UNIVERSIDAD CARLOS III DE MADRID
working papers

Working Paper 02-18
Business Economics Series 12
May 2002

Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 123
28903 Getafe (Spain)
Fax (34) 91 624 9608

ON THE TIME VALUE OF RUIN IN THE DISCRETE TIME RISK MODEL

Shuanming Li [1] and José Garrido [2] *

Abstract

Using an approach similar to that of Gerber and Shiu (1998), a recursive formula is given for the expected discounted penalty due at ruin, in the discrete time risk model. With it the joint distribution of three random variables is obtained; time to ruin, the surplus just before ruin and the deficit at ruin.

The time to ruin is analyzed through its probability generating function (p.g.f.). The joint distribution for the compound binomial model is derived in Cheng et al. (2000) using martingale techniques and a duality argument. Here we find a recursive formula for the p.g.f. of ruin time T; the discounted moments of the deficit at ruin and the surplus just before ruin. A detailed discussion is given in the case u = 0 and when the claim size in a unit time is geometrically distributed.

[1] Department of Mathematics and Statistics Concordia University, Canada
[2] Business Department, University Carlos III of Madrid, Spain

* This research was funded by the Natural Sciences and Engineering Council of Canada (NSERC) operating grant OGP0036860 and the Spanish Department of Education, Sports and Culture.

On the Time Value of Ruin in the Discrete Time Risk Model

Shuanming Li[1] and José Garrido[1,2*]
[1]Department of Mathematics and Statistics
Concordia University, Canada
[2]Department of Business Administration
University Carlos III of Madrid, Spain

Abstract

Using an approach similar to that of Gerber and Shiu (1998), a recursive formula is given for the expected discounted penalty due at ruin, in the discrete time risk model. With it the joint distribution of three random variables is obtained; time to ruin, the surplus just before ruin and the deficit at ruin.

The time to ruin is analyzed through its probability generating function (p.g.f.). The joint distribution for the compound binomial model is derived in Cheng et al. (2000) using martingale techniques and a duality argument. Here we find a recursive formula for the p.g.f. of ruin time T, the discounted moments of the deficit at ruin and the surplus just before ruin. A detailed discussion is given in the case $u = 0$ and when the claim size in a unit time is geometrically distributed.

1 Introduction

Problems associated with the calculation of ultimate ruin probabilities, for the continuous time risk model, have received considerable attention in recent years.

*This research was funded by the Natural Sciences and Engineering Council of Canada (NSERC) operating grant OGP0036860 and the Spanish Department of Education, Sports and Culture.

These include studies of the distribution of the ruin time (finite-time ruin probabilities), the surplus before ruin and the deficit at ruin, as well as moments of these variables.

We explore analog problems, but in the discrete time risk model. Section 2 gives a recursive formula for the expected discounted penalty due at ruin, using an approach based on Gerber and Shiu (1998), but here with a discrete model. This discounted penalty depends on the deficit at ruin and the surplus just before ruin. Hence, our recursive formula yields the joint distribution of the three random variables time to ruin, the surplus just before ruin and the deficit at ruin.

Given the discrete nature of our model, probability generating functions (p.g.f.) are used throughout to analyze the time of ruin and its associated random variables. The joint distribution for the compound binomial model is derived in Cheng et al. (2000) using martingale techniques and a duality argument. Instead, in Section 4 we find a recursive formula for the p.g.f. of ruin time T, the discounted moments of the deficit at ruin and the surplus just before ruin. Section 3 gives a detailed discussion for the case $u = 0$ and when the claim size in a unit time period is geometrically distributed.

These results are of independent interest and can give a better understanding of the analogous results in the continuous time model, as a limit case of the discrete time model.

2 Model Description and Notations

Consider the discrete time surplus process

$$U(t) = u + t - \sum_{i=1}^{t} X_i \,, \quad t = 1, 2, \dots \,,$$

where $u \in \mathbb{N}$ is the initial reserve. The X_i are i.i.d. random variables with common probability function (p.f.) $p(k) = P(X = k)$, for $k = 0, 1, 2, \dots$, denoting the total claim amount in period i, occuring at the end of the period. Denote by $\mu_k = E[X^k]$ the k-th moment of X and by $\hat{p}(s) = \sum_{i=0}^{\infty} s^i p(i)$ its p.g.f..

Without loss of generality we assume that the premium income per unit time is 1, and, in order to have a positive loading factor, that $\mu_1 < 1$.

Now define (the possibly defective random variable)

$$T = \min\{t = 1, 2, \dots ;\ U(t) \le 0\}$$

to be the ruin time,

$$\Psi(u) = P(T < \infty \,|\, U(0) = u) \ , \quad u \in \mathbb{N} \ ,$$

to be the ultimate ruin probability and

$$\psi(u, t) = P(T = t \,|\, U(0) = u) \ , \quad t = 1, 2, 3, \dots \ ,$$

to be the ruin probability at time t.

Consider $f_3(x, y, t \,|\, u) = P\{U(T-1) = x,\ |U(T)| = y,\ T = t \,|\, U(0) = u\}$, $x, y \in \mathbb{N}$, the joint probability function of the surplus just before ruin, deficit at ruin and ruin time. Let $v \in (0, 1)$ be the (constant) discount factor over one period and define $f_2(x, y \,|\, u) = \sum_{t=1}^{\infty} v^t f_3(x, y, t \,|\, u)$ as a discounted joint p.d.f. of $U(T-1)$ and $|U(T)|$. Similarly, denote by $f(x \,|\, u) = \sum_{y=0}^{\infty} f_2(x, y \,|\, u)$. The usual conditional probability formulas give the following relation:

$$f_2(x, y \,|\, u) = f(x \,|\, u) \frac{p(x+y+1)}{\sum_{k=x+1}^{\infty} p(k)} \ , \quad x, y \in \mathbb{N} \ .$$

One of our goals is to find $f_2(x, y \,|\, 0)$ and $f_2(x, y \,|\, u)$ for any $x, y \in \mathbb{N}$. In Cheng et al. (2000), the authors find $f_2(x, y \,|\, 0)$ using martingale properties and a duality argument for the compound binomial model.

Let $w(x, y)$, $x, y = 0, 1, 2, \dots$ be the non-negative values of a penalty function. For $0 < v < 1$, define

$$\phi(u) = E\left[v^T w\big(U(T-1), |U(T)|\big) I(T < \infty) \;\Big|\; U(0) = u\right] \ , \quad u \in \mathbb{N} \ . \qquad (1)$$

The quantity $w\big(U(T-1), |U(T)|\big)$ can be interpreted as the penalty at the time of ruin for the surplus $U(T-1)$ and deficit $|U(T)|$. Then $\phi(u)$ is the expected discounted penalty if v is viewed as a discount rate.

We will use a technique similar to that in Gerber and Shiu (1998) to give a recursive expression for $\phi(u)$.

Theorem 1

$$\phi(0) = v \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \rho^x w(x, y) p(x+y+1) \ , \qquad (2)$$

and for $u \in \mathbb{N}^+$,

$$\phi(u) = v \sum_{x=0}^{u-1} \phi(u-x) \sum_{y=0}^{\infty} \rho^y p(x+y+1) + v\rho^{-u} \sum_{x=u}^{\infty} \rho^x \sum_{y=0}^{\infty} w(x,y) p(x+y+1) \ ,$$

where $0 < \rho < 1$ is the root of the equation

$$q(s) := \frac{\hat{p}(s)}{s} = \frac{1}{v} \ . \tag{3}$$

Proof: Conditioning on the first claim X_1, we have $\phi(u) = vE[\phi(u+1-X_1)]$, for $u \in \mathbb{N}$. Hence

$$\phi(0) = p(0)v\phi(1) + v \sum_{x=1}^{\infty} w(0, x-1)p(x) \tag{4}$$

and

$$\begin{aligned} \phi(u) &= p(0)v\phi(u+1) + v \sum_{x=1}^{u} \phi(u+1-x)p(x) \\ &\quad + v \sum_{x=u+1}^{\infty} w(u, x-u-1)p(x) \ , \quad u \in \mathbb{N}^+ \ . \end{aligned} \tag{5}$$

For $0 < \rho < 1$, define $\phi_\rho(u) = \rho^u \phi(u)$. Then for $u \in \mathbb{N}^+$ we have

$$\phi_\rho(u) = \frac{vp(0)}{\rho}\phi_\rho(u+1) + v \sum_{x=1}^{u} \rho^{x-1} \phi_\rho(u+1-x)p(x) + v\rho^u \sum_{x=u+1}^{\infty} w(u, x-u-1)p(x) \ ,$$

Rearranging terms

$$\begin{aligned} \phi_\rho(u+1) &= \frac{\rho}{vp(0)}\phi_\rho(u) - \frac{1}{p(0)} \sum_{x=1}^{u} \rho^x \phi_\rho(u+1-x)p(x) \\ &\quad - \frac{1}{p(0)}\rho^{u+1} \sum_{x=u+1}^{\infty} w(u, x-u-1)p(x) \end{aligned}$$

and hence $\Delta\phi_\rho(u) := \phi_\rho(u+1) - \phi_\rho(u)$ is given by

$$\begin{aligned} \Delta\phi_\rho(u) &= \phi_\rho(u) \left[\frac{\rho/v - p(0)}{p(0)}\right] - \frac{1}{p(0)} \left\{ \sum_{x=1}^{u} \rho^x \phi_\rho(u+1-x)p(x) \right. \\ &\quad \left. - \rho^{u+1} \sum_{x=u+1}^{\infty} w(u, x-u-1)p(x) \right\} \ , \quad u \in \mathbb{N}^+ \ . \end{aligned} \tag{6}$$

Now assume there is a $\rho \in (0,1)$ such that $\frac{\hat{p}(\rho)}{\rho} = \frac{1}{v}$ (its existence is discussed after the proof). Then (6) becomes

$$\Delta\phi_\rho(u) = \phi_\rho(u)\left[\frac{\hat{p}(\rho)-p(0)}{p(0)}\right] - \frac{1}{p(0)}\left\{\sum_{x=1}^{u}\rho^x\phi_\rho(u+1-x)p(x) - \rho^{u+1}\sum_{x=u+1}^{\infty} w(u,x-u-1)p(x)\right\} . \quad (7)$$

Summing from $u=1$ to $m-1$ and rearranging terms, we obtain

$$\begin{aligned}\phi_\rho(m) &= \phi_\rho(1) + \frac{1}{p(0)}\left\{\sum_{u=1}^{\infty}\rho^u p(u)\sum_{x=1}^{m-1}\phi_\rho(x) - \sum_{x=1}^{m-1}\sum_{u=x}^{m-1}\rho^{u+1-x}p(u+1-x)\phi_\rho(x) - \sum_{u=1}^{m-1}\rho^{u+1}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\} \\ &= \phi_\rho(1) + \frac{1}{p(0)}\left\{\sum_{x=1}^{m-1}\phi_\rho(x)\sum_{u=m-x+1}^{\infty}\rho^u p(u) - \sum_{u=1}^{m-1}\rho^{u+1}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\} . \quad (8)\end{aligned}$$

Since $\lim_{m\to\infty}\phi_\rho(m) = 0$, for $0<\rho<1$, we have

$$\begin{aligned}\phi_\rho(1) &= \frac{1}{p(0)}\sum_{u=1}^{\infty}\rho^{u+1}\sum_{x=u}^{\infty}w(u,x-u)p(x+1) , \\ \phi(1) &= \frac{1}{p(0)}\sum_{u=1}^{\infty}\rho^{u}\sum_{x=u}^{\infty}w(u,x-u)p(x+1) . \quad (9)\end{aligned}$$

From (4) and (9) we have

$$\begin{aligned}\phi(0) &= v\sum_{u=1}^{\infty}\rho^u\sum_{x=u}^{\infty}w(u,x-u)p(x+1) + v\sum_{x=1}^{\infty}w(0,x-1)p(x) \\ &= v\sum_{u=0}^{\infty}\rho^u\sum_{x=u}^{\infty}w(u,x-u)p(x+1) = v\sum_{u=0}^{\infty}\rho^u\sum_{x=0}^{\infty}w(u,x)p(x+u+1)\end{aligned}$$

and in turn (8) implies that

$$\phi_\rho(m) = \frac{1}{p(0)}\left\{\sum_{x=1}^{m-1}\phi_\rho(x)\sum_{u=m-x+1}^{\infty}\rho^u p(u) + \sum_{u=m}^{\infty}\rho^{u+1}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\} .$$

Hence

$$\begin{aligned}
\phi(m) &= \frac{1}{p(0)}\left\{\sum_{x=1}^{m-1}\phi(x)\sum_{u=m-x+1}^{\infty}\rho^{u-m+x}p(u)\right.\\
&\qquad\left.+\sum_{u=m}^{\infty}\rho^{u+1-m}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\}\\
&= \frac{1}{p(0)}\left\{\sum_{y=1}^{m-1}\phi(m-y)\sum_{x=0}^{\infty}\rho^{x+1}p(x+y+1)\right.\\
&\qquad\left.+\sum_{u=m}^{\infty}\rho^{u+1-m}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\}\\
&= \frac{1}{p(0)}\left\{\sum_{y=0}^{m-1}\phi(m-y)\sum_{x=0}^{\infty}\rho^{x+1}p(x+y+1)-\phi(m)\sum_{x=0}^{\infty}\rho^{x+1}p(x+1)\right.\\
&\qquad\left.+\sum_{u=m}^{\infty}\rho^{u+1-m}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\}\ .
\end{aligned}$$

Finally,

$$\begin{aligned}
\phi(m) &= \frac{\rho}{\hat{p}(\rho)}\left\{\sum_{y=0}^{m-1}\phi(m-y)\sum_{x=0}^{\infty}\rho^{x}p(x+y+1)\right.\\
&\qquad\left.+\sum_{u=m}^{\infty}\rho^{u-m}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\right\}\\
&= v\sum_{y=0}^{m-1}\phi(m-y)\sum_{x=0}^{\infty}\rho^{x}p(x+y+1)+v\sum_{u=m}^{\infty}\rho^{u-m}\sum_{x=u}^{\infty}w(u,x-u)p(x+1)\\
&= v\sum_{y=0}^{m-1}\phi(m-y)\sum_{x=0}^{\infty}\rho^{x}p(x+y+1)+v\sum_{u=m}^{\infty}\rho^{u-m}\sum_{x=0}^{\infty}w(u,x)p(x+u+1)\ .
\end{aligned}$$

This completes the proof. □

Remark: If instead, the ruin time is defined to be the first time that the surplus drops to a strictly negative value, $T^* = \min\{t = 1, 2, \dots\ ;\ U(t) < 0\}$, then it is easily seen that $\phi^*(u) := E[v^{T^*}w\big(U(T^*-1), |U(T^*)|\big)I(T^* < \infty)\ \big|\ U(0) = u]$ satisfies the following recursive formula, slightly different of that in Theorem 1:

$$\phi^*(0) = \frac{1}{p(0)}\rho\sum_{x=0}^{\infty}\rho^{x}\sum_{y=1}^{\infty}w(x,y)p(x+y+1)\ ,$$

and for $u \in \mathbb{N}^+$

$$\phi^*(u) = v \sum_{x=0}^{u} \phi^*(u-x) \sum_{y=0}^{\infty} \rho^y p(x+y+1) + v\rho^{-u} \sum_{x=u}^{\infty} \rho^x \sum_{y=1}^{\infty} w(x,y) p(x+y+1) \, .$$

By definition, q in (3) is strictly convex. Hence equation (3) has at most two solutions. Further, $\lim_{s\to 0+} q(s) = +\infty$ and $q(1) = 1 < \frac{1}{v}$, implying that for a fixed $v \in (0,1)$, $q(s) = \frac{\hat{p}(s)}{s} = \frac{1}{v}$ has a unique solution between 0 and 1, say $s = \rho$.

If X is sufficiently regular, then equation (3) also has a solution greater than 1, say $s = R$. When $v = 1$, this R is the classical *adjustment coefficient* for the discrete model.

Now, here the solution ρ of (3) is a function of the discount factor v. Call it $\rho(v)$ and consider the following properties.

Lemma 1 If $\rho(v) \in (0,1)$ is the solution of $q(s) = \frac{\hat{p}(s)}{s} = \frac{1}{v}$, for $v \in (0,1)$, then

a) $0 < \rho(v) < v$,

b) $\lim_{v\to 0} \rho(v) = 0$ and $\lim_{v\to 1} \rho(v) = 1$,

c) Derivatives of ρ of any order exist, especially

$$\rho'(v) = \frac{\hat{p}(\rho)}{1 - v\hat{p}'(\rho)} \, , \quad \rho''(v) = \frac{v\hat{p}''(\rho)[\rho'(v)]^2 + 2\hat{p}'(\rho)\rho'(v)}{1 - v\hat{p}'(\rho)} \, ,$$

where $\lim_{v\to 0} \rho'(v) = p(0)$, $\lim_{v\to 1} \rho'(v) = \frac{1}{1-\mu_1}$, $\lim_{v\to 0} \rho''(v) = 2p(1)p(0)$ and $\lim_{v\to 1} \rho''(v) = \frac{\mu_2 + \mu_1(1-2\mu_1)}{(1-\mu_1)^3}$.

Further discussion of the properties of ρ and R, under zero and negative loading factors, can be found in Cheng et al. (2000). For simplicity, we have denoted throughout the paper $\rho(v)$ by ρ, when $v \in (0,1)$ is fixed.

3 Analysis of the $u = 0$ case

Since $f_3(x, y, t \,|\, 0)$ is the joint p.f. of $U(T-1)$, $|U(T)|$ and T, then

$$\begin{aligned}
\phi(0) &= E\left[v^T w\big(U(T-1), |U(T)|\big) I(T<\infty) \;\big|\; U(0)=0\right] \\
&= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\sum_{t=1}^{\infty} v^t w(x,y) f_3(x,y,t \,|\, 0) \\
&= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} w(x,y) \sum_{t=1}^{\infty} v^t f_3(x,y,t \,|\, 0) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} w(x,y) f_2(x,y \,|\, 0) \;.
\end{aligned}$$

Comparing with Theorem 1, we find that $f_2(x, y \,|\, 0) = v\rho^x p(x + y + 1)$, implying that $g(y \,|\, 0) = \sum_{x=0}^{\infty} f_2(x, y \,|\, 0) = \sum_{x=0}^{\infty} v\rho^x p(x + y + 1)$ and $f(x \,|\, 0) = \sum_{y=0}^{\infty} f_2(x, y \,|\, 0) = v\rho^x \sum_{y=x+1}^{\infty} p(y)$.

Now

$$\begin{aligned}
E\left[v^T I(T<\infty) \;\big|\; U(0)=0\right] &= \sum_{y=0}^{\infty} g(y \,|\, 0) = v \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} \rho^x p(x+y+1) \\
&= v \sum_{x=0}^{\infty}\sum_{y=x+1}^{\infty} \rho^x p(y) = v \sum_{y=1}^{\infty}\sum_{x=0}^{y-1} \rho^x p(y) = v \sum_{y=1}^{\infty} p(y) \frac{1-\rho^y}{1-\rho} \\
&= \frac{v[1-\hat{p}(\rho)]}{1-\rho} = \frac{v-\rho}{1-\rho} = 1 - \frac{1-v}{1-\rho} \;, \qquad (10)
\end{aligned}$$

and hence by Lemma 1-(c) we reproduce the well known result (Gerber, 1988):

$$\begin{aligned}
\Psi(0) &= P\left[T<\infty \;\big|\; U(0)=0\right] \\
&= \lim_{v\to 1} E\left[v^T I(T<\infty) \;\big|\; U(0)=0\right] = \lim_{v\to 1} \frac{1-\rho'(v)}{-\rho'(v)} = \mu_1 \;. \qquad (11)
\end{aligned}$$

On the other hand, $E\left[v^T I(T < \infty) \,|\, U(0) = 0\right] = \sum_{n=1}^{\infty} v^n P[T = n \,|\, U(0) = 0]$ and hence

$$P\left[T=n \;\big|\; U(0)=0\right] = \frac{\frac{d^n}{dv^n}\left(\frac{v-1}{1-\rho}\right)\Big|_{v=0}}{n!} \;.$$

After some simplifications, we have

$$\begin{aligned}
P\left[T=1 \;\big|\; U(0)=0\right] &= 1-\rho'(0) = 1-p(0) \;, \\
P\left[T=2 \;\big|\; U(0)=0\right] &= \frac{-\rho''(0)-2\rho'(0)^2+2\rho'(0)}{2} = p(0)-p(0)^2-p(0)p(1) \;.
\end{aligned}$$

In general, it is difficult to obtain $P[T = n \,|\, U(0) = 0]$ in closed form. Only for some special distributions of X can explicit expressions be found.

Example 1 (Random Walk)
If $P(X = 0) = p$ and $P(X = 2) = 1 - p = q$, then $\hat{p}(\rho) = p + q\rho^2$ and $\rho = \frac{1-\sqrt{1-4q(1-q)v^2}}{2qv}$. Hence

$$E[v^T I(T < \infty) \,\big|\, U(0) = 0] = 1 + \frac{v-1}{1-\rho} = 1 + \frac{(2qv-1) - \sqrt{1-4q(1-q)v^2}}{2} \,.$$

Now, since

$$\begin{aligned}\sqrt{1-4q(1-q)v^2} &= 1 + \frac{1}{2}[-4q(1-q)v^2] + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}[-4q(1-q)v^2]^2 + \cdots \\ &+ \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}[-4q(1-q)v^2]^n + \cdots \qquad (12)\end{aligned}$$

as expected $P[T = 1 \,|\, U(0) = 0] = q$, $P[T = 2 \,|\, U(0) = 0] = q(1-q)$ and

$$\begin{aligned} P[T = 2n-1 \,\big|\, U(0) = 0] &= 0 \,, \quad n = 2, 3, \ldots \\ P[T = 2n \,\big|\, U(0) = 0] &= \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2 \times n!}[2q(1-q)]^n \,. \end{aligned}$$

Example 2 (Geometric Claims)
If $X \sim$ geometric(θ), i.e. $P(X = k) = (1-\theta)\theta^k$, for $k \in \mathbb{N}$, then $\hat{p}(\rho) = \frac{1-\theta}{1-\rho\theta}$ and $\rho = \frac{1-\sqrt{1-4\theta(1-\theta)v}}{2\theta}$. Hence

$$E[v^T I(T < \infty) \,\big|\, U(0) = 0] = \frac{v-\rho}{1-\rho} = \frac{1}{2(1-\theta)}[1 - \sqrt{1-4\theta(1-\theta)v}] \,.$$

An argument similar to (12), for θ and v rather than q and v^2, gives

$$\begin{aligned} P[T = 1 \,\big|\, U(0) = 0] &= \theta \,, \\ P[T = n \,\big|\, U(0) = 0] &= \frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{n!} 2^n \theta^n (1-\theta)^{n-1} \,, \quad n = 2, 3, \ldots \end{aligned}$$

3.1 Moments and covariances

Using Theorem 1 we now obtain the joint and marginal moments of $U(T-1)$, $|U(T)|$ and T when $u = 0$, by specifying $w(x, y)$. To simplify calculations the equilibrium distribution of X is introduced. It will play a critical role in what follows.

3.1.1 Equilibrium distribution of a discrete random variable

Let X be the non-negative r.v. taking integer values of Section 2, with its p.f. denoted $p(x)$, its k-th moment μ_k and its survival function $\bar{P}(x) = P(X > x) = \sum_{y=x+1}^{\infty} p(y)$, $x \in \mathbb{N}$. Consider the *equilibrium distribution* of p, defined as

$$p_1(x) := \frac{\bar{P}(x)}{\mu_1} = \frac{\sum_{y=x+1}^{\infty} p(y)}{\mu_1}, \quad x \in \mathbb{N} .$$

Now define $\mu_{(n)} := E[X^{(n)}]$ to be the n-th factorial moment of X, where here $x^{(n)} = x(x-1)\cdots(x-n+1)$ denotes the n-th factorial power of x and $x^{(0)} = 1$. It is well known in summation calculus (see e.g. Hamming, 1973, p.182) that $\sum_{k=x}^{y} k^{(n)} = \frac{(y+1)^{(n+1)} - x^{(n+1)}}{n+1}$, for $n \in \mathbb{N}^+$, $x, y \in \mathbb{N}$, and $x \leq y$. Hence the n-th factorial moment $\mu_{1:(n)}$ of p_1 is given by

$$\begin{aligned} \mu_{1:(n)} &= \sum_{x=1}^{\infty} x^{(n)} p_1(x) = \frac{1}{\mu_1} \sum_{x=1}^{\infty} x^{(n)} \sum_{y=x+1}^{\infty} p(y) = \frac{1}{\mu_1} \sum_{y=2}^{\infty} p(y) \sum_{x=1}^{y-1} x^{(n)}, \quad n \geq 1, \\ &= \frac{1}{\mu_1} \sum_{y=2}^{\infty} p(y) \frac{y^{(n+1)}}{n+1} = \frac{\mu_{(n+1)}}{\mu_1 (n+1)} = \frac{\mu_{(n+1)}}{\mu_{(1)}(n+1)}, \quad \text{for } \mu_1 = \mu_{(1)} . \end{aligned} \tag{13}$$

Similarly, the p.g.f. of the equilibrium distribution p_1 is given by

$$\hat{p}_1(s) = \sum_{x=0}^{\infty} s^x p_1(x) = \frac{1 - \hat{p}(s)}{\mu_1 (1-s)}, \quad -1 \leq s \leq 1 ,$$

with $\hat{p}_1(1) = 1$, and its survival function is

$$\begin{aligned} \bar{P}_1(x) &= \sum_{y=x+1}^{\infty} p_1(y) = \frac{1}{\mu_1} \sum_{y=x+1}^{\infty} \sum_{k=y+1}^{\infty} p(k), \quad x \in \mathbb{N} , \\ &= \frac{1}{\mu_1} \sum_{k=x+2}^{\infty} p(k)[k - (x+1)] = \frac{1}{\mu_1} \sum_{k=x+1}^{\infty} p(k)[k - (x+1)] . \end{aligned} \tag{14}$$

Now define the equilibrium distribution of p_1, or equivalently, the *second order equilibrium distribution* of p :

$$p_2(x) = \frac{\bar{P}_1(x)}{\mu_{1:1}} = \frac{1}{\mu_{1:1}\mu_1} \sum_{y=x+1}^{\infty} [y-(x+1)]p(y) \ , \quad x \in \mathbb{N} \ ,$$

where $\mu_{1:1}$ is the first order moment of p_1. The factorial moments of p_2 are obtained as in (13) to be

$$\mu_{2:(n)} = \frac{\mu_{1:(n+1)}}{\mu_{1:1}(n+1)} = \frac{\mu_{1:(n+1)}}{\mu_{1:(1)}(n+1)} \ .$$

Then the p.g.f. of $p_2(x)$ is given by

$$\hat{p}_2(s) = \sum_{x=0}^{\infty} s^x p_2(x) = \frac{1-\hat{p}_1(s)}{\mu_{1:1}(1-s)} \ , \quad -1 \leq s \leq 1 \ ,$$

with $\hat{p}_2(1) = 1$, and the corresponding survival function:

$$\begin{aligned}
\bar{P}_2(x) &= \sum_{k=x+1}^{\infty} p_2(k) = \frac{1}{\mu_{1:1}\mu_1} \sum_{k=x+1}^{\infty} \sum_{y=k+1}^{\infty} [y-(k+1)]p(y) \\
&= \frac{1}{\mu_{1:1}\mu_1} \sum_{y=x+2}^{\infty} p(y) \sum_{k=x+1}^{y-1} [y-(k+1)] \\
&= \frac{1}{\mu_{1:1}\mu_1} \sum_{y=x+2}^{\infty} p(y) \{1 + \cdots + [y-(x+1)-1]\} \\
&= \frac{1}{\mu_{1:1}\mu_1} \sum_{y=x+2}^{\infty} p(y) \{\frac{1}{2}[y-(x+1)]^2 + \frac{1}{2}[y-(x+1)]\} \\
&= \frac{1}{2\mu_{1:1}\mu_1} \sum_{y=x+1}^{\infty} p(y)[y-(x+1)]^2 - \frac{1}{2}p_2(x) \ .
\end{aligned}$$

Define similarly the subsequent equilibrium distributions of p, from the third order $p_3(x) = \frac{1}{\mu_{2:1}}\bar{P}_2(x)$ up to the n-th order $p_n(x) = \frac{1}{\mu_{n-1:1}}\bar{P}_{n-1}(x)$ for $x \in \mathbb{N}$, where the following lemma gives an expression for $\bar{P}_n(x)$.

Lemma 2 The survival function $\bar{P}_n(x)$, of the n-th order equilibrium distribution

p_n can be expressed as

$$\bar{P}_n(x) = \frac{1}{n!\prod_{l=0}^{n-1}\mu_{l:1}} \sum_{y=x+1}^{\infty} p(y)[y-(x+1)]^{(n)} , \tag{15}$$

$$= \frac{1}{\mu_{(n)}} \sum_{y=x+1}^{\infty} p(y)[y-(x+1)]^{(n)} , \tag{16}$$

where $\mu_{l:1}$ is the mean of l-th order equilibrium distribution and $\mu_{0:1} = \mu_1$ is the mean of p (or 0-th order equilibrium distribution).

Proof: (14) shows that (15) holds for $n = 1$. By induction, assume that (15) holds for any n, then

$$\begin{aligned}
\bar{P}_{n+1}(x) &= \sum_{t=x+1}^{\infty} p_{n+1}(t) = \sum_{t=x+1}^{\infty} \frac{1}{\mu_{n:\,1}} \bar{P}_n(t) \\
&= \frac{1}{n!\prod_{l=0}^{n}\mu_{l:\,1}} \sum_{y=x+2}^{\infty} p(y) \sum_{t=x+1}^{y-1} [y-(t+1)]^{(n)} \\
&= \frac{1}{n!\prod_{l=0}^{n}\mu_{l:\,1}} \sum_{y=x+2}^{\infty} p(y) \sum_{k=0}^{y-(x+2)} k^{(n)} \\
&= \frac{1}{(n+1)!\prod_{l=0}^{n}\mu_{l:\,1}} \sum_{y=x+1}^{\infty} p(y)[y-(x+1)]^{(n+1)} ,
\end{aligned}$$

verifies (15) also for $n+1$. Further, since $\bar{P}_n(-1) = 1$, we conclude from (15) that

$$n! \prod_{l=0}^{n-1} \mu_{l:1} = \mu_{(n)} , \quad n \in \mathbb{N}^+ , \tag{17}$$

and hence $\bar{P}_n(x)$ is also given by (16). □

To obtain additional properties of higher order equilibrium distributions of p, we need the following identity.

Identity 1 For $n \in \mathbb{N}$ and $y \in \mathbb{N}^+$,

$$\sum_{x=0}^{y-1} x^{(n)} s^x = n! \frac{s^n(1-s^y)}{(1-s)^{n+1}} - \sum_{k=1}^{n} \frac{n!}{k!} \frac{s^{n-k}}{(1-s)^{n-k+1}} y^{(k)} s^y . \tag{18}$$

Proof: Let $I_n = \sum_{x=0}^{y-1} x^{(n)} s^x$. Clearly $I_n = n\frac{s}{1-s} I_{n-1} - \frac{1}{1-s} y^{(n)} s^y$ and $I_0 = \frac{1-s^y}{1-s}$. The result follows by mathematical induction. □

Now the following lemma gives an expression for $\hat{p}_{n+1}(s) = \sum_{x=0}^{\infty} s^x p_{n+1}(x)$, the p.g.f. of p_{n+1}.

Lemma 3 For $n \in \mathbb{N}^+$ and $-1 \leq s \leq 1$

$$\hat{p}_{n+1}(s) = \frac{(-1)^n (n+1)!}{\mu_{(n+1)}} \frac{1-\hat{p}(s)}{(1-s)^{n+1}} + \frac{(n+1)!}{\mu_{(n+1)}} \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k!} \frac{\mu_{(k)}}{(1-s)^{n-k+1}} , \quad (19)$$

with $\hat{p}_{n+1}(1) = 1$.

Proof: Since $p_{n+1}(x) = \frac{\bar{P}_n(x)}{\mu_{n:1}} = \frac{n+1}{\mu_{(n+1)}} \sum_{y=x+1}^{\infty} p(y)[y-(x+1)]^{(n)}$, by (16) and (17) we have

$$\begin{aligned}
\hat{p}_{n+1}(s) &= \frac{n+1}{\mu_{(n+1)}} \sum_{x=0}^{\infty} s^x \sum_{y=x+1}^{\infty} p(y)[y-(x+1)]^{(n)} \\
&= \frac{n+1}{\mu_{(n+1)}} \sum_{y=1}^{\infty} p(y) \sum_{x=0}^{y-1} s^x [y-(x+1)]^{(n)} \\
&= \frac{n+1}{\mu_{(n+1)}} \sum_{y=1}^{\infty} p(y) s^{y-1} \sum_{x=0}^{y-1} s^{-x}\, x^{(n)} \\
&= \frac{n+1}{\mu_{(n+1)}} \sum_{y=1}^{\infty} p(y) s^{y-1} \left[(-1)^n n! \frac{1-s^y}{(1-s)^{n+1}} \left(\frac{1}{s}\right)^{y-1} \right. \\
&\quad \left. + \sum_{k=1}^{n} \frac{(-1)^{n-k} n!}{k!} \frac{y^{(k)}}{(1-s)^{n-k+1}} \left(\frac{1}{s}\right)^{y-1} \right] ,
\end{aligned}$$

where the last equality holds by (18), which in turn yields (19). □

Lemma 4 For m and $n \in \mathbb{N}^+$ we have that $\mu_{n:(m)} = \frac{n!\,m!}{(m+n)!} \frac{\mu_{(n+m)}}{\mu_{(n)}}$.

Proof: By induction and noting that $\mu_{n+1:(m)} = \frac{\mu_{n:(m+1)}}{(m+1)\mu_{n:1}}$. □

3.1.2 Moments and covariances of $U(T-1)$, $|U(T)|$ and v^T

From the above results we see that if the penalty function in Theorem 1 is chosen to be $w(x,y) = x^{(m)}y^{(n)}$, for $m, n \in \mathbb{N}$, then

$$\begin{aligned}
&E[v^T \big(U(T-1)\big)^{(m)} \big(|U(T)|\big)^{(n)} I(T<\infty) \mid U(0)=0] \\
&= v\sum_{x=0}^{\infty} \rho^x x^{(m)} \sum_{y=x+1}^{\infty} [y-(x+1)]^{(n)} p(y) \\
&= n!\, v \prod_{l=0}^{n} \mu_{l:1} \sum_{x=m}^{\infty} \rho^x x^{(m)} p_{n+1}(x) \\
&= n!\, v\rho^m \prod_{l=0}^{n} \mu_{l:1} \frac{d^m}{d\rho^m} \hat{p}_{n+1}(\rho) = \frac{\mu_{(n+1)}}{n+1} v\rho^m \frac{d^m}{d\rho^m} \hat{p}_{n+1}(\rho) \ ,
\end{aligned}$$

also

$$\begin{aligned}
E[v^T |U(T)|^{(n)} I(T<\infty) \mid U(0)=0] &= v\sum_{x=0}^{\infty} \rho^x \sum_{y=x+1}^{\infty} [y-(x+1)]^{(n)} p(y) \\
= v\sum_{x=0}^{\infty} \rho^x p_{n+1}(x)\, n! \prod_{l=0}^{n} \mu_{l:1} &= v\, n! \prod_{l=0}^{n} \mu_{l:1}\, \hat{p}_{n+1}(\rho) = \frac{\mu_{(n+1)}}{n+1} v\hat{p}_{n+1}(\rho)
\end{aligned}$$

and

$$\begin{aligned}
E[v^T \left(U(T-1)\right)^{(m)} I(T<\infty) \big| U(0)=0] &= v\sum_{x=0}^{\infty} \rho^x x^{(m)} \sum_{y=x+1}^{\infty} p(y) \\
= v\rho^m \sum_{x=m}^{\infty} \rho^{(x-m)} x^{(m)} \mu_1 p_1(x) &= v\rho^m \mu_1 \frac{d^m}{d\rho^m} \hat{p}_1(\rho) \ .
\end{aligned}$$

Only the exact results in the simplest cases, $w(x,y) = x^{(1)}$, $w(x,y) = y^{(1)}$ or when $w(x,y) = x^{(1)}y^{(1)}$, are given here.

1. The expected discount value of $|U(T)|$ is obtained using $w(x,y) = y^{(1)}$:

$$\begin{aligned}
E[v^T |U(T)| I(T<\infty) \mid U(0)=0] &= v\sum_{x=0}^{\infty} \rho^x \sum_{y=x+1}^{\infty} [y-(x+1)] p(y) \\
= v\mu_{1:1}\mu_1 \sum_{x=0}^{\infty} \rho^x p_2(x) = \frac{\mu_{(2)}}{2} v\, \hat{p}_2(\rho) &= v\left[\frac{\mu_1(1-\rho) - 1 + \hat{p}(\rho)}{(1-\rho)^2}\right] \ .
\end{aligned}$$

Since, from (11), here $\Psi(0) = P[T < \infty \mid U(0) = 0] = \mu_1$, then

$$E[v^T |U(T)| \bigm| T < \infty, U(0) = 0] = \frac{\mu_{(2)}}{2\mu_{(1)}} v\hat{p}_2(\rho) = v \left[\frac{\mu_1(1-\rho) - 1 + \hat{p}(\rho)}{\mu_1(1-\rho)^2} \right] ,$$

and in particular $E[|U(T)| \bigm| T < \infty, U(0) = 0] = \mu_{1:1}\hat{p}_2(\rho) = \mu_{1:1} = \frac{\mu_{(2)}}{2\mu_1}$.

2. By contrast, $w(x, y) = x^{(1)}$ gives the expected discount value of $U(T-1)$:

$$E[v^T U(T-1) I(T < \infty) \bigm| U(0) = 0] = v \sum_{x=0}^{\infty} x\rho^x \sum_{y=x+1}^{\infty} p(y)$$

$$= v\mu_1 \sum_{x=0}^{\infty} x\rho^x p_1(x) = v\mu_1\rho\hat{p}_1'(\rho) = v\rho \left[\frac{1 - \hat{p}(\rho) - \hat{p}'(\rho)(1-\rho)}{(1-\rho)^2} \right] .$$

Similarly $\Psi(0) = \mu_1$ implies that

$$E[v^T U(T-1) \bigm| T < \infty, U(0) = 0] = v\rho\hat{p}_1'(\rho) = v\rho \left[\frac{1 - \hat{p}(\rho) - \hat{p}'(\rho)(1-\rho)}{\mu_1(1-\rho)^2} \right],$$

and again the same expectation is obtained:

$$E[U(T-1) \bigm| T < \infty, U(0) = 0] = \hat{p}_1'(1) = \mu_{1:1} = \frac{\mu_{(2)}}{2\mu_1} .$$

Finally, denote by $Z = U(T-1) + |U(T)| + 1$ the claim amount causing ruin. Using the above results it follows that

$$E[Z \bigm| T < \infty, U(0) = 0] = 1 + E[U(T-1) \bigm| T < \infty, U(0) = 0]$$

$$+ E[|U(T)| \bigm| T < \infty, U(0) = 0] = 1 + \frac{\mu_2 - \mu_1}{\mu_1} = \frac{\mu_2}{\mu_1} \geq \mu_1 ,$$

showing that the expectation of the claim amount causing ruin is greater than the expectation of a single claim X.

3. The expected discount value of $U(T-1)$ and $|U(T)|$, jointly, is obtained using $w(x, y) = x^{(1)} y^{(1)}$:

$$E[v^T U(T-1)|U(T)| I(T < \infty) \bigm| U(0) = 0] = v\mu_1\mu_{1:1} \sum_{x=0}^{\infty} x\rho^x p_2(x)$$

$$= v\mu_1\mu_{1:1}\rho\hat{p}_2'(\rho) = \frac{\mu_{(2)}}{2} v\rho\hat{p}_2'(\rho) ,$$

which implies

$$E[v^T U(T-1)|U(T)| \bigm| T<\infty, U(0)=0] = v\mu_{1:1}\rho\hat{p}_2'(\rho) = \frac{\mu_{(2)}}{2\mu_1}v\rho\hat{p}_2'(\rho) \ ,$$

and, in particular $E[U(T-1)|U(T)| \bigm| T<\infty, U(0)=0] = \mu_{1:1}\hat{p}_2'(1) = \mu_{1:1}\mu_{2:1} = \frac{\mu_{(3)}}{3!\,\mu_1}$. Finally, from the above results it follows that

$$\begin{aligned}\mathrm{Cov}[U(T-1), |U(T)| \bigm| T<\infty, U(0)=0] &= \mu_{1:1}\mu_{2:1} - (\mu_{1:1})^2 \\ &= \frac{\mu_{(3)}}{6\mu_1} - \left(\frac{\mu_{(2)}}{2\mu_1}\right)^2 \ .\end{aligned}$$

Remarks:

1. Clearly, if $\mu_{2:1} > (<)\ \mu_{1:1}$, then $U(T-1)$ and $|U(T)|$ are positively (resp. negatively) correlated, while if $\mu_{2:1} = \mu_{1:1}$, then $U(T-1)$ and $|U(T)|$ are uncorrelated. A sufficient condition for $\mu_{2:1} > (<,=)\ \mu_{1:1}$ to hold is that p_1 be a DFR (resp. IFR, constant failure rate) distribution, or equivalently, that p be an IMRL (resp. DMRL, constant mean residual life) distribution.
2. If $X \sim p$ is geometrically distributed, then the equilibrium distribution of any order $p_n = p$, i.e. the same distribution as that of X, and hence $|U(T)|$ and $U(T-1)$ are uncorrelated. Furthermore, due to the lack of memory property, $|U(T)|$ is independent of $U(T-1)$ and of T.

3.1.3 Moments of $U(T-1)$, $|U(T)|$ and the ruin time T

To obtain the mean of $U(T-1)$, $|U(T)|$ and T, jointly, we need the following:

Lemma 5

$$\begin{aligned}&E[Tw\big(U(T-1), |U(T)|\big)I(T<\infty) \bigm| U(0)=0] \\ &\quad = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} w(x,y)p(x+y+1) + \frac{1}{1-\mu_1}\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} xw(x,y)p(x+y+1) \ .\end{aligned}$$

Proof: Consider $\phi(0) = E[v^T w\big(U(T-1), |U(T)|\big)I(T<\infty) \bigm| U(0)=0]$ in (2) of Theorem 1. Taking derivatives with respect to v on both sides and setting $v=1$ (which implies $\rho=1$), completes the proof. □

Now using $w(x,y) = x^{(m)}y^{(n)}$, in Lemma 5, gives

$$
\begin{aligned}
&E[T\big(U(T-1)\big)^{(m)}|U(T)|^{(n)}I(T<\infty) \bigm| U(0)=0] \\
&= \sum_{x=0}^{\infty} x^{(m)} \sum_{y=x+1}^{\infty} [y-(x+1)]^{(n)}p(y) + \sum_{x=0}^{\infty} \frac{x\,x^{(m)}}{1-\mu_1} \sum_{y=x+1}^{\infty} [y-(x+1)]^{(n)}p(y) \\
&= n!\prod_{l=0}^{n} \mu_{l:1} \left[\sum_{x=0}^{\infty} x^{(m)}p_{n+1}(x) + \sum_{x=0}^{\infty} \frac{x^{(m+1)}p_{n+1}(x)}{1-\mu_1} + \sum_{x=0}^{\infty} \frac{mx^{(m)}\,p_{n+1}(x)}{1-\mu_1} \right] \\
&= \frac{\mu_{(n+1)}}{n+1} \left[\mu_{n+1:\,(m)} + \frac{\mu_{n+1:\,(m+1)}}{(1-\mu_1)} + \frac{m}{(1-\mu_1)} \mu_{n+1:\,(m)} \right] \\
&= \frac{m!\,n!}{(m+n+1)!} \left(1 + \frac{m}{1-\mu_1} \right) \mu_{(n+m+1)} + \frac{n!\,(m+1)!}{(1-\mu_1)(n+m+2)!} \mu_{(n+m+2)}.
\end{aligned}
$$

Since $E[I(T<\infty)\,|\,U(0)=0] = P[T<\infty\,|\,U(0)=0] = \Psi(0) = \mu_1$, here the following special cases are immediate:

$$
\begin{aligned}
&E[TU(T-1)|U(T)| \bigm| T<\infty, U(0)=0] \\
&\quad = \frac{E[TU(T-1)|U(T)|I(T<\infty) \bigm| U(0)=0]}{E[I(T<\infty) \bigm| U(0)=0]} \\
&\quad = \mu_{1:1}\mu_{2:1} + \frac{\mu_{1:1}[\mu_{2:\,(2)} + \mu_{2:\,(1)}]}{1-\mu_1} = \frac{\mu_{(3)}}{6\mu_1} + \frac{\mu_{(4)} + 2\mu_{(3)}}{12\,\mu_1(1-\mu_1)} .
\end{aligned}
$$

Similarly

$$
\begin{aligned}
E[T|U(T)| \bigm| T<\infty, U(0)=0] &= \frac{E[T|U(T)|I(T<\infty) \bigm| U(0)=0]}{E[I(T<\infty) \bigm| U(0)=0]} \\
= \mu_{1:1} + \frac{\mu_{1:1}\mu_{2:1}}{1-\mu_1} &= \frac{\mu_{(2)}}{2\mu_1} + \frac{\mu_{(3)}}{6\mu_1(1-\mu_1)} ,
\end{aligned}
$$

while

$$
\begin{aligned}
E[TU(T-1) \bigm| T<\infty, U(0)=0] &= \frac{E[TU(T-1)I(T<\infty) \bigm| U(0)=0]}{E[I(T<\infty) \bigm| U(0)=0]} \\
= \mu_{1:1} + \frac{\mu_{1:\,(2)} + \mu_{1:\,(1)}}{1-\mu_1} &= \mu_{1:\,1} + \frac{\mu_{1:\,2}}{1-\mu_1} = \frac{\mu_{(2)}}{2\mu_1} + \frac{2\mu_{(3)} + 3\mu_{(2)}}{6\mu_1\,(1-\mu_1)} .
\end{aligned}
$$

Finally

$$
\begin{aligned}
E[T \bigm| T<\infty, U(0)=0] &= \frac{E[TI(T<\infty) \bigm| U(0)=0]}{E[I(T<\infty) \bigm| U(0)=0]} = 1 + \frac{\mu_{1:1}}{1-\mu_1} \\
= 1 + \frac{\mu_{(2)}}{2\mu_1(1-\mu_1)} .
\end{aligned}
$$

With the above moments, the covariances between $U(T-1)$, $|U(T)|$ and T can be obtained as:

$$\begin{aligned}\text{Cov}[T, |U(T)| \mid T<\infty, U(0)=0] &= \mu_{1:1} + \frac{\mu_{1:1}\mu_{2:1}}{1-\mu_1} - \left(1+\frac{\mu_{1:1}}{1-\mu_1}\right)\mu_{1:1} \\ &= \frac{\mu_{1:1}(\mu_{2:1}-\mu_{1:1})}{1-\mu_1} = \frac{1}{1-\mu_1}\text{Cov}[U(T-1), |U(T)| \mid T<\infty, U(0)=0] \ ,\end{aligned}$$

and

$$\begin{aligned}\text{Cov}[T, U(T-1) \mid T<\infty, U(0)=0] &= \mu_{1:1} + \frac{\mu_{1:2}}{1-\mu_1} - \left(1+\frac{\mu_{1:1}}{1-\mu_1}\right)\mu_{1:1} \\ &= \frac{\mu_{1:2}-\mu_{1:1}^2}{1-\mu_1} = \frac{1}{1-\mu_1}\left[\frac{2\mu_{(3)}+3\mu_{(2)}}{6\mu_1} - \left(\frac{\mu_{(2)}}{2\mu_1}\right)^2\right] \geq 0 \ .\end{aligned}$$

Remarks:

1. T and $U(T-1)$ are positively correlated, since the later ruin occurs, the bigger the surplus becomes.
2. The covariance of T and $|U(T)|$ is proportional to that of $U(T-1)$ and $|U(T)|$.
3. T and $|U(T)|$ are positively or negatively correlated, depending on whether $\mu_{2:1} > \mu_{1:1}$ or $\mu_{2:1} < \mu_{1:1}$. The same holds for $U(T-1)$ and $|U(T)|$.

4 The case $u > 0$

The results of the previous section are extended here to the case $u > 0$. Consider $\phi(0)$ in (2) of Theorem 1 and rewrite $\phi(u)$ as

$$\phi(u) = \sum_{z=0}^{u-1} \phi(u-z) g(z \mid 0) + A(u) \ , \quad \text{for } u \in \mathbb{N}^+ \tag{20}$$

$$= \frac{1}{1+\beta}\sum_{z=0}^{u-1} \phi(u-z) l(z) + \frac{1}{1+\beta} M(u) \ , \tag{21}$$

where $M(u) = (1+\beta)A(u)$ and

$$\begin{aligned} A(u) &= v\rho^{-u}\sum_{z=u}^{\infty}\rho^z\sum_{y=0}^{\infty}w(z,y)p(z+y+1)\ , \quad \text{for } u\in\mathbb{N} \\ &= \phi(0) - v\rho^{-u}\sum_{z=0}^{u-1}\rho^z\sum_{y=0}^{\infty}w(z,y)p(z+y+1)\ , \quad \text{with } A(0)=\phi(0)\ . \end{aligned} \tag{22}$$

Here β is defined as $\frac{1}{1+\beta} := \sum_{z=0}^{\infty} g(z\,|\,0) = \frac{v-\rho}{1-\rho}$, by (10). Note that then $l(z) = (1+\beta)g(z\,|\,0)$ is a proper p.f. on $\mathbb{N}$.

This is a recursive formula for $\phi(u)$, which can be calculated starting from $\phi(0)$, for any penalty function $w(x,y)$.

Now define the compound geometric p.f. $k(u) = \sum_{n=0}^{\infty}\frac{\beta}{1+\beta}(\frac{1}{1+\beta})^n l^{*n}(u)$, for $u\in\mathbb{N}$. Then the following explicit formula can be given for $\phi(u)$.

Theorem 2

$$\phi(u) = \frac{1}{\beta}\sum_{z=0}^{u-1}M(u-z)k(z)\ , \quad u\in\mathbb{N}^+\ . \tag{23}$$

Proof: Consider (21) and let $\hat{\phi}(s) = \sum_{u=0}^{\infty}s^u\phi(u)$, $\hat{l}(s) = \sum_{z=0}^{\infty}s^z l(z)$ and $\hat{M}(s) = \sum_{u=0}^{\infty}s^u M(u)$ be the transforms of ϕ, l and M, respectively. Then the transform of (21) is given by

$$\hat{\phi}(s) = \frac{\hat{M}(s)-\phi(0)\hat{l}(s)}{1+\beta-\hat{l}(s)}\ .$$

Now $\hat{k}(s) = \sum_{u=0}^{\infty}s^u k(u) = \frac{\beta}{1+\beta-\hat{l}(s)}$ implies that $\hat{\phi}(s) = \frac{\hat{M}(s)-\phi(0)\hat{l}(s)}{\beta}\frac{\beta}{1+\beta-\hat{l}(s)} = \frac{\hat{M}(s)-\phi(0)\hat{l}(s)}{\beta}\hat{k}(s)$. Inverting yields (23). □

Since k is a compound geometric p.f., it can be obtained recursively by Panjer's formula, and $\phi(u)$ is calculated with (23) in Theorem 2. In particular, for the case $v=1$, Lemma 1 implies that $\rho=1$, $\beta=(1-\mu_1)/\mu_1$, $g(z\,|\,0)=p_1(z)$ and $\phi(u) = E[w\big(U(T-1),|U(T)|\big)I(T<\infty)\,|\,U(0)=u]$, yielding the following result.

Corollary 1 For $u\in\mathbb{N}^+$

$$E[w\big(U(T-1),|U(T)|\big)I(T<\infty)\ \big|\ U(0)=u] = \frac{\mu_1}{1-\mu_1}\sum_{z=0}^{u-1}M(u-z)k(z)\ ,$$

where $M(u) = \frac{1}{\mu_1} \sum_{x=u}^{\infty} \sum_{y=0}^{\infty} w(x,y)p(x+y+1)$ and $k(z) = (1-\mu_1) \sum_{n=0}^{\infty} \mu_1^n p_1^{*n}(z)$.

As a first application of Theorem 1, we derive the joint p.f. $f_2(x, y \,|\, u)$.

Theorem 3 For $u, x \in \mathbb{N}^+$ and $y \in \mathbb{N}$, we have

$$f_2(x, y \,|\, u) = \sum_{z=0}^{u-1} f_2(x, y \,|\, u - z) g(z \,|\, 0) + \rho^{-u} f_2(x, y \,|\, 0) I(x \geq u) \ . \qquad (24)$$

Proof: Consider (20) and (22). For $x \in \mathbb{N}^+$ and $y \in \mathbb{N}$ fixed, set

$$w(z, s) = \begin{cases} 1 & \text{if } z = x, s = y \\ 0 & \text{otherwise} \end{cases}$$

then, in this particular case, $\phi(u)$ becomes

$$\begin{aligned} f_2(x, y \,|\, u) &= \begin{cases} \sum_{z=0}^{u-1} f_2(x, y \,|\, u - z) g(z \,|\, 0) & \text{if } x < u \\ \sum_{z=0}^{u-1} f_2(x, y \,|\, u - z) g(z \,|\, 0) + v\rho^{x-u} p(x + y + 1) & \text{if } x \geq u \end{cases} \\ &= \sum_{z=0}^{u-1} f_2(x, y \,|\, u - z) g(z \,|\, 0) + \rho^{-u} f_2(x, y \,|\, 0) I(x \geq u) \ . \end{aligned}$$

□

This theorem gives a recursive formula for the joint p.d.f. $f_2(x, y \,|\, u)$, starting at $f_2(x, y \,|\, 1) = \frac{f_2(x,y \,|\, 0) I(x \geq 1)}{\rho[1 - g(0 \,|\, 0)]}$. Once $f_2(x, y \,|\, u)$ has been obtained, $f_2(x \,|\, u)$ and $g(y \,|\, u)$ follow easily. Using renewal arguments, Cheng et al. (2000) give the same result in the compound binomial case.

An application of Theorem 2 yields an alternative expression for $f_2(x, y \,|\, u)$.

Theorem 4 For $x \in \mathbb{N}^+$, and $y \in \mathbb{N}$

$$f_2(x, y \,|\, u) = \gamma(u) f_2(x, y \,|\, 0) \ , \quad u \in \mathbb{N}^+ \ , \qquad (25)$$

where

$$\gamma(u) = \begin{cases} \frac{1}{\beta} \sum_{z=0}^{u-1} (1 + \beta) \rho^{z-u} k(z) & \text{if } 1 \leq u \leq x \\ \frac{1}{\beta} \sum_{z=u-x}^{u-1} (1 + \beta) \rho^{z-u} k(z) & \text{if } u > x \end{cases} \ . \qquad (26)$$

Proof: Let $\gamma(u)=\sum_{z=0}^{u-1}\gamma(u-z)g(z\,|\,0)+\rho^{-u}I(x\geq u)$, for $u\in\mathbb{N}^+$. From (24) it follows that $f_2(x,y\,|\,u)=\gamma(u)f_2(x,y\,|\,0)$. On the other hand, from Theorem 2 we see that for the above choice of penalty function, $\gamma(u)$ can also be expressed as $\gamma(u)=\frac{1}{\beta}\sum_{z=0}^{u-1}(1+\beta)\rho^{-(u-z)}I(x\geq u-z)k(z)$ and (26) follows. □

Remark: (25) can be viewed as a generalized Dickson formula in a discrete setting. Again if $v=1$ then $\rho=1$, $l(z)=p_1(z)$ and $1+\beta=\frac{1}{\mu_1}$. In turn, $k(z)=\sum_{n=0}^{\infty}(1-\mu_1)\mu_1^n p_1^{*n}(z)$ and $K(u-1)=\sum_{z=0}^{u-1}k(z)=\sum_{n=0}^{\infty}(1-\mu_1)\mu_1^n P_1^{*n}(u-1)=1-\Psi(u)$. Hence from Theorem 4, $\gamma(u)$ becomes

$$\frac{f_2(x,y\,|\,u)}{f_2(x,y\,|\,0)}=\begin{cases}\frac{K(u-1)}{1-\mu_1}=\frac{1-\Psi(u)}{1-\Psi(0)} & \text{if } 1\leq u\leq x\\ \frac{[K(u-1)-K(u-x-1)]}{1-\mu_1}=\frac{\Psi(u-x)-\Psi(u)}{1-\Psi(0)} & \text{if } u>x\end{cases}.$$

This gives Dickson's classical formula for $f_2(x,y\,|\,u)$ in the discrete model.

As a second application of Theorem 1, consider a constant $w(x,y)=1$ to derive $\phi_T(u)=E[v^T I(T<\infty)\mid U(0)=u]$, the p.g.f. of the ruin time T with initial reserve $u\in\mathbb{N}^+$:

$$\phi_T(u)=\sum_{z=0}^{u-1}\phi_T(u-z)g(z\,|\,0)+H(u)\ , \tag{27}$$

where $g(z\,|\,0)=\sum_{x=0}^{\infty}v\rho^x p(x+z+1)$, as before, and $H(u)$ is:

$$\begin{aligned}H(u)&=v\rho^{-u}\sum_{z=u}^{\infty}\rho^z\sum_{y=0}^{\infty}p(z+y+1)=v\mu_1\rho^{-u}\sum_{z=u}^{\infty}\rho^z p_1(z)\\&=v\mu_1\rho^{-u}[\hat{p}_1(\rho)-\sum_{z=0}^{u-1}\rho^z p_1(z)]=\sum_{z=u}^{\infty}g(z\,|\,0)=\phi_T(0)-\sum_{z=0}^{u-1}g(z\,|\,0)\ .\end{aligned}$$

This is a recursive formula for $\phi_T(u)$, starting at $\phi_T(1)=\frac{H(1)}{1-g(1\,|\,0)}\frac{\phi_T(0)-g(0\,|\,0)}{1-g(1\,|\,0)}$

Remark: If $\phi_{T^*}(u)=E[v^{T^*}I(T^*<\infty)\mid U(0)=u]$ is the p.g.f. of the modified ruin time T^*, considered in the first remark, we can easily see that

$$\phi_{T^*}(u)=\sum_{z=0}^{u}\phi_{T^*}(u-z)g(z\,|\,0)+\sum_{z=u+1}^{\infty}g(z\,|\,0)\ .$$

Now consider the p.g.f. transforms $\hat{\phi}_T(s)=\sum_{u=0}^{\infty}s^u\phi_T(u)$, $\hat{g}(s)=\sum_{z=0}^{\infty}s^z g(z\,|\,0)$ and $\hat{H}(s)=\sum_{u=0}^{\infty}s^u H(u)$. From (27)

$$\hat{\phi}_T(s)=\phi_T(0)+\frac{\hat{H}(s)-H(0)}{1-\hat{g}(s)}\ , \tag{28}$$

where $\phi_T(0) = H(0) = \frac{v-\rho}{1-\rho}$. Simplifying terms gives $\hat{g}(s) = v\left[\frac{\hat{p}(\rho)-\hat{p}(s)}{\rho-s}\right]$, and

$$\hat{H}(s) - H(0) = v\mu_1 s \frac{\hat{p}_1(\rho) - \hat{p}_1(s)}{\rho - s} = \frac{vs}{\rho - s}\left[\frac{1-\hat{p}(\rho)}{1-\rho} - \frac{1-\hat{p}(s)}{1-s}\right] .$$

Replacing these two expressions into (28) finally gives

$$\hat{\phi}_T(s) = \phi_T(0) + \frac{s[v(1-\rho)\hat{p}(s) + \rho(1-s) + v(\rho - s)]}{(1-\rho)(1-s)[v\hat{p}(s) - s]} . \tag{29}$$

Remarks:

1. Recall that equation $q(s) = \frac{\hat{p}(s)}{s} = \frac{1}{v}$ has at most two solutions, $0 < \rho < 1$ and $R > 1$. It is easily seen that for these values $\hat{g}(\rho) = v\hat{p}'(\rho)$ and $\hat{g}(R) = 1$.
2. For some distributions of X, $\phi_T(u)$ can be obtained by inverting $\hat{\phi}_T$.

Example 3 (Geometric Claims)

If $X \sim$ geometric(θ), as in Example 2, then $\rho = \frac{1-\sqrt{1-4\theta(1-\theta)v}}{2\theta}$, $R = \frac{1+\sqrt{1-4\theta(1-\theta)v}}{2\theta}$, $\phi_T(0) = \frac{1-\theta R}{1-\theta}$, $\hat{g}(s) = \frac{v\theta}{(1-\rho\theta)}\frac{s\theta}{(1-s\theta)} = \frac{vs\theta}{R(1-s\theta)}$, $\hat{H}(s) - H(0) = \frac{v\theta}{(1-\rho\theta)}\frac{s\theta}{(1-s\theta)} = \frac{vs\theta}{R(1-s\theta)}$,

$$\hat{\phi}_T(s) - \phi_T(0) = \frac{\hat{H}(s) - H(0)}{1 - \hat{g}(s)} = \frac{v}{R}\frac{\frac{R^2\theta}{R-v(1-\theta)}\frac{s}{R}}{1 - \frac{R^2\theta}{R-v(1-\theta)}\frac{s}{R}} = \frac{v}{R}\frac{\frac{s}{R}}{1-\frac{s}{R}} = \frac{v}{R}\sum_{u=1}^{\infty} s^u R^{-u}$$

and, in turn

$$\phi_T(u) = \frac{v}{R}R^{-u} = vR^{-(u+1)} = \left[\frac{1-\theta R}{1-\theta}\right]R^{-u} = \phi_T(0)R^{-u} , \quad u \in \mathbb{N} .$$

The special case $v = \rho = 1$ yields $\phi_T(u) = \Psi(u) = \frac{\theta}{1-\theta}R^{-u} = \Psi(0)R^{-u}$, which is a discrete analog of the continuous Poisson/exponential model. In this case ϕ_T can also be derived by the martingale optional sampling theorem and the lack of memory property of the geometric distribution, as in Cheng et al. (2000).

In this simpler case, it is possible to obtain the distribution of the ruin time $P(T = n \,|\, U(0) = u)$. Since $\frac{1}{R} = \frac{1}{2(1-\theta)v}[1 - \sqrt{1 - 4\theta(1-\theta)v}] = \sum_{n=0}^{\infty} b_n v^n$, where

$b_0 = \theta$ and $b_n = \frac{1\times 3\times\cdots(2n-1)}{(n+1)!} 2^{n+1}\theta^{n+1}(1-\theta)^n$, then $\left(\frac{1}{R}\right)^{u+1} = \sum_{n=0}^{\infty} c_n v^n$, where $c_0 = b_0^{u+1}$ and $c_n = \frac{1}{nb_0}\sum_{k=1}^{n}[(u+2)k-n]c_{n-k}b_k$. This implies that

$$\phi_T(u) = \sum_{n=1}^{\infty} v^n P(T=n \mid U(0)=u) = vR^{-(u+1)} = \sum_{n=1}^{\infty} c_{n-1}v^n \ .$$

Finally $P(T=n \mid U(0)=u) = c_{n-1}$ is given by

$$\begin{aligned} c_{n-1} &= \begin{cases} c_0 & \text{if } n=1 \\ \frac{1}{(n-1)b_0}\sum_{k=1}^{n-1}[(u+2)k-n+1]c_{n-1-k}b_k & \text{if } n \geq 2 \end{cases} \\ &= \begin{cases} \theta^{u+1} & \text{if } n=1 \\ \frac{1}{\theta(n-1)}\sum_{k=1}^{n-1}[(u+2)k-n+1]b_k P(T=n-k \mid U(0)=u) & \text{if } n \geq 2 \end{cases} . \end{aligned}$$

To complete this example, consider the factorial moments of T, which are obtained through successive derivatives of ϕ_T at $v=1$. For instance

$$E[T \bigm| U(0)=u] = \frac{d}{dv} vR^{-(u+1)}\Big|_{v=1} = R(1)^{-(u+1)}\left[1 + \frac{u+1}{R(1)}\frac{1-\theta}{\sqrt{1-4\theta(1-\theta)}}\right] ,$$

where $R(1) = \frac{1+\sqrt{1-4\theta(1-\theta)}}{2\theta}$. Similarly, the second order factorial moment is:

$$\begin{aligned} &E[T(T-1)I(T<\infty) \bigm| U(0)=u] = \frac{d^2}{dv^2} vR^{-(u+1)}\Big|_{v=1} \\ &= R(1)^{-(u+1)}\left[\frac{2(u+1)(1-\theta)}{\sqrt{1-4\theta(1-\theta)}} + \frac{(u+1)(u+2)(1-\theta)^2}{R(1)[1-4\theta(1-\theta)]} + \frac{2(u+1)\theta(1-\theta)^2}{[1-4\theta(1-\theta)]^{\frac{3}{2}}}\right] . \end{aligned}$$

Next, we show that in general, $\phi_T(u)$ can be expressed as a compound geometric tail. Since $\phi_T(u) = \sum_{z=0}^{u-1}\phi_T(u-z)g(z \mid 0) + \sum_{z=u}^{\infty} g(z \mid 0)$, it can be re-written as

$$\phi_T(u) = \frac{1}{1+\beta}\sum_{z=0}^{u-1}\phi_T(u-z)l(z) + \frac{1}{1+\beta}\sum_{z=u}^{\infty} l(z) \ , \quad u \in \mathbb{N} \ , \qquad (30)$$

where $\frac{1}{1+\beta} = \sum_{z=0}^{\infty} g(z \mid 0) = \phi_T(0) = \frac{v-\rho}{1-\rho}$ and $l(z) = \frac{g(z \mid 0)}{\sum_{z=0}^{\infty} g(z \mid 0)} = (1+\beta)g(z \mid 0)$ is a proper p.f., while $\bar{L}(u) = \sum_{z=u+1}^{\infty} l(z)$ is the tail probability of l.

Theorem 5 $\phi_T(u)$ can be expressed as a compound geometric sum

$$\begin{aligned} \phi_T(u) &= \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\beta}\right)^n \bar{L}^{*n}(u-1) , \quad u \in \mathbb{N} \\ &= \phi_{T^*}(u) + \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\beta}\right)^n l^{*n}(u) , \end{aligned} \tag{31}$$

where $\bar{L}^{*n}$ and l^{*n} are the n-th convolution of $\bar{L}$ and l, while $\bar{L}(-1) = \bar{L}^{*n}(-1) = 1$.

Proof: Here (27) can be re-written as $1-\phi_T(u) = \frac{1}{1+\beta}\sum_{z=0}^{u-1}[1-\phi_T(u-z)]l(z)+\frac{\beta}{1+\beta}$ for $u \in \mathbb{N}^+$. Theorem 2 then gives $1-\phi_T(u) = \sum_{z=0}^{u-1} k(z)$ and hence

$$1-\phi_T(u) = \frac{\beta}{1+\beta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\beta}\right)^n \sum_{z=0}^{u-1} l^{*n}(z) = \frac{\beta}{1+\beta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\beta}\right)^n L^{*n}(u-1) ,$$

which in turn implies $\phi_T(u) = \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\beta}\right)^n \bar{L}^{*n}(u-1)$, for $u \in \mathbb{N}^+$. With $\bar{L}^{*n}(-1) = 1$ this formula also holds for $u = 0$.

Similarly, the remark to (27) implies $\phi_{T^*}(u) = \frac{1}{1+\beta}\sum_{z=0}^{u}\phi_{T^*}(u-z)l(z)+\frac{1}{1+\beta}\bar{L}(u)$, for $u \in \mathbb{N}$. Subtracting from (30), we have

$$\phi_T(u)-\phi_{T^*}(u) = \frac{1}{1+\beta}\sum_{z=0}^{u}[\phi_T(u-z)-\phi_{T^*}(u-z)]l(z) + \frac{\beta}{(1+\beta)^2}l(u) .$$

Using transforms, it is easily seen that $\phi_T(u)-\phi_{T^*}(u)$ is compound geometric:

$$\phi_T(u)-\phi_{T^*}(u) = \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}(\frac{1}{1+\beta})^n l^{*n}(u) , \quad u \in \mathbb{N} ,$$

which implies that $\phi_{T^*}(u) = \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}(\frac{1}{1+\beta})^n \bar{L}^{*n}(u)$, to complete the proof. □

Remarks:

1. Theorem 5 gives a generalized Beekman's convolution formula for the discrete time model. If $v = \rho = 1$, then it simplifies to Beekman's classical convolution formula $\Psi(u) = (1-\mu_1)\sum_{n=1}^{\infty}\mu_1^n \bar{P}_1^{*n}(u-1)$.

2. Since $\phi_T(u)$ admits a compound geometric form and $l(z)$ is discrete, Panjer's recursive formula can also be used here. Further, any properties (e.g. two-sided bounds) of compound geometric distribution also apply to $\phi_T(u)$.

3. The theorem also gives the relationship between the p.g.f. transforms of two differently defined ruin times T and T^*.

A recursive p.g.f. formula for the ruin time T implies that moments of T can be obtained recursively. Denote the first and second factorial moments by $E_1(u) = E[TI(T<\infty)\,|\,U(0)=u]$ and $E_{(2)}(u) = E[T^{(2)}I(T<\infty)\,|\,U(0)=u]$.

Theorem 6

$$\begin{aligned} E_1(u) &= \mu_1 + \frac{\mu_{(2)}}{2\,(1-\mu_1)} + \mu_1 \sum_{z=0}^{u-1} E_1(u-z)p_1(z) + \mu_1 \sum_{z=0}^{u-1} [\Psi(u-z)-1]\,p_1(z) \\ &\quad + \frac{\mu_{(2)}}{2\,(1-\mu_1)} \sum_{z=0}^{u-1} [\Psi(u-z)-1]\,p_2(z)\,, \quad u \in \mathbb{N}^+ \end{aligned} \tag{32}$$

and

$$\begin{aligned} E_{(2)}(u) &= \sum_{z=0}^{u-1} E_{(2)}(u-z)\mu_1\,p_1(z) + \sum_{z=0}^{u-1} E_1(u-z)\left[2\mu_1\,p_1(z) + \frac{\mu_{(2)}}{(1-\mu_1)}p_2(z)\right] \\ &\quad + \sum_{z=0}^{u-1} [\Psi(u-z)-1]\left[\frac{\mu_{(2)}(\mu_2-3\mu_1+2)}{2(1-\mu_1)^3}p_2(z) + \frac{\mu_{(3)}p_3(z)}{3(1-\mu_1)^2}\right] \\ &\quad + \frac{\mu_{(2)}(\mu_2-3\mu_1+2)}{2(1-\mu_1)^3} + \frac{\mu_{(3)}}{3(1-\mu_1)^2}\,, \quad u \in \mathbb{N}^+\,. \end{aligned} \tag{33}$$

Proof: By definition $\frac{\partial}{\partial v}\phi_T(u)|_{v=1} = E[TI(T<\infty)\,|\,U(0)=u] = E_1(u)$. Differentiating $\phi_T(u) = \sum_{z=0}^{u-1}\phi_T(u-z)\,g(z|0) + \sum_{z=u}^{\infty} g(z|0)$ on both sides then gives

$$\frac{\partial}{\partial v}\phi_T(u) = \sum_{z=0}^{u-1}\frac{\partial}{\partial v}\phi_T(u-z)g(z\,|\,0) + \sum_{z=0}^{u-1}\phi_T(u-z)\frac{\partial}{\partial v}g(z\,|\,0) + \frac{\partial}{\partial v}\sum_{z=u}^{\infty} g(z\,|\,0)\,. \tag{34}$$

Substitute $\frac{\partial}{\partial v} g(z \,|\, 0) = \sum_{x=0}^{\infty} \rho^x p(x+z+1) + \sum_{x=0}^{\infty} v x \rho^{x-1} \rho'(v) p(x+z+1)$ in (34) and set $v = \rho = 1$. Then $\phi_T(u) = \Psi(u)$, $g(z \,|\, 0) = \mu_1 p_1(z)$ and $\rho'(1) = \frac{1}{1-\mu_1}$ give

$$
\begin{aligned}
E_1(u) &= \mu_1 \sum_{z=0}^{u-1} E_1(u-z) p_1(z) + \mu_1 \sum_{z=0}^{u-1} \Psi(u-z) p_1(z) \\
&\quad + \frac{1}{1-\mu_1} \sum_{z=0}^{u-1} \Psi(u-z) \sum_{x=0}^{\infty} x p(x+z+1) + \sum_{z=u}^{\infty} \sum_{x=0}^{\infty} p(x+z+1) \\
&\quad + \frac{1}{1-\mu_1} \sum_{z=u}^{\infty} \sum_{x=0}^{\infty} x p(x+z+1) \\
&= \mu_1 \sum_{z=0}^{u-1} E_1(u-z) p_1(z) + \mu_1 \sum_{z=0}^{u-1} \Psi(u-z) p_1(z) \\
&\quad + \frac{1}{1-\mu_1} \sum_{z=0}^{u-1} \Psi(u-z) \sum_{x=z+1}^{\infty} [x-(z+1)] p(x) \\
&\quad + \sum_{z=u}^{\infty} \sum_{x=z+1}^{\infty} p(x) + \frac{1}{1-\mu_1} \sum_{z=u}^{\infty} \sum_{x=z+1}^{\infty} [x-(z+1)] p(x) \\
&= \mu_1 \sum_{z=0}^{u-1} E_1(u-z) p_1(z) + \mu_1 \sum_{z=0}^{u-1} \Psi(u-z) p_1(z) \\
&\quad + \frac{\mu_{1:1}\mu_1}{1-\mu_1} \sum_{z=0}^{u-1} \Psi(u-z) p_2(z) + \mu_1 \sum_{z=u}^{\infty} p_1(z) + \frac{\mu_{1:1}\mu_1}{1-\mu_1} \sum_{z=u}^{\infty} p_2(z) \\
&= \mu_1 \sum_{z=0}^{u-1} E_1(u-z) p_1(z) + \mu_1 \sum_{z=0}^{u-1} \Psi(u-z) p_1(z) \\
&\quad + \frac{\mu_{(2)}}{2(1-\mu_1)} \sum_{z=0}^{u-1} \Psi(u-z) p_2(z) + \mu_1 \bar{P}_1(u-1) + \frac{\mu_{(2)}}{2(1-\mu_1)} \bar{P}_2(u-1)\,,
\end{aligned}
$$

which yields the recursive formula for $E_1(u)$. To obtain $E_{(2)}(u)$, differentiate both sides of (34), set $v = \rho = 1$ and note that $\rho''(1) = \frac{\mu_2 - \mu_1 - 2\mu_1^2}{(1-\mu_1)^3}$. $\square$

As a final application of Theorems 1 and 2, we give expressions for the discounted moments of $U(T-1)$ and $|U(T)|$.

Denote by $\phi^{(n)}(u; v) = E[v^T |U(T)|^{(n)} I(T < \infty) \,|\, U(0) = u]$ and by $\phi_{(m)}(u; v) = E[v^T \big(U(T-1)\big)^{(m)} I(T < \infty) \,|\, U(0) = u]$.

Corollary 2 For $u \in \mathbb{N}^+$

$$\phi^{(n)}(u;v) = \phi^{(n)}(0;v) + \sum_{z=0}^{u-1} \phi^{(n)}(u-z;v)g(z\,|\,0) + \frac{\mu_{(n+1)}}{n+1} v \sum_{z=0}^{u-1} \rho^{z-u} p_{n+1}(z) \;,$$

where $\phi^{(n)}(0;v) = \frac{\mu_{(n+1)}}{n+1} v \hat{p}_{n+1}(\rho)$ is given in Section 3.1.2. In particular when $v=1$ then $\phi^{(n)}(u;1) = E[|U(T)|^{(n)} I(T<\infty)\,|\,U(0)=u]$ is given recursively by

$$\phi^{(n)}(u;1) = \phi^{(n)}(0;1) + \mu_1 \sum_{z=0}^{u-1} \phi^{(n)}(u-z;1)p_1(z) + \frac{\mu_{(n+1)}}{n+1} P_{n+1}(u-1) \;.$$

Corollary 3 For $u \in \mathbb{N}^+$

$$\phi_{(m)}(u;v) = \phi_{(m)}(0;v) + \sum_{z=0}^{u-1} \phi_{(m)}(u-z;v)g(z\,|\,0) + v\mu_1 \sum_{z=0}^{u-1} \rho^{z-u} x^{(m)} p_1(z) \;,$$

where $\phi_{(m)}(0;v) = v\rho^m \mu_1 \frac{d^m}{d\rho^m}\hat{p}_1(\rho)$ is given in Section 3.1.2. In particular when $v=1$ then $\phi_{(m)}(u;1) = E[\big(U(T-1)\big)^{(m)} I(T<\infty)\,|\,U(0)=u]$ is given recursively:

$$\phi_{(m)}(u;1) = \phi_{(m)}(0;1) + \mu_1 \sum_{z=0}^{u-1} \phi_{(m)}(u-z;1)p_1(z) + \mu_1 \sum_{z=0}^{u-1} z^{(m)} p_1(z) \;.$$

Conclusion

The above results further illustrate the usefulness of the penalty function. Our formulas for the discrete time risk model can help better understand their analogous counterparts in the continuous time model, which is a limiting case. Hopefully, these results will also prove to be of independent interest.

References

[1] Cheng, S., Gerber, H.U., Shiu, E.S.W. (2000) "Discounted probabilities and ruin theory in the compound binomial model." *Insurance: Mathematics and Economics*, **26**, 239-250.

[2] Delbaen, F. (1990) "A remark on the moments of ruin time in classical risk theory." *Insurance: Mathematics and Economics*, **9**, 121-126.

[3] Dickson, D.C.M. (1994) "Some comments on the compound binomial model." *ASTIN Bulletin*, **24**, 33-45.

[4] Egídio dos Reis, A.D. (2000) " On the moments of ruin and recovery times." *Insurance: Mathematics and Economics*, **27**, 331-343.

[5] Gerber, H.U. (1988) "Mathematical fun with compound binomial process." *ASTIN Bulletin*, **18**, 161-168.

[6] Gerber, H.U., Shiu, E.S.W. (1997) "The join distribution of the time of ruin, the surplus immediately before ruin and the deficit at ruin." *Insurance: Mathematics and Economics*, **21**, 129-137.

[7] Gerber, H.U., Shiu, E.S.W. (1998) "On the time value of ruin." *North American Actuarial Journal*, **2**, 48-78.

[8] Hamming, R.W. (1979) *Numerical methods for scientists and Engineers*. 2nd edition, Dover, New-York.

[9] Lin, X.S., Willmot, G.E. (1999) "Analysis of a defective renewal arising in ruin theory." *Insurance: Mathematics and Economics*, **25**, 63-84.

[10] Lin, X.S., Willmot, G.E. (2000) "The moments of the time of ruin, the surplus before ruin, and the deficit at ruin." *Insurance: Mathematics and Economics*, **27**, 19-44.

[11] Panjer, H., Willmot, G.E. (1992) *Insurance risk models*. Society of Actuaries, Schaumburg, IL.

[12] Picard, P.H., Lefevre, C. (1998) "The moments of ruin time in classical risk model with discrete claim size distribution." *Insurance: Mathematics and Economics*, **23**, 157-172 (with corregidum, *I:ME*, **25**, 105-107).

[13] Shiu, E.S.W. (1989) "The probability of eventual ruin in the compound binomial model." *ASTIN Bulletin*, **19**, 179-190.

[14] Willmot, G.E. (1993) "Ruin probabilities in the compound binomial model." *Insurance: Mathematics and Economics*, **12**, 133-142.

[15] Willmot, G.E., Cai, J. (2001) " Aging and other distributional properties of discrete compound geometric distributions." *Insurance: Mathematics and Economics*, **28**, 361-379.