Working Paper 01-15
Business Economics Series 05
March 2001

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PROJECTIVE SYSTEM APPROACH TO THE MARTINGALE CHARACTERIZATION OF THE ABSENCE OF ARBITRAGE[1]

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Abstract

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Keywords: Arbitrage, Martingale Measure, Asset Pricing, Radon Measure, Projective System

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[1] Supported by grant BEC2000 – 1388 – C04 from Spanish Ministry of Science and Technology.

Projective system approach to the martingale characterization of the absence of arbitrage *

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1 Introduction

For a frictionless securities market, several authors have proved different versions of the, so called, Fundamental Theorem of Asset Pricing (see for instance [4], [5], [8], [9], [10] or[12]). In the case of a finite number of assets and a finite discrete time, this result simply states that the absence of arbitrage characterizes the existence of an equivalent martingale measure.

But things go wrong if one passes to infinite time (see [1]) or to infinitely many securities (see [9]). In both situations, the characterization of an equivalent martingale measure for the price process of the assets needs notions such as "no free lunch" or "no free lunch with bounded risk", generalizing the concept of "no arbitrage".

The purpose of this paper is to formulate the problem for infinite discrete time in a different mathematical setting, in order to obtain a theorem of asset pricing which may be phrased using only the classical notion of "no arbitrage".

*Supported by grant $BEC2000-1388-C04$ from Spanish Ministry of Science and Technology.

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In securities market models with a countable number of trading dates and a finite number of assets, the absence of arbitrage allows us to construct a projective system of topological spaces and a projective system of regular measures. Under financially sound mild assumptions, the projective limit measure is a martingale measure on the projective limit space. Since, in general, the projective limit space strictly contains the initial space of states of nature the initial probability measure and the martingale measure are not equivalent. However, we show that the projections of both measures on every instant of time are equivalent. Therefore, there exist strictly positive Radon-Nikodym derivatives between the corresponding projections.

The paper is outlined as follows. Section 2 summarizes some basic notions and properties of projective systems of Radon measures and introduces the securities market model. Section 3 develops the construction of the projective model. Relying on the Prokhorov's theorem on the existence of projective limits of projective systems of measures, we prove in Section 4 our main result, Theorem 2, that characterizes the absence of arbitrage by the existence of a projectively equivalent martingale measure. Finally, Section 5 concludes the paper.

2 Preliminaries

First, we recall the concepts of a projective system of topological spaces, a Radon measure and a projective system of Radon measures (see [2] or [11] for further details).

Let Y be an arbitrary Hausdorff topological space and β its Borel σ-algebra. A Radon measure m on Y is a positive measure on β satisfying that m is locally finite (every point has a neighborhood which has a finite m-measure) and m is inner regular on β (for every $B \in \beta$, $m(B) = \sup\{m(K) : K \subset B, K \text{ compact}\}$).

If m is a Radon measure on Y, Z is a Hausdorff topological space and $f\colon Y \to Z$ is a continuous map, the image measure $f(m)$ given by $f(m)(B) = m(f^{-1}(B))$, for all Borel-measurable set B on Z, is a Radon measure on Z.

Let $(I, \leq)$ be a directed set. Consider a family of Hausdorff topological spaces $(X_i)_{i\in I}$ and the continuous maps $\pi_{ij}\colon X_j \longrightarrow X_i,\ i,j \in I,\ i \leq j$. We say that $(X_i)_{i\in I}$ is a projective system of Hausdorff topological spaces with maps π_{ij}, if $\pi_{ik} = \pi_{ij} \circ \pi_{jk}$ for all $i, j, k \in I$, $i \leq j \leq k$.

The projective limit of the projective system $(X_i)_{i\in I}$ is the set

$$X = \lim_{i\in I} X_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i : x_j = \pi_{jk}(x_k) \text{ if } j, k \in I,\ j \leq k \right\}$$

endowed with the relative product topology.

For each $i \in I$, the natural projection $\pi_i\colon X \longrightarrow X_i$ is continuous. Moreover, $\pi_i = \pi_{ij} \circ \pi_j$ for all $i, j \in I$, $i \leq j$.

A projective system of Radon measures is a family of Radon measures m_i on X_i, $i \in I$, such that $\pi_{ij}(m_j) = m_i$ if $i, j \in I$, $i \leq j$. A Radon measure m on the projective limit X is the projective limit of the measures $(m_i)_{i\in I}$, denoted by $m = \lim_{i\in I} m_i$, if $\pi_i(m) = m_i$ for all $i \in I$.

The following existence result is adapted from [11].

Theorem 1 (Prokhorov) *A projective system $(m_i)_{i\in I}$ of finite Radon measures has a projective limit if and only if given $\varepsilon > 0$ there exists a compact set $K \subset X$, such that $m_i(X_i \setminus \pi_i(K)) \leq \varepsilon$, for all $i \in I$.*

An important consequence is given in the following corollary whose proof may be also found in [2].

Corollary 1 *A countable projective system $(m_n)_{n\in\mathbb{N}}$ of finite Radon measures has a projective limit.*

Now, we describe our model of a frictionless financial market with a finite number of assets $n \in \mathbb{N}$ and a countable set of trading dates T. For convenience, let us take $T = \mathbb{N}$. The information available to the agents at every time is described by a probability space (Ω, Σ, μ) and an increasing family of sub-σ-algebras of Σ, $\{\Sigma_t\}_{t\in T}$, whose union generates Σ.

The prices of the risky stocks are given by a stochastic process $(P(t, \_))_{t\in T}$, with values in $\mathbb{R}^n$, adapted to the filtration $\{\Sigma_t\}_{t\in T}$. Naturally, for every $j = 1\ldots, n$, $\omega \in \Omega$ and $t \in T$, $P_j(t, \omega)$ is the price of Asset j if the true state of nature revealed in t is ω.

We will suppose that the first security is a riskless bond. Without loss of generality, see [5], assume that the prices of the stocks have been discounted by the price of the bond, that is, take $P_1(t, \_) = 1$ for all $t \in T$.

The market is said to satisfy the absence of arbitrage if for every $t \in T$, $t \geq 1$, and every bounded Σ_{t-1}-measurable function $x\colon \Omega \longrightarrow \mathbb{R}^n$,

$$\langle x(\omega), P(t, \omega) - P(t-1, \omega)\rangle = 0, \ \mu \text{ almost surely}$$

whenever

$$\langle x(\omega), P(t, \omega) - P(t-1, \omega)\rangle \geq 0, \ \mu \text{ almost surely;}$$

where $\langle ., .\rangle$ denotes the inner product on $\mathbb{R}^n$.

The absence of arbitrage prevents the existence of zero cost portfolios with positive return. Any reasonable model of a financial market should satisfy this condition, because, otherwise, some astute agent would take advantage of the arbitrage opportunities making riskless profits without investment.

3 The discrete time projective model

We start by imposing an additional condition on the filtration $\{\Sigma_t\}_{t\in T}$. Suppose that for every $t \in T$, there exists a countable partition $(A_t^j)_{j=1}^\infty$ of Ω, formed by events of positive probability, that generates Σ_t. Because the filtration $\{\Sigma_t\}_{t\in T}$ is increasing, the partition $(A_t^j)_{j=1}^\infty$, $t \in T$, $t \geq 1$, can be chosen to satisfy the following properties:

P1) Given $j \in \mathbb{N}$ and $t \in \mathbb{N}$, there is a unique $\Phi_t(j) \in \mathbb{N}$ such that $A_t^j \subset A_{t-1}^{\Phi_t(j)}$.

P2) $A_{t-1}^h = \bigcup_{j \in \Phi_t^{-1}(h)} A_t^j$, for every $h \in \mathbb{N}$, and every $t \in \mathbb{N}$.

Next, we see that our infinite discrete time model leads in a natural way to a projective system of spaces and of measures.

Consider, for every $t \in T$, the countable set $\Omega_t = \{A_t^j : j \in \mathbb{N}\}$ endowed with the discrete topology. Obviously, every Ω_t is a metrizable Hausdorff space whose Borel σ-algebra is just Σ_t. For every $t \in T$, $t \geq 1$, we define the continuous map $\pi_{t-1,t} \colon \Omega_t \longrightarrow \Omega_{t-1}$ by $\pi_{t-1,t}(A_t^j) = A_{t-1}^{\Phi_t(j)}$. Hence, the family $(\Omega_t)_{t \in T}$ is a projective system of Hausdorff topological spaces with maps $\pi_{t-1,t}$. Let us denote by $\bar{\Omega}$ its projective limit and by $\pi_t \colon \bar{\Omega} \longrightarrow \Omega_t$ the natural projection of $\bar{\Omega}$ in Ω_t, $t \in T$. Trivially, $\bar{\Omega}$ is a metrizable Hausdorff space and all the projections π_t, $t \in T$, are continuous. Denote by $\bar{\Sigma}$ the Borel σ-algebra of $\bar{\Omega}$.

Since, for every $t \in T$, the family $(A_t^j)_{j=1}^{\infty}$ is a partition of Ω, given $\omega \in \Omega$ there exists a unique $t(\omega) \in \mathbb{N}$ such that $\omega \in A_t^{t(\omega)}$. Therefore, the map $I \colon \Omega \to \bar{\Omega}$, $I(\omega) = \left(A_t^{t(\omega)}\right)_{t=0}^{\infty}$ is well defined. The map I identifies every state of nature $\omega \in \Omega$ with the "path" or "trajectory" in $\bar{\Omega}$ formed by the events $A_t^{t(\omega)}$ to which ω belongs in each instant of time $t \in T$.

Given $\bar{\omega} = (A_t^{\bar{\omega}})_{t=0}^{\infty} \in \bar{\Omega}$, one easily obtains that $I^{-1}(\bar{\omega}) = \bigcap_{t=0}^{\infty} A_t^{\bar{\omega}}$. Therefore, I is one to one if and only if for every $\omega, \omega' \in \Omega$, $\omega \neq \omega'$, there exist $t \in T$ and $i, j \in \mathbb{N}, i \neq j$, for which $\omega \in A_t^i$ and $\omega' \in A_t^j$. The map I fails to be one to one if there are two different states of nature with identical "paths" over time. From the financial point of view, if two states of nature are indistinguishable over time, one can consider that they are the same. So, we will suppose, from now on, that I is an injection.

For every $t \in \mathbb{N}$, $\varphi_t = \pi_t \circ I$ is the natural projection of Ω on Ω_t. Clearly, $A_t^j = \varphi_t^{-1}(A_t^j)$ for every $A_t^j \in \Omega_t$.

Now, we replicate the filtration structure in the projective system. For every $t \in T$, let $\bar{\Sigma}_t$ be the σ-algebra on $\bar{\Omega}$ generated by the countable partition of $\bar{\Omega}$,

$$\{\pi_{t-1}^{-1}(A_{t-1}^h) : A_{t-1}^h \in \Omega_{t-1}\}.$$

The family $\{\bar{\Sigma}_t\}_{t=0}^{\infty}$ is a filtration on $\bar{\Omega}$ and $\pi_{t-1}^{-1}(A_{t-1}^h) = \bigcup_{j \in \Phi_t^{-1}(h)} \pi_t^{-1}(A_t^j)$, if $A_{t-1}^h \in \Omega_{t-1}$.

Let us show that the injection I is consistent with the filtration structures on Ω and $\bar{\Omega}$. More explicitly, I is a Σ-$\bar{\Sigma}$ measurable map. Indeed, for any $s, j \in \mathbb{N}$

$$\bigcap_{t=0}^{s} \pi_t^{-1}(A_t^j) \in \Sigma$$

because

$$I^{-1}\left(\bigcap_{t=0}^{s} \pi_t^{-1}(A_t^j)\right) = \bigcap_{t=0}^{s} \varphi_t^{-1}(A_t^j) = \bigcap_{t=0}^{s} A_t^j \in \Sigma_s.$$

Furthermore, I is a Σ_t-$\bar{\Sigma}_t$ measurable map for every $t \in T$. Indeed, it suffices to prove that, for each $j \in \mathbb{N}$, the inverse image by I of the set $\pi_t^{-1}(A_t^j)$ is Σ_t-measurable. But, $I^{-1}\left(\pi_t^{-1}(A_t^j)\right) = \varphi_t^{-1}(A_t^j) = A_t^j$, and this set, obviously, belongs to Σ_t. Moreover, since π_t is continuous, $\varphi_t = \pi_t \circ I$ is Σ_t-Σ_t measurable.

The next step is to introduce measures in the projective system. First of all, as I is Σ-$\bar{\Sigma}$ measurable, $\bar{\mu} = I(\mu)$ is a probability measure on $(\bar{\Omega}, \bar{\Sigma})$. On the other hand, fix $t \in T$ and consider the image measure $\mu_t = \varphi_t(\mu)$ on (Ω_t, Σ_t). Certainly, μ_t coincides with the restriction of μ to Σ_t. Straightforward computations show that $(\mu_t)_{t=0}^{\infty}$ is a projective system of Radon measures whose projective limit is $\bar{\mu}$. So, $\bar{\mu} = \varprojlim_{t\in\mathbb{N}} \mu_t$ is a Radon measure and can be view as the extension of μ to $\bar{\Omega}$.

Finally, we associate with every Σ_t-measurable function $h\colon \Omega \longrightarrow \mathbb{R}^n$ a $\bar{\Sigma}_t$-measurable function $\bar{h}\colon \bar{\Omega} \to \mathbb{R}^n$ in the following way. Since h is Σ_t-measurable and Σ_t is generated by the countable partition $(A_t^j)_{j=1}^{\infty}$, h must be constant on every $A_t^j \in \Omega_t$. Thus, we define the function $\bar{h}\colon \bar{\Omega} \to \mathbb{R}^n$ by $\bar{h}(\bar{\omega}) = h(\pi_t(\bar{\omega}))$. Obviously, $\bar{h}$ extends the function h, in fact, $\bar{h}(I(\omega)) = h(\omega)$, for all $\omega \in \Omega$. In particular, the stocks price process $(P(t, \_))_{t\in T}$ gives rise to an stochastic process $\left(\bar{P}(t, \_)\right)_{t=0}^{\infty}$ on $\bar{\Omega}$ adapted to the filtration $\left\{\bar{\Sigma}_t\right\}_{t=0}^{\infty}$.

In summary, we have enlarged our original securities market model: the probability space (Ω, Σ, μ), the filtration $\{\Sigma_t\}_{t=0}^{\infty}$ and the price process $(P(t, \_))_{t=0}^{\infty}$, to a new one, with the corresponding probability space $(\bar{\Omega}, \bar{\Sigma}, \bar{\mu})$, filtration $\left\{\bar{\Sigma}_t\right\}_{t=0}^{\infty}$ and process $\left(\bar{P}(t, \_)\right)_{t=0}^{\infty}$. This new description of the discrete-time financial market will be called the projective model.

4 Characterization of the absence of arbitrage in the projective model

We begin this section by translating the notion of absence of arbitrage to projective terms.

Proposition 1 *The market satisfies the absence of arbitrage if for each $t \in T$, $t \geq 1$, and every bounded Σ_{t-1}-measurable function $x\colon \Omega \to \mathbb{R}^n$,*

$$\langle \bar{x}(\bar{\omega}), \bar{P}(t, \bar{\omega}) - \bar{P}(t-1, \bar{\omega}) \rangle = 0, \ \bar{\mu} \text{ almost surely}$$

whenever

$$\langle \bar{x}(\bar{\omega}), \bar{P}(t, \bar{\omega}) - \bar{P}(t-1, \bar{\omega}) \rangle \geq 0, \ \bar{\mu} \text{ almost surely.}$$

Definition 1 *We say that a measure $\bar{\lambda}$ on $\bar{\Omega}$ is projectively equivalent to $\bar{\mu}$ if $\lambda_t = \pi_t(\bar{\lambda})$ is equivalent to μ_t for all $t \in T$, i.e. if λ_t and μ_t have the same null events.*

We would like to note that two measures $\bar{\lambda}$ and $\bar{\mu}$ on $\bar{\Omega}$ can be projectively equivalent without being equivalent. An instance of this situation can be found in Example 1.

Definition 2 *A risk-neutral projective probability measure (or projectively equivalent martingale measure) is a probability measure $\bar{\lambda}$ on $\bar{\Omega}$, projectively equivalent to $\bar{\mu}$, such that the stochastic process $\left(\bar{P}(t,\_)\right)_{t=0}^{\infty}$ is a martingale under $\bar{\lambda}$, i.e. for every $t \in T$, $t \geq 1$, $E_{\bar{\lambda}}[\bar{P}(t,\_)|\bar{\Sigma}_{t-1}] = \bar{P}(t-1,\_)$. Here, $E_{\bar{\lambda}}$ denotes the conditional expectation operator associated with $\bar{\lambda}$.*

We can already state and prove the main result of this paper: the equivalence between the absence of arbitrage and the existence of a projectively equivalent martingale measure.

Theorem 2 *The market satisfies the absence of arbitrage if and only if there exists a risk-neutral projective probability measure.*

Proof: First, we prove the necessity. Suppose that the market is arbitrage free. The Fundamental Theorem of Asset Pricing, see for instance [3], [6], [7], [8] or [9], asserts that for any time interval of finite length, the absence of arbitrage implies the existence of an equivalent martingale measure. In particular, for every $t-1 \in T$, $t \geq 1$, there exists a probability measure θ_t on Σ_t, equivalent to μ_t, such that $(P(t-1,\_), P(t,\_))$ is a martingale under θ_t and the filtration (Σ_{t-1}, Σ_t), i.e. $E_{\theta_t}[P(t,\_)|\Sigma_{t-1}] = P(t-1,\_)$. In addition, θ_t can be chosen such that the density $f_t = \frac{d\theta_t}{d\mu} > 0$ is bounded, see [9]. Therefore, we can write

$$P(t-1,\_)E[f_t|\Sigma_{t-1}] = E[f_t P(t,\_)|\Sigma_{t-1}]. \tag{1}$$

The relation $1 = E\left[\frac{f_t}{E[f_t|\Sigma_{t-1}]} \mid \Sigma_{t-1}\right]$ holds because $\frac{f_t}{E[f_t|\Sigma_{t-1}]} \in L^1(\Sigma_t)$.

Consequently, for every Σ_{t-1}- measurable function $g\colon \Omega \to \mathbb{R}$, one has

$$\int_{A_{t-1}^h} g d\mu = \int_{A_{t-1}^h} g \frac{f_t}{E[f_t|\Sigma_{t-1}]} d\mu, \quad A_{t-1}^h \in \Omega_{t-1}. \tag{2}$$

So, in particular,

$$P(t-1,\_) = E\left[P(t,\_)\frac{f_t}{E[f_t|\Sigma_{t-1}]} \mid \Sigma_{t-1}\right]. \tag{3}$$

The idea of this part of the proof is to build a projective system of Radon measures $(\lambda_t)_{t=0}^{\infty}$ whose projective limit, $\bar{\lambda}$, is a measure on $\bar{\Sigma}$ with all the wanted properties.

Given $t \in T$, take the function $q_t \in L^1(\Sigma_t)$ defined by

$$q_t = \begin{cases} 1 & \text{if } t = 0 \\ \dfrac{\prod_{j=1}^{t} f_j}{\prod_{j=1}^{t} E[f_j|\Sigma_{j-1}]} & \text{if } t \in T, t \geq 1 \end{cases}.$$

Let λ_t be the measure on Σ_t whose Radon-Nikodym derivative with respect to μ_t is q_t, *i.e.*, $q_t = \frac{d\lambda_t}{d\mu_t}$. Obviously, λ_t and μ_t are equivalent measures and, therefore, since μ_t is a Radon measure, λ_t is also a Radon measure.

Let us prove, step by step, that $(\lambda_t)_{t=0}^{\infty}$ leads to a risk neutral projective probability measure.

Step 1 The family $(\lambda_t)_{t=0}^{\infty}$ *is a projective system of Radon measures.*

We have to check that for every $t \in T$, $t \geq 1$,

$$\pi_{t-1,t}(\lambda_t) = \lambda_{t-1}. \tag{4}$$

Given any $A_{t-1}^h \in \Omega_{t-1}$,

$$\begin{aligned} \pi_{t-1,t}(\lambda_t)(A_{t-1}^h) = \lambda_t(\pi_{t-1,t}^{-1}(A_{t-1}^h)) &= \sum_{j\in\Phi_t^{-1}(h)} \lambda_t(A_t^j) \\ &= \int_{\bigcup_{j\in\Phi_t^{-1}(h)} A_j^t} q_t d\mu = \int_{\bigcup_{j\in\Phi_t^{-1}(h)} A_j^t} q_{t-1} d\mu. \end{aligned} \tag{5}$$

Note that the last identity comes from (2), because $\bigcup A_t^j{}_{j\in\Phi_t^{-1}(h)} \in \Sigma_{t-1}$ and $q_t = \frac{f_t}{E[f_t|\Sigma_{t-1}]} q_{t-1}$.

On the other hand,

$$\lambda_{t-1}(A_{t-1}^h) = \lambda_{t-1}\left(\bigcup_{j\in\Phi_t^{-1}(h)} A_t^j\right) = \int_{\bigcup_{j\in\Phi_t^{-1}(h)} A_t^j} q_{t-1} d\mu. \tag{6}$$

Combining (5) and (6) we obtain (4).

Step 2 For every $t \in T$, $t \geq 1$, λ_{t-1} *is the restriction of* λ_t *to* Σ_{t-1}.

Fix $t \in T$, $t \geq 1$. We will prove that λ_t and λ_{t-1} coincide on Σ_{t-1}. Given $A_{t-1}^h \in \Omega_{t-1}$,

$$\lambda_{t-1}(A_{t-1}^h) = \int_{A_{t-1}^h} q_{t-1} d\mu = \sum_{j\in\Phi_t^{-1}(h)} \int_{A_t^j} q_{t-1} d\mu$$

According to (2),

$$\begin{aligned} \sum_{j\in\Phi_t^{-1}(h)} \int_{A_t^j} q_{t-1} d\mu &= \sum_{j\in\Phi_t^{-1}(h)} \int_{A_t^j} q_{t-1} \frac{f_t}{E[f_t|\Sigma_{t-1}]} d\mu \\ &= \int_{\bigcup_{j\in\Phi_t^{-1}(h)} A_j^t} q_t d\mu = \lambda_t(A_{t-1}^h). \end{aligned}$$

Then, truly, $\lambda_{t-1}(A_{t-1}^h) = \lambda_t(A_{t-1}^h)$.

Step 3 For every $t \in T$, $t \geq 1$, *the finite process* $(P(j, \_))_{j=0}^{t}$ *is a martingale under* λ_t.

Multiplying (3) by q_{t-1} we arrive to the relation

$$P(t-1, \_) q_{t-1} = E\left[P(t, \_) q_t | \Sigma_{t-1}\right].$$

Therefore,

$$\int_{A_{t-1}^h} P(t-1, \_) d\lambda_{t-1} = \int_{A_{t-1}^h} P(t, \_) d\lambda_t, \quad A_{t-1}^h \in \Omega_{t-1}. \tag{7}$$

However, from Step 3 we derive the relation

$$\int_{A^h_{t-1}} P(t-1,\_)d\lambda_{t-1} = \int_{A^h_{t-1}} P(t-1,\_)d\lambda_t, \quad A^h_{t-1} \in \Omega_{t-1}. \tag{8}$$

Finally, the equalities (7) and (8) yield $P(t-1,\_) = E_{\lambda_t}[P(t,\_)|\Sigma_{t-1}]$.

Step 4 The projective limit $\bar{\lambda}$ of $(\lambda_t)_{t=0}^{\infty}$ is a Radon measure on $\bar{\Sigma}$ projectively equivalent to $\bar{\mu}$.

According to the corollary following the Prokhorov theorem, the projective limit of any countable projective system of Radon measures exists and is a Radon measure. Consequently, there exists $\bar{\lambda} = \lim_{t \in T} \lambda_t$, the projective limit of $(\lambda_t)_{t=0}^{\infty}$, and $\bar{\lambda}$ is a Radon probability measure on $\bar{\Omega}$. As λ_t and μ_t are equivalent for all $t \in T$, $t \geq 1$, the measures $\bar{\lambda}$ and $\bar{\mu}$ are projectively equivalent.

Step 5 The process $\left(\bar{P}(t,\_)\right)_{t=0}^{\infty}$ is a martingale with respect to $\bar{\lambda}$ and the filtration $\left\{\bar{\Sigma}_t\right\}_{t=0}^{\infty}$.

We have to prove that

$$\int_{\pi^{-1}_{t-1}(A^h_{t-1})} \bar{P}(t-1,\_)d\bar{\lambda} = \int_{\pi^{-1}_{t-1}(A^h_{t-1})} \bar{P}(t,\_)d\bar{\lambda}, \quad A^h_{t-1} \in \Omega_{t-1}. \tag{9}$$

Clearly, given $A^j_t \in \Omega_t$, the restriction of $\bar{P}(t,\_)$ to $\pi_t^{-1}(A^j_t)$ coincides with the restriction of $P(t,\_)$ to A^j_t. Analogously, the restriction of $\bar{P}(t-1,\_)$ to $\pi^{-1}_{t-1}(A^h_{t-1})$ equals the restriction of $P(t-1,\_)$ to A^h_{t-1}, for every $A^h_{t-1} \in \Omega_{t-1}$. Since $\pi_t(\bar{\lambda}) = \lambda_t$ and $\pi_{t-1}(\bar{\lambda}) = \lambda_{t-1}$, relation (9) is equivalent to (8), that has been already proven.

In summary, $\bar{\lambda}$ is a risk-neutral projective probability measure and the necessity part is completed.

We turn now to the sufficiency. Let $\bar{\lambda}$ be a projectively equivalent martingale measure. Then, for every $\bar{\Sigma}_{t-1}$-measurable bounded function $\bar{h} \colon \bar{\Omega} \to \mathbb{R}^n$,

$$E_{\bar{\lambda}}[\langle \bar{h}, \bar{P}(t,\_) - \bar{P}(t-1,\_)\rangle] = 0. \tag{10}$$

If, in addition, $\langle \bar{h}(\bar{\omega}), \bar{P}(t,\bar{\omega}) - \bar{P}(t-1,\bar{\omega})\rangle \geq 0$, $\bar{\mu}$ almost surely, then we have that $\langle \bar{h}(\bar{\omega}), \bar{P}(t,\bar{\omega}) - \bar{P}(t-1,\bar{\omega})\rangle \geq 0$, $\bar{\lambda}$ almost surely, because $\bar{\mu}$ and $\bar{\lambda}$ are projectively equivalent. Then, from (10) and Proposition 1 we conclude that the absence of arbitrage holds in the market.

Example 1 (Back and Pliska) *Let us examine, under our method, the example of Back and Pliska [1]. Imagine the random experiment of rolling a fair die until the first number different from* 6 *comes out. Denote by $\omega \in \mathbb{N}$ the number of the roll when this occurs. Clearly, the probability of every event $\omega \in \mathbb{N}$ is $\mu(\omega) = \frac{5}{6}(\frac{1}{6})^{\omega-1}$. Suppose that only two securities can be sold and bought every time $t \in \mathbb{N}$ that we roll the die. The first one is the riskless bond. The price process of the second security is*

$$P_2^t(\omega) = \begin{cases} 1 & \text{if } t = 0 \\ \left(\frac{1}{2}\right)^t & \text{if } 0 < t < \omega \\ (\omega^2 + 2\omega + 2)\left(\frac{1}{2}\right)^{\omega} & \text{if } t \geq \omega \end{cases}.$$

This market has no arbitrage but no measure on $\mathbb{N}$ is an equivalent martingale measure to μ.

It is easy to check that for this example $\bar{\Omega} = \mathbb{N} \cup \{\infty\}$, the Alexandroff compactification of $\mathbb{N}$. The projective model just adds one more event corresponding to the point of infinity: "Number 6 comes out in all the rolls". Obviously, the point of infinity is a null event, i.e. $\bar{\mu}(\infty) = 0$. Since the market is arbitrage free, there must be a projectively equivalent martingale measure. Following the constructive procedure of Theorem 2, one finds that the measure $\bar{\lambda}(\omega) = \frac{1}{2\omega(\omega+1)}$, $\bar{\lambda}(\infty) = \frac{1}{2}$, is a risk-neutral projective probability measure.

Observe that $\bar{\lambda}$ assigns positive probability to the $\bar{\mu}$-null event ∞ and, consequently, $\bar{\mu}$ and $\bar{\lambda}$ are not equivalent measures. However, as pointed out by Theorem 2, $\bar{\mu}$ and $\bar{\lambda}$ are projectively equivalent.

5 Conclusions

For an infinite number of trading dates the characterization of the arbitrage absence by the existence of equivalent martingale measures presents some difficulties, and the price process of the assets needs notions such as "no free lunch" or "no free lunch with bounded risk", generalizing the concept of "no arbitrage".

This paper has formulated the problem for infinite discrete time in a different mathematical setting, and it has obtained a theorem of asset pricing which may be phrased using only the classical notion of "no arbitrage".

The martingale measure is built as a projective limit of Radon measures and extends the initial probability space. Both the martingale measure and the initial probability measure generate equivalent projections.

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