1** holds. Property **R2** holds because h is non-decreasing. The homogeneity of $h(\mathfrak{r}(\cdot))$ is evident if h is homogeneous. If $\mathfrak{r}$ is coherent and (4.3) holds, then

$$h\big(\mathfrak{r}(X)\big) + h\big(\mathfrak{r}(Y)\big) \preccurlyeq h\big(\mathfrak{r}(X) + \mathfrak{r}(Y)\big) \preccurlyeq h\big(\mathfrak{r}(X+Y)\big)\,.$$

□

As an immediate consequence of Proposition 4.5 we deduce that every linear non-decreasing map is coherent.

If a risk transformation h satisfies $h(\bigvee G) = \bigvee h(G)$ for all $G \subset \mathbb{G}$ and $\mathbb{G}$ is sup-generating, then $h(\mathbb{G})$ is also sup-generating. Notice that $h(\mathbb{G})$ is a subset of $\mathbb{C}'$ and thus the maximum operations in G and in $h(G)$ might be different. Since h is increasing, $\bigvee h(G) \preccurlyeq h(\bigvee G)$ always holds, so that the only real requirement is the reversed inequality.

Proposition 4.6. *If $\mathbb{G} \subset \mathbb{C}$ is a sup-generating family, $h : \mathbb{C} \mapsto \mathbb{C}'$ is increasing and $h(\bigvee G) = \bigvee h(G)$ for all $G \subset \mathbb{G}$, then $h(\mathbb{G}) \subset h(\mathbb{C})$ is a sup-generating family.*

Proof. Let $b \in \mathbb{C}$, we have

$$\begin{aligned}\bigvee\{h(a) \in h(\mathbb{G}) :\ h(a) \preccurlyeq h(b)\} &= h\Big(\bigvee\{a \in \mathbb{G} :\ h(a) \preccurlyeq h(b)\}\Big)\\ &\succcurlyeq h\Big(\bigvee\{a \in \mathbb{G} :\ a \preceq b\}\Big) = h(b)\end{aligned}$$

and since the reversed inequality trivially holds, we conclude

$$h(b) = \bigvee\{h(a) :\ h(a) \preccurlyeq h(b)\}\,.$$

□

Particularly important instances of transformations of risks arise if $\mathbb{C}$ is a family of convex closed subsets of $\mathbb{R}^d$ ordered by inclusion and $\mathbb{C}'$ is $\mathbb{R}^d$ with the inverse coordinatewise order.

Example 4.7 (Vector-valued risk measures from set-valued ones). Let A be a convex cone in $\mathbb{R}^d$ such that $\mathbb{R}^d_+ \subset A$. Consider the cone $\mathbb{C}$ from Example 3.4. Define $g(y) = f(y) = -y + A$ for $y \in \mathbb{R}^d$. Let $h(F)$ denote the minimum of the set F with respect to the ordering in $\mathbb{R}^d$ generated by A. If $\mathfrak{r}$ is a $\mathbb{C}$-valued risk measure, then $h(\mathfrak{r}(\cdot))$ is a vector-valued risk measure. Indeed, the map h is monotone and homogeneous. Since

$$h(F - y + A) = h(F - y) = h(F) - y = h(F) + h(-y + A)\,,$$

h it is linear on $\mathbb{G}$. Finally, h satisfies (4.3), since $x = h(F_1)$ and $y = h(F_2)$ imply that $F_1 + F_2 \subset (x_1 + x_2) + A$.

5 Depth-trimmed regions

Depth functions assign to a point its degree of centrality with respect to the distribution of a random vector, see Zuo and Serfling [21]. The higher the depth of a point is, the more central this point is with respect to the distribution of the random vector. Depth-trimmed (or central) regions are sets of central points associated with a random vector. Given a depth function, depth-trimmed regions can be obtained as its level sets. With a d-dimensional random vector X associate the family of *depth-trimmed regions*, i.e. sets $\mathfrak{D}^\alpha(X)$, $\alpha \in (0, 1]$, such that the following properties hold for all α and all $X \in \mathcal{L}^\infty_d$:

D0 if $X = x$ a.s., then $\mathfrak{D}^\alpha(X) = \{x\}$ for all α;

D1 $\mathfrak{D}^\alpha(X + y) = \mathfrak{D}^\alpha(X) + y$ for all $y \in \mathbb{R}^d$;

D2 $\mathfrak{D}^\alpha(tX) = t\mathfrak{D}^\alpha(X)$ for all $t > 0$;

D3 $\mathfrak{D}^\alpha(X) \subset \mathfrak{D}^\beta(X)$ if $\alpha \geq \beta$;

D4 $\mathfrak{D}^\alpha(X)$ is connected and compact.

These properties are similar to those discussed by Zuo and Serfling [22, Th. 3.1]. Property **D0** is not usually considered, but it follows the spirit of depth-trimmed regions. Moreover, [22] requires one further property:

D1′ $\mathfrak{D}^\alpha(AX + y) = A\mathfrak{D}^\alpha(X) + y$ holds for any $d \times d$ nonsingular matrix A and any $y \in \mathbb{R}^d$.

We will consider two additional properties of depth-trimmed regions, that, to our knowledge, have not been studied in the literature so far,

D5 if $X \geq Y$ a.s., then $\mathfrak{D}^\alpha(X) \subset \mathfrak{D}^\alpha(Y) + \mathbb{R}^d_+$;

D6 $\mathfrak{D}^\alpha(X + Y) \subset \mathfrak{D}^\alpha(X) + \mathfrak{D}^\alpha(Y)$.

Observe that depth-trimmed regions take values in the cone of compact subsets of $\mathbb{R}^d$ and the addition operation in **D5** and **D6** is the Minkowski (or elementwise) addition.

Example 5.1 (Halfspace trimming). The *halfspace trimmed regions* are built as the intersection of closed halfspaces whose probability is not smaller than a given value:

$$\begin{aligned}\mathrm{HD}^\alpha(X) &= \big\{x \in \mathbb{R}^d : \mathbf{P}\{X \in H\} > \alpha,\ \forall \text{ closed halfspace } H \ni x\big\}\\ &= \bigcap \big\{H : H \text{ closed halfspace with } \mathbf{P}\{X \in H\} \geq 1-\alpha\big\}.\end{aligned}$$

It is well known that the halfspace trimmed regions satisfy **D0**–**D4** and **D1′**. The new property **D5** is not hard to derive. Property, **D6** does not hold in general. Univariate examples where **D6** fails can be built from examples for which the value at risk does not satisfy **R4**.

The mentioned definition of the halfspace trimmed region is taken from Massé and Theodorescu [17]. Alternatively, the strict inequality in the definition of HD^α is swapped with the non-strict one, see Rousseeuw and Ruts [20]. However the definition of Massé and Theodorescu [17] for halfspace trimmed regions leads to a simpler relationship between the value at risk and the univariate halfspace trimming, see Section 6.

Example 5.2 (Zonoid trimming). Koshevoy and Mosler [16] defined *zonoid trimmed regions* in terms of expectations of certain random vectors. Let X be an integrable random vector in $\mathbb{R}^d$, i.e. $X \in \mathcal{L}^1_d$. For $\alpha \in (0,1]$ define

$$\mathrm{ZD}^\alpha(X) = \big\{\mathbf{E}[Xl(X)] : l : \mathbb{R}^d \mapsto [0, \alpha^{-1}] \text{ measurable and } \mathbf{E}l(X) = 1\big\}. \tag{5.1}$$

Properties **D1**–**D4** and **D1′** are already derived in [16], while **D0** is trivial. The proofs of **D5** and **D6** do not involve serious technical difficulties.

Example 5.3 (Expected convex hull trimming). *Expected convex hull regions* of a random vector X at level n^{-1} for $n \geq 1$ are defined by Cascos [5] as the selection (or Aumann) expectation of the convex hull of n independent copies $X_1, \dots, X_n$ of X, see [18, Sec. 2.1] for the definition of expectation for random sets. The expected convex hull region can be given implicitly in terms of its support function as

$$s(\mathrm{CD}^{1/n}(X), u) = \mathbf{E}\max\{\langle X_1, u\rangle, \langle X_2, u\rangle, \dots \langle X_n, u\rangle\} \quad \text{for all } u \in \mathbb{R}^d\,,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product. Note that for any $K \subset \mathbb{R}^d$ its support function is given by $s(K, u) = \sup\{\langle x, u\rangle : x \in K\}$ for $u \in \mathbb{R}^d$. The expected convex hull regions satisfy properties **D0**–**D6** and also **D1′**.

Example 5.4 (Integral trimming). Let $\mathcal{F}$ be a family of measurable functions from $\mathbb{R}^d$ into $\mathbb{R}$. Cascos and López-Díaz [6] defined the family of integral trimmed regions as

$$\begin{aligned}\mathrm{D}^\alpha_{\mathcal{F}}(X) &= \bigcup_{Y \in \mathcal{P}_\alpha(X)} \Big\{x \in \mathbb{R}^d : \mathfrak{f}(x) \leq \mathbf{E}\,\mathfrak{f}(Y) \text{ for all } \mathfrak{f} \in \mathcal{F}\Big\}\\ &= \bigcup_{Y \in \mathcal{P}_\alpha(X)} \bigcap_{\mathfrak{f} \in \mathcal{F}} \mathfrak{f}^{-1}\big((-\infty, \mathbf{E}\,\mathfrak{f}(Y)]\big)\,,\end{aligned}$$

where $\mathcal{P}_\alpha(X)$ is defined in Section 4.3 as the set of all random vectors Y whose probability distribution satisfies $\mathbf{P}\{Y \in B\} \leq \alpha^{-1}\mathbf{P}\{X \in B\}$ for each Borel set $B \subset \mathbb{R}^d$.

All families of integral trimmed regions satisfy **D3**. Further, if for any $\mathfrak{f} \in \mathcal{F}$, $t > 0$ and $z \in \mathbb{R}^d$, the function $\mathfrak{f}_{t,z}$ defined as $\mathfrak{f}_{t,z}(x) = \mathfrak{f}(tx+z)$ belongs to $\mathcal{F}$, then the integral trimmed regions generated by $\mathcal{F}$ satisfy properties **D1** and **D2**.

If $\mathcal{F} = \{\mathfrak{f}_{t,z} : t > 0, z \in \mathbb{R}^d\}$ for a decreasing (with respect to the coordinatewise order) and continuous function $\mathfrak{f}$, then

$$\mathrm{D}^\alpha_{\mathcal{F}}(X) = \bigcup_{Y \in \mathcal{P}_\alpha(X)} \bigcap_{t>0,\, z \in \mathbb{R}^d} \frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY+z)\big) - z\Big) + \mathbb{R}^d_+\,. \tag{5.2}$$

Notice that in Section 4.3 risk measures were build as the infimum of the risks associated to random vectors in $\mathcal{P}_\alpha(X)$ and in the framework of set-valued risk measures this infimum would become the intersection. The reason we take a union running over $\mathcal{P}_\alpha(X)$ for the integral trimming will become clear from (7.1), which generates risk measures from integral trimmed regions.

Hereafter we will assume that all depth-trimmed regions satisfy **D0**–**D5**.

6 Risk measures generated by depth-trimmed regions

As a motivation for the following, note that for a random variable X, $\alpha \in (0, 1/2]$ and $n \geq 1$, we have

$$\begin{aligned} \mathrm{V@R}_\alpha(X) &= -\min \mathrm{HD}^\alpha(X)\,,\\ \mathrm{ES}_\alpha(X) &= -\min \mathrm{ZD}^\alpha(X)\,,\\ \mathrm{EM}_{1/n}(X) &= -\min \mathrm{CD}^{1/n}(X)\,. \end{aligned}$$

The following example provides another argument showing relationships between depth-trimmed regions and risk measures.

Example 6.1 (Depth-trimmed regions as set-valued risk measures). Observe that any depth-trimmed region that satisfies **D0**–**D5** can be easily transformed into a set-valued risk measure from Definition 2.1. It is easy to show that $\mathfrak{r}(X) = \mathfrak{D}^\alpha(X) + \mathbb{R}^d_+$ it is a risk measure in the cone $\mathbb{C}$ of closed subsets of $\mathbb{R}^d$ with the addition operation being the closure of the Minkowski addition and the reversed inclusion order. The functions f and g are given by $f(x) = g(x) = x + \mathbb{R}^d_+$.

In our framework, a random portfolio X will be acceptable or not depending on the depth-trimmed region of level α associated with X. Since the depth-trimmed regions are subsets of the space $\mathbb{R}^d$ where X takes its values, we need to map it into the space $\mathbb{C}$ where risk measures take their values. This map is provided by the function f from Definition 2.1. Then

$$\mathfrak{d}^\alpha(X) = f(\mathfrak{D}^\alpha(X))$$

is a subset of $\mathbb{C}$. Recall that the acceptance cone $\mathbb{A}$ is a subset of $\mathbb{C}$ that characterises the acceptable values of the risk measure.

Definition 6.2. *The* acceptance set at level α *associated with the depth-trimmed region* $\mathfrak{D}^\alpha(\cdot)$ *and function* f *is defined as*

$$\mathcal{A}_\alpha^{\mathfrak{d}} = \{X \in \mathcal{L}_d^\infty : \mathfrak{d}^\alpha(X) \subset \mathbb{A}\}.$$

In the simplest case, $X = x$ is a degenerated random variable. Then $\mathfrak{d}^\alpha(X) = \{f(x)\}$ and $\{x : f(x) \in \mathbb{A}\}$ are acceptable deterministic values.

Before discussing the main properties of the acceptance set $\mathcal{A}_\alpha^{\mathfrak{d}}$, we will prove an auxiliary result.

Lemma 6.3. *If* $\mathfrak{d}^\alpha(X) + f(0) \subset \mathbb{A}$*, then* $X \in \mathcal{A}_\alpha^{\mathfrak{d}}$*.*

Proof. By Lemma 2.2, $f(x) + f(0) \preceq f(x)$ for all $x \in \mathfrak{D}^\alpha(X)$. Since $f(x) + f(0) \in \mathbb{A}$ and $\mathbb{A}$ is upper with respect to $\preceq$, we have $f(x) \in \mathbb{A}$. Thus $\mathfrak{d}^\alpha(X) \subset \mathbb{A}$. □

Theorem 6.4. *The acceptance sets associated with depth-trimmed regions satisfy the following properties:*

(i) $0 \in \mathcal{A}_\alpha^{\mathfrak{d}}$ *for all* α*;*

(ii) if $\alpha \geq \beta$*, then* $\mathcal{A}_\beta^{\mathfrak{d}} \subset \mathcal{A}_\alpha^{\mathfrak{d}}$*;*

(iii) if $X \in \mathcal{A}_\alpha^{\mathfrak{d}}$*, then* $tX \in \mathcal{A}_\alpha^{\mathfrak{d}}$ *for all* $t > 0$*;*

(iv) if $X \in \mathcal{A}_\alpha^{\mathfrak{d}}$ *and* $\mathfrak{r}(x) = f(x) \in \mathbb{A}$*, then* $x + X \in \mathcal{A}_\alpha^{\mathfrak{d}}$*;*

(v) if $Y \in \mathcal{A}_\alpha^{\mathfrak{d}}$ *and* $Y \leq X$ *a.s., then* $X \in \mathcal{A}_\alpha^{\mathfrak{d}}$*;*

(vi) if $X, Y \in \mathcal{A}_\alpha^{\mathfrak{d}}$ *and* ***D6*** *holds, then* $X + Y \in \mathcal{A}_\alpha^{\mathfrak{d}}$*.*

Proof. (i) By (3.1), $f(0) \in \mathbb{A}$, while **D0** implies $\mathfrak{d}^\alpha(0) = \{f(0)\}$. Thus $0 \in \mathcal{A}_\alpha^{\mathfrak{d}}$ for all α.
(ii) By **D3**, $\mathfrak{D}^\alpha(X) \subset \mathfrak{D}^\beta(X)$ whenever $\alpha \geq \beta$. Thus $\mathfrak{d}^\alpha(X) \subset \mathfrak{d}^\beta(X)$ and $\mathcal{A}_\beta^{\mathfrak{d}} \subset \mathcal{A}_\alpha^{\mathfrak{d}}$ trivially holds.
(iii) By **D2** and the homogeneity of f, we have $\mathfrak{d}^\alpha(tX) = t\mathfrak{d}^\alpha(X)$ for all $t > 0$. Since $\mathbb{A}$ is a cone $\mathfrak{d}^\alpha(tX) \subset \mathbb{A}$ if and only if $\mathfrak{d}^\alpha(X) \subset \mathbb{A}$.
(iv) Let $\mathfrak{r}(x) \in \mathbb{A}$. By **D1**, we have $\mathfrak{d}^\alpha(X + x) = f(\mathfrak{D}^\alpha(X) + x)$ and by (2.5), we have $\mathfrak{d}^\alpha(X + x) + f(0) = \mathfrak{d}^\alpha(X) + \mathfrak{r}(x) \subset \mathbb{A}$ because $\mathbb{A}$ is a (convex) cone. By Lemma 6.3, $X + x \in \mathcal{A}_\alpha^{\mathfrak{d}}$.
(v) Note that $f(\mathfrak{D}^\alpha(Y)) \subset \mathbb{A}$. Since $\mathbb{A}$ is upper, $f(\mathfrak{D}^\alpha(Y) + \mathbb{R}_+^d) \subset \mathbb{A}$. By **D5**, $f(\mathfrak{D}^\alpha(X)) \subset \mathbb{A}$.
(vi) If **D6** holds, then $\mathfrak{D}^\alpha(X + Y) \subset \mathfrak{D}^\alpha(X) + \mathfrak{D}^\alpha(Y)$, in view of **D2** which is assumed for all depth-trimmed regions. Together with (2.5), this means that

$$\mathfrak{d}^\alpha(X + Y) + f(0) \subset f(\mathfrak{D}^\alpha(X) + \mathfrak{D}^\alpha(Y)) + f(0) = \mathfrak{d}^\alpha(X) + \mathfrak{d}^\alpha(Y) \subset \mathbb{A}\,.$$

Finally, Lemma 6.3 yields that $X + Y \in \mathcal{A}_\alpha^{\mathfrak{d}}$. □

Similarly to the construction used in Section 3, we measure the risk of a portfolio X in terms of the collection of deterministic portfolios x that *cancel* the risk induced by X and make $X + x$ acceptable.

Definition 6.5. *The risk measure induced by a family of depth-trimmed regions $\mathfrak{D}^\alpha$ at level α is given by*

$$\mathfrak{s}_\alpha(X) = \bigvee \{f(y) : \ f(\mathfrak{D}^\alpha(X) - y) \subset \mathbb{A}\,, \ y \in \mathbb{R}^d\}\,. \tag{6.1}$$

By **D1**, $\mathfrak{s}_\alpha(X)$ can be given alternatively in terms of the acceptance set at level α as

$$\mathfrak{s}_\alpha(X) = \bigvee \{f(y) : \ X - y \in \mathcal{A}_\alpha^{\mathfrak{d}}\,, \ y \in \mathbb{R}^d\}\,.$$

Theorem 6.6. *The function $\mathfrak{s}_\alpha(X)$ from (6.1) is a homogeneous risk measure, i.e. it satisfies* ***R1*** *and* ***R2*** *and also* ***R3****. If the family of depth trimmed regions satisfies* ***D6****, then $\mathfrak{s}_\alpha(X)$ is a coherent risk measure.*

Proof. By (2.3) and **D1** we deduce that

$$\begin{aligned}
\mathfrak{s}_\alpha(X - y) + g(y) &= \bigvee \{f(z) + g(y) : \ f(\mathfrak{D}^\alpha(X - y) - z) \subset \mathbb{A}\,, \ z \in \mathbb{R}^d\} \\
&= \bigvee \{f(y + z) : \ f(\mathfrak{D}^\alpha(X) - (y + z)) \subset \mathbb{A}\,, z \in \mathbb{R}^d\} \\
&= \mathfrak{s}_\alpha(X)\,,
\end{aligned}$$

so **R1** holds. Furthermore, let $Y \leq X$ a.s. and $y \in \mathbb{R}^d$ such that $Y - y \in \mathcal{A}_\alpha^{\mathfrak{d}}$. Since $Y - y \leq X - y$ a.s., by Theorem 6.4(v), we have $X - y \in \mathcal{A}_\alpha^{\mathfrak{d}}$. Consequently

$$\{y \in \mathbb{R}^d : \ Y - y \in \mathcal{A}_\alpha^{\mathfrak{d}}\} \subset \{y \in \mathbb{R}^d : \ X - y \in \mathcal{A}_\alpha^{\mathfrak{d}}\}$$

and finally $\mathfrak{s}_\alpha(Y) \preceq \mathfrak{s}_\alpha(X)$.

Property **R3** follows directly from Theorem 6.4(iii) and the homogeneity of f:

$$\begin{aligned}
\mathfrak{s}_\alpha(tX) &= \bigvee \{f(y) : \ tX - y \in \mathcal{A}_\alpha^{\mathfrak{d}}\,, \ y \in \mathbb{R}^d\} \\
&= \bigvee \{f(y) : \ X - t^{-1}y \in \mathcal{A}_\alpha^{\mathfrak{d}}\,, \ y \in \mathbb{R}^d\} \\
&= \bigvee \{f(ty) : \ X - y \in \mathcal{A}_\alpha^{\mathfrak{d}}\,, \ y \in \mathbb{R}^d\} \\
&= t\mathfrak{s}_\alpha(X).
\end{aligned}$$

If **D6** holds, Theorem 6.4(vi) implies that $X + Y - (x + y) \in \mathcal{A}_\alpha^{\mathfrak{d}}$ whenever $X - x, Y - y \in \mathcal{A}_\alpha^{\mathfrak{d}}$, $x, y \in \mathbb{R}^d$. Consequently using **D2**

$$\{x \in \mathbb{R}^d : \ X - x \in \mathcal{A}_\alpha^{\mathfrak{d}}\} + \{y \in \mathbb{R}^d : \ Y - y \in \mathcal{A}_\alpha^{\mathfrak{d}}\} \subset \{z \in \mathbb{R}^d : \ X + Y - z \in \mathcal{A}_\alpha^{\mathfrak{d}}\}\,,$$

whence $\mathfrak{s}_\alpha(X) + \mathfrak{s}_\alpha(Y) \preceq \mathfrak{s}_\alpha(X + Y)$, which together with **R3** yield **R4**. $\square$

Theorem 6.7. *The risk measure induced by a family of depth-trimmed regions satisfies*

$$\mathfrak{s}_\alpha(X) = f(0) + \bigvee \{a \in \mathbb{G} : \ \mathfrak{d}^\alpha(X) \subset \mathbb{A} + a\} .$$

If $\mathbb{A} = \{a \in \mathbb{C} : \ f(0) \preceq a\}$, $f(0) = \mathbf{e}$, $\mathbb{G}$ is sup-generating and $\mathbb{C}$ is inf-closed (i.e. closed for $\wedge$, see [11]), then

$$\mathfrak{s}_\alpha(X) = \bigwedge \mathfrak{d}^\alpha(X) .$$

Proof. It is easy to see that $f(x-y) = f(x) + g(-y)$. Then

$$\begin{aligned}\mathfrak{s}_\alpha(X) &= \bigvee \{f(y) : \ f(\mathfrak{D}^\alpha(X)) + g(-y) \subset \mathbb{A}\} \\ &= f(0) + \bigvee \{g(y) : \ f(\mathfrak{D}^\alpha(X)) \subset \mathbb{A} + g(y)\} .\end{aligned}$$

For the second statement, note that

$$\begin{aligned}\mathfrak{s}_\alpha(X) &= f(0) + \bigvee \{g(y) : \ c \in \mathbb{A} + g(y) \ \text{ for all } \ c \in \mathfrak{d}^\alpha(X)\} \\ &= \bigvee \{g(y) : \ f(0) + g(y) \preceq c \ \text{ for all } \ c \in \mathfrak{d}^\alpha(X)\} \\ &= \bigvee \{a \in \mathbb{G} : \ a \preceq \bigwedge \mathfrak{d}^\alpha(X)\} .\end{aligned}$$

Then the statement follows from Theorem 3.3. □

7 Basic risk measures associated with depth-trimmed regions

Let us now specialise the constructions from Section 6 for several basic definitions of depth-trimmed regions. Let $X = (X_1, X_2, \dots, X_d) \in \mathcal{L}_d^\infty$. We give only constructions of set-valued risk measures with values in the cone $\mathbb{C}$ described in Example 3.4. Recall that A is a convex cone in $\mathbb{R}^d$ such that $\mathbb{R}_+^d \subset A$, $\mathbb{C}$ is the family of closed convex sets F such that $F = F + A$, and the acceptance cone is $\mathbb{A} = \{F \in \mathbb{C} : \ A \subset F\}$. The acceptance cone $\mathbb{A}$ is compatible with the inclusion, since

$$A + \bigvee \{a + A : \ a \in \mathbb{R}^d, \ F \supset A + a\} = F$$

for all $F \in \mathbb{C}$. Indeed, it suffices to check that every such F is the union of $a + A$ for all a such that $a + A \subset F$. By Theorem 6.7, $\mathfrak{s}_\alpha(X)$ is the intersection of all sets from $\mathfrak{d}^\alpha(X)$. Thus, $\mathfrak{s}_\alpha(X)$ is a subset of $\mathbb{R}^d$ such that $\mathfrak{s}_\alpha(X) + A = \mathfrak{s}_\alpha(X)$.

Risk measures generated by halfspace trimming. The halfspace trimming induces homogeneous risk measures, i.e. risk measures that satisfy **R3**. This set-valued risk measure is given by $\mathfrak{s}_\alpha(X_1) = \big[\mathrm{V@R}_\alpha(X_1), +\infty\big)$ in the univariate case. In general,

$$\begin{aligned}\mathfrak{s}_\alpha(X_1, X_1, \dots, X_1) =& \big\{\big(\mathrm{V@R}_\alpha(X_1), \mathrm{V@R}_\alpha(X_1), \dots, \mathrm{V@R}_\alpha(X_1)\big)\big\} + A , \\ \mathfrak{s}_\alpha(X) \supset& \big\{\big(\mathrm{V@R}_\alpha(X_1), \mathrm{V@R}_\alpha(X_2), \dots, \mathrm{V@R}_\alpha(X_d)\big)\big\} + A .\end{aligned}$$

Risk measures generated by zoonoid trimming. The zonoid trimming induces coherent risk measures. Then $\mathfrak{s}_\alpha(X_1) = \big[\mathrm{ES}_\alpha(X_1), +\infty\big)$ and in the multivariate setting one has

$$\begin{aligned} \mathfrak{s}_\alpha(X_1, X_1, \dots, X_1) &= \big\{\big(\mathrm{ES}_\alpha(X_1), \mathrm{ES}_\alpha(X_1), \dots, \mathrm{ES}_\alpha(X_1)\big)\big\} + A\,, \\ \mathfrak{s}_\alpha(X) &\supset \big\{\big(\mathrm{ES}_\alpha(X_1), \mathrm{ES}_\alpha(X_2), \dots, \mathrm{ES}_\alpha(X_d)\big)\big\} + A\,, \end{aligned}$$

where equality holds if $A = \mathbb{R}^d_+$.

Risk measures generated by expected convex hull trimming. The expected convex hull trimming induces coherent risk measures. Then $\mathfrak{s}_{1/n}(X_1) = \big[\mathrm{EM}_{1/n}(X_1), +\infty\big)$ and

$$\begin{aligned} \mathfrak{s}_{1/n}(X_1, X_1, \dots, X_1) &= \big\{\big(\mathrm{EM}_{1/n}(X_1), \mathrm{EM}_{1/n}(X_1), \dots, \mathrm{EM}_{1/n}(X_1)\big)\big\} + A\,, \\ \mathfrak{s}_{1/n}(X) &\supset \big\{\big(\mathrm{EM}_{1/n}(X_1), \mathrm{EM}_{1/n}(X_2), \dots, \mathrm{EM}_{1/n}(X_d)\big\} + A\,, \end{aligned}$$

where equality holds if $A = \mathbb{R}^d_+$.

Integral trimmed risk measures. The integral trimmed regions generate new multivariate risk measures. If $\mathcal{F} = \{\mathfrak{f}(tx + z) :\ t > 0,\ z \in \mathbb{R}^d\}$, where $\mathfrak{f}$ is continuous and decreasing and $A = \mathbb{R}^d_+$, then

$$\mathfrak{s}_\alpha(X) = - \inf_{Y \in \mathcal{P}_\alpha(X)} \sup_{t>0,\, z \in \mathbb{R}^d} \frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY + z)\big) - z\Big) + \mathbb{R}^d_+\,. \tag{7.1}$$

This risk measure appears from the worst conditioning construction applied to the risk measure

$$\mathfrak{s}_1(X) = \inf_{t>0,\, z \in \mathbb{R}^d} \frac{1}{t}\Big(- \mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tX + z)\big) + z\Big) + \mathbb{R}^d_+\,. \tag{7.2}$$

This risk measure satisfies **R1**–**R3** and results from the reversed order homogenisation construction analogous to (4.1) and (4.2) applied to the risk measure

$$\mathfrak{r}(X) = -\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(X)\big) + \mathbb{R}^d_+\,. \tag{7.3}$$

Notice that this homogenisation would also preserve **R2**, but not **R4**. The idea of constructing scalar risk measures using real-valued functions of vector portfolios appears also [4]. Alternatively, it is possible to take infimum in (7.2) over $t > 0$ or over $z \in \mathbb{R}^d$ only, which results in a risk measure that satisfies **R3** or **R1** respectively.

Example 7.1. The function $\mathfrak{f}(t) = e^{-t/\gamma}$ yields the risk measure $\mathfrak{r}(X) = \gamma \log(\mathbf{E} e^{-X/\gamma})$ by (7.3) in $\mathbb{C} = \mathbb{R}$ with the reversed order and $f(x) = g(x) = -x$. The properties **R1** and **R2** evidently holds, while (2.1) follows from the Hölder inequality, i.e. $\mathfrak{r}$ is a convex risk measure, which does not satisfy **R3**. Since **R1** already holds, there is no need to take infimum over $z \in \mathbb{R}^d$ in (7.2). The corresponding convex risk measure is called the *entropic risk measure* with γ being the risk tolerance coefficient.

If we attempt to produce a homogeneous (and thereupon coherent) risk measure from $\mathfrak{r}$, we need to apply (4.1), which in view of the reversed order on the real line turns into

$$\mathfrak{r}_{\mathrm{h}}(X) = \sup_{t>0} \, t^{-1}\,\mathfrak{r}(tX) = \sup_{t>0} \, t^{-1} \, \log(\mathbf{E}e^{-tX}) = \sup_{t>0} \, \log((\mathbf{E}Y^t)^{1/t})$$

for $Y = e^{-X}$. Since $(\mathbf{E}Y^t)^{1/t}$ is an increasing function of $t > 0$, we have

$$\mathfrak{r}_{\mathrm{h}}(X) = \lim_{t\to\infty} \frac{\log(\mathbf{E}e^{-tX})}{t} \,.$$

It is easy to see that the limit equals $(-\operatorname{essinf} X)$, so a coherent variant of $\mathfrak{r}$ is not particularly interesting.

Vector-valued risk measures derived from their set-valued counterparts. The cone $A \supset \mathbb{R}^d_+$ generates a partial order on $\mathbb{R}^d$ by setting $y \preceq x$ if and only if $y \in x + A$. By taking the infimum of any set-valued risk measure $\mathfrak{s}_\alpha$ with respect to this partial order we arrive at a vector-valued risk measure, see Example 4.7. Notice that this vector-valued risk measure has values in $\mathbb{C} = \mathbb{R}^d$ with the reversed ordering to the one generated by A and $f(x) = g(x) = -x$.

Alternatively, a vector-valued risk measure is constructed by taking $\mathbb{C} = \mathbb{R}^d$ with the vector addition and the partial order generated by $\mathbb{A} = A$. If $f(x) = g(x) = x$ for all x, then $\mathbb{A}$ is compatible with the partial order. The corresponding vector-valued risk measure is generated by a family of depth-trimmed regions as

$$\mathfrak{s}_\alpha(X) = \bigvee \{y \in \mathbb{R}^d : \; \mathfrak{D}^\alpha(X) \subset A + y\} \,.$$

It is easy to confirm that the maximum is well-defined. For this, it suffices to consider the case $\mathfrak{D}^\alpha(X) = \{0\}$. Then $\vee\{-y : \; y \in A\}$ is not greater than 0, since $0 \in A - y$ for all $y \in A$. This risk measure equals the previous vector-valued risk measure, where we have reversed the signs and thus the ordering.

8 Depth-trimmed regions generated by risk measures

Consider a family of homogeneous risk measures $\mathfrak{r}_\alpha$ for $\alpha \in (0, 1]$ such that

$$\mathfrak{r}_\alpha \succeq \mathfrak{r}_\beta \,, \quad \alpha \geq \beta \,, \tag{8.1}$$

which are associated with functions f and g according to **R1**. Such family of risk measures can be generated using the worst conditioning construction from Section 4.3.

Definition 8.1. *The* depth-trimmed regions generated by the family of risk measures *are defined as*

$$\mathfrak{D}^\alpha(X) = \{y \in \mathbb{R}^d : \; \mathfrak{r}_\alpha(X - y) \preceq f(0)\} \,.$$

Notice that for any given risk measure, functions f and g are implicit. We recall that function f is the risk associated to a degenerate random variable and function g controls the translations of risks. The depth-trimmed regions generated by a family of risk measures are alternatively given by

$$\begin{aligned}\mathfrak{D}^\alpha(X) &= \{y \in \mathbb{R}^d :\ \mathfrak{r}_\alpha(X) + g(-y) \preceq f(0)\} \\ &= \{y \in \mathbb{R}^d :\ \mathfrak{r}_\alpha(X) \preceq f(y)\}\,. \end{aligned} \tag{8.2}$$

Recall that a function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is called *upper semicontinuous* if $\{x \in \mathbb{R}^d :\ a \preceq f(x)\}$ is closed for every $a \in \mathbb{C}$.

Theorem 8.2. *The depth-trimmed regions generated by a family of risk measures satisfy*

*(i) properties **D1**, **D2**, **D3** and **D5**;*

(ii) are convex if the risk measure is convex;

(iii) are closed if f is upper semicontinuous.

Proof. *(i)* Properties **D1** and **D2** trivially hold by **R1** and **R3** respectively. Meanwhile the nesting property of depth-trimmed regions, **D3**, is a consequence of (8.1). We will show that **D5** follows from **R2**. If $Y \le X$ a.s., then **R2** yields that $\mathfrak{r}_\alpha(Y) \preceq \mathfrak{r}_\alpha(X)$. Then

$$\{y \in \mathbb{R}^d : \mathfrak{r}_\alpha(Y) \preceq f(y)\} \supset \{y \in \mathbb{R}^d : \mathfrak{r}_\alpha(X) \preceq f(y)\}$$

and by (8.2) we have $\mathfrak{D}^\alpha(Y) \supset \mathfrak{D}^\alpha(X)$ and finally $\mathfrak{D}^\alpha(X) \subset \mathfrak{D}^\alpha(Y) + \mathbb{R}^d_+$.
(ii) Given $y, z \in \mathfrak{D}^\alpha(X)$ and $t \in [0, 1]$, by Lemma 2.2

$$\mathfrak{r}_\alpha(X) \preceq tf(y) + (1-t)f(z) \preceq f(ty + (1-t)z)$$

and finally $ty + (1-t)z \in \mathfrak{D}^\alpha(X)$.
(iii) If f is upper semicontinuous, the set $\{y \in \mathbb{R}^d :\ \mathfrak{r}_\alpha(X) \preceq f(y)\}$ is closed. □

Under mild conditions, it is possible to recover a risk measure from the depth-trimmed regions it generated.

Definition 8.3. *If $\mathbb{F} \subset \mathbb{C}$, we say that $\mathbb{F}$ is an* inf-generating family *[11] if*

$$b = \bigwedge\{a \in \mathbb{F} :\ b \preceq a\}, \quad \textit{for all}\, b \in \mathbb{C}\,.$$

In order to recover a risk measure from the depth-trimmed regions it generated, it is convenient that $g = f$ and thus $\mathbb{G} = \mathbb{F}$, further such a set should be sup-generating and inf-generating.

If $\mathbb{A} = \{a \in \mathbb{C} :\ f(0) \preceq a\}$, $f(0) = \mathbf{e}$, $\mathbb{G} = \mathbb{F}$ is sup-generating and inf-generating and $\mathbb{C}$ is inf-closed, then by Theorem 6.7, equation (8.2) and Definition 8.3, we have

$$\begin{aligned}\mathfrak{s}_\alpha(X) &= \bigwedge \mathfrak{d}^\alpha(X) \\ &= \bigwedge\{f(y) :\ \mathfrak{r}_\alpha(X) \preceq f(y)\} \\ &= \mathfrak{r}_\alpha(X)\,.\end{aligned}$$

Example 8.4 (Expected convex hull trimming revisited). The expected minimum can be formulated as a *spectral risk measure*, see [1], in terms of the expression

$$\mathrm{EM}_{1/n}(X) = -\int_0^1 n(1-t)^{n-1} F_X^{-1}(t)dt\,, \tag{8.3}$$

where $n \geq 1$. For any $\alpha \in (0,1]$, define $\mathrm{EM}_\alpha(X)$ substituting n by α^{-1} in (8.3). The risk measures EM_α generates a family of depth-trimmed regions for $X \in \mathcal{L}_1^\infty$ with a continuous parameter $\alpha \in (0,1]$. Applying Definition 8.1, we obtain $\mathfrak{D}^\alpha(X) = [-\mathrm{EM}_\alpha(X), +\infty)$.

Depth-trimmed regions treat all directions in the same way, and the regions $\mathfrak{D}^\alpha$ must be slightly modified so that they coincide with the expected convex hull trimmed regions when $\alpha = 1/n$. Define

$$\begin{aligned}
\mathrm{CD}^\alpha(X) &= \mathfrak{D}^\alpha(X) \,\cap\, \big(-\mathfrak{D}^\alpha(-X)\big) \\
&= [-\mathrm{EM}_\alpha(X)\,,\, \mathrm{EM}_\alpha(-X)] \\
&= \left[\alpha^{-1}\int_0^1 (1-t)^{\alpha^{-1}-1} F_X^{-1}(t)dt\,,\, \alpha^{-1}\int_0^1 t^{\alpha^{-1}-1} F_X^{-1}(t)dt\right].
\end{aligned}$$

In this formulation, we can assume that the parameter α takes any value in $(0,1]$ and thus, we obtain an extension of the univariate expected convex hull trimmed regions.

9 Duality results

The dual space to $\mathcal{L}_d^\infty$ is the family of finitely additive bounded *vector* measures $\mu = (\mu_1,\dots,\mu_d)$, which act on $X \in \mathcal{L}_d^\infty$ as $\mathbf{E}_\mu(X) = \mathbf{E}_{\mu_1}(X_1) + \cdots + \mathbf{E}_{\mu_d}(X_d)$. Let $|\mu_1|,\dots,|\mu_d|$ be the total mass of $\mu_1,\dots,\mu_d$ respectively. The polar set to the cone of acceptable portfolios can be written as

$$\mathcal{A}^* = \bigcap_{X\in\mathcal{A}} \{\mu :\, \mathbf{E}_\mu(X) \geq 0\}\,.$$

As in [14], we can apply the bipolar theorem to show that

$$\mathcal{A} = \bigcap_{\mu\in\mathcal{A}^*} \{X :\, \mathbf{E}_\mu(X) \geq 0\}\,.$$

Since all random vectors taking values in $\mathbb{R}_+^d$ are acceptable, all elements in $\mathcal{A}^*$ are positive, i.e. $\mu(F) \in \mathbb{R}_+^d$ for all Borel $F \subset \mathbb{R}^d$. Thus, a set-valued risk measure with values in the cone of convex closed sets in $\mathbb{R}^d$ with the inclusion order can be represented as

$$\mathfrak{r}(X) = \bigcap_{\mu\in\mathcal{P}} \{x \in \mathbb{R}^d :\; \langle \mu, x\rangle \geq \mathbf{E}_\mu(-X)\}\,,$$

where $\langle \mu, x\rangle = |\mu_1|x_1 + \cdots + |\mu_d|x_d$ and $\mathcal{P}$ is a certain set of finitely additive bounded vector measures with positive total masses. If the risk measure satisfies the Fatou property, then all measures from $\mathcal{P}$ can be chosen to be σ-additive. Recall that the Fatou property means that the risk measure is lower semicontinuous in probability, i.e. the lower limit (in the Painlevé-Kuratowski sense [18, Def. B.5]) of $\mathfrak{r}(X_k)$ is not greater than $\mathfrak{r}(X)$ if X_k converges in probability to X.

By applying to $\mathfrak{r}$ the worst conditioning construction, we obtain

$$\begin{aligned}\mathfrak{r}_\alpha(X) &= \bigwedge_{Y \in \mathcal{P}_\alpha(X)} \bigwedge_{\mu \in \mathcal{P}} \{x \in \mathbb{R}^d : \langle \mu, x \rangle \geq \mathbf{E}_\mu(-Y)\} \\ &= \bigwedge_{\mu \in \mathcal{P}} \left\{x \in \mathbb{R}^d : \langle \mu, x \rangle \geq (-\mathbf{E}_\mu)_\alpha(X)\right\} ,\end{aligned}$$

where

$$(-\mathbf{E}_\mu)_\alpha(X) = (-\mathbf{E}_{\mu_1})_\alpha X_1 + \cdots + (-\mathbf{E}_{\mu_d})_\alpha X_d \,.$$

Thus $\mathfrak{r}_\alpha$ also admits the dual representation, where instead of the expectation $\mathbf{E}_\mu(-X)$ we take the expected shortfall of X with respect to the measure μ. Definition 8.1 yields then a dual representation for the family of depth-trimmed regions.

10 Conclusions

It is likely that further results from the morphological theory of lattices [11] have applications in the framework of risk measures. In particular, it would be interesting to find an interpretation for dilation mappings that commute with supremum, erosions that commute with infimum, and pairs of erosions and dilations that are called adjunctions.

It is possible to consider a more general ordering for multivariate portfolios. Instead of designating $X \leq Y$ in case X is less than Y coordinatewisely, it is possible to assume that X is less than Y if TX is coordinatewisely less than TY for a certain linear transformation T. From the economical viewpoint, this linear transformation would correspond, e.g. to exchanges of various currencies, cf [15].

It is possible to consider a variant of **R2** where $Y \leq X$ is understood with respect to any other chosen order on $\mathcal{L}_d^\infty$. The consistency issues for scalar risk measures for vector portfolios are investigated in [4] and in [3] for the one-dimensional case.

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