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DEBT REFINANCING AND CREDIT RISK

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Abstract

Many firms choose to refinance their debt. We investigate the long run effects of this extended practice on credit ratings and credit spreads. We find that debt refinancing generates systematic rating downgrades unless a minimum firm value growth is observed. Deviations from this growth path imply asymmetric results: A lower value growth generates downgrades and a higher value growth upgrades as expected. However, downgrades will tend to be higher in absolute terms. On the other hand, credit spreads will be independent of the risk free interest rate in the short run, but positively correlated with this rate in the long run.

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1 Introduction:

Many different reasons may be behind the decision of a particular firm to issue new debt: Financing a new investment project, getting founds to operate in a period of low earnings, or simply refinancing existing debt. The purpose of the issue is not irrelevant. An example is provided by Gande et al. (1997), who examine differences in debt securities underwritten by Section 20 subsidiaries of bank holding companies relative to those underwritten by investment houses. Among other results, they find that when debt is used to refinance existing debt, the credit spread is on average 14 basis points above the one that results considering "other purposes". Intuitively, if the purpose of the issue is to finance a new investment project that will increase the expected earnings of the firm, and its market value, the risk premium should be lower than in the case in which debt proceeds are used to refinance existing debt, because in this situation no added value is created.[1] Refinancing current debt on the other hand seems to be one of the most important, if not the first, reason to issue new debt. The mentioned article for instance considers a sample in which 43.5% of the issues had the purpose to refinance existing debt. More evidence in this line is given by Hansen and Crutchley (1990), who investigate the relationship between corporate earnings and sales of common stock, convertible bonds, and straight bonds. In this case, 64% of straight bond issues were used at least partially to refinance existing debt. This ratio grows up to 72% when they consider convertible debt.

In spite of the fact that debt refinancing appears as an extending practice, we know little about how this can potentially affect the credit standing of a firm in the long run. The present article represents a first attempt in this direction. We introduce the concept of *refinancing contract*, modeling dividend rates, maturities, and nominal debt payments, as part of this contract. We then describe the credit spreads faced by the firm to refinance as a function of the firm characteristics and the specific contract selected, and

[1] The theoretical argument given by Gande et al. (1997) to justify different credit spreads depending on the purpose of the issue is nevertheless not the same we provide here. They argue that when a firm has a bank loan, and wants to refinance it with public debt, potential buyers may expect that the firm has been induced by the bank to take this decision because the loan is at risk. However, significant differences are found even when new issues are classified as "investment grade". This indicates that debt refinancing is not a practice that firms use only in case of, or to avoid, credit distress.

analyze how the fact that firms choose to refinance their debt can potentially affect the credit rating and the credit spreads of those firms in the long run. We find that debt refinancing generates systematic credit rating downgrades unless a minimum firm value growth is observed. Deviations from such a firm value growth path imply on the other hand asymmetric results: While a lower value growth results in downgrades and a higher value growth results in upgrades as expected, the same deviation will have a higher effect in absolute terms when it is negative than when it is positive. This means that we will expect that among those firms that choose to refinance their debt, downgrades will tend to be stronger than upgrades. We also predict that credit spreads will not be affected by variations in the risk free interest rate in the short run, but will be positively correlated with this rate in the long run.

The rest of the article is organized as follows: Section 2 introduces the concept of *refinancing contract*, and describes when, and how, a contract of this type with an arbitrary number of future payment dates n, can be designed. Section 3 analyzes the effects of debt refinancing under the specific cases of $n = 1$ and $n = 2$.[2] Finally section 4 summarizes the main findings of the paper.

2 The General Case

We will assume the following theoretical framework:

A1: There are no taxes, problems concerning indivisibility, bankruptcy costs, transactions costs, or agency costs.

A2: Trading takes place continuously.

A3: There exits a risk free asset with constant interest rate r, that applies for borrowing and lending, and for any maturity.

A4: Every individual acts as if she can buy or sell as much of any security as she wishes without affecting the market price.

A5: Individuals may take short positions in any security, including the risk free asset, and receive the proceeds of the sale. Restitution is required for payouts made to securities held short.

[2] This last case can be seen as a simplification to short term debt and long term debt.

A6: Modigliani-Miller Theorem obtains, that is, the firm value is independent of its capital structure.

A7: The firm value, V, follows the diffusion process given by

$$dV = (\mu - \delta) V dt + \sigma V dz \tag{1}$$

where μ is the expected rate of return on V, δ is the constant rate of firm value which is paid to equity holders as dividends, σ is the volatility of the rate of return which will be assumed to be constant, and z is a standard Brownian motion.

No assumption is made at this moment about the profile of nominal payments that constitute the corporate debt. We simply assume that a debt contract was signed at some period prior to current period t. Under this contract, at least a certain debt payment has to be satisfied at some future period $\tau > t$. This, and any posterior debt payment, is to be financed by issuing additional equity. Under these conditions the equity and debt values will be a function of the firm value and time. Denote then the equity value as $S(V, t)$, and the debt value as $F(V, t)$.

We start by defining the general form of any refinancing contract.

Definition 1: *A refinancing contract between the firm and the debt holders at τ, is a vector $\Theta \equiv (\delta, \Psi, \Upsilon) \in \Re \times \Re^n \times \Re^n$, with $n < \infty$, by which:*

a) The firm, which is assumed to maximize equity holders' wealth, promises (under limited liability) the payment of Ψ at Υ, that is, the payment of ψ_i at τ_i, where $\psi_i \in \Psi$, $\tau_i \in \Upsilon$, $i = 1, ..., n$, and $\tau_1 > \tau$.

b) The firm also restricts itself to apply a dividend rate equal to δ, and loses the right to issue new debt. These restrictions apply until Θ has been canceled, either by satisfying nominal payments regularly (issuing new equity), or by means of a posterior debt refinancing contract.

c) The debt holders renounce to $F(V, \tau)$.

We say that Θ is feasible, if and only if the firm and the debt holders are willing to sign Θ. The set of feasible Θ is denoted by Θ^F.

A refinancing contract (RC) is therefore similar to a standard debt contract. The main difference is that debt holders do not provide cash to the firm at issuance, but the renounce to the payment of current debt (covenant *c)*). In addition, we include an agrement on dividends (covenant *b)*). This agrement prevent equity holders from extracting a higher share of the firm value (with the implied reduction for debt holders), by increasing the dividend rate after signing the contract.

The following lemma establishes a necessary condition for a feasible set of refinancing contracts to exist.

Lemma: *Let $S(V, \Theta, \tau)$ and $F(V, \Theta, \tau)$ denote the equity and debt value at τ when the value of the firm is V, the debt profile consists on the payment of Ψ at Υ, and the dividend rate is δ. Then, $\Theta \in \Theta^F$ if and only if $S(V, \Theta, \tau) = S(V, \tau)$, implying $S(V, \tau) > 0$ as a necessary condition for a feasible Θ to exist.*

Proof: See appendix.

Although a formal statement of the proof is in the appendix, the intuition is straightforward: Modigliani and Miller's theorem implies that no value is created nor destroyed in the firm by refinancing its debt. As a consequence, equity holders can neither gain, nor lose due to refinancing. If they are worst off with the contract they will simply refuse it, but if they are better off this means that debt holders are worst off, and in this case they will be who refuse the contract. This allows us to identify the set of feasible refinancing contracts with the set of contracts that leave equity holders with the same value. Limited liability on the other hand makes the equity value to be strictly positive if no current debt payment has to be satisfied, which is the case after signing the contract. This makes $S(V, \tau) > 0$ finally to be a necessary condition for a feasible contract to exists.

One implication is that equity and debt can still be valued assuming that debt will be payed by issuing additional equity. The reason is that the possibility of refinancing will not alter their welfare with respect to this situation in any sense. We set up this argument as follows:

Remark 1: Refinancing does not alter neither equity holders, nor debt holders wealth. This implies that $S(V, \Theta, \tau)$ and $F(V, \Theta, \tau)$ can be valued assuming that debt payments are to be financed by issuing new equity, even if this never happens, that is, even if the firm always chooses to refinance its debt.

Searching for a feasible contract implies at this point searching for $2n+1$ elements. Assumptions B1-B2 below will allow us to reduce the dimension of the problem to 3. Basically, we will impose some restrictions on the relation between debt payments, and on the time spread between these payments. B1-B2 joint with B3 will also describe the behavior of the equity value as a function of these 3 elements. Although this behavior is stated as an assumption, it will be proved to hold later on for the specific cases of $n = 1$ and $n = 2$.

Assumption B1: Let $\Psi = \psi_1 \Phi$, where Φ is the n-dimensional vector which first element ϕ_1 equals 1, and the remaining are some fixed values $\phi_i > 0$ $\forall i \geq 2$. $S(V, \Theta, \tau)$ is then assumed to be a continuous and strictly decreasing function in ψ_1 ($CSD(\psi_1)$), with $S(V, \Theta, \tau) \mid_{\psi_1=0} = V$, and $\lim_{\psi_1 \to \infty} S(V, \Theta, \tau) = E^{R.N} \int_\tau^{\tau_1} \delta V(s) e^{-r(s-\tau)} ds = V\left(1 - e^{-\delta T_1}\right)$.[3]

$\Psi = \psi_1 \Phi$ reflects the ratio between nominal debt payments. In the case of a loan, for instance, we will have $\phi_i = 1$ $\forall i$. B1 asserts that the equity value is a continuous and strictly decreasing function in the nominal payments that equity holders have to satisfy. $S(V, \Theta, \tau) \mid_{\psi_1=0} = V$ recognizes that if there is no debt, then the equity holders own the firm. $\lim_{\psi_1 \to \infty} S(V, \Theta, \tau) = E^{R.N} \int_\tau^{\tau_1} \delta V(s) e^{-r(s-\tau)} ds$ indicates that as nominal debt tends to infinity, default at τ_1 becomes unavoidable, and the unique value associated to equity is the value of the dividends that will be received until the first debt payment is required. Standard arguments allow us to use risk neutral valuation.

Assumption B2: Let $\Pi = (\tau_1 - \tau) \Lambda$, where Π denotes the n-dimensional vector which first element π_1 equals $(\tau_1 - \tau)$, and π_i equals $(\tau_i - \tau_{i-1})$ $\forall i \geq 2$. Λ on the other hand denotes the n-dimensional vector which first element η_1 equals 1, and the remaining are some fixed values $\eta_i > 0$ $\forall i \geq 2$. $S(V, \Theta, \tau)$ is then assumed to be a continuous and strictly increasing function in τ_1 ($CSI(\tau_1)$), with $\lim_{\tau_1 \to \tau} S(V, \Theta, \tau) = Max\left\{0, V - \sum_{i=1}^n \psi_i\right\}$,[4] and $\lim_{\tau_1 \to \infty} S(V, \Theta, \tau) = V$. Denote $\hat{\psi}_1$ the ψ_1 value such that $S(V, \tau) = V - \psi_1 \sum_{i=1}^n \phi_i$, that is, $\hat{\psi}_1 = \frac{F(V,\tau)}{\sum_{i=1}^n \phi_i}$.

[3] $T_1 = \tau_1 - \tau$.

[4] $V - \sum_{i=1}^n \psi_i = V - \psi_1 \sum_{i=1}^n \phi_i$

$\Pi = (\tau_1 - \tau)\Lambda$ describes the time spread between payment dates. If $\eta_i = 1\ \forall i$ for instance, then time between one debt payment and the following is always the same. As τ_1 tends to τ, new corporate debt tends to consist in a single payment satisfied at τ. As τ_1 tends to infinity, the present value of future debt payments, $F(V, \Theta, \tau)$, tends to zero, and the equity value tends to the firm value.

We may denote $\Delta \equiv (n, \Phi, \Lambda)$ the vector that describes the corporate debt structure that results from a given RC Θ, and $\Theta^F \mid \Delta$ the subset of Θ^F that satisfies some given corporate debt structure Δ. Searching for a feasible RC, that is, searching for an element in $\Theta^F \mid \Delta$, reduces now to search for the possible values of δ, ψ_1 and τ_1 that make the equity value to be equal before and after the contract. The following assumption completes the description of the dependence of $S(V, \Theta, \tau)$ on these three elements.

Assumption B3: $S(V, \Theta, \tau)$ is a continuous and strictly increasing function in δ $(CSI(\delta))$, with $\lim_{\delta\to\infty} S(V, \Theta, \tau) = V$.

$\lim_{\delta\to\infty} S(V, \Theta, \tau) = V$ reflects that for any $\tau_1 > \tau$, in the limit case of $\delta = \infty$, the equity holders liquidate the firm before any debt payment can be required. Note also that $S(V, \Theta, \tau)\mid_{\delta=0}$ coincides with the case presented in Geske (1979).

The following definition and the subsequent remark will help us to proceed in the search of an element in $\Theta^F \mid \Delta$.

Definition 2: *The sequence $\alpha - \beta - \gamma$ is an order of choice in δ, ψ_1 and τ_1.*

Remark 2: An order of choice in δ, ψ_1 and τ_1 implies an order of choice in δ, Ψ and Υ, given Δ.

All of the above allows us finally to establish the following theorem:

Theorem: *Suppose $S(V, \tau) > 0$, and let $\varphi^\delta \equiv \Re_+$, $\varphi^{\psi_1} \equiv \left(\hat{\psi}_1, \infty\right)$, $\varphi^{\tau_1} \equiv (\tau, \infty)$. Consider any sequence $\alpha-\beta-\gamma$, where α is chosen in φ^α, and define $\varphi^{\beta|\alpha}$ as the subset of φ^β for which $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\gamma \in \varphi^\gamma$, given α. For any Δ, $\varphi^{\beta|\alpha}$ is a non empty set. Moreover, for any $\alpha \in \varphi^\alpha$, and $\beta \in \varphi^{\beta|\alpha}$, there is only one $\gamma \in \varphi^\gamma$ such that $S(V, \Theta, \tau) = S(V, \tau)$.*

Proof: See appendix.

Corollary 1: $\Theta^F \neq \emptyset$ if and only if $S(V,\tau) > 0$.

Proof: This holds given lemma and theorem above.■

The theorem asserts that, whenever $S(V,\tau) > 0$, a feasible RC with any arbitrary capital structure can be generated, and also describes how can it be constructed. Although this feasible Θ is not unique for a given Δ, and there is actually an infinite number of elements in $\Theta^F \mid \Delta$, not everything is possible. Choosing one element in $\Theta^F \mid \Delta$ could be seen as a matter of priority. Take for instance the sequence $\tau_1 - \delta - \psi_1$: The maturity of the first payment, τ_1, is freely chosen in the interval (τ, ∞), however, this election restricts the range of dividend rates, δ, that can be selected in $[0, \infty)$, and the election of the dividend rate in the restricted interval, finally determines a unique first debt payment, ψ_1, in $\left(\hat{\psi}_1, \infty\right)$. On the other hand, corollary 1 implies that refinancing is feasible under the same conditions it is feasible to issue new equity to pay the debt. Note also that $S(V,t) > 0\ \forall t < \tau$ given limited liability, what would allow the firm to refinance at any $t < \tau$.

We end this section with three additional corollaries that follow directly from the proof of the theorem.

Corollary 2: Suppose $S(V,\tau) > 0$, and fix Δ. Then for a given $(\tau_1, \delta) \in \varphi^{\tau_1} \times \varphi^{\delta}$ such that $\delta T_1 < \ln\left[\frac{V}{F(V,\tau)}\right]$ there exists one, and only one $\psi_1 \in \varphi^{\psi_1}$, such that the RC generated in this way is feasible.

Proof: See appendix.

Corollary 2 establishes a joint restriction on possible values of τ_1 and δ.

Corollary 3: Suppose $S(V,\tau) > 0$, and fix Δ. Then for a given $(\tau_1, \psi_1) \in \varphi^{\tau_1} \times \varphi^{\psi_1}$ such that $S(V,\Theta,\tau)\mid_{\delta=0} \leq S(V,\tau)$, there exists one, and only one $\delta \in \varphi^{\delta}$, such that the RC generated in this way is feasible.

Proof: See appendix.

Corollary 3 this time indicates the joint condition that τ_1 and ψ_1 have to satisfy in any feasible RC. Although the proof of the theorem makes clear that such a pair of values exists in $(\tau_1, \psi_1) \in \varphi^{\tau_1} \times \varphi^{\psi_1}$, no explicit expression is available in this case.

Corollary 4: Suppose $S(V,\tau) > 0$, and fix Δ. Then for a given $(\delta, \psi_1) \in \varphi^{\delta} \times \varphi^{\psi_1}$ there exists one, and only one $\tau_1 \in \varphi^{\tau_1}$, such that the RC generated in this way is feasible.

Proof: See appendix.

Finally, corollary 4 asserts that the election of δ in φ^{δ}, does not restrict the set of ψ_1 values in φ^{ψ_1} that can be selected, and vice versa. This means that for any pair $(\delta, \psi_1) \in \varphi^{\delta} \times \varphi^{\psi_1}$, there will always exist a $\tau_1 \in \varphi^{\tau_1}$ such that the resulting vector (δ, τ_1, ψ_1) represents a feasible RC.

The fact that any possible debt structure can be chosen, joint with the freedom to select any sequence $\alpha - \beta - \gamma$, gives the RC the possibility to be as "imaginative" as one may desire. We may, nevertheless, describe how to design two types of contracts that are commonly used in practice: Loans which are repaid in equal monthly installments, and coupon bonds.

In our notation, the type of loan described can be represented by a vector Δ, where n is equal to the number of years times 12, and Φ as well as Λ, are given by a vector of dimension n with all elements equal to 1. In order to guarantee that we obtain monthly payments, we may start by choosing τ_1: If we fix τ equal to zero, and if we assume that μ and σ are in annual terms, τ_1 equal to $\frac{1}{12}$ generates the monthly payments desired. Corollary 2 finally ensures that for any δ lower than $12 \times \ln\left[\frac{V}{F(V,0)}\right]$, we will be able to find a ψ_1 value in the interval $\left(\frac{F(V,0)}{n}, \infty\right)$, such that the resulting contract allows the firm to refinance. A contract that implies equal monthly installments.

A coupon bond will require a little bit more of elaboration. Clearly, this way to refinance needs the *debt principal* to be equal to $F(V,\tau)$. If coupons are payed annually, then n will be the number of years, and Λ will be a vector of ones of dimension n. Φ this time will be given by a vector of dimension n, with all elements equal to 1 but the element n, which will be equal to $(1+p)$. Consider again current period equals zero, and choose $\tau_1 = 1$ to guarantee annual payments. Again we can use Corollary 2 to ensure that there exits a ψ_1 in the interval $\left(\frac{F(V,0)}{n+p}, \infty\right)$, that allows the firm to refinance for any δ lower than $\ln\left[\frac{V}{F(V,0)}\right]$. According to Φ, we will have equal *coupon* payments between periods 1 and $n-1$, and the payment of coupon plus $\psi_1 p$ at the final date n. But we need a pair of values (ψ_1, p), such that the equality $\psi_1 p = F(V,0)$ holds. As a first step we may prove that such a pair exists. Note that this equality, joint with restriction $\psi_1 > \frac{F(V,0)}{n+p}$, leads to $\psi_1 n > 0$. In short, equity holders could pay $F(V,0)$ at current period zero, or defer this payment to a future period. Condition $\psi_1 n > 0$ simply says that at least a coupon payment ($n > 0$) is needed in order to charge the interests that debt holders will require for this deferment. The problem is that in general, $\psi_1 p$ will not be equal to $F(V,0)$. We could however use the following algorithm: Guess p_0, and evaluate $\psi_1(p_0)$. If $\psi_1 p_0 < F(V,0)$, guess a higher p_1, if $\psi_1 p_0 > F(V,0)$, guess a lower p_1, and repeat up to the point in which $\psi_1 p = F(V,0)$. At the end we will have designed a coupon bond that allows the firm to refinance.

We next analyze the specific cases of $n = 1$ and $n = 2$. These will be useful to provide the basic implications of the refinancing strategy.

3 Particular Cases:

3.1 $n = 1$

Although any possible initial debt structure could be considered, we will assume in this case that n remains constant along time. This means that a single zero coupon bond, with nominal ψ and maturity at τ, is replaced by a single zero coupon bond, with some nominal ψ_1 and some maturity $\tau_1 > \tau$, whenever $S(V,\tau) > 0$.

In order to show that a feasible RC exists, we need to describe how $S(V,\Theta,\tau)$ is to be valued. Specifically, we need to find the equity value at τ, when the corporate debt consists in the payment of ψ_1 at $\tau_1 > \tau$, and the dividend rate is δ, that is, $S(V,\Theta,\tau)$ for $\Theta \equiv (\delta, \psi_1, \tau_1)$.

$S(V,\Theta,\tau)$ has two sources of value: On one hand the value associated to the dividends that will be received between τ and τ_1, $D(V,\tau)$. On the other hand the option value, $O(V,\tau)$, that comes from the possibility of acquiring the firm at τ_1 by paying ψ_1. Applying risk neutral valuation we find that

$$D(V,\tau) = V\left(1 - e^{-\delta T_1}\right)$$

and[5]

[5] $N(\cdot)$ denotes the standard normal cumulative distribution function.

$$O(V,\tau) = Ve^{-\delta T_1} N(d_1) - \psi_1 e^{-rT_1} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{V}{\psi_1}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T_1}{\sigma\sqrt{T_1}} \\ d_2 &= d_1 - \sigma\sqrt{T_1} \end{aligned}$$

Finally

$$S(V,\Theta,\tau) = V\left(1 - e^{-\delta T_1}\right) + Ve^{-\delta T_1} N(d_1) - \psi_1 e^{-rT_1} N(d_2) \qquad (2)$$

The fact that assumptions B1-B3 hold in this case is established in the following proposition.

Proposition 1: *B1-B3 hold for $n = 1$, making a RC with $n = 1$ feasible whenever $S(V,\tau) > 0$.*

Proof: See appendix.

Figure 1 describes two of the six alternative ways to design a RC with $n = 1$: Those associated to sequences $\tau_1 - \delta - \psi_1$ and $\tau_1 - \psi_1 - \delta$, respectively.[6]

Consider first $\tau_1 - \delta - \psi_1$: In principle, any dividend rate between zero and infinity would be feasible in a RC. However, for a given maturity strictly greater than current period τ, the set of possible dividend rates reduces to the interval $[0, \delta_2)$. We may say that the higher the debt maturity, the lower the dividend rate that debt holders will be willing to accept to refinance existing debt. Now let assume that a debt maturity has been chosen. Figure 1 makes clear that the higher δ, the higher the debt payment, or in other words, the higher the yield that debt holders will charge to the firm to refinance its debt.

[6] $S(V,\Theta,\tau)_{\psi_1\psi_1} > 0$, implying that $S(V,\Theta,\tau)$ is a strictly convex function in ψ_1.

Sequence $\tau_1 - \psi_1 - \delta$ can also be analyzed using Figure 1: In this specific case of $n = 1$, it is clear that $\hat{\psi}_1 = \psi$. Any debt maturity strictly greater than τ, will nevertheless lead to a strictly positive yield reflected in a higher ψ_1. For a fixed maturity, we observe on the other hand that the higher the yield charged to equity holders, that is, the higher ψ_1, the higher also the dividend rate. In short, equity holders demand a higher dividend rate as compensation for bearing a higher yield to maturity.

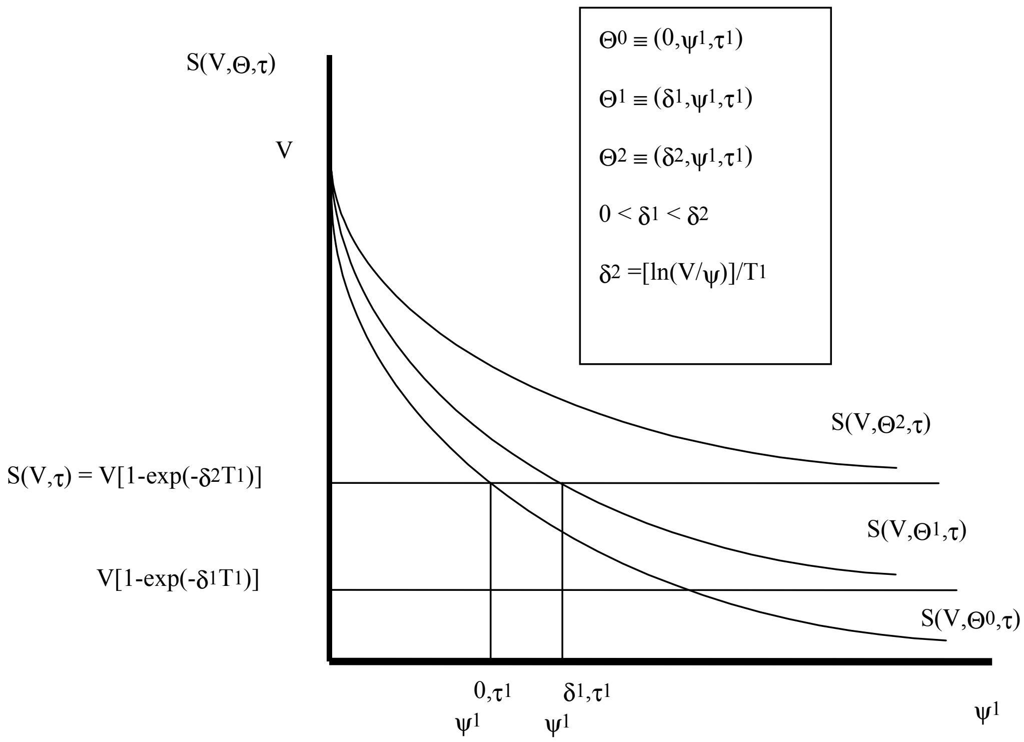

Figure 1: Sequences $\tau_1 - \delta - \psi_1$ and $\tau_1 - \psi_1 - \delta$ for $n = 1$.

An interesting aspect is that in fact, the credit spread ($C.S$) on corporate debt that results from refinancing, will depend on the specific RC chosen in the feasible set; A feasible set that at the same time depends on the current firm value, the current nominal debt, the firm return volatility, and the risk free interest rate. We know that for any element Θ of this set, the new equity value, $S(V,\Theta,\tau)$, equals the previous one, $S(V,\tau)$. This can be expressed as $V-\psi_1 e^{-RT_1}=V-\psi$, where R is the interest rate associated to the new corporate debt. Then it is straightforward to show that

$$C.S=\frac{\ln\left(\frac{\psi_1}{\psi}\right)}{T_1}-r$$

Although the dividend rate does not explicitly appear in the expression above, it does through its influence on ψ_1 and T_1.

Given the firm characteristics and the risk free rate, that is, given a vector (V,ψ,σ,r), the $C.S$ will be a function of the specific contract chosen, which we represent by a vector (δ,ψ_1,τ_1). We have seen however that only two of these three elements are "freely" chosen. Consider for instance (δ,τ_1) are selected according to the restriction imposed by corollary 2, then the credit spread will be a function of $(V,\psi,\sigma,r,\delta,\tau_1)$, with $\delta T_1<\ln\left(\frac{V}{\psi}\right)$. In order to make some comparative statics with respect to the $C.S$, we need to derive how ψ_1 depends on these variables and parameters. Let define $\Gamma(V,\Theta,\tau)=S(V,\Theta,\tau)-S(V,\tau)$. Then, Θ belongs to the feasible set if and only if $\Gamma(V,\Theta,\tau)=0$, and the derivative of ψ_1 with respect to variable or parameter j will be given by

$$(\psi_1)_j=-\frac{\Gamma(V,\Theta,\tau)_j}{\Gamma(V,\Theta,\tau)_{\psi_1}}$$

what leads to $(\psi_1)_V<0$, $(\psi_1)_\psi>0$, $(\psi_1)_\sigma>0$, $(\psi_1)_r>0$, $(\psi_1)_\delta>0$, $(\psi_1)_{\tau_1}>0$ (see appendix for details). We may now describe the dependence of the $C.S$ on the firm value and the nominal debt to refinance:

$$C.S_V=\frac{(\psi_1)_V}{\psi_1 T_1}<0$$

$$C.S_\psi=\frac{(\psi_1)_\psi\,\psi-\psi_1}{\psi_1\psi T_1}>0$$

A higher leverage ratio (a lower V or a higher ψ), means a higher credit risk, and the result is a higher $C.S$ faced by the firm to refinance.

Figure 2 represents the $C.S$ as a function of the maturity date for different firm values. The base case in this and other simulations is $V = 100$, $\psi = 30$, $\sigma = 0.2$, $r = 0.05$ and $\delta = 0.01$. As mentioned above, the lower the firm value, the higher the credit risk of the firm, and the higher the $C.S$ that it has to face to refinance.

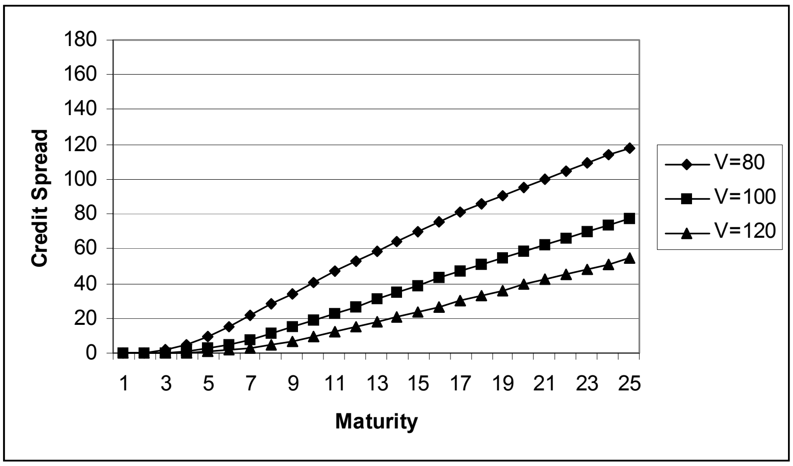

Figure 2: $C.S$ as a function of the maturity date for different firm values. Base case: $\psi = 30$, $\sigma = 0.2$, $r = 0.05$ and $\delta = 0.01$.

Figure 3 provides the credit spread as a function of the maturity date for different current nominal debt payments. We can interpret the effect of a higher ψ in the same way we interpreted the effect of a lower V.

On the other hand

$$C.S_{\sigma} = \frac{(\psi_1)_{\sigma}}{\psi_1 T_1} > 0$$

It is also reasonable to observe that the higher the firm risk, the higher the credit spread on the firm debt. Figure 5 plots the $C.S$ as a function of the maturity date for different firm return volatilities, showing that it is an increasing function of the firm risk.

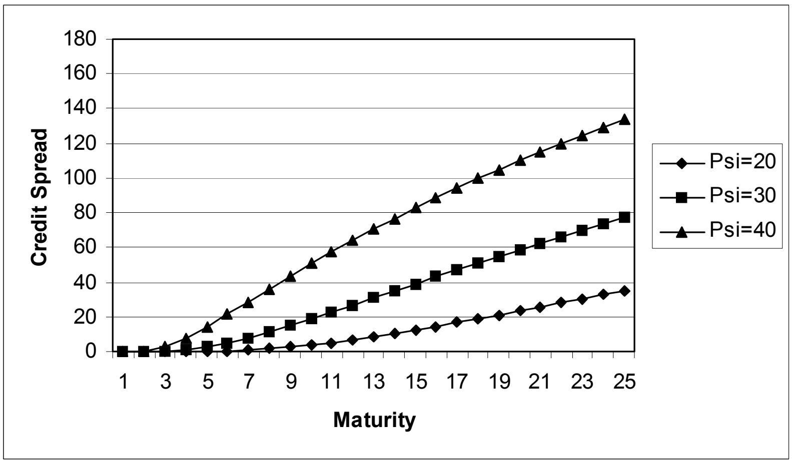

Figure 3: $C.S$ as a function of the maturity date for different nominal debt payments. Base case: $V = 100$, $\sigma = 0.2$, $r = 0.05$ and $\delta = 0.01$.

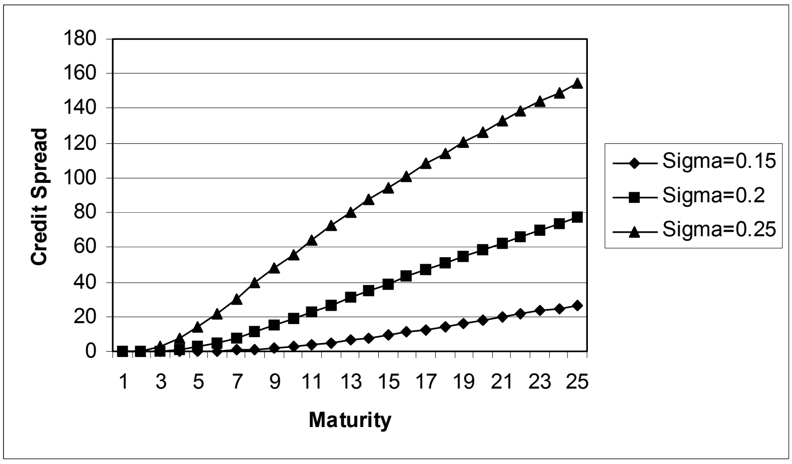

Figure 4: $C.S$ as a function of the maturity date for different firm return volatilities. Base case: $V = 100$, $\psi = 30$, $r = 0.05$ and $\delta = 0.01$.

The credit spread shows however to be independent of the risk free interest rate:

$$C.S_r = 0$$

This result may appear inconsistent with that derived in Merton (1974), where the credit spread results a decreasing function of the risk free interest rate. In our case however, we are analyzing the credit spread of a new issued debt at the moment it is issued, and with the goal of refinancing current debt, which is not the case in Merton (1974).

The dependence of the credit spread on the dividend rate is described by

$$C.S_\delta = \frac{(\psi_1)_\delta}{\psi_1 T_1} > 0$$

A higher dividend rate implies a lower expected firm value growth and a higher default probability, what leads to a higher credit spread. Figure 4 again represents the $C.S$ as a function of the maturity date, but now alternative dividend rates are considered. It reflects the important effect that the dividends have on the credit spread.

Finally

$$C.S_{\tau_1} = \frac{(\psi_1)_{\tau_1} - \psi_1 R}{\psi_1 T_1}$$

Although no formal proof for the fact that $C.S_{\tau_1} > 0$ can be provided, this holds in all (reported and unreported) simulations performed. Nevertheless, it again may seem inconsistent with the results in Merton (1974). It must be pointed out the substantial difference in the analysis of the time dependence followed by Merton and the one we drive here (not only the inclusion of a dividend rate): In fact, he sets the so called "quasi debt-to-firm value ratio" constant. In order to keep that ratio equal to a fixed q for a given firm value and interest rate, ψ_1 should be determined as qVe^{rT}. In our case, however, we impose that the implied ψ_1 value is consistent with a feasible RC, what

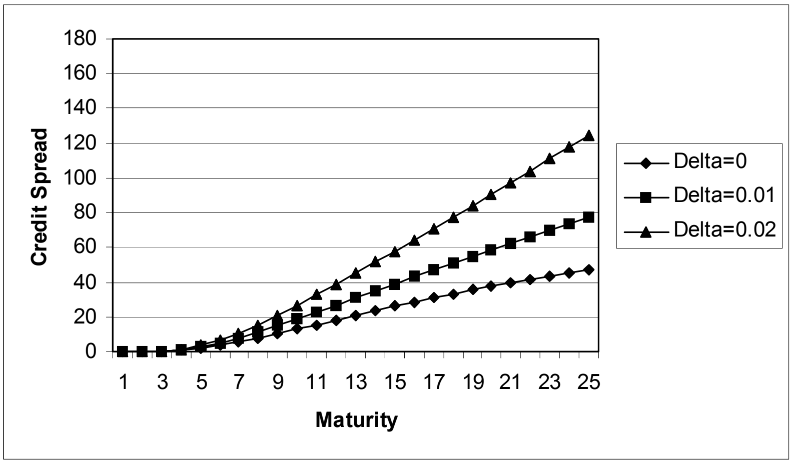

Figure 5: $C.S$ as a function of the maturity date for different dividend rates. Base case: $V = 100$, $\psi = 30$, $\sigma = 0.2$ and $r = 0.05$.

makes the ratio q to move from values below 1 to values above 1 for different maturities.[7]

Implications of the Refinancing Strategy for $n = 1$.

The fact that a firm chooses to refinance its debt has also several implications in terms of the future evolution of credit ratings and credit spreads.

To start with we may represent the new nominal debt payment that results from refinancing at τ, as a function of the firm value.[8] This is done in picture 6: As the firm value tends to the default point ψ, the new nominal payment (and the credit spread that the firm has to face to refinance) tends to infinity. As the firm value tends to infinity on the other hand, the new payment tends to the previous payment capitalized at the risk free interest rate.

Let now assume that the credit rating of the firm is measured at any time the firm refinances, as the ratio current firm value to new nominal debt. Picture 7 follows directly from Picture 6, and represents the credit rating

[7]Merton finds that the sign of $C.S_{\tau_1}$depends on whether q is higher, equal or lower than 1.

[8]To simplify the exposition we assume a non dividend paying firm. The strict convexity follows from $(\psi_1)_V < 0$ and $(\psi_1)_{VV} > 0$.

at τ, $V(\tau)/\psi_1$, as a function of the ratio $V(\tau)/\psi$. Let $A = V(\tau_{-1})/\psi$ be the credit rating of the firm at issuance of ψ. Then, if the firm value stays constant between τ_{-1} and τ, the credit rating falls to a. The fact that the firm refinances makes possible to observe a downgrade in the credit rating of the firm even if it does not lose market value, and as Picture 7 indicates, even with a strictly positive growth (point C). In order to keep its credit rating a firm value growth large enough (point B), has to take place. In summary, any ratio $V(\tau)/\psi$ lower that B will be followed by a downgrade, while any value of this ratio above B will lead to an upgrade. Note also that deviations from B will have a different impact on the credit rating depending on whether this deviation is positive or negative. A negative deviation will have a higher effect in absolute terms than an equivalent positive deviation. Debt refinancing therefore is expected to generate stronger downgrades than upgrades.

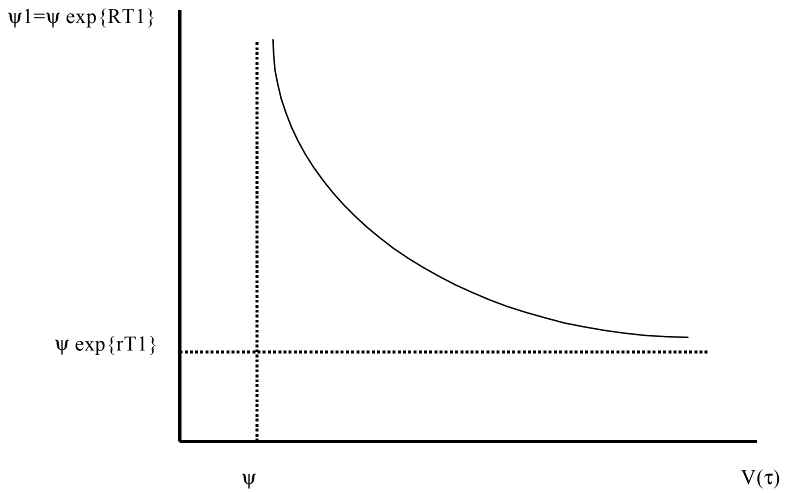

Figure 6: ψ_1 as a function of $V(\tau)$.

Refinancing also generates another testable implication in terms of credit spreads: As has been pointed out, the credit spread will not depend on the risk free interest rate. Nevertheless, this null influence will hold only in the short run. The new nominal debt payment ψ_1 is an increasing function of the risk free interest rate. As a consequence, the credit spread that the firm will

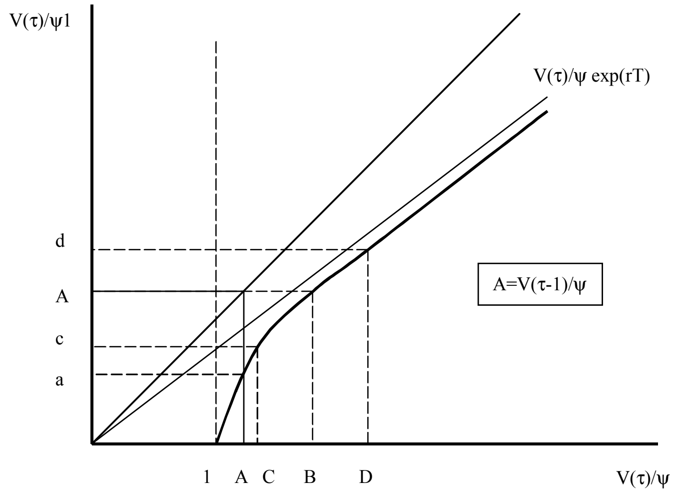

Figure 7: $V(\tau)/\psi_1$ as a function of $V(\tau)/\psi$.

have to face at τ_1 to refinance ψ_1 will tend to be higher the higher the risk free interest at τ, given that this risk premium is an increasing function of the nominal debt to be refinanced. Some evidence of this "lagged effect" has already been provided by Guha, Hiris and Visviki (2000). Specifically, they find that credit spreads on bonds rated by Moody's, are positively correlated with the two years lagged long term Government Bond Yield.

3.2 $n = 2$

We have analyzed the case in which the firm always maintains a single zero coupon bond as corporate debt. The main implications derived from assuming that the firm refinances its debt in terms of credit ratings and credit spreads appear in this simple case. Exploring the situation in which the firm always refinances with $n = 2$ is interesting however for several reasons: First, it can be seen as a simplification to short and long term debt, what better represents the debt structure of a firm. Second, it incorporates the fact that equity holders do not only care about the debt that currently matures at the time of deciding whether or not satisfying it, but also about all future debt remaining. This makes for instance the current bankruptcy-triggering firm value to diverge from the current nominal debt payment, something that does not happen with $n = 1$.

Proposition below ensures that at any time the firm has to satisfy a debt payment, refinancing with $n = 2$ is feasible.

Proposition 2: *B1-B3 hold for $n = 2$, making a refinancing contract with $n = 2$ feasible whenever $S(V, \tau) > 0$.*

Proof: See appendix.

Proposition 2 does not assume any specific initial debt structure. However, we may think in a model in which the firm maintains an stable corporate debt structure with short and long term debt, keeping the ratio short term debt/long term debt, and the time spread between them, constant. These can be associated to the specific industry in which the firm operates.

This stable corporate debt structure translates into a vector $\Delta \equiv (n, \Phi, \Lambda)$, where $n = 2$, $\Phi = (1, \phi)$ and $\Lambda = (1, \eta)$. Assume again that the firm does not pay dividends. As a result, it is always possible to consider that the firm refinances not only under a constant Δ (as stated in the theorem), but also with some fixed constant T_1 (given corollary 2).

The following proposition implies that the effect of debt refinancing on the evolution of the credit rating of the firm described for $n = 1$, also applies in this case.

Proposition 3: *Let $\bar{V}$ and $\bar{V}_1$ denote the bankruptcy-triggering firm value at τ and τ_1 respectively. Then $\bar{V}_1$ is a strictly decreasing and strictly convex function in $V(\tau)$, with $\lim_{V(\tau)\to\bar{V}} = \infty$ and $\lim_{V(\tau)\to\infty} = \bar{V}e^{rT_1}$.*

Proof: See appendix

The shape of $\bar{V}_1$ as a function of V will be therefore analogous to the one we found for ψ_1 with respect to ψ under $n = 1$. The credit rating at τ will be now described by the ratio $V(\tau)/\bar{V}_1$, and the same analysis with made for $V(\tau)/\psi_1$ applies here.

Refinancing makes $\bar{V}_1$ to be the new bankruptcy-triggering firm value, a critical threshold that will evolve over time as the firm refinances its debt repeatedly. Although no explicit solution for it can be provided, it can be shown that it belongs to the same range in which KMV Corporation finds typically to be the *default point*.

Proposition 4: *Let ψ_1 and ψ_2 be the new short and long term debt resulting from refinancing at τ, and let T be the time spread between these payments. Then for any $\sigma \in (0, \infty)$, $\bar{V}_1 \in \left(\psi_1, \psi_1 + \psi_2 e^{-rT}\right)$. Moreover $\lim_{\sigma \to 0} \bar{V}_1 = \psi_1 + \psi_2 e^{-rT}$ and $\lim_{\sigma \to \infty} \bar{V}_1 = \psi_1$.*

Proof: See appendix.

With a database over 100.000 company-years of data and over 2.000 incidents of default or bankruptcy, KMV has found that firms generally default when the firm value lies somewhere between short term debt and total debt in nominal terms.[9] Clearly, $\bar{V}_1 \in \left(\psi_1, \psi_1 + \psi_2 e^{-rT}\right)$ implies that $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2)$.

4 Conclusions:

Refinancing existing debt seems to be one of the most important, if not the first, reason to issue new debt. We investigate the long run effects of this extended practice on credit ratings and credit spreads. Debt refinancing generates systematic rating downgrades unless a minimum firm value growth is observed. Deviations from this growth path imply asymmetric results: A lower firm value growth generates downgrades and a higher firm value growth upgrades as expected. However, downgrades will tend to be higher in absolute terms. Finally, credit spreads and risk free interest rate will be independent in the short run, but positively correlated in the long run.

5 Appendix:

5.1 Proof of lemma:

$S(V, \Theta, \tau)$ is what equity holders get after signing Θ, therefore they will be willing to sign if and only if $S(V, \Theta, \tau) \geq S(V, \tau)$. $F(V, \Theta, \tau)$ is what debt holders have after signing Θ, therefore they will be willing to sign if and only if $F(V, \Theta, \tau) \geq F(V, \tau)$. At the same time $S(V, \Theta, \tau) + F(V, \Theta, \tau) = S(V, \tau) + F(V, \tau) = V$. $S(V, \Theta, \tau) = S(V, \tau)$ implies that $F(V, \Theta, \tau) = F(V, \tau)$ and $\Theta \in \Theta^F$. On the other hand $\Theta \in \Theta^F$ implies $S(V, \Theta, \tau) \geq S(V, \tau)$.

[9]Crosbie, Peter J. (1999).

Suppose $S(V,\Theta,\tau) > S(V,\tau)$, then $F(V,\Theta,\tau) < F(V,\tau)$ and $\Theta \notin \Theta^F$, what is a contradiction, proving the first argument of the lemma. Finally $S(V,\Theta,\tau) > 0 \ \forall\Theta \mid \tau_1 > \tau$,[10] implying $S(V,\tau) > 0$ as a necessary condition for a feasible Θ to exist.

5.2 Proof of theorem:

We have six possible sequences $\alpha - \beta - \gamma$. Let analyze each one of them:

Case 1: $\tau_1 - \delta - \psi_1$

Given $\tau_1 \in \varphi^{\tau_1}$, $S(V,\Theta,\tau)$ is $CSD(\psi_1)$, with $\lim_{\psi_1 \to \hat{\psi}_1} S(V,\Theta,\tau) > S(V,\tau) \ \ \forall\delta \in \varphi^{\delta}$, and $\lim_{\psi_1 \to \infty} S(V,\Theta,\tau) = V\left(1 - e^{-\delta T_1}\right)$. Therefore, $S(V,\Theta,\tau) = S(V,\tau)$ reaches a solution for at least one $\psi_1 \in \varphi^{\psi_1}$, if and only if $V\left(1 - e^{-\delta T_1}\right) < S(V,\tau)$, if and only if $\delta < \frac{\ln\left[\frac{V}{F(V,\tau)}\right]}{T_1}$. As a result, $\varphi^{\delta|\tau_1} \equiv \left[0, \frac{\ln\left[\frac{V}{F(V,\tau)}\right]}{T_1}\right) \neq \emptyset$. Because $S(V,\Theta,\tau)$ is $CSD(\psi_1)$, we finally have that for any $\tau_1 \in \varphi^{\tau_1}$, and $\delta \in \varphi^{\delta|\tau_1}$, there is a unique $\psi_1 \in \varphi^{\psi_1}$ such that $S(V,\Theta,\tau) = S(V,\tau)$.

Case 2: $\tau_1 - \psi_1 - \delta$

Given $\tau_1 \in \varphi^{\tau_1}$, $S(V,\Theta,\tau)$ is $CSI(\delta)$, with $\lim_{\delta\to\infty} S(V,\Theta,\tau) = V \ \forall\psi_1 \in \varphi^{\psi_1}$. Therefore, $S(V,\Theta,\tau) = S(V,\tau)$ reaches a solution for at least one $\delta \in \varphi^{\delta}$, if and only if $S(V,\Theta,\tau) \mid_{\delta=0} \leq S(V,\tau)$, if and only if $\psi_1 \geq \psi_1^{0,\tau_1}$, where $\psi_1^{0,\tau_1} > \hat{\psi}_1$ is the ψ_1 value such that $S(V,\Theta,\tau) \mid_{\delta=0} = S(V,\tau)$.[11] As a result, $\varphi^{\psi_1|\tau_1} \equiv \left[\psi_1^{0,\tau_1}, \infty\right) \neq \emptyset$. Because $S(V,\Theta,\tau)$ is $CSI(\delta)$, we finally have that for any $\tau_1 \in \varphi^{\tau_1}$, and $\psi_1 \in \varphi^{\psi_1|\tau_1}$, there is a unique $\delta \in \varphi^{\delta}$ such that $S(V,\Theta,\tau) = S(V,\tau)$.

Case 3: $\delta - \tau_1 - \psi_1$

Given $\delta \in \varphi^{\delta}$, $S(V,\Theta,\tau)$ is $CSD(\psi_1)$, with $\lim_{\psi_1 \to \hat{\psi}_1} S(V,\Theta,\tau) > S(V,\tau) \ \ \forall\tau_1 \in \varphi^{\tau_1}$, and $\lim_{\psi_1 \to \infty} S(V,\Theta,\tau) = V\left(1 - e^{-\delta T_1}\right)$. Therefore, $S(V,\Theta,\tau) = S(V,\tau)$ reaches a solution for at least one $\psi_1 \in \varphi^{\psi_1}$, if and

[10]Limited liability makes $S(V,\Theta,\tau) > 0$ when no payment has to be currently satisfied.

[11]B1 implies that ψ_1^{0,τ_1} exists and is unique, and joint with B2 also implies that $\psi_1^{0,\tau_1} > \hat{\psi}_1$.

only if $V\left(1-e^{-\delta T_1}\right) < S\left(V,\tau\right)$, if and only if $\tau_1 < \tau + \frac{\ln\left[\frac{V}{F(V,\tau)}\right]}{\delta}$. As a result, $\varphi^{\tau_1|\delta} \equiv \left(\tau, \tau + \frac{\ln\left[\frac{V}{F(V,\tau)}\right]}{\delta}\right) \neq \emptyset$. Because $S\left(V,\Theta,\tau\right)$ is $CSD\left(\psi_1\right)$, we finally have that for any $\delta \in \varphi^{\delta}$, and $\tau_1 \in \varphi^{\tau_1|\delta}$, there is a unique $\psi_1 \in \varphi^{\psi_1}$ such that $S\left(V,\Theta,\tau\right) = S\left(V,\tau\right)$.

Case 4: $\delta - \psi_1 - \tau_1$

Given $\delta \in \varphi^{\delta}$, $S\left(V,\Theta,\tau\right)$ is $CSI\left(\tau_1\right)$, with $\lim_{\tau_1\to\tau} S\left(V,\Theta,\tau\right) = V - \psi_1 \sum_{i=1}^{n} \phi_i < S\left(V,\tau\right)\ \forall\psi_1 \in \varphi^{\psi_1}$, and $\lim_{\tau_1\to\infty} S\left(V,\Theta,\tau\right) = V$. As a result, $\varphi^{\psi_1|\delta} \equiv \varphi^{\psi_1} \neq \emptyset$. Because $S\left(V,\Theta,\tau\right)$ is $CSI\left(\tau_1\right)$, we finally have that for any $\left(\delta,\psi_1\right) \in \varphi^{\delta} \times \varphi^{\psi_1}$, there is a unique $\tau_1 \in \varphi^{\tau_1}$ such that $S\left(V,\Theta,\tau\right) = S\left(V,\tau\right)$.

Case 5: $\psi_1 - \tau_1 - \delta$

Given $\psi_1 \in \varphi^{\psi_1}$, $S\left(V,\Theta,\tau\right)$ is $CSI\left(\delta\right)$, with $\lim_{\delta\to\infty} S\left(V,\Theta,\tau\right) = V$ $\forall\tau_1 \in \varphi^{\tau_1}$. Therefore, $S\left(V,\Theta,\tau\right) = S\left(V,\tau\right)$ reaches a solution for at least one $\delta \in \varphi^{\delta}$, if and only if $S\left(V,\Theta,\tau\right)|_{\delta=0} \leq S\left(V,\tau\right)$, if and only if $\tau_1 \leq \tau_1^{0,\psi_1}$, where $\tau_1^{0,\psi_1} > \tau$ is the τ_1 value such that $S\left(V,\Theta,\tau\right)|_{\delta=0} = S\left(V,\tau\right)$.[12] As a result, $\varphi^{\tau_1|\psi_1} \equiv \left(\tau, \tau_1^{0,\psi_1}\right] \neq \emptyset$. Because $S\left(V,\Theta,\tau\right)$ is $CSI\left(\delta\right)$, we finally have that for any $\psi_1 \in \varphi^{\psi_1}$, and $\tau_1 \in \varphi^{\tau_1|\psi_1}$, there is a unique $\delta \in \varphi^{\delta}$ such that $S\left(V,\Theta,\tau\right) = S\left(V,\tau\right)$.

Case 6: $\psi_1 - \delta - \tau_1$

Given $\psi_1 \in \varphi^{\psi_1}$, $S\left(V,\Theta,\tau\right)$ is $CSI\left(\tau_1\right)$, with $\lim_{\tau_1\to\tau} S\left(V,\Theta,\tau\right) = V - \psi_1 \sum_{i=1}^{n} \phi_i < S\left(V,\tau\right)\ \forall\delta \in \varphi^{\delta}$, and $\lim_{\tau_1\to\infty} S\left(V,\Theta,\tau\right) = V$. As a result, $\varphi^{\delta|\psi_1} \equiv \varphi^{\delta} \neq \emptyset$. Because $S\left(V,\Theta,\tau\right)$ is $CSI\left(\tau_1\right)$, we finally have that for any $\left(\psi_1,\delta\right) \in \varphi^{\psi_1} \times \varphi^{\delta}$, there is a unique $\tau_1 \in \varphi^{\tau_1}$ such that $S\left(V,\Theta,\tau\right) = S\left(V,\tau\right)$.■

5.3 Proof of corollary 2:

This holds given the proof of case 1 and case 3.■

[12] B2 implies that τ_1^{0,ψ_1} exists and is unique. B2 also implies that $\tau_1^{0,\psi_1} > \tau$.

5.4 Proof of corollary 3:

This holds given the proof of case 2 and case 5.■

5.5 Proof of corollary 4:

This holds given the proof of case 4 and case 6.■

5.6 Proof of proposition 1:

$S(V,\Theta,\tau)$ is clearly a continuous function in ψ_1, with[13]

$$\begin{aligned} S(V,\Theta,\tau)_{\psi_1} &= -e^{-rT_1}N(d_2) < 0 \\ S(V,\Theta,\tau) &\mid \ _{\psi_1=0} = V \\ \lim_{\psi_1\to\infty} S(V,\Theta,\tau) &= V\left(1-e^{-\delta T_1}\right) \end{aligned}$$

and B1 holds. On the other hand, $S(V,\Theta,\tau)$ is a continuous function in τ_1, with[14]

$$S(V,\Theta,\tau)_{\tau_1} = \delta V e^{-\delta T_1}\left[1-N(d_1)\right]+$$

$$+Ve^{-\delta T_1} f(d_1)\tfrac{\sigma}{2\sqrt{T_1}} + r\psi_1 e^{-rT_1}N(d_2) > 0$$

$$\begin{aligned} \lim_{\tau_1\to\infty} S(V,\Theta,\tau) &= V \\ \lim_{\tau_1\to\tau} S(V,\Theta,\tau) &= \begin{cases} V-\psi_1 & if \quad V>\psi_1 \\ 0 & if \quad V\leq\psi_1 \end{cases} \\ &= Max\left\{0, V-\psi_1\right\} \end{aligned}$$

[13] $S(V,\Theta,\tau)_j$ denotes the first derivative of $S(V,\Theta,\tau)$ with respect to j.

[14] $f(\cdot)$ denotes the standard normal density function.

and B2 also holds. Finally, $S(V,\Theta,\tau)$ is a continuous function in δ, with

$$\begin{aligned} S(V,\Theta,\tau)_\delta &= T_1 V e^{-\delta T_1}\left[1-N(d_1)\right] > 0 \\ \lim_{\delta\to\infty} S(V,\Theta,\tau) &= V \end{aligned}$$

and our last condition B3 holds as well■.

5.7 Partial derivatives:

$$(\psi_1)_V = -\frac{e^{-\delta T_1}[1-N(d_1)]}{e^{-rT_1}N(d_2)} < 0$$

$$(\psi_1)_\psi = \frac{1}{e^{-rT_1}N(d_2)} > 0$$

$$(\psi_1)_\sigma = \frac{Ve^{-\delta T_1}f(d_1)\sqrt{T_1}}{e^{-rT_1}N(d_2)} > 0$$

$$(\psi_1)_r = \psi_1 T_1 > 0$$

$$(\psi_1)_\delta = \frac{T_1 V e^{-\delta T_1}[1-N(d_1)]}{e^{-rT_1}N(d_2)} > 0$$

$$(\psi_1)_{\tau_1} = \frac{\delta V e^{-\delta T_1}[1-N(d_1)] + Ve^{-\delta T_1} f(d_1)\frac{\sigma}{2\sqrt{T_1}} + r\psi_1 e^{-rT_1}N(d_2)}{e^{-rT_1}N(d_2)} > 0$$

5.8 Proof of proposition 2:

We need to derive what is the equity value when the debt structure of the firm is composed by short term debt ψ_1, and long term debt ψ_2; τ_1 and τ_2 are respectively short and long term debt maturities. Suppose first an asset $C(V,t)$ with two sources of value: On one hand it pays the firm dividends between τ_1 and τ_2. On the other hand it gives the right to buy the firm at τ_2 by paying ψ_2. In the same way we derived (2), it can be shown that at any $t \le \tau_1$

$$C(V,t) = Ve^{-rT_1}\left(1-e^{-\delta T}\right) + Ve^{-\delta T_2}N(b_1) - \psi_2 e^{-rT_2}N(b_2)$$

where

$$
\begin{aligned}
b_1 &= \frac{\ln\left(\frac{V}{\psi_2}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T_2}{\sigma\sqrt{T_2}} \\
b_2 &= b_1 - \sigma\sqrt{T_2}
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= \tau_1 - t \\
T_2 &= \tau_2 - t \\
T &= \tau_2 - \tau_1
\end{aligned}
$$

If current and/or new potential equity holders pay ψ_1 at τ_1, they acquire precisely this asset. As long as they have the option of refusing this payment

$$S(V, \Theta, \tau_1) = Max\{0, C(V, \tau_1) - \psi_1\}$$

$C(V, \tau_1)$ is a strictly increasing function in V, with $\lim_{V \to 0} C(V, \tau_1) = 0$ and $\lim_{V \to \infty} C(V, \tau_1) = \infty$, therefore there exists a unique $\bar{V}_1 \in (0, \infty)$ such that $S(V, \Theta, \tau_1) > 0 \ \forall \ V > \bar{V}_1$. This will be the implicit solution to

$$S\left(\bar{V}_1, \Theta, \tau_1\right) = \bar{V}_1\left(1 - e^{-\delta T}\right) + \bar{V}_1 e^{-\delta T} N(c_1) - \psi_2 e^{-rT} N(c_2) - \psi_1 = 0$$

where

$$
\begin{aligned}
c_1 &= \frac{\ln\left(\frac{\bar{V}_1}{\psi_2}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \\
c_2 &= c_1 - \sigma\sqrt{T}
\end{aligned}
$$

At $\tau < \tau_1$ the equity also finds two sources of value. On one hand the value associated to dividends received from τ to τ_1, $D(V, \tau)$; On the other hand the option value that appears due to the possibility of buying the described asset at τ_1, $O(V,\tau)$. The first is

$$D(V,\tau) = V\left(1 - e^{-\delta T_1}\right)$$

The option value will be[15]

$$\begin{aligned} O(V,\tau) &= Ve^{-\delta T_1}\left(1 - e^{-\delta T}\right) N(a_1) + \\ &\quad + Ve^{-\delta T_2} N_2(a_1, b_1; \rho) - \psi_2 e^{-rT_2} N_2(a_2, b_2; \rho) - \psi_1 e^{-rT_1} N(a_2) \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{\ln\left(\frac{V}{V_1}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T_1}{\sigma\sqrt{T_1}} \\ a_2 &= a_1 - \sigma\sqrt{T_1} \\ \rho &= \sqrt{\frac{T_1}{T_2}} \end{aligned}$$

Finally

$$\begin{aligned} S(V,\Theta,\tau) &= O(V,\tau) + D(V,\tau) \\ &= V\left(1 - e^{-\delta T_1}\right) + Ve^{-\delta T_1}\left(1 - e^{-\delta T}\right) N(a_1) + \qquad \text{(A8)} \\ &\quad + Ve^{-\delta T_2} N_2(a_1, b_1; \rho) - \psi_2 e^{-rT_2} N_2(a_2, b_2; \rho) - \psi_1 e^{-rT_1} N(a_2) \end{aligned}$$

[15] This expression can be derived following the methodology applied to the valuation of compound options. For a detailed exposition see Kwok [15].
$N_2(a, b; \rho)$ represents the cumulative standard bivariate normal distribution function, with integration limits a and b, and correlation coefficient ρ.

(**??**) converges to the expression given in Geske (1979) for two periods as δ tends to zero. The first term on the r.h.s. is the value of the dividends that will be received from τ to τ_1, the second is the value of the dividends that will be received from τ_1 to τ_2 if the firm does not default at τ_1, and the last three terms represent the compound option on the firm.

According to the notation used in B1 and B2, we could express ψ_2 as $\phi\psi_1$, and T_2 as $(1+\eta)\,T_1$. Then

$$S\left(V,\Theta,\tau\right) = V\left(1-e^{-\delta T_1}\right) + Ve^{-\delta T_1}\left(1-e^{-\delta\eta T_1}\right)N\left(a_1\right) +$$

$$+Ve^{-\delta(1+\eta)T_1}N_2\left(a_1,b_1;\rho\right) - \phi\psi_1 e^{-r(1+\eta)T_1}N_2\left(a_2,b_2;\rho\right) - \psi_1 e^{-rT_1}N\left(a_2\right)$$

where

$$\begin{aligned} b_1 &= \frac{\ln\left(\frac{V}{\phi\psi_1}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)(1+\eta)\,T_1}{\sigma\sqrt{(1+\eta)\,T_1}} \\ b_2 &= b_1 - \sigma\sqrt{(1+\eta)\,T_1} \\ \rho &= \sqrt{\frac{1}{1+\eta}} \end{aligned}$$

At the same time, $\bar{V}_1$ will be the firm value that satisfies

$$S\left(\bar{V}_1,\Theta,\tau_1\right) = \bar{V}_1\left(1-e^{-\delta\eta T_1}\right) + \bar{V}_1 e^{-\delta\eta T_1}N\left(c_1\right) - \phi\psi_1 e^{-r\eta T_1}N\left(c_2\right) - \psi_1 = 0$$

where

$$\begin{aligned} c_1 &= \frac{\ln\left(\frac{\bar{V}_1}{\phi\psi_1}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)\eta T_1}{\sigma\sqrt{\eta T_1}} \\ c_2 &= c_1 - \sigma\sqrt{\eta T_1} \end{aligned}$$

Proving that B1 holds requires to obtain the sign of $S(V,\Theta,\tau)_{\psi_1}$. In the process of differentiating $N_2(a,b;\rho)$ it is useful to use the following expression when applying Liebnitz's rule:

$$N_2(a,b;\rho)=\int_{-\infty}^{a} f(x)\,N\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right)dx$$

On the other hand, it must be taken into account that $\bar{V}_1$ will change with ψ_1. In fact

$$\left(\bar{V}_1\right)_{\psi_1}=-\frac{S\left(\bar{V}_1,\Theta,\tau_1\right)_{\psi_1}}{S\left(\bar{V}_1,\Theta,\tau_1\right)_{\bar{V}_1}}=\frac{\left[1+\phi e^{-r\eta T_1}N(c_2)\right]}{1-e^{-\delta\eta T_1}\left[1-N(c_1)\right]}>0$$

At the same time, it can be easily proved that $c_i=\frac{b_i-\rho a_i}{\sqrt{1-\rho^2}}$ for $i=1,2$. All this information, joint with the following definitions

$$\begin{aligned} m &= (a_1)_{\psi_1}=(a_2)_{\psi_1} \\ n &= (b_1)_{\psi_1}=(b_2)_{\psi_1} \end{aligned}$$

lead to

$$S(V,\Theta,\tau)_{\psi_1}=Ve^{-\delta T_1}\left(1-e^{-\delta\eta T_1}\right)f(a_1)\,m+$$

$$+Ve^{-\delta(1+\eta)T_1}\left[f(a_1)\,N(c_1)\,m+f(b_1)\,N\left(\tfrac{a_1-\rho b_1}{\sqrt{1-\rho^2}}\right)n\right]-$$

$$-\phi e^{-r(1+\eta)T_1}N_2(a_2,b_2;\rho)-$$

$$-\phi\psi_1 e^{-r(1+\eta)T_1}\left[f(a_2)\,N(c_2)\,m+f(b_2)\,N\left(\tfrac{a_2-\rho b_2}{\sqrt{1-\rho^2}}\right)n\right]-$$

$$-e^{-rT_1}N(a_2)-\psi_1 e^{-rT_1}f(a_2)\,m$$

If we now apply the following identities:

$$\begin{aligned} \frac{a_1 - \rho b_1}{\sqrt{1-\rho^2}} &= \frac{a_2 - \rho b_2}{\sqrt{1-\rho^2}} \\ V e^{-\delta T_1} f(a_1) &= \bar{V}_1 e^{-rT_1} f(a_2) \\ V e^{-\delta(1+\eta)T_1} f(b_1) &= \phi \psi_1 e^{-r(1+\eta)T_1} f(b_2) \end{aligned}$$

we finally get

$$S(V,\Theta,\tau)_{\psi_1} = e^{-rT_1} f(a_2)\, mS\left(\bar{V}_1,\Theta,\tau_1\right) -$$

$$-\phi e^{-r(1+\eta)T_1} N_2(a_2, b_2; \rho) - e^{-rT_1} N(a_2)$$

$$= -\left[\phi e^{-r(1+\eta)T_1} N_2(a_2, b_2; \rho) + e^{-rT_1} N(a_2)\right] < 0$$

$S(V,\Theta,\tau)$ is then a continuous function in ψ_1, with $S(V,\Theta,\tau)_{\psi_1} < 0$. This, joint with

$$S(V,\Theta,\tau)\mid_{\psi_1=0} = V$$

$$\lim_{\psi_1\to\infty} S(V,\Theta,\tau) = V\left(1 - e^{-\delta T_1}\right)$$

proves that assumption B1 holds.

On the other hand, $S(V,\Theta,\tau)$ is a continuous function in τ_1, with[16]

[16]Use

$$S\left(\bar{V}_1,\Theta,\tau_1\right)_{\tau_1} = \delta\eta\bar{V}_1 e^{-\delta\eta T_1}\left[1 - N(c_1)\right] + r\eta\phi e^{-r\eta T_1} N(c_2) + \phi\psi_1 e^{-r\eta T_1} f(c_2)\frac{\sigma}{2\sqrt{\eta T_1}} > 0,$$

and the same arguments applied to $S(V,\Theta,\tau)_{\psi_1}$ to derive $S(V,\Theta,\tau)_{\tau_1}$.

$$S(V,\Theta,\tau)_{\tau_1} = \delta V e^{-\delta T_1}\left[1 - N(a_1)\right] +$$

$$+\delta(1+\eta)Ve^{-\delta(1+\eta)T_1}\left[N(a_1) - N_2(a_1,b_1;\rho)\right] +$$

$$+r(1+\eta)\phi\psi_1 e^{-r(1+\eta)T_1}N_2(a_2,b_2;\rho) +$$

$$+\phi\psi_1 e^{-r(1+\eta)T_1}f(a_2)N(c_2)\tfrac{\sigma}{2\sqrt{T_1}} +$$

$$+\phi\psi_1 e^{-r(1+\eta)T_1}f(b_2)N\left(\tfrac{a_2-\rho b_2}{\sqrt{1-\rho^2}}\right)\tfrac{\sigma}{2\sqrt{(1+\eta)T_1}} +$$

$$+r\psi_1 e^{-rT_1}N(a_2) + \psi_1 e^{-rT_1}f(a_2)\tfrac{\sigma}{2\sqrt{T_1}} > 0$$

$$\lim_{\tau_1\to\infty} S(V,\Theta,\tau) = V$$

$$\lim_{\tau_1\to\tau} S(V,\Theta,\tau) = \begin{cases} V - \psi_1 - \phi\psi_1 & if \quad V > \psi_1 + \phi\psi_1 \\ 0 & if \quad V \le \psi_1 + \phi\psi_1 \end{cases}$$

$$= Max\left\{0, V - \psi_1 - \phi\psi_1\right\}$$

proving that assumption B2 also holds. Finally, $S(V,\Theta,\tau)$ is a continuous function in δ, with[17]

$$S(V,\Theta,\tau)_{\delta} = T_1 V e^{-\delta T_1}\left[1 - N(a_1)\right] +$$

$$(1+\eta)T_1 V e^{-\delta(1+\eta)T_1}\left[N(a_1) - N_2(a_1,b_1;\rho)\right] > 0$$

$$\lim_{\delta\to\infty} S(V,\Theta,\tau) = V$$

and assumption B3 is satisfied.■

[17] Use

$$S\left(\bar{V}_1,\Theta,\tau_1\right)_{\delta} = \eta T_1 \bar{V} e^{-\delta\eta T_1}\left[1 - N(c_1)\right] > 0,$$

and again the same arguments applied to $S(V,\Theta,\tau_1)_{\psi_1}$ to get $S(V,\Theta,\tau_1)_{\delta}$.

It is possible to analyze graphically case 1 for $n = 2$. Let denote $S(V, \Theta, \tau)$ simply as S, and let show that S is strictly convex in (ψ_1, ψ_2), where we do not impose the restriction $\psi_2 = \phi\psi_1$.

$$\begin{aligned} S_{\psi_1} &= -e^{-rT_1} N(a_2) \\ S_{\psi_2} &= -e^{-rT_2} N(a_2, b_2; \rho) \end{aligned}$$

$$S_{\psi_1\psi_1} = \frac{e^{-rT_1} f(a_2)}{\bar{V}_1 \sigma \sqrt{T_1} \left\{1 - e^{-\delta T}\left[1 - N(c_1)\right]\right\}} > 0$$

$$S_{\psi_2\psi_2} = \frac{e^{-r(T_2+T)} f(a_2) \left[N(c_2)\right]^2}{\bar{V}_1 \sigma \sqrt{T_1} \left\{1 - e^{-\delta T}\left[1 - N(c_1)\right]\right\}} + \frac{e^{-rT_2} f(b_2) N\left(\frac{a_2 - \rho b_2}{\sqrt{1-\rho^2}}\right)}{\psi_2 \sigma \sqrt{T_2}} > 0$$

$$S_{\psi_1\psi_2} = \frac{e^{-rT_2} f(a_2) N(c_2)}{\bar{V}_1 \sigma \sqrt{T_1} \left\{1 - e^{-\delta T}\left[1 - N(c_1)\right]\right\}}$$

$$S_{\psi_1\psi_1} S_{\psi_2\psi_2} - \left(S_{\psi_1\psi_2}\right)^2 = \frac{e^{-r(T_1+T_2)} f(a_2) f(b_2)}{\bar{V}_1 \psi_2 \sigma^2 \sqrt{T_1} \sqrt{T_2}} \frac{N\left(\frac{a_2 - \rho b_2}{\sqrt{1-\rho^2}}\right)}{\left\{1 - e^{-\delta T}\left[1 - N(c_1)\right]\right\}} > 0$$

The strict convexity of S, joint with $S\,|_{(\psi_1,\psi_2)=(0,0)} = V$, and $\lim_{(\psi_1,\psi_2)\to(\infty,\infty)} S = V\left(1 - e^{-\delta T_1}\right)$, leads to figure 8.

5.9 Proof of proposition 3:

Let ψ be the debt payment to be satisfied at τ. Then

$$S(V, \tau) = VN(w_1) - \phi\psi e^{-r\eta T_1} N(w_2) - \psi$$

where

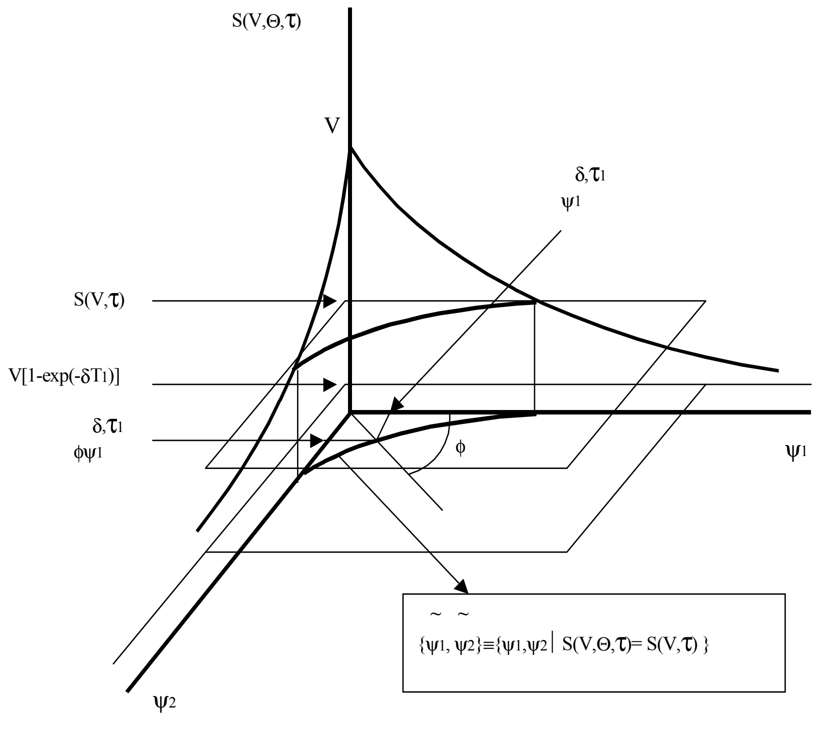

Figure 8: Sequence $\tau_1 - \delta - \psi_1$ for $n = 2$.

$$
\begin{aligned}
w_1 &= \frac{\ln\left(\frac{V}{\phi\psi}\right) + \left(r + \frac{\sigma^2}{2}\right)\eta T_1}{\sigma\sqrt{\eta T_1}} \\
w_2 &= w_1 - \sigma\sqrt{\eta T_1}
\end{aligned}
$$

and $S(V,\tau) > 0 \ \forall V > \bar{V}$, being $\bar{V}$ the implicit solution to $S\left(\bar{V},\tau\right) = 0$. On the other hand

$$
S(V,\Theta,\tau) = VN_2(k_1,l_1;\rho) - \phi\psi_1 e^{-r(1+\eta)T_1}N_2(k_2,l_2;\rho) - \psi_1 e^{-rT_1}N(k_2)
$$

where

$$
\begin{aligned}
k_1 &= \frac{\ln\left(\frac{V}{\bar{V}_1}\right) + \left(r + \frac{\sigma^2}{2}\right)T_1}{\sigma\sqrt{T_1}} \\
k_2 &= k_1 - \sigma\sqrt{T_1} \\
l_1 &= \frac{\ln\left(\frac{V}{\phi\psi_1}\right) + \left(r + \frac{\sigma^2}{2}\right)(1+\eta)T_1}{\sigma\sqrt{(1+\eta)T_1}} \\
l_2 &= l_1 - \sigma\sqrt{(1+\eta)T_1} \\
\rho &= \sqrt{\frac{1}{1+\eta}}
\end{aligned}
$$

and $\bar{V}_1$ is the implicit solution to

$$
S\left(\bar{V}_1,\Theta,\tau_1\right) = \bar{V}_1 N(h_1) - \phi\psi_1 e^{-r\eta T_1}N(h_2) - \psi_1 = 0
$$

with

$$
\begin{aligned}
h_1 &= \frac{\ln\left(\frac{\bar{V}_1}{\phi\psi_1}\right)+\left(r+\frac{\sigma^2}{2}\right)\eta T_1}{\sigma\sqrt{\eta T_1}} \\
h_2 &= h_1-\sigma\sqrt{\eta T_1}
\end{aligned}
$$

If we define $\theta=\frac{\bar{V}_1}{\psi_1}$, then previous expressions can be written as

$$
\theta N\left(h_1\right)-\phi e^{-r\eta T_1}N\left(h_2\right)-1=0
$$

where

$$
h_1=\frac{\ln\left(\frac{\theta}{\phi}\right)+\left(r+\frac{\sigma^2}{2}\right)\eta T_1}{\sigma\sqrt{\eta T_1}}
$$

The result is that θ is a constant, that is, θ will not depend on the firm value at τ (although $\bar{V}_1$ and ψ_1 will). Note also that $\theta=\frac{\bar{V}}{\psi}$. We can express condition $\Gamma\left(V,\Theta,\tau\right)=S\left(V,\Theta,\tau\right)-S\left(V,\tau\right)=0$ as

$$
\begin{aligned}
\Gamma\left(V,\Theta,\tau\right)=&\left[VN_2\left(k_1,l_1;\rho\right)-\bar{V}_1\tfrac{\phi}{\theta}e^{-r(1+\eta)T_1}N_2\left(k_2,l_2;\rho\right)-\bar{V}_1\tfrac{1}{\theta}e^{-rT_1}N\left(k_2\right)\right]- \\
&-\left[VN\left(w_1\right)-\bar{V}\tfrac{\phi}{\theta}e^{-r\eta T_1}N\left(w_2\right)-\bar{V}\tfrac{1}{\theta}\right]=0
\end{aligned}
\tag{A9}
$$

where

$$
\begin{aligned}
w_1 &= \frac{\ln\left(\frac{V}{\bar{V}}\right)+\ln\left(\frac{\theta}{\phi}\right)+\left(r+\frac{\sigma^2}{2}\right)\eta T_1}{\sigma\sqrt{\eta T_1}} \\
l_1 &= \frac{\ln\left(\frac{V}{\bar{V}_1}\right)+\ln\left(\frac{\theta}{\phi}\right)+\left(r+\frac{\sigma^2}{2}\right)\left(1+\eta\right)T_1}{\sigma\sqrt{\left(1+\eta\right)T_1}}
\end{aligned}
$$

(A9) implies that as V tends to $\bar{V}$, $\bar{V}_1$ tends to infinity. At the same time, $\lim_{V\to\infty}\bar{V}_1 = \bar{V}e^{rT_1}$.[18] It can also be proved that $\bar{V}_1$ is a strictly decreasing function in V. In fact[19]

$$\begin{aligned}
\left(\bar{V}_1\right)_V &= -\frac{\Gamma\left(V,\Theta,\tau\right)_V}{\Gamma\left(V,\Theta,\tau\right)_{\bar{V}_1}} \\
&= -\frac{N\left(w_1\right) - N_2\left(k_1,l_1;\rho\right)}{\frac{\phi}{\theta}e^{-r(1+\eta)T_1}N_2\left(k_2,l_2;\rho\right) + \frac{1}{\theta}e^{-rT_1}N\left(k_2\right)} < 0
\end{aligned}$$

Figure 9 represents $S\left(V,\tau\right)$ and $S\left(V,\Theta,\tau\right)$ as a function of V.[20] Clearly, for any $V > \bar{V}$, $N\left(w_1\right) > N_2\left(k_1,l_1;\rho\right)$, given that these are the derivatives of $S\left(V,\tau\right)$ and $S\left(V,\Theta,\tau\right)$ with respect to V. This implies $\left(\bar{V}_1\right)_V < 0$. $\left(\bar{V}_1\right)_{VV} > 0$ finally follows from the fact that $\frac{\phi}{\theta}e^{-r(1+\gamma)T_1}N_2\left(k_2,l_2;\rho\right) + \frac{1}{\theta}e^{-rT_1}N\left(k_2\right)$ is a strictly increasing function in V, while $N\left(w_1\right) - N_2\left(k_1,l_1;\rho\right)$ is strictly decreasing $\forall V > \bar{V}$ as Figure 9 indicates, and actually tends to zero as V grows.■

5.10 Proof of proposition 4:

Let $Z\left(V\right) = VN\left(g_1\right) - \psi_2 e^{-rT}N\left(g_2\right) - \psi_1$

where

$$\begin{aligned}
g_1 &= \frac{\ln\left(\frac{V}{\psi_2}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
g_2 &= g_1 - \sigma\sqrt{T}
\end{aligned}$$

then $\bar{V}_1 \equiv V \mid Z\left(V\right) = 0$.

[18] Note that this implies that $\lim_{V\to\infty}\psi_1 = \psi e^{rT_1}$ and $\lim_{V\to\infty}\phi\psi_1 = \phi\psi e^{rT_1}$, that is, as the default risk tends to zero, new debt payments tend to current debt payments capitalized at the riskfree interest rate.

[19] Use the arguments in Appendix 5.1 to derive $S\left(V,\Theta,\tau\right)_V$ and $S\left(V,\Theta,\tau\right)_{\bar{V}_1}$.

[20] $x = V - \bar{V}\frac{\phi}{\theta}e^{-r\eta T_1} - \bar{V}\frac{1}{\theta}$ and $y = V - \bar{V}_1\frac{\phi}{\theta}e^{-r(1+\eta)T_1} - \bar{V}_1\frac{1}{\theta}e^{-rT_1}$. $\bar{V}\frac{\phi}{\theta}e^{-r\eta T_1} + \bar{V}\frac{1}{\theta} < \bar{V}_1\frac{\phi}{\theta}e^{-r(1+\eta)T_1} + \bar{V}_1\frac{1}{\theta}e^{-rT_1}$ given $\bar{V}_1 > \bar{V}e^{rT_1}$.

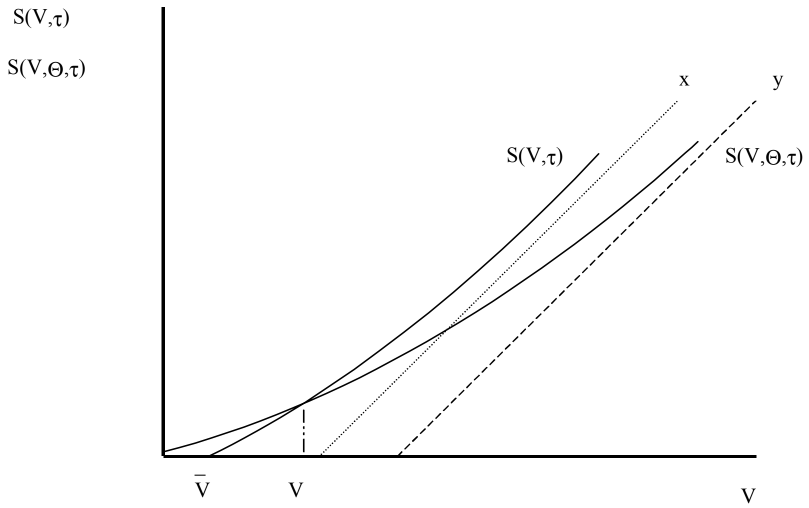

Figure 9: $S(V,\tau)$ and $S(V,\Theta,\tau)$ as a function of V.

Suppose first $\sigma \in (0,\infty)$ and $V = \psi_1$, then

$$\begin{aligned} Z(\psi_1) &= \psi_1 N(g_1) - \psi_2 e^{-rT} N(g_2) - \psi_1 \\ &= -\psi_1 [1 - N(g_1)] - \psi_2 e^{-rT} N(g_2) < 0 \end{aligned}$$

this, joint with $Z_V = N(g_1) > 0$, implies that $\bar{V}_1 > \psi_1 \ \forall \sigma \in (0,\infty)$. We have that $\lim_{\sigma\to\infty} N(g_1) = 1$ and $\lim_{\sigma\to\infty} N(g_2) = 0$, therefore $\lim_{\sigma\to\infty} Z(V) = V - \psi_1 = 0 \Leftrightarrow V = \psi_1$, proving $\lim_{\sigma\to\infty} \bar{V}_1 = \psi_1$.

On the other hand, consider $V = \psi_1 + \psi_2 e^{-rT}$, then

$$Z\left(\psi_1 + \psi_2 e^{-rT}\right) = -\psi_1 [1 - N(g_1)] + \psi_2 e^{-rT} [N(g_1) - N(g_2)]$$

$\lim_{\sigma\to 0} Z\left(\psi_1 + \psi_2 e^{-rT}\right) = 0$ because $\lim_{\sigma\to 0} N\left(g_1 \mid V = \psi_1 + \psi_2 e^{-rT}\right) = \lim_{\sigma\to 0} N\left(g_2 \mid V = \psi_1 + \psi_2 e^{-rT}\right) = 1$. As a result, $\bar{V}_1 = \psi_1 + \psi_2 e^{-rT}$ in this limit case. Given that $Z_\sigma > 0$ we also have that $Z\left(\psi_1 + \psi_2 e^{-rT}\right) > 0 \ \forall \sigma \in (0,\infty)$. This, joint again with $Z_V = N(g_1) > 0$, implies $\bar{V}_1 < \psi_1 + \psi_2 e^{-rT}$ $\forall \sigma \in (0,\infty)$, and concludes the proof.■

5.11 Default probabilities:

It is possible to derive how default probabilities implicit in Merton (1974) and Geske (1977), depend on those variables and parameters that affect them. Non of these articles include the possibility of a positive dividend rate, so we will consider this simplified case.

5.11.1 Merton (1974):

The equity value in this case will be given by

$$S\left(V,t\right)=VN\left(d_1\right)-\psi e^{-rT}N\left(d_2\right)$$

where

$$\begin{aligned}
d_1 &= \frac{\ln\left(\frac{V}{\psi}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\\
d_2 &= d_1-\sigma\sqrt{T}\\
T &= \tau-t
\end{aligned}$$

There is only one possibility of default: Defaulting at τ, and default will take place if and only if the firm value falls bellow ψ at that moment. As a result, the default probability will be

$$P=1-N\left(s\right) \tag{A11.1}$$

where

$$s=\frac{\ln\left(\frac{V}{\psi}\right)+\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

It is straightforward to show that P is a decreasing function in V and μ, and an increasing function in ψ. More interesting is how P depends on r, t and σ: In this simple case the risk free rate has no effect on the default probability. On the other hand, the sign of the derivative of P with respect to t and σ, will depend on V. In fact

$$P_t \gtreqless 0 \; as \; V \lesseqgtr \tilde{V}_M \tag{A11.2}$$

$$\tilde{V}_M = \psi e^{\left(\mu - \frac{\sigma^2}{2}\right)T} \tag{A11.3}$$

$$P_\sigma \gtreqless 0 \; as \; V \lesseqgtr \hat{V}_M \tag{A11.4}$$

$$\hat{V}_M = \psi e^{-\left(\mu + \frac{\sigma^2}{2}\right)T} \tag{A11.5}$$

Expression (A11.2) describes the dependence of P on t: If $V > \psi e^{\left(\mu - \frac{\sigma^2}{2}\right)T}$, then $V > \psi$.[21] The closer is t to maturity in this case, the lower the probability of an unfavorable firm value change before τ. Consider now $V < \psi$, then $V < \psi e^{\left(\mu - \frac{\sigma^2}{2}\right)T}$. As time goes by, the probability of a favorable firm value change given the expected growth in V along time, falls. Obviously it is also possible to have $V < \psi e^{\left(\mu - \frac{\sigma^2}{2}\right)T}$ and $V > \psi$. We could interpret this situation in the following sense: The difference between V and ψ, although positive, is small enough for t to have a negative effect, given the lower expected growth in V before maturity.

On the other hand, (A11.4) indicates that if V is low enough, then an increase in the firm business risk may reduce the default probability given an increased probability of a favorable firm value change (now due to shocks). Note that this does not imply that debt value increases: Given that equity value will always grow with the firm risk in this case, the debt value has to fall. The reason is that although increasing the volatility may reduce the default probability in some cases, this has the additional effect of reducing the expected value for debt holders in the event of default. The debt value is given by

[21] We assume $\left(\mu - \frac{\sigma^2}{2}\right) > 0$.

$$F(V,t) = V[1 - N(d_1)] + \psi e^{-rT} N(d_2)$$

The first term on the r.h.s. represents the debt value associated to default. $[1 - N(d_1)]$ is a decreasing function in σ if and only if $V < \psi e^{-\left(r-\frac{\sigma^2}{2}\right)T}$, which holds $\forall\ V < \hat{V}_M$.[22]

5.11.2 Geske (1977):

Consider the simplest version of Geske (1977) in which the corporate debt consists only on the payment of ψ_1 at τ_1 (short-run), and the payment of ψ_2 at τ_2 (long-run). Under these conditions the equity value will be

$$S(V,t) = VN_2(a_1, b_1; \rho) - \psi_2 e^{-rT_2} N_2(a_2, b_2; \rho) - \psi_1 e^{-rT_1} N(a_2)$$

where

$$\begin{aligned}
a_1 &= \frac{\ln\left(\frac{V}{\bar{V}}\right) + \left(r + \frac{\sigma^2}{2}\right)T_1}{\sigma\sqrt{T_1}} \\
a_2 &= a_1 - \sigma\sqrt{T_1} \\
b_1 &= \frac{\ln\left(\frac{V}{\psi_2}\right) + \left(r + \frac{\sigma^2}{2}\right)T_2}{\sigma\sqrt{T_2}} \\
b_2 &= b_1 - \sigma\sqrt{T_2} \\
T_1 &= \tau_1 - t
\end{aligned}$$

[22](A11.4) has been derived *ceteris paribus*. This means that we have not taken into account the effect of a higher risk in terms of the rate of return μ, which would require additional assumptions about how these variables relate to each other.

$$
\begin{aligned}
T_2 &= \tau_2 - t \\
\rho &= \sqrt{\frac{T_1}{T_2}}
\end{aligned}
$$

and $\bar{V}$ is the implicit solution to

$$S\left(\bar{V}, \tau_1\right) = \bar{V} N\left(c_1\right) - \psi_2 e^{-rT} N\left(c_2\right) - \psi_1 = 0 \tag{A11.6}$$

where

$$
\begin{aligned}
c_1 &= \frac{\ln\left(\frac{\bar{V}}{\psi_2}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \\
c_2 &= c_1 - \sigma\sqrt{T} \\
T &= \tau_2 - \tau_1
\end{aligned}
$$

In section 3.2 we saw that no explicit solution exists for $\bar{V}$. It is possible however to derive how this bankruptcy-triggering firm value depends on different variables and parameters, even with the absence of such an explicit solution.

First note that $\bar{V}$ does not depend on current firm value, V, expected rate of return on V, μ, or time to maturity of the first debt payment, τ_1, as long as they do not appear in (A11.6). The reason is that $\bar{V}$ is actually the firm value that makes an option on the firm at τ_1, with maturity at τ_2, and strike equal to ψ_2 to worth exactly ψ_1. This option value does not depend on previous (to τ_1) firm values, previous periods, or μ (this last case by standard option valuation arguments).

$\bar{V}$ will be a function of those variables and parameters that do appear in (A11.6): Formally, $\bar{V} = \bar{V}\left(r, \psi_1, \psi_2, T, \sigma\right)$. The derivative of $\bar{V}$ with respect to variable or parameter j, $\bar{V}_j$, can be found as $\bar{V}_j = -\frac{S\left(\bar{V},\tau_1\right)_j}{S\left(\bar{V},\tau_1\right)_{\bar{V}}}$.[23] Then

[23]The equality $\frac{\psi_2 e^{-rT} f(c_2)}{\bar{V} f(c_1)} = 1$ is used. $f(\cdot)$ denotes the standard normal density function.

$$\bar{V}_r = -\frac{T\psi_2 e^{-rT} N(c_2)}{N(c_1)} < 0 \quad \text{(A11.7)}$$

$$\bar{V}_{\psi_1} = \frac{1}{N(c_1)} > 0 \quad \text{(A11.8)}$$

$$\bar{V}_{\psi_2} = \frac{e^{-rT} N(c_2)}{N(c_1)} > 0 \quad \text{(A11.9)}$$

$$\bar{V}_T = -\frac{\bar{V} f(c_1) \frac{\sigma}{2\sqrt{T}} + r\psi_2 e^{-rT} N(c_2)}{N(c_1)} < 0 \quad \text{(A11.10)}$$

$$\bar{V}_\sigma = -\frac{\bar{V} f(c_1) \sqrt{T}}{N(c_1)} < 0 \quad \text{(A11.11)}$$

Not surprisingly given the interpretation made about $\bar{V}$, its value behaves in the opposite direction of the call option does with respect to changes in risk free interest rate, strike, time to maturity and volatility. Finally, the higher ψ_1, the higher the firm value that makes the call option to worth exactly ψ_1. A special remark can be done with respect to (A11.11): Higher volatility reduces the short-run bankruptcy-triggering firm value, given the positive effect on the call option. This means that, ceteris paribus, riskier firms support lower V values in the short-run without defaulting. This fact drive us to the question of what is the net effect of a higher volatility on the short-run default probability. As we will see later on, the dependence of $\bar{V}$ on σ is a key element for determining whether or not entering riskier business has a positive or negative effect on this probability.

Short-Run Default Probability

The default probability in the short-run is the probability of observing at τ_1 the firm value to fall below $\bar{V}$. If we denote by P^1 this probability, then

$$P^1 = 1 - N(n) \quad \text{(A11.12)}$$

where

$$n = \frac{\ln\left(\frac{V}{\bar{V}}\right) + \left(\mu - \frac{\sigma^2}{2}\right) T_1}{\sigma\sqrt{T_1}}$$

The problem of analyzing the dependence of P^1 on those variables and parameters that affect it, seems quite similar to that made in the case of only one payment, however, we should take into account two important differences: First, the bankruptcy-triggering firm value in this case is not constant as before, that is, it changes if the variables or parameters change. Second, the time dependence is substantially complicated. In fact, for the case of only one payment, where the variables related to time were t, and τ, it was enough to obtain P_t, given that $P_T = P_\tau = -P_t$, and all the time dependence was summarized in P_t. Now we should analyze three alternatives:

a) P^1_t: Time to maturity of the short and long-term debt reduces, but time between short and long-term debt remains constant.

b) $P^1_{\tau_1}$: Time to maturity of the short-term debt increases, and time between short and long-term debt reduces (note how the presence of long-term debt makes a) and b) not to be formally the same as before).

c) $P^1_{\tau_2}$: Time to maturity of the long-term debt increases, and time between short and long-term debt also increases.

What we need to have in mind is whether or not T changes, because this has a direct influence on $\bar{V}$, and therefore on P^1 (and P^2). Specifically $P^1 = P^1\left(V, \mu, \bar{V}, T_1, \sigma\right) = P^1\left(V, r, \mu, \psi_1, \psi_2, t, \sigma\right)$. We next describe the relationship of these variables and parameters with P^1; Results (A11.7) - (A11.11) are used:

$$P^1_V = -\frac{f(n)}{V\sigma\sqrt{T_1}} < 0 \tag{A11.13}$$

The higher the current firm value, the higher the expected firm value at maturity and the lower the default probability.

$$P^1_r = \frac{f(n)}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_r < 0 \tag{A11.14}$$

(A11.14) links P^1 with the risk free interest rate, finding an inverse relation.

$$P^1_\mu = -\frac{f(n)}{\sigma}\sqrt{T_1} < 0 \qquad \text{(A11.15)}$$

Higher μ implies higher expected firm value at τ_1, and therefore lower probability of observing this value below $\bar{V}$.

$$P^1_{\psi_1} = \frac{f(n)}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_{\psi_1} > 0 \qquad \text{(A11.16)}$$

$$P^1_{\psi_2} = \frac{f(n)}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_{\psi_2} > 0 \qquad \text{(A11.17)}$$

(A11.16) and (A11.17) are the consequence of the influence of ψ_1 and ψ_2 on $\bar{V}$.

$$P^1_t = -f(n)\frac{\ln\left(\frac{V}{\bar{V}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T_1}{2\sigma(T_1)^{3/2}} \gtreqqless 0 \; as \; V \lesseqqgtr \tilde{V}_{G1} \qquad \text{(A11.18)}$$

where

$$\tilde{V}_{G1} = \bar{V}e^{\left(\mu - \frac{\sigma^2}{2}\right)T_1} \qquad \text{(A11.19)}$$

We can interpret $\tilde{V}_{G1}$ in the same way we interpreted $\tilde{V}_M$: Basically, for V low enough, the closer is t to τ_1, the lower the chances of a growth in V large enough for it to excess the critical threshold $\bar{V}$ at τ_1.

$$P^1_{\tau_1} = f(n)\frac{\ln\left(\frac{V}{\bar{V}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T_1 - 2\varepsilon_{\bar{V},T}\left(\frac{\rho^2}{1-\rho^2}\right)}{2\sigma(T_1)^{3/2}} \lesseqqgtr 0 \qquad \text{(A11.20)}$$

$$as \; V \lesseqqgtr \tilde{V}_{G1}e^{2\varepsilon_{\bar{V},T}\left(\frac{\rho^2}{1-\rho^2}\right)} \qquad \text{(A11.21)}$$

$\varepsilon_{\bar{V},T}$ is the elasticity of $\bar{V}$ with respect to T, that could be obtained from (A11.10). Note that $P^1_{\tau_1}$ is not exactly $-P^1_t$. In this case it is not enough that $V \lesseqqgtr \tilde{V}_{G1}$ to observe that a higher τ_1 reduces the short-run default probability, because higher τ_1 increases now $\bar{V}$, and this tends to increase this probability. We should have an even lower firm value.[24]

$$P^1_{\tau_2} = \frac{f(n)}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_T < 0 \tag{A11.22}$$

Higher τ_2 reduces P^1 because of its influence on $\bar{V}$.

$$P^1_\sigma = \frac{f(n)}{\sigma}\left(\frac{\varepsilon_{\bar{V},\sigma}}{\sigma\sqrt{T_1}} + n + \sigma\sqrt{T_1}\right) \lesseqqgtr 0 \ as \ V \lesseqqgtr \hat{V}_{G1} \tag{A11.23}$$

where

$$\hat{V}_{G1} = \bar{V}e^{-\left[\left(\mu+\frac{\sigma^2}{2}\right)T_1+\varepsilon_{\bar{V},\sigma}\right]} \tag{A11.24}$$

and $\varepsilon_{\bar{V},\sigma} < 0$ is the elasticity of $\bar{V}$ with respect to σ, which can be easily derived from (A11.11). Note that $\varepsilon_{\bar{V},\sigma}$ increases $\hat{V}_{G1}$, that is, the higher the effect of σ on $\bar{V}$, the easier to find that increasing the firm return volatility reduces the short-run default probability. Again this does not mean that total debt value increases with volatility under $V < \hat{V}_{G1}$: σ has a positive effect on the equity value, and assuming σ does not alter V, the consequence is a negative effect on total debt value.

Long-Run Default Probability

The probability of defaulting the payment of ψ_2, P^2, is the probability of non observing $V(\tau_1) > \bar{V}$ and $V(\tau_2) > \psi_2$. Formally

$$P^2 = 1 - N_2(n, m; \rho) \tag{A11.25}$$

[24] $\tilde{V}_{G1}e^{2\varepsilon_{\bar{V},T}\left(\frac{\rho^2}{1-\rho^2}\right)} < \tilde{V}_{G1}$.

where

$$m = \frac{\ln\left(\frac{V}{\psi_2}\right) + \left(\mu - \frac{\sigma^2}{2}\right) T_2}{\sigma\sqrt{T_2}}$$

P^2 is a function of the same variables and parameters than P^1, that is, $P^1 = P^1\left(V, r, \mu, \psi_1, \psi_2, t, \sigma\right)$. Let denote

$$\begin{aligned} J &= f(n)\, N\left(\frac{m - \rho n}{\sqrt{1-\rho^2}}\right) = f(n)\, N(A) > 0 \\ H &= f(m)\, N\left(\frac{n - \rho m}{\sqrt{1-\rho^2}}\right) = f(m)\, N(B) > 0 \end{aligned}$$

then

$$P_V^2 = -\frac{1}{V\sigma}\left(\frac{J}{\sqrt{T_1}} + \frac{H}{\sqrt{T_2}}\right) < 0 \qquad \text{(A11.26)}$$

The higher the current firm value, the higher the expected firm value both in the short and in the long-run, and the lower the probability of defaulting the payment of ψ_2.

$$P_r^2 = \frac{J}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_r < 0 \qquad \text{(A11.27)}$$

In the same way it happened with P^1, we find an inverse relation between the risk free interest rate and the long-run default probability. In both cases due to the negative effect on $\bar{V}$.

$$P_\mu^2 = -\frac{1}{\sigma}\left(J\sqrt{T_1} + H\sqrt{T_2}\right) < 0 \qquad \text{(A11.28)}$$

(A11.28) reflects that the higher the expected growth in V, the lower the probability of observing this value below critical thresholds $\bar{V}$ and ψ_2.

$$
\begin{aligned}
P_{\psi_1}^2 &= \frac{J}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_{\psi_1} < 0 && \text{(A11.29)}\\
P_{\psi_2}^2 &= \frac{J}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_{\psi_2} + \frac{H}{\psi_2\sigma\sqrt{T_2}} > 0 && \text{(A11.30)}
\end{aligned}
$$

(A11.29) and (A11.30) have a clear interpretation: Higher nominal payments imply higher threshold values $\bar{V}$ and ψ_2, what leads to a higher long-run default probability.

$$
\begin{aligned}
P_t^2 &= -J\left[\frac{\ln\left(\frac{V}{\bar{V}}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T_1}{2\sigma\left(T_1\right)^{3/2}}\right]\\
&\quad -H\left[\frac{\ln\left(\frac{V}{\psi_2}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T_2}{2\sigma\left(T_2\right)^{3/2}}\right]\\
&\quad +\frac{1-\rho^2}{2\rho T_2}f_2\left(n,m;\rho\right)\\
&= \sqrt{1-\rho^2}f_2\left(n,m;\rho\right)\left\{\frac{\sqrt{1-\rho^2}}{2\rho T_2}\right.\\
&\quad -\frac{N\left(A\right)}{f\left(A\right)}\left[\frac{\ln\left(\frac{V}{\bar{V}}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T_1}{2\sigma\left(T_1\right)^{3/2}}\right]\\
&\quad \left.-\frac{N\left(B\right)}{f\left(B\right)}\left[\frac{\ln\left(\frac{V}{\psi_2}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T_2}{2\sigma\left(T_2\right)^{3/2}}\right]\right\} \qquad \text{(A11.31)}
\end{aligned}
$$

$P_t^2 \gtreqless 0$ as $V \lesseqgtr \tilde{V}_{G2t}$.[25] A detailed inspection of (A11.31) reveals that in

[25] $f_2\left(n,m;\rho\right)$ denotes the bivariate normal density function.

fact P_t^2 is positive for V lower that some threshold $\tilde{V}_{G2t}$, and negative for V higher than this value. To see this first note that second term in (A11.31) is positive for $V < \tilde{V}_{G1}$, and the third term is positive $V < \psi_2 e^{\left(\mu - \frac{\sigma^2}{2}\right)T_2}$. P_t^2 is then positive for V low enough. On the other hand these two terms are continuous and strictly decreasing functions in V,[26] that in addition tend to $-\infty$ as V tends to infinity. As a result there is a critical value $\tilde{V}_{G2t}$ below which P_t^2 is positive and above which P_t^2 is negative. No explicit expression exists for $\tilde{V}_{G2t}$, and it cannot even be argued if it is higher or lower than $\tilde{V}_{G1}$. The interpretation however is clear: For V low enough, as time goes by, the long-run default probability increases because this means lower time for V to grow above critical values $\bar{V}$ and ψ_2.

$$
\begin{aligned}
P_{\tau_1}^2 &= J\left[\frac{\ln\left(\frac{V}{\bar{V}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T_1 - 2\varepsilon_{\bar{V},T}\left(\frac{\rho^2}{1-\rho^2}\right)}{2\sigma\left(T_1\right)^{3/2}}\right] - \frac{1}{2\rho T_2} f_2\left(n,m;\rho\right) \\
&= f_2\left(n,m;\rho\right) \\
&\quad \left\{\frac{\sqrt{1-\rho^2}N\left(A\right)}{f\left(A\right)}\left[\frac{\ln\left(\frac{V}{\bar{V}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T_1 - 2\varepsilon_{\bar{V},T}\left(\frac{\rho^2}{1-\rho^2}\right)}{2\sigma\left(T_1\right)^{3/2}}\right]\right. \\
&\quad \left. - \frac{1}{2\rho T_2}\right\} \qquad \text{(A11.32)}
\end{aligned}
$$

$P_{\tau_1}^2 \lesseqgtr 0$ as $V \lesseqgtr \tilde{V}_{G2\tau_1}$. It could be derived an explicit solution for $\tilde{V}_{G2\tau_1}$, but no clear interpretation for this specific expression can be provided. It could be said however that, for V low enough, longer time to maturity of the short-term debt reduces the long-run default probability, essentially through the reduction in the short-run default probability.

---

[26] $\frac{N(x)}{f(x)}$ is positive and strictly increasing $\forall x$. A does not depend on V and B is strictly increasing in V. Therefore $\frac{N(A)}{f(A)}$ does not depend on V and $\frac{N(B)}{f(B)}$ is strictly increasing in V.

$$
\begin{aligned}
P^2_{\tau_2} &= J\frac{J}{\bar{V}\sigma\sqrt{T_1}}\bar{V}_T + H\left[\frac{\ln\left(\frac{V}{\psi_2}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T_2}{2\sigma\left(T_2\right)^{3/2}}\right] \\
&\quad +\frac{\rho}{2T_2}f_2\left(n,m;\rho\right) \\
&= \sqrt{1-\rho^2}f_2\left(n,m;\rho\right)\left\{\frac{N\left(A\right)}{f\left(A\right)}\frac{J}{\bar{V}\sigma\sqrt{T_1}}\right. \\
&\quad +\frac{N\left(B\right)}{f\left(B\right)}\left[\frac{\ln\left(\frac{V}{\psi_2}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T_2}{2\sigma\left(T_2\right)^{3/2}}\right] \\
&\quad \left. +\frac{\rho}{2T_2\sqrt{1-\rho^2}}\right\} \qquad \text{(A11.33)}
\end{aligned}
$$

$P^2_{\tau_2}$ may be positive or negative. The first term in (A11.33) is negative and constant in V, while the third term is positive and also constant. Finally, the second term is strictly increasing in V, and tends to infinity as V tends to infinity. These arguments allow us to ensure that $P^2_{\tau_2} > 0$ if V is large enough. However, it is not possible to argue that a threshold firm value determining the sign of $P^2_{\tau_2}$ exists in general (the limit of the second term as the firm value tends to zero is not determined).

$$
P^2_\sigma = \frac{1}{\sigma}\left[J\left(\frac{\varepsilon_{\bar{V},\sigma}}{\sigma\sqrt{T_1}}+n+\sigma\sqrt{T_1}\right)+H\left(m+\sigma\sqrt{T_2}\right)\right] \qquad \text{(A11.34)}
$$

$P^2_\sigma \lesseqqgtr 0$ as $V \lesseqqgtr \hat{V}_{G2}$. For V low enough, higher volatility reduces the long-run default probability due to higher chances of favorable firm value changes due to shocks.

We summarize results in Table 1.

	P	P^1	P^2
V	< 0	< 0	< 0
r	0	< 0	< 0
μ	< 0	< 0	< 0
ψ	> 0	$-$	$-$
ψ_1	$-$	> 0	> 0
ψ_2	$-$	> 0	> 0
t	$\gtreqqless 0$ *as* $V \lesseqqgtr \tilde{V}_M$	$\gtreqqless 0$ *as* $V \lesseqqgtr \tilde{V}_{G1}$	$\gtreqqless 0$ *as* $V \lesseqqgtr \tilde{V}_{G2t}$
τ_1	$-$	$\lesseqqgtr 0$ *as* $V \lesseqqgtr \tilde{V}_{G1} e^{2\varepsilon_{\bar{V},T}\left(\frac{\rho^2}{1-\rho^2}\right)}$	$\lesseqqgtr 0$ *as* $V \lesseqqgtr \tilde{V}_{G2\tau_1}$
τ_2	$-$	< 0	$\lesseqqgtr 0$ *as* $V \lesseqqgtr \tilde{V}_{G2\tau_2}$
σ	$\lesseqqgtr 0$ *as* $V \lesseqqgtr \hat{V}_M$	$\lesseqqgtr 0$ *as* $V \lesseqqgtr \hat{V}_{G1}$	$\lesseqqgtr 0^*$

Table 1: Comparative statics. $^* > 0$ for V large enough.

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