# Stochastic Optimal Control and Regime Switching: Applications in Economics

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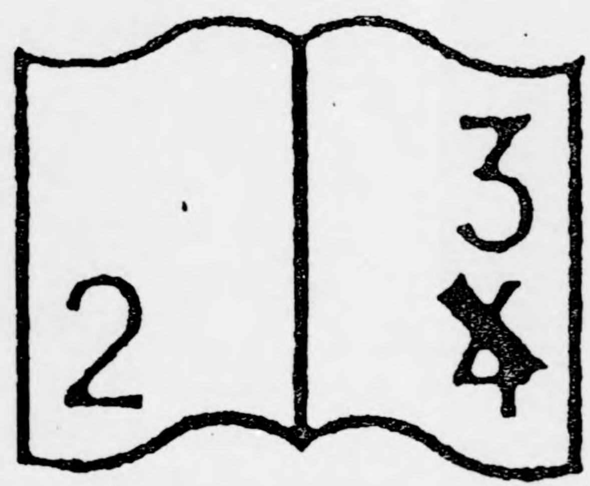


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# Abstract

Economic decisions under uncertainty generally involve a change of stochastic regime. This thesis examines the formal conditions for optimizing such decisions and looks at applications to exchange rate intervention, physical investment and consumption behaviour. Many of these economic regime switchings can be mathematically formulated as stopping problems. Global optimality is achieved by applying Hamilton-Jacobi-Bellman equations in each regime, together with the joining conditions at the switching boundaries. Chapter 1 establishes the framework for optimisation and provides various boundary conditions for different switching cases.

Chapter 2 applies optimal stopping techniques to derive optimal "time-consistent" exchange rate target zones in the presence of proportional/lump sum intervention costs. It further shows that such discretionary equilibria can be improved upon by a credible commitment to an exchange rate mechanism (such as ERM).

Chapter 3 characterises the irreversible oil investment decision in the North Sea as an optimal regime switching problem. In the absence of Petroleum Revenue Tax (PRT), it shows how the optimal development decision will be deferred when real oil prices follow a geometric Brownian motion.

In chapter 4, an intertemporal partial equilibrium model of investment is used to assess the effects of stochastic capital depreciation on optimal investment behaviour, in a context where a sales constraint effectively decomposes the problem into two distinct regimes. The presence of the uncertainty about depreciation reduces firm's demand

for investment; and increasing the variability of capital depreciation further reduces investment. The uncertainty also makes investment "smoother" than that under certainty.

Finally, chapter 5 and 6 deal with optimal consumption/portfolio decisions in a two-asset model with shortselling and borrowing restrictions imposed. Chapter 5 formulates a regime switching problem due to the presence of the borrowing constraint and specifies the corresponding boundary conditions. Chapter 6 characterises optimal solutions to various combinations of parameters for constant relative and constant absolute risk aversion utility functions. In many cases, if labour income is fully diversifiable, the borrowing constraint only binds when the wealth level falls below a threshold, and risk taking behaviour at the low level of wealth is associated with a convex portion of the indirect utility function (value function). In such regime-switch cases, the introduction of the borrowing constraint makes consumption more volatile relative to income. It also generates the precautionary motive for saving.

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# Declaration

Chapters 2 and 3 are adopted from the joint papers with Marcus Miller. Chapter 3 was disseminated as a working paper in Department of Economics, Warwick University (Working Paper No. 9240). No other materials in this thesis have been published in any form.

# Introduction

Economic decisions under uncertainty generally involve a change of stochastic regime. This thesis examines the formal conditions for optimizing such decisions and looks at applications to exchange rate intervention, physical investment and consumption behaviour. Many of these economic regime switchings can be mathematically formulated as stopping problems. This applies whether the switch is induced by the changes of the underlying forcing processes (chapters 3 and 4) or by state contingent constraints (chapters 5–7). If the costs associated with the controls are proportional and/or lump sum (chapter 2–4), the controls are always exercised at boundaries (Dixit 1991, 1991b). So to determine an optimal control is equivalent to choose an optimal stopping (or switching) boundary. If the costs are convex (chapters 5–7), the optimal control in each separate regime has to be determined simultaneously with the stopping (switching) boundary (Whittle 1983; Dixit 1991, 1991b). So in both cases, characterising boundary conditions for various switching cases is crucial. Chapter 1 summarises basic properties of stochastic optimal control and presents regime switching frameworks for both linear and convex costs of controls. Various switching conditions are characterised as boundary conditions. The applications of optimal stopping in economics include two parts: linear and convex control costs. The former cost structure is found in chapters 2–3, and latter in chapters 4–6.

The positive analysis of target zones shows how anticipated intervention at the edges of a currency band influences the rate inside the band (Krugman, 1991). But it does not explain why a policy of marginal (or intra-marginal) intervention might be chosen in the first place. If fundamentals are driven by a Wiener process, and the monetary authorities seek to minimise the deviation of exchange rate from a target value subject to intervention costs proportional to the size of intervention (and/or lump sum), the problem of finding the time-consistent optimal managed exchange rate can be formulated as an optimal stopping problem. Chapter 2 deals precisely with this case. Since the exchange rate is forward looking and expectations play a crucial role, chapter 2 further investigates whether the time-consistent equilibria can be improved upon by adopting rules.

Investment under future price fluctuations will normally be delayed because investment involves sunk costs which are costly to reverse and the firms which carry out the projects generally have the option to wait (Pindyck 1991, Dixit 1992). Such "irreversibility" is particularly apparent in the case of a North Sea oil field development since capital expenditures on erecting platforms and establishing pipelines are large and abandonment involving environmental cleaning and severance pay for workers is expensive. Because of the presence of the irreversibility, the investment opportunity is analogous to a financial call option and abandonment is equivalent to a financial put option (Pindyck, 1991), so this case can also be characterised as an optimal stopping problem. Chapter 3 applies this technique to assess the effects of various development costs and development lags on firm's development decisions under price uncertainty.

A linear cost structure normally produces elegant results, but many economic applications involve convex costs of control. In this case, deviations from the optimal path will be corrected continuously since larger deviations require higher costs of control (Dixit, 1991). When regimes are induced by state contingent constraints, the optimal policies have to be determined simultaneously with the selection of the switching points. Chapter 4 looks at the optimal investment decisions in a partial equilibrium model where adjustment costs are a convex function of investment. The effect of stochastic capital depreciation on the behaviour of investment is examined in circumstances where an upper limit on sales is imposed as a constraint.

Another example of applying continuous control is found in a two-asset model of consumption and portfolio decisions. In chapters 5 and 6, the version of Karatzas *et al* (1986) is used to assess the impact of introducing liquidity constraints on consumption and portfolio behaviour. The the phenomena of consumption smoothing and precautionary saving are also discussed not as they arise from the volatility of income but with reference to the presence of liquidity constraints.

# Chapter 1

# Mathematical Techniques

## 1.1 Introduction

Stochastic control theory is a relatively new theory which shows very important applications in economics. However, actual economic systems subject to control do not admit a strictly deterministic analysis in view of random factors of various kinds which influence their behaviour (Malliaris and Brock,1983). Such factors include, for example, fluctuation in the price level, in exchange rate and in supply and demand of commodities. The stochastic control theory takes the random nature of the behaviour of a system into account. In such cases, it is natural, when choosing a control strategy, to proceed from the average expected results, taking note of all the possible variants of the behaviour of a controlled system.

The economic applications of adopting continuous stochastic optimal control can be found in Merton (1969, 1971) in treating optimal consumption and portfolio choices when the individual facing portfolio decision of spanning his wealth over risky and riskfree securities. Subsequent developments of using an intertemporal optimisation framework treating option pricing are summarised in the book by Merton (1990). The applications of using discrete optimal control models (impulse and/or instantaneous control) can be found in many aspects, such as Scarf's inventory model (1959), Bertola (1989) and Pindyck (1988) in irreversible investment, Dixit (1989) in entry and exit problems. The aim of this chapter is to summarise some properties of continuous and discrete stochastic optimal control theory which will lead to the establishment of a mathematical model to treat stochastic regime switching problems.

The standard approach to these control problems is dynamic programming developed by Bellman (1957). This is due to the fact that there exist profound relationships between the stochastic representations and certain types of partial differential equations, so the variational methods applied to these partial differential equations can yield some satisfactory results. Many results in this area are well-known. The value function of the continuous controlled diffusion is twice continuously differentiable in the domain considered for given regularity conditions. The problem of exit from a domain through fixed boundary is a Dirichlet problem where value matching condition is satisfied. The problem of optimal exit from a domain through a fixed boundary has one more condition, i.e., a smooth pasting condition. In the stochastic regime switching case, for prescribed boundary, value matching and smooth pasting conditions are satisfied (Krylov 1980, Whittle 1983); for an optimally chosen boundary second order smooth pasting condition is satisfied (Whittle 1983). In the infinite time horizon case, transversality condition will ensure the uniqueness of the solution. In the discrete control case, all the controls are boundary controls. If the control is instantaneous, the boundary is a reflecting one; if the control is impulse, then we may have one-sided or two-sided (s,S) controls.

This chapter has three sections. Section 2 describes various properties regarding the stochastic processes. Section 3 states some basic results in continuous control theory and a model for treating stochastic regime switching is developed. Section 4 summarises certain features of the discrete control problems, and section 5 concludes this chapter.

## 1.2 Basic Properties of Stochastic Processes

In this section, we shall state some preliminary requirements which will be useful to continuous and discrete stochastic optimal control problems. Most results are already in the textbooks by Friedman (1975), Karatzas and Shreve (1988). We shall rearrange these important results in order to suit the development of the mathematical model treating stochastic regime switching problems in this thesis.

First part is those of the basic concepts and properties of stochastic processes, Brownian motion, stochastic integral and differential. Itô's lemma is provided at the end of this part. The second part describes the relationship between Markov processes and the solutions to the stochastic differential equations. The final part shows the linkage between the stochastic representations and certain types of partial differential equations, various boundary value problems are described.

## 1.2.1 Some Preliminaries

### 1.2.1.1 Stochastic Processes and $\sigma$-algebras

The theory of probability deals with observations or experiments that can be repeated many times under the 'identical' conditions and where we inquire into various numerical characteristics of the phenomenon being studied. Thus we are interested in quantities taking on values that depend on the particular outcome of the observation. These are random variables. If these outcomes depend on time, then they are stochastic processes. Precisely speaking, a stochastic process is a mathematical model for the occurrence, at each moment after the initial time, of a random phenomenon. The randomness is captured by the introduction of a measurable space $(\Omega, \mathcal{F})$, called the sample space, on which probability measures can be placed. In this case $\Omega$ is the sample space which contains every elementary point of random outcomes, $\mathcal{F}$ is a $\sigma$-algebra which contains all the complement, union and intersection of the elementary points given in the sample space and it is closed under all countable set operations. Thus, a stochastic process is a collection of random variable $X = \{X_t : 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F})$, which take values in a second measurable space $(S, \varphi)$, called the state space. Here, we specify the state space to be a d-dimensional Euclidean space equipped with the $\sigma$-fields of Borel set, i.e., $S = \Re^d$, $\varphi = \mathcal{B}(\Re^d)$, where $\mathcal{B}(\mathcal{U})$ is to denote the smallest $\sigma$-field containing all the open sets of a topological space $\mathcal{U}$. The index $t \in [0, \infty)$ of the random variable $X_t$ admits a convenient interpretation as time.

For a fixed sample point in the sample space $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$; $t \geq 0$ is the sample path of the process $X$ associated with

$\omega$. It provides the mathematical description for the random process whose outcome can be observed continuously in time.

The introduction of the $\sigma$-field is due to the reason of applying the theory of Lebesgue integration. Probability measures are defined on $\sigma$-fields and random variables are assumed to be measurable with respect to these $\sigma$-fields. Thus, implicitly in the statement that a random process $X_t$ is a collection of $(\Re^d, \mathcal{B}(\Re^d))$-valued random variable on $(\Omega, \mathcal{F})$ is the assumption that each $X_t$ is $\mathcal{F}/\mathcal{B}(\Re^d)$-measurable. The nontechnical reason to include $\sigma$-fields in the study of stochastic process is that these $\sigma$-fields can be used to keep track of information. The temporal feature of a stochastic process suggests a flow of time, in which, at every moment $t \geq 0$, we can have time reference as past, present and future and the amount of information brought about can then be compared at different time. Having noticed this feature of $\sigma$-fields we demand that the $\sigma$-field generated by a stochastic process is also time dependent and nondecreasing, i.e., for $\{\mathcal{F}_t; t \geq 0\}$ we have $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_\infty$ for any $0 \leq s < t < \infty$ and $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$.

We say a stochastic process $X_t$ is adapted to the $\sigma$-field $\{\mathcal{F}_t\}$ if $X_t$ is $\mathcal{F}_t$-measurable random variable. If the process is progressively measurable with respect to the product $\sigma$-field $\mathcal{B}([0,t,]) \otimes \mathcal{F}_t$, then the process is progressively measurable. We say the two stochastic processes are stochastically equivalent if every sample path of these two processes agrees with probability one. If any sample path of a stochastic process is continuous, then the process is continuous.

#### 1.2.1.2 Stopping Time

Sometimes, we would like to know when will a random event manifests itself, this type of time which depends on the random event is itself a random variable. We then define a stopping time $\tau$ with respect to a stochastic process $X_t$ if $X_t$ is a stochastic process adapted to $\mathcal{F}_t$. For $0 \leq t < T$, we have the random time $\{\tau \leq t\} \in \mathcal{F}_t$. If the corresponding process is a Markov process, then the stopping time is a Markov time.

#### 1.2.1.3 Brownian Motion

Many application of Brownian motions can be found in economics and finance, such as exchange rate target zone and option pricing. Mathematically, Brownian motions are the basic elements from which stochastic integrals and stochastic differential equations are developed. The integration with respect to a Brownian motion gives us a unifying representation for a large class of martingales and diffusion processes (Karatzas 1988). Diffusion processes represented in this way exhibit a rich connection with the theory of partial differential equations.

A Brownian motion is a continuous, adapted process $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$[1], defined on some probability space $(\Omega, \mathcal{F}, P)$, with the properties that $B_0 = 0$ almost surely. And for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\sigma$-field $\mathcal{F}_s$ and is normally distributed with mean zero and variance $t - s$. So constructed Brownian motion is a square integrable martingale (Friedman, 1975), and it is Hölder continuous with any exponent $\alpha < 1/2$; for $\alpha > 1/2$, almost all sample paths are nowhere Hölder continuous, therefore it is nowhere differentiable

[1] We shall later denote it by $W_t$.

(Friedman 1975). The Brownian motion after a stopping time is still a Brownian motion, so a Brownian motion starts afresh at any stopping time (Friedman, 1975). The order of a Brownian motion with respect to $t$ is approximately $\sqrt{t}$, so the sample path of a Brownian motion can be explosive if $t \longrightarrow \infty$, namely,

$$\overline{\lim_{t\to\infty}} \frac{B(t)}{\sqrt{2t\log\log t}} = 1 \quad \text{a.s.} \tag{1.2.1}$$

$$\underline{\lim}_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log t}} = -1 \quad \text{a.s.} \tag{1.2.2}$$

where $B(t)$ is a Brownian motion, $\overline{lim}$ denotes the upper limit, $\underline{lim}$ denotes the lower limit, a.s. means almost surely.

#### 1.2.1.4 Stochastic Integral

Stochastic calculus grew out of the need to assign meaning to ordinary differential equations involving continuous stochastic processes. Since the most important such process, Brownian motion, cannot be differentiated, stochastic calculus takes the task opposite to that of classical calculus: the stochastic integral is defined first and then the stochastic differential is given meaning through the fundamental "theorem" of calculus. This "theorem" is really a definition in stochastic calculus, because the differential has no meaning apart from that assigned to it when it becomes an integral. For this theory to achieve its full potential, it must have some simple rules for computation. These are contained in the change of variable formula (Itô's rule), which is the counterpart of the chain rule from the classical calculus.

A stochastic integral is defined for a step function with respect to a Brownian motion. Let f(t) be a step function in $L^2_w[\alpha,\beta]$, $f(t) = f_i$ if $t_i \le t < t_{i+1}$, $0 \le i \le r-1$, where $\alpha = t_0 < t_1 < \cdots < t_r = \beta$. The random variable

$$\sum_{k=0}^{r-1} f(t_k)[B(t_{k+1}) - B(t_k)] \tag{1.2.3}$$

is denoted by

$$\int_\alpha^\beta f(t)dB(t). \tag{1.2.4}$$

Apparently, for such defined stochastic integral, we have

$$E\int_\alpha^\beta f(t)dB(t) = 0, \tag{1.2.5}$$

$$E|\int_\alpha^\beta f(t)dB(t)|^2 = E\int_\alpha^\beta f^2(t)dt, \tag{1.2.6}$$

where $E$ denotes mathematical expectation.

A stochastic indefinite integral is defined as

$$I(t) = \int_0^t f(s)dB(s), \quad 0 \le s \le t, \tag{1.2.7}$$

and it is not difficult to prove that $I(t)$ is a continuous martingale.

If we have a process $\xi(t)$ $(0 \le t \le T)$ such that for any $0 \le t_1 \le t_2 \le T$

$$\xi(t_2) - \xi(t_1) = \int_{t_1}^{t_2} a(t)dt + \int_{t_1}^{t_2} b(t)dB(t), \tag{1.2.8}$$

where $a \in L^1_w[0,T], b \in L^2_w[0,T]$. Then we say that $\xi(t)$ has stochastic differential $d\xi$, on $[0,T]$, given by

$$d\xi(t) = a(t)dt + b(t)dB(t). \tag{1.2.9}$$

Therefore, the stochastic differential defined in this sense comes from the stochastic integral.

For a well-defined function we can seek its stochastic differential by using Itô's lemma. Let $dx_i(t) = a_i(t)dt + b_i(t)dB$ and let $f(x_1, \cdots, x_m, t)$ be a continuous function in $(X, t)$ where $X = (x_1, x_2, \cdots, x_m) \in \Re^m, t \geq 0$, together with its first $t$-derivative and second $x$-derivatives. Then, $f(x_1(t), \cdots, x_m(t), t)$ has a stochastic differential given by

$$\begin{aligned} df(X(t), t) = [f_t(X(t), t) + \sum_{i=1}^{m} f_{x_i}(Z(t), t)a_i(t) \\ + \frac{1}{2} \sum_{i,j=1}^{m} f_{x_i x_j}(X(t), t)b_i(t)b_j(t)]dt \\ + \sum_{i=1}^{m} f_{x_i}(X(t), t)b_i(t)dB(t), \end{aligned} \tag{1.2.10}$$

where

$$X(t) = (x_1(t), \cdots, x_m(t)).$$

## 1.2.2 Markov Processes and Stochastic Differential Equations

### 1.2.2.1 Markov Processes

A Markov process is a process whose probability density function satisfying Chapman-Kolmogorov equation

$$P(s, x, t, A) = \int_{\Re^d} P(s, x, \lambda, dy)p(\lambda, y, t, A), \tag{1.2.11}$$

for any $0 \leq s < \lambda < t < \infty$, $x \in \Re^d$, $A \in \mathcal{B}_d$.

The Markov process only has one period memory, the conditional expectation of a Markov functional only depends on the $\sigma$-field which the expectation conditions on, namely

$$E_{x,s}\{f(x(t+h))|\mathcal{F}_t^s\} = E_{x(t),t}f(x(t+h)) \quad \text{a.s.} \tag{1.2.12}$$

where $x(t)$ is a Markov process, $h > 0$, and $\mathcal{F}_t^s$ is the $\sigma$-fields from $s$ to $t$. This property is called Markov property. We notice that a Brownian motion is also a Markov process (Karatzas, 1988). If we let $h$ to be a Markov time (a stopping time), then the above property is strong Markov property (Friedman 1975). If a right continuous Markov process satisfies Feller property (Friedman, 1975 p. 23), then it satisfies strong Markov property.

Markov and strong Markov properties are important for characterising stochastic processes, because a solution to a stochastic differential equation is a Markov process, a continuous solution has strong Markov property. The corresponding stopping time to the Markov process is a Markov time: the first exit time to a close set, the first entry time to an open set are all Markov times. If the process satisfies strong Markov property, the first exit time from an open set or first entry time to a closed set are also Markov times (Friedman 1975). These factors are important for treating optimal stopping problems.

#### 1.2.2.2 Stochastic Differential Equation

A stochastic differential equation is a combination of a stochastic differential and its initial condition. Let $b(x,t) = (b_1(x,t), \cdots, b_n(x,t))$, $\sigma(x,t) = (\sigma_{ij}(x,t))_{i,j=1}^n$ and suppose the functions $b_i(x,t), \sigma_{ij}(x,t)$ are measurable in $(x,t) \in \Re^n \times [0,T]$. If $\xi(t)$ $(0 \leq t \leq T)$ is a stochastic process such that

$$d\xi(t) = b(\xi(t), t) + \sigma(\xi(t), t)dB(t), \tag{1.2.13}$$

$$\xi(0) = \xi_0 \quad \text{a.s.} \tag{1.2.14}$$

then $\xi(t)$ satisfying the system of stochastic differential equation (1.2.13) and the initial condition (1.2.14). Here, it is implicitly assumed that $b(\xi(t), t) \in L^1_w[0, T]$ and $\sigma(\xi(t), t) \in L^2_w[0, T]$.

An existence and uniqueness theorem for fixed time horizon $t \in [0, T]$ holds if $b(x, t), \sigma(x, t)$ are measurable and satisfy Lipschitz conditions, namely

$$|b(x,t) - b(\bar{x},t)| \leq K^1_*|x - \bar{x}|, \quad |\sigma(x,t) - \sigma(\bar{x},t)| \leq K^2_*|x - \bar{x}|$$
$$|b(x,t)| \leq K^1(1 + |x|), \qquad |\sigma(x,t)| \leq K^2(1 + |x|) \tag{1.2.15}$$

where $K^i$, $K^i_*$ $(i = 1, 2.)$ are positive constants. $\xi_0$ is any n-dimensional random vector independent of $\mathcal{F}(B(t), 0 \leq t \leq T)$, such that $E|\xi_0|^2 < \infty$, then the solution to equations (1.2.13) and (1.2.14) exists and is unique in $M^2_w[0, T]$.

A stronger local existence and uniqueness theorem holds for a stochastic differential equation provided that $b(x, t), \sigma(x, t)$ are measurable in a bounded domain and satisfy Lipschitz conditions, the solution has a continuous version where every sample path agrees almost surely and the first exit time matched almost surely (Friedman, p.103, 1975).

The solution to stochastic differential equations (1.2.13) and (1.2.14) is a Markov process, satisfies strong Markov property (Friedman pp.111-112, 1975). The solution to (1.2.13) and (1.2.14) is also a diffusion process with drift $b(x,t)$, and diffusion matrix $a = \sigma(x,t)\sigma^*(x,t)$ (Friedman p.115, 1975).

### 1.2.3 Expected Functional as a Solution to a Partial Differential Equation

In order to consider the stochastic representation of the solutions to certain partial differential equations, first we introduce two partial differential operators and their related boundary value problems. Let us define the first partial differential operator as

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d_1} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + C(x)u. \tag{1.2.16}$$

with all the coefficients of the operator are real in domain D, $L$ is elliptic at a point $x^0$ if the matrix $(a_{ij}(x))$ is positive definite, i.e., for any real vector $\xi \neq 0$,

$$\sum a_{ij}(x^0)\xi_i\xi_j > 0. \tag{1.2.17}$$

The following boundary problem is called Dirichlet problem or first boundary value problem if

$$Lu(x) = f(x) \quad \text{in } D \tag{1.2.18}$$

$$u(x) = \phi(x) \quad \text{on } \partial D \tag{1.2.19}$$

where $\partial D$ is the boundary of the bounded domain $D$.

Another partial differential operator is

$$Mu = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x,t)\frac{\partial u}{\partial x_i} + C(x,u) - \frac{\partial u}{\partial t} \tag{1.2.20}$$

with real coefficients defined in an $(d+1)$-dimensional domain $Q$. If

$$\sum a_{ij}(x^0, t^0)\xi_i\xi_j > 0 \quad \text{for} \quad \text{any} \quad \xi \in \Re_d, \xi \neq 0, \tag{1.2.21}$$

then $M$ is a parabolic operator at $(x^0, t^0)$. $M$ is uniformly parabolic in $Q$ if there is a positive constant $\mu$, such that

$$\sum a_{ij}(x,t)\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{for all} \quad (x,t) \in Q, \xi \in \Re_d, \xi \neq 0. \tag{1.2.22}$$

The initial value problem for operator $M$ is a Cauchy problem if

$$Mu(x,t) = f(x,t) \quad \text{in} \quad D \times (0,T], \tag{1.2.23}$$

with initial condition

$$u(x,0) = \phi(x) \quad \text{on} \quad D, t = 0. \tag{1.2.24}$$

The first initial-boundary value problem generated by this operator is

$$Mu(x,t) = f(x,t) \quad \text{in} \quad D \times (0,T], \tag{1.2.25}$$

with initial condition

$$u(x,0) = \phi(x) \quad \text{on} \quad D, \quad \text{when} \quad t = 0, \tag{1.2.26}$$

and boundary condition

$$u(x,t) = g(x,t) \quad \text{on} \quad \partial D \quad \text{for} \quad t \in (0,T]. \tag{1.2.27}$$

The existence and uniqueness theorems for these boundary value problems are provided in Friedman (pp.134-139, 1975).

Under certain conditions the following stochastic representation is the solution to the Dirichlet problem (1.2.18)-(1.2.19):

$$U(x) = E_x\phi(\xi(\tau))\exp[\int_0^\tau C(\xi(s))ds] - E_x\int_0^t f(\xi(t))\exp[\int_0^t C(\xi(s))ds]dt, \qquad (1.2.28)$$

where $\tau$ is the first exit time (Markov time) from D, $\tau < \infty$ almost surely.

The following stochastic representation is the solution to the Cauchy problem (1.2.23) and (1.2.24):

$$U(x,t) = E_{x,t}\phi(\xi(T))\exp[\int_t^T C(\xi(s),s)ds] - E_{x,t}\int_t^T f(\xi(s),s)\exp[\int_\tau^s C(\xi(\lambda),\lambda),d\lambda]ds. \qquad (1.2.29)$$

Finally, the following stochastic representation is the solution to the first initial boundary problem (1.2.25)-(1.2.27):

$$U(x,t) = E_{x,t}g(\xi(\tau),\tau)\exp[\int_t^\tau C(\xi(s),s)ds]\chi_{\tau<T} + E_{x,t}\phi(\xi(T))\exp[\int_t^\tau C(\xi(s),s)ds]\chi_\tau - E_{x,t}\int_t^\tau f(\xi(s),s)\exp[\int_\tau^s C(\xi(\lambda),\lambda),d\lambda]ds, \qquad (1.2.30)$$

where $\chi$ is an indicator.

## 1.3 Continuous-Time Stochastic Optimal Control

Stochastic control theory is a relatively young branch of mathematics. The beginning of its intensive development falls in the late and early 60s. During that period an extensive literature appeared on optimal stochastic control using the quadratic performance criterion (cf Wonham, 1970). At the same time, Girsanov (1961) and Howard (1960) made the first steps in constructing a general theory, based on Bellman's technique of dynamic programming. The optimal stochastic control in an infinite horizon with discounting is also established in Kushner (1967, 1971).

Control techniques often involve rules for stopping the processes. A general and rather sophisticated theory of optimal stopping rules for Markov chains and Markov processes, developed by many authors, is described by Shiryayev (1978). The development of such technique makes it possible to treat the regime switching problems under uncertainty, where the optimal policies together with the optimally chosen boundary to stop the process in each regime have to be determined simultaneously. The motivation in this section is to develop a mathematical model treating stochastic regime switching problems by using optimal stopping techniques. It is essential to clarify various conditions characterising boundary behaviour. In this section, boundary conditions for prescribed boundaries are provided by Krylov (1980), the optimality boundary conditions are from Whittle (1983).

The first part of this section is to state the stochastic optimal control problem and its corresponding Bellman equation, second part describes an infinite horizon optimal control problem and its asymptotical condition, third one treats the fixed boundary problem, the last part develops a free boundary problem and a model for stochastic regime switching.

### 1.3.1 Description of the Stochastic Optimal Control Problem and Bellman Equation

Let us consider a system which performs according to the following stochastic differential equation:

$$dx_s = b(\alpha_s, x_s)ds + \sigma(\alpha_s, x_s)dB_s \tag{1.3.1}$$

with the initial condition

$$x(0) = x, \tag{1.3.2}$$

where $x_s, x \in \Re^d$, $\alpha_s$ is a control parameter, $B_s$ is $d_1$-dimensional Brownian motion, $b$ is a d-dimensional vector, $\sigma$ is a $d_1 \times d$ matrix.

Let $\mathcal{A}$ be the set of all admissible controls, then choosing appropriately the random process $\alpha_s$ with value in $\mathcal{A}$ we can obtain various solutions to equation (1.3.1). We can then control this diffusion process.

Suppose that the criterion of the control is given by a cost functional for evaluating the control performance. Suppose also that the cost rate incurred for a given control and state is $f^{\alpha_s}(x_s)$, then the total costs used in an infinite horizon is

$$\rho^\alpha = \int_0^\infty f^{\alpha_s}(x_s)ds. \tag{1.3.3}$$

In this case, we would like to find a strategy $\alpha = \alpha^0$ such that the expected loss is kept minimum.

$$V(x) = V^{\alpha^0}(x) = \inf_{\alpha \in \mathcal{A}} E_x \int_0^\infty f^{\alpha_s}(x_s) ds, \tag{1.3.4}$$

where $E_x$ is an expectation operator conditional on initial state $x$, $V(x)$ is the value function.

In order to find such strategy, we shall apply Bellman's principle, namely,

$$V(x) = \inf_{\alpha \in \mathcal{A}} E_x \{ \int_0^t f^{\alpha_s}(x_s) ds + V(x_t) \}, \tag{1.3.5}$$

where the first term on the right hand side of (1.3.5) is the cost lost up to time $t$, the second term is the minimum cost incurred if the process starts from $x_t$. So, to find the optimal policy which controls the overall performance of the value function reduces to find the optimal policy which controls the temporary behaviour of the value function.

If we suppose that $V(x)$ is sufficiently smooth. Applying Itô's lemma we derive

$$V(x) = E_x V(x_t) - E_x \int_0^t L^{\alpha_s}(x_s) V(x_s) ds, \tag{1.3.6}$$

where

$$L^\alpha(x) = \sum_{i,j=1}^{d} a_{ij}(\alpha, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_i(\alpha, x) \frac{\partial}{\partial x_i}, \tag{1.3.7}$$

and

$$a_{ij}(\alpha, x) = \frac{1}{2} \sum_{k=1}^{d_1} \sigma_{ik}(\alpha, x) \sigma_{kj}(\alpha, x). \tag{1.3.8}$$

Therefore, if follows from the Bellman's principle that

$$
\begin{aligned}
&\inf_{\alpha\in\mathcal{A}}\{E_x[\int_0^t f^{\alpha_s}(x_s)ds + V(x_s)] - V(x)\} \\
&= \inf_{\alpha\in\mathcal{A}} E_x[\int_0^t f^{\alpha_s}(x_s)ds + L^{\alpha_s}V(x_s)ds] \\
&= 0. \qquad (1.3.9)
\end{aligned}
$$

Divide all the expression by $t$ and let $t \to 0$, obtaining thereby the equation

$$\inf_{\alpha\in\mathcal{A}}[L^\alpha(x)V(x) + f^\alpha(x)] = 0. \qquad (1.3.10)$$

We notice that such constructed strategy determining the choice of a control at $t$ on the basis of the instantaneous value of $x_t$ rather than the entire history of the trajectory $x_{[0,t]}$. Intuitive reasoning suggests that we could have restricted ourselves to the aforementioned strategies from the very beginning. Indeed, the knowledge of how the trajectory has arrived at the point $x_t$ cannot help us, by any means, to influence the 'future' behaviour of the trajectory because increments of the process $B_t$, which determine this behaviour, do not depend on the 'past' (see previous section for the property of the Brownian motion). Furthermore, the cost we have to pay after the trajectory has arrived at the point $x_t$ is not a function of the preceding segment of the trajectory. If it is therefore advantageous, for any reason, to use a control at least once after the trajectory has reached the point $x_t$, it will be advantageous, for the same reasons, to use this strategy each time when the trajectory reaches $x_t$.

### 1.3.2 Infinite Horizon Problem

Consider an infinite horizon stochastic optimal control problem, where the value function is the expected discounted costs

$$V^{\alpha}(x) = E_x \int_0^{\infty} f^{\alpha_s}(x_s) \exp(-\int_0^s C^{\alpha_t}(x_t)dt)ds, \qquad (1.3.11)$$

where $C^{\alpha_s}(x_s)$ is the discount rate.

The dynamics of the system is governed by the following stochastic differential equation (SDE),

$$dx_s = b(\alpha_s, x_s)ds + \sigma(\alpha_s, x_s)dB_s, \qquad (1.3.12)$$

where $\alpha$ is a control parameter, $x_s \in \Re^d$ for $s \in [o, \infty)$, $B_s$ is $d_1$-dimensional Brownian motion, $b$ is d-dimensional vector, $\sigma$ is a $d_1 \times d$ matrix. The initial condition for this SDE is

$$x_0 = x. \qquad (1.3.13)$$

Then the optimal control problem is described as

$$\inf_{\alpha \in \mathcal{A}} V^{\alpha}(x) \qquad (1.3.14)$$

$$\text{s.t.} \quad (1.3.12) \text{ and } (1.3.13)$$

where $\mathcal{A}$ is the admissible control set.

It is not difficult to guess the partial differential equation for the stochastic representation (1.3.11) (see equations (1.2.18), (1.2.19) and

(1.2.28)), namely.

$$\inf_{\alpha \in \mathcal{A}} [L^\alpha(x)V(x) - C^\alpha(x)V(x) + f^\alpha(x)] = 0. \tag{1.3.15}$$

The boundary condition in this case is replaced by a transversality condition

$$\lim_{t \to \infty} e^{-Ct} V(x) = 0. \tag{1.3.16}$$

### 1.3.3 Fixed Boundary Problem (One-Dimensional Case)

In order to avoid unnecessary difficulties, we only discuss one-dimensional control case in this part. Multi-dimensional cases are provided in Krylov (chapters 2 and 3, 1980). Before proceeding to list all the available conditions for the fixed boundary problem, we first state the requirements for all the coefficients.

Let $\mathcal{A}$ be a nonempty convex set of some Euclidean space, and $\sigma(\alpha, x)$, $b(\alpha, x)$, $C^\alpha(x)$, $f^\alpha(x)$ be real functions given for $\alpha \in \mathcal{A}$, $x \in \Re$. Assume that $C^\alpha \geq 0$; $\sigma(\alpha, x)$, $b(\alpha, x)$, $c^\alpha(x)$, $f^\alpha(x)$ are bounded and satisfy a Lipschitz condition with respect to $(\alpha, x)$, namely, there exist constant $K$ and $K'$ such that for all $\alpha, \beta \in \mathcal{A}$, $x, y \in \Re$,

$$|a(\alpha, x)| + |b(\alpha, x)| + |c^\alpha(x)| + |f^\alpha(x)| \leq K, \tag{1.3.17}$$

$$\begin{aligned} |\sigma(\alpha, x) - \sigma(\beta, y)| + |b(\alpha, x) - b(\beta, y)| + |C^\alpha(x) - C^\beta(y)| \\ + |f^\alpha(x) - f^\beta(y)| \quad \leq K'(|x - y| + |\alpha - \beta|), \end{aligned} \tag{1.3.18}$$

where

$$a(\alpha, x) = \frac{1}{2}\sigma^2(\alpha, x). \quad (1.3.19)$$

Furthermore, we assume that the controlled processes are uniformly non-degenerated, i.e., for some constant $\delta > 0$ and $\alpha \in \mathcal{A}, x \in \Re$

$$a(\alpha, x) \geq \delta. \quad (1.3.20)$$

Let a Brownian motion $(B_t, \mathcal{F}_t)$ be given on some complete probability space $(\Omega, \mathcal{F}, P)$ and $\sigma$-algebras of $\mathcal{F}_t$ be complete with respect to measure $P$.

By a strategy we mean a random process $\alpha_t(\omega)$ with value in $\mathcal{A}$, which is progressively measurable with respect to the system of $\sigma$-algebras of $\{\mathcal{F}_t\}$. We denote $\mathcal{U}$ the set of all admissible strategies.

To each strategy $\alpha \in \mathcal{U}$ and a point $x$ we set into correspondence a solution $x_t^{\alpha,x}$ of the equation

$$x_t = x + \int_0^t b(\alpha_s, x_s)ds + \int_0^t \sigma(\alpha_s, x_s)dB_s. \quad (1.3.21)$$

By Itô's theorem (Karatzas and Shreve, 1988) the solution is unique. We fix numbers $r_1 < r_2$ to form a bounded regime and a function $g(x)$ given for $x = r_1, x = r_2$.

Let us denote $\tau^{\alpha,x}$ the first exit time of $x_t^{\alpha,x}$ from $(r_1, r_2)$, $\tau^{\alpha,x}$ is a Markov time, and we set

$$\begin{aligned} V^\alpha(x) = E_x\{\int_0^{\tau^{\alpha,x}} f^{\alpha_t}(x_t^{\alpha,x}) \exp[-\int_0^t C^{\alpha_s}(x_s^{\alpha,x})ds]dt \\ +g(x_{\tau^{\alpha,x}}^{\alpha,x}) \exp[-\int_0^{\tau^{\alpha,x}} C^{\alpha_s}(x_s^{\alpha,x})ds]\}, \end{aligned} \quad (1.3.22)$$

and

$$V(x) = \sup_{\alpha \in \mathcal{U}} V^{\alpha}(x). \qquad (1.3.23)$$

We now suppress some super-indices provided it does not cause ambiguity, and in addition, let

$$\varphi_t^{\alpha,x} = \int_0^t C^{\alpha_s}(x_s^{\alpha,x})ds. \qquad (1.3.24)$$

Then (1.3.22) can be simplified as

$$V^{\alpha}(x) = E_x[\int_0^{\tau} f^{\alpha_t}(x_t)e^{-\varphi_t}dt + g(x_{\tau})e^{-\varphi_{\tau}}]. \qquad (1.3.25)$$

Now, we have the following theorem (Krylov, p.25, 1980).

**Theorem 1.1** *For $x \in [r_1, r_2]$, $V(r_1) = g(r_1)$, $V(r_2) = g(r_2)$,and its derivatives up to and including the second order are continuous on $[r_1, r_2]$, and $V''(x)$ satisfies a Lipschitz condition on $[r_1, r_2]$. For all $x \in [r_1, r_2]$.*

$$\sup_{\alpha \in \mathcal{A}}[a(\alpha, x)V''(x) + b(\alpha, x)V'(x) - C^{\alpha}(x)V(x) + f^{\alpha}(x)] = 0. \quad (1.3.26)$$

*Furthermore, $V$ is the unique solution of (1.3.26) in the class of functions which are twice continuously differentiable on $[r_1, r_2]$ and equal to $g$ at the end points of this interval.*

The above theorem is from Krylov (1980, p.25). If there is only one-sided bounded boundary, i.e., $x \in (-\infty, r_2]$ or $x \in [r_1, \infty)$ the condition on the other side is a transversality condition.

We notice that equation (1.3.26) is a second order ordinary differential equation (ODE), two boundary conditions will uniquely pin down the solution $V(x)$, which, according to the theorem, is the optimal value function. The variational method used to seek the optimal function for $\alpha$ in equation (1.3.26) will determine the optimal policy. In the above case, we demand that the exit time is finite $\tau^{\alpha,x} < \infty$ .

### 1.3.4 Free Boundary Problem and Stochastic Regime Switching

#### 1.3.4.1 Free Boundary Problem

Sometimes, apart from determining the optimal policy, we have to, at the same time, determine when to stop the process optimally. If the problem is time homogeneous, it is equivalent to say where to stop the process. In this case, we have to seek the optimal stopping boundary. Consider the problem in the bounded regime (it can be extended into the entire space) $x \in [r_1, r_2], g(x)$ is defined and twice continuously differentiable, $\nu$ is the optimal Markov time to stop the process, then the value function is

$$V^{\alpha,\nu}(x) = E_x\{\int_0^{\nu\wedge\tau} e^{-\varphi_t} f^{\alpha_t}(x_t)dt + e^{-\varphi_{\tau\wedge\nu}} g(x_{\nu\wedge\tau})\}. \qquad (1.3.27)$$

The optimal value function is

$$W(x) = \sup_{\alpha\in\mathcal{U},\nu} V^{\alpha,\nu}(x), \qquad (1.3.28)$$

where

$$\nu \wedge \tau = \min\{\nu, \tau\}. \qquad (1.3.29)$$

Let us also define

$$F[W] = \sup_{\alpha \in \mathcal{U}, \nu} [L^{\alpha}(x)V(x) - C^{\alpha}(x)V(x) + f^{\alpha}(x)] \tag{1.3.30}$$

Then we have the following theorem (Krylov, 1980, p.39).

**Theorem 1.2**

**(a)** *$W$ together with its derivatives is continuous on $[r_1, r_2]$, $W'$ is absolutely continuous, and $W''$ is bounded on $[r_1, r_2]$. The function $W''$ satisfies Lipschitz condition outside the set $\Gamma = \{x \in [r_1, r_2] : W(x) = g(x)\}$.*

**(b)** *$W \geq g, W(r_i) = g(r_i), F[W] \leq 0$ (a.s.) and $F[W] = 0$ on $[r_1, r_2] \backslash \Gamma$.*

**(c)** *$W' = g'$ on the set $\Gamma \cap [r_1, r_2]$.*

We notice, from section 1.3.3, that the value matching conditions are always satisfied for given boundaries. Suppose we have the case where $\nu < \infty$ almost surely and $[r_1^*, r_2^*]$ is the boundary where exit occurs, applying the theorem we have the following condition

$$F[W] = 0 \quad x \in (r_1^*, r_2^*), \tag{1.3.31}$$

$$W(r_i^*) = g(r_i^*), \;\; i = 1, 2, \quad x = r_i^*, \tag{1.3.32}$$

and

$$W'(r_i^*) = g'(r_i^*), \;\; i = 1, 2, \quad x = r_i^*. \tag{1.3.33}$$

So the smooth pasting conditions are satisfied for the optimally chosen boundaries, while the outside regimes are stopping regimes. To uniquely determine the solution to (1.3.31) for given boundaries, condition (1.3.32) are enough. But optimal stopping require us to determine also the optimal boundaries, so we need condition (1.3.33).

#### 1.3.4.2 Stochastic Regime Switching

In order to consider the regime switching problem, first we separate the whole real space by $b$ ($b$ is to be determined), where regime one is given by $(-\infty, b)$, regime two is given by $(b, +\infty)$. Then we introduce two sets of state dynamics into these two regimes, separate them by different sub-indices.

If the process is initially in regime one, $\tau_1$ is first exit time the process exit to regime two, then value function in regime one is

$$V_1^{\alpha,\tau_1}(x) = E_x\{\int_0^{\tau_1} e^{-\varphi_t} f^{\alpha_t}(x_t)dt + e^{-\varphi_{\tau_1}} V_2(x_{\tau_1})\}. \tag{1.3.34}$$

The optimal value function in regime one for prescribed $b$ is

$$V_1(x) = \sup_{\alpha \in \mathcal{U}} V_1^{\alpha,\tau_1}(x), \tag{1.3.35}$$

where $x_{\tau_1} = b$, $\tau_1 < \infty$ almost surely.

Applying Theorem 1.1, we have

$$V_1(b) = V_2(b) \tag{1.3.36}$$

Suppose in regime two, we also have $\tau_2 < \infty$ almost surely, then apply boundary condition from Whittle (1983):

$$V_1'(b) = V_2'(b). \tag{1.3.37}$$

Unlike the optimal stopping case, the smooth pasting condition here is due to the two-sided or reversible switching (Whittle, 1983). The other conditions are provided by transversality conditions at the two ends.

Suppose if it is possible to find a boundary $b^*$ such that the switching in both regimes are optimal, applying Theorem 1.2, we have the boundary conditions:

$$V_1(b^*) = V_2(b^*), \tag{1.3.38}$$

and

$$V_1'(b^*) = V_2'(b^*). \tag{1.3.39}$$

Applying also Whittle's boundary condition, then

$$V_1''(b^*) = V_2''(b^*). \tag{1.3.40}$$

Here, the second order smooth pasting condition is due to that the boundary $b^*$ is optimally chosen (Whittle, 1983). The other conditions are two-side transversality conditions. We notice that for both prescribed regime switching and optimal regime switching, these conditions will uniquely determine the solutions in both regimes.

## 1.4 Discrete Stochastic Optimal Control

Many authors have investigated the problems of regulating Brownian motions where the costs linearly depend on the regulations or controls, the control which regulates the continuous Brownian motion shows the impulse behaviour. The word 'discrete' is used to describe the control rather than the state dynamics. Many economic applications adopting such mathematical model can be found in Scarf's (s,S) inventory theory (1959), irreversible investment (Bertola 1989, Pindyck 1988), entry and exit problem (Dixit 1989) and exchange rate target zone (Krugman 1991, Miller and Weller 1991).

Impulse control was first recognised by Bensoussan and Lions (1975) who considered the finite horizon problem with fixed control costs by using optimal stopping to a diffusion process in $\Re^n$. In their case, Bensoussan and Lions find that the optimal control policy is one of 'impulse control', where the control is used at a series of stopping times to instantaneously move the state of the system by a finite amount. This jump type of control is, of course, necessitated by the incursion of a fixed cost every time the control is applied. Bensoussan and Lions restricted themselves to the case where all costs are bounded and, in particular, rule out the case where holding costs rise linearly with the state of the system and the costs of control rise in proportion to the magnitude of the control.

Richard (1977) introduced a optimal control problem where the state of a system is modelled by a homogeneous diffusion process in $\Re^1$. Each time the system is controlled, a fixed cost is incurred as well as a cost which is proportional to the magnitude of the control applied. In addition to the costs of control, there are holding or carrying costs incurred which are a function of the state of the system. Richard found that the sufficient conditions to determine the optimal control both in an infinite horizon case with discounting and a finite horizon case. In both cases the optimal policy is one of 'impulse' control originally introduced by Bensoussan and Lions. But the existence of such control is failed to be addressed. Following the same line, Constantinides and Richard (1977) formulated a continuous-time, infinite horizon cash management model with both fixed and proportional transaction costs and with linear holding costs. In their model, the controlled diffusion process is simplified to be a Brownian motion with drift. They show that all the sufficient conditions provided in Richard (1977) are met, so the optimal impulse controls exist. They also show that if the proportional transaction cost of decreasing the cash balance is sufficiently high, it is never optimal to decrease the cash balance. Then the cash management model[2] degenerates to the inventory model[3].

Afterward, Harrison and Taksar (1983) put forward an approach to treat the regulated Brownian motion where the control is only proportional, and state dynamics in the absence of control is a Brownian motion with drift in $\Re^1$. They successfully provide both existence and uniqueness conditions to their optimal control problem. They also verify that the optimal control for given upper and lower boundaries is instantaneous, and the control boundaries are determined simultaneously with the optimal policy. The boundary conditions ensuring the uniqueness of the solution are first and second order smooth pasting conditions, namely, the first order derivative of the value function with respect to state variable equals the unit cost of the control, the second order derivative of the value function at the boundary is zero.

At the same time, Harrison, Sellke and Taylor (1983) solved the impulse control of Brownian motion by proving the existence and uniqueness theorems. The diagrams for the derivative of the value function developed there is particularly useful to treat the problem. In their

[2]Two-sided (s,S) model.
[3]One-sided (s,S) model.

case, the holding costs are continuously incurred at a rate proportional to the storage level[4], and the control may cause the storage level to jump by any desired amount at any time except that the storage level should be kept nonnegative[5]. Both positive and negative jumps entail fixed plus proportional cost, and the optimal control policy is one that enforces an upward jump to q whenever level zero is hit, this side of the control boundary is a natural boundary; and enforces a downward jump to Q whenever level S is hit[6] to avoid higher holding costs. Various boundary conditions are provided to ensure the uniqueness of the solution to the Bellman equation and to determine the free boundaries.

Recently, Dixit (1991) using discrete Markov chain approximation to the Brownian motion with drift has verified various boundary conditions for instantaneous and impulse control problems. He shows that the value matching conditions hold for any given control parameters (i.e., the controls occurs at any prescribed boundary), and the smooth pasting conditions hold for the optimal control (i.e., for the optimally chosen boundary). Up to now, we are fully equipped to treat the regulated Brownian Motion problems.

In previous section we show that the optimal stopping approach can be used to treat the case when the diffusion process is to be stopped at the boundary or the case where continuous switching occurs. In those cases value matching and smooth pasting conditions are satisfied for any given prescribed boundary (Krylov 1980, Whittle 1983), second order smooth pasting hold for optimally chosen boundary (Whittle, 1983).

[4]Which follows a Brownian motion.
[5]Natural boundary.
[6]$0 < q < Q < S$

The higher order contacting conditions provided there are due to the fact that the state evolves continuously, there are no discrete controls allowed. But the spirit in these two areas is similar (Karatzas and Shreve, 1984), because for the discrete control case, every time when the discrete control is applied, it is a optimal stopping problem. If all the controls only depend on the state variables, the stopping times will define the corresponding stopping boundary where the discrete controls are applied.

In this section, we shall digress the well developed literature on instantaneous and impulse control for Brownian motion and list all the available boundary conditions.

### 1.4.1 Instantaneous Control of Brownian Motion

Harrison and Taksar (1983) consider an instantaneous control problem where the controller can continuously monitor the content of a storage system. In the absence of control, the content process $Z_t$ fluctuates as a Brownian motion with drift $\mu$ and variance $\sigma^2$, the holding costs are continuously incurred at rate $h(Z_t)$. In order to avoid excessive holding costs, the controller may at any time increase the content by any amount desired, incurring a proportional cost $r$ time the size of the increase; or he may decrease the content by any amount desired, incurring a proportional cost $l$ time the size of the decrease. The controller's objective is to find a policy that minimises the expected discounted sum of holding costs control and costs over an infinite time horizon, where the costs are continuously discounted at interest rate $\gamma$. The formulation of the model is given as follow.

Let $S = [\alpha, \beta]$ be a compact space, $h : S \mapsto \Re$ be a convex holding cost function, the control cost parameters $r$ and $l$ and interest rate $\gamma$ are all positive. Let $X_t$ be a Brownian motion defined on the space $(\Omega, \mathcal{F}, P)$ where $\Omega$ is the space of all the continuous function on $\Re^+ = [0, \infty)$, $\mathcal{F}$ is its $\sigma$-field, $P$ is the probability measure. Suppose $X_t$ is adapted to $\mathcal{F}_t$ where $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$. $X_t$ is a Brownian motion with drift $\mu$ and variance $\sigma^2$. We denote $E_x$ the conditional expectation operator on $x \in S$, and the control processes $R = (R_t, t \geq 0)$ and $L = (L_t, t \geq 0)$ satisfy:

$$R(\omega), L(\omega) \text{ are right continuous, nonnegative and nondecreasing for all } \omega \in \Omega; \qquad (1.4.1)$$

$$R_t \text{ and } L_t \text{ are } \mathcal{F}_t - \text{measurable for all } t \geq 0. \qquad (1.4.2)$$

Then we have that $R_t$, $L_t$ are all adapted to $\{\mathcal{F}_t\}$.

Let the controlled process $Z = X + R - L$, then $(R, L)$ is feasible if

$$P_x(Z_t \in S \text{ for all } t \geq 0) = 1 \text{ for all } x \in S, \qquad (1.4.3)$$

$$E_x[\int_0^\infty e^{-\gamma t} dR_t] < \infty \text{ a.s. } x, \qquad (1.4.4)$$

$$E_x[\int_0^\infty e^{-\gamma t} dL_t] < \infty \text{ a.s. } x. \qquad (1.4.5)$$

Associated to the feasibility conditions, the cost function is given by

$$V(x) = E_x\{\int_0^\infty e^{-\gamma t}[h(Z_t)dt + rdR_t + ldL_t]\}, \quad x \in S. \qquad (1.4.6)$$

The state dynamics is simply

$$dZ_t = \mu dt + \sigma dW_t + dR_t - dL_t. \tag{1.4.7}$$

The controller has to minimise the expected cost (1.4.6) subject to (1.4.7).

For so constructed problem, we have the following theorems:

**Theorem 1.3** *For some given barriers $\alpha \leq a < b \leq \beta$, the controlled process is always bounded within $\{a, b\}$, i.e.,*

$$a \leq Z_t \leq b, \quad t \leq 0 \tag{1.4.8}$$

*and the value function satisfies the following conditions:*

$$\Gamma V(x) - \nu V(x) + h(x) = 0, \quad a \leq x \leq b, \tag{1.4.9}$$

$$V'(x) + r = 0, \quad \alpha \leq x \leq a, \tag{1.4.10}$$

$$V'(x) - l = 0, \quad b \leq x \leq \beta \tag{1.4.11}$$

$$-r \leq V'(x) \leq l, \tag{1.4.12}$$

*where*

$$\Gamma = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}. \tag{1.4.13}$$

This theorem actually ensures the properties obtained by Dixit for the arbitrary chosen barriers. (For proof, see Propositions 5.7 and 5.11 in Harrison and Taksar (1983).)

**Theorem 1.4** *For the optimally chosen barriers $a^*$ and $b^*$, the necessary and sufficient conditions are those from Theorem 1.3 and*

$$V''(x) \rightarrow 0 \text{ as } x \downarrow a^* \text{ if } a^* > \alpha \tag{1.4.14}$$

$$V'''(x) \rightarrow 0 \text{ as } x \uparrow b^* \text{ if } b^* < \beta \qquad (1.4.15)$$

(For proof, see Propositions 5.7, 5.11 and 6.6 in Harrison and Taksar (1983).)

Now, by applying these conditions, we can draw the diagram for the derivative of the value function. Figure 1.1 shows the behaviour of $V'(x)$. $V'(x)$ is a $S$-shaped curve between $a$ and $b$, it smooth pasts at $V'(x) = -r$, $l$ respectively.

### 1.4.2 Impulse Control of Brownian Motion

To introduce the impulse control, it is necessary to suppose that the control costs are fixed plus proportional. If the jump size is $\xi$, then the control costs are:

$$\phi(\xi) = \begin{cases} K + k\xi & \text{if } \xi > 0 \\ 0 & \xi = 0 \\ L - l\xi & \xi < 0 \end{cases} \qquad (1.4.16)$$

In this case, the control policy consists of a sequence of stopping times $\{\tau_0, \tau_1, \cdots\}$, and a sequence of jump variables $\{\xi_0, \xi_1, \cdots, \xi_n, \cdots\}$ such that

$$P_x(0 = \tau_0 < \tau_1 < \cdots < \xi_n, < \cdots \rightarrow \infty) = 1 \;\; a.s. \;\; x \in \Re, \qquad (1.4.17)$$

and

$$\xi_n \in \mathcal{F}_{\tau_n} \text{ for all } n = 0, 1, \cdots \qquad (1.4.18)$$

The objective function is thus given by

$$V(x) = E_x[\int_0^\infty h(Z_t)e^{-\gamma t}dt + \sum_{n=0}^{\infty} e^{-\gamma\tau_n}\phi(\xi_n)]. \qquad (1.4.19)$$

The state dynamics is

$$dZ_t = \mu dt + \sigma dW_t \text{ for } \tau_{n-1} < t \leq \tau_n, \qquad (1.4.20)$$

and

$$Z_{\tau_n^+} = Z_{\tau_n} + \xi_{\tau_n}. \qquad (1.4.21)$$

The optimal conditions to (1.4.19) subject to (1.4.20) and (1.4.21) are summarised in the following theorem.

**Theorem 1.5** *The sufficient conditions ensuring the optimality are*

$$\Gamma V(x) - \nu V(x) + h(x) = 0 \quad s \leq x \leq S, \qquad (1.4.22)$$

$$V(q) - V(s) = K + k(q-s) \quad s < q, \qquad (1.4.23)$$

$$V(S) - V(Q) = L - l(Q-S) \quad Q < S, \qquad (1.4.24)$$

$$V'(q) = V'(s) = -k, \qquad (1.4.25)$$

$$V'(S) = V'(Q) = l, \qquad (1.4.26)$$

*where* $q < Q$.

(For proof, see Richard (1977), Harrison *et al* (1983) and Dixit (1989).)

The optimality conditions demand that the controlled Brownian motion be bounded in the regime $x \in [s, S]$, and the jump control will be applied if the process hits the lower bound $s$, it will be brought back to $q$; if the process hits the upper bound $S$, it will be brought back to

$Q$.

The diagram for the derivative of the value function is shown in Figure 1.2, where $V'(x)$ at $s$ and $S$ are the same as at $q$ and $Q$. The shaded areas are associated with the costs of exercising the lump sum controls at $s$ and $S$ respectively.

### 1.4.3 Mixed Impulse and Instantaneous Control

Various impulse and instantaneous control problems for the Brownian motion have be shown in Dixit (1989), in this part we only show the case where one side is controlled instantaneously and the other side has an impulse control.

Consider a storage system, where the storage level is kept above a given lower bound. When the storage level increases, the holding costs are also increasing, so the impulse control will be applied if the holding costs are two high. All the conditions and notations adopted in this part have the same meaning given in the previous two parts if otherwise stated. The value function in this case is

$$V(x) = E_x\{\int_0^\infty e^{-\gamma t}[h(Z_t)dt + rdR_t] + \sum_{n=0}^{\infty} e^{-\gamma\tau_n}\psi(\xi_n)\}, \qquad (1.4.27)$$

where

$$\psi(\xi_n) = L - l\xi. \qquad (1.4.28)$$

The state dynamics is

$$dZ_t = \mu dt + \sigma dw_t + dR_t \quad \text{for} \quad \tau_{n-1} < t \leq \tau_n, \qquad (1.4.29)$$

and

$$Z_{\tau_n^+} = Z_{\tau_n} + \xi_{\tau_n}, \quad n = 0, 1, \cdots \tag{1.4.30}$$

Applying Theorems 1.3 and 1.4, we obtain the Bellman equation:

$$\Gamma V(x) - \nu V(x) + h(x) = 0 \quad \text{for} \quad x \in [a, S]. \tag{1.4.31}$$

Boundary conditions for the side instantaneously controlled are:

$$V'(a) = -r, \tag{1.4.32}$$

$$V''(a) = 0. \tag{1.4.33}$$

The boundary conditions for the impulse control side are:

$$V(S) - V(Q) = L - l(Q - S), \tag{1.4.34}$$

$$V'(S) = V'(Q) = l, \quad \alpha < a < Q < S. \tag{1.4.35}$$

Conditions (1.4.32)-(1.4.35) will uniquely determine the solution to (1.4.31) and the boundary parameters $a, Q, S$.

The optimal policy is when the process hits the lower bound $a$, it will be brought back by infinitesimal amount immediately; if the process hits the upper bound S, it will be brought to Q. Unlike the continuous control case, the rate of change of the discrete control with respect to time is infinite, then the discrete control is a kind of singular control.

The diagram for $V'$ in this case is given in Figure 1.3. The instantaneous control is applied at $a$, so $V'(x)$ smooth pasts to $V'(a) = -r$. Jump control is exercised at $S$ where $V'(x)$ matches that at $Q$ because of the optimality, and the shaded area is the costs of lump sum.

## 1.5 Conclusion

As discussed above, stochastic regime switching, whatever the control is continuous or discrete, can be characterised as a optimal stopping problem. Instead of stopping the processes all together when they cross a boundary, the switching normally allows them to continue in a different regime which may be caused by different driving processes (chapter 3) or by different state dependent constraints (chapters 4–6). The sufficient condition which ensure the optimality in each separate regime is the Bellman equation, while the global optimality is achieved by joining these Bellman equations using appropriate boundary conditions.

In chapter 2, where there is no regime switching, a simple optimal stopping method is used to derive the optimal exchange rate target zones. In chapter 3, where switches between regimes are allowed but irreversible, the discrete control technique is used to determine the optimal oil investment in the North Sea. Finally, from chapter 4 to chapter 6, where the regimes are defined by state dependent constraints (output constraint in chapter 4, borrowing and shortselling constraints from chapter 5 to 6), continuous control techniques are used to deduce the optimal investment policy when capital depreciation is stochastic (chapter 4) and the optimal consumption/portfolio decisions in a two asset model.

## 1.6 FIGURES

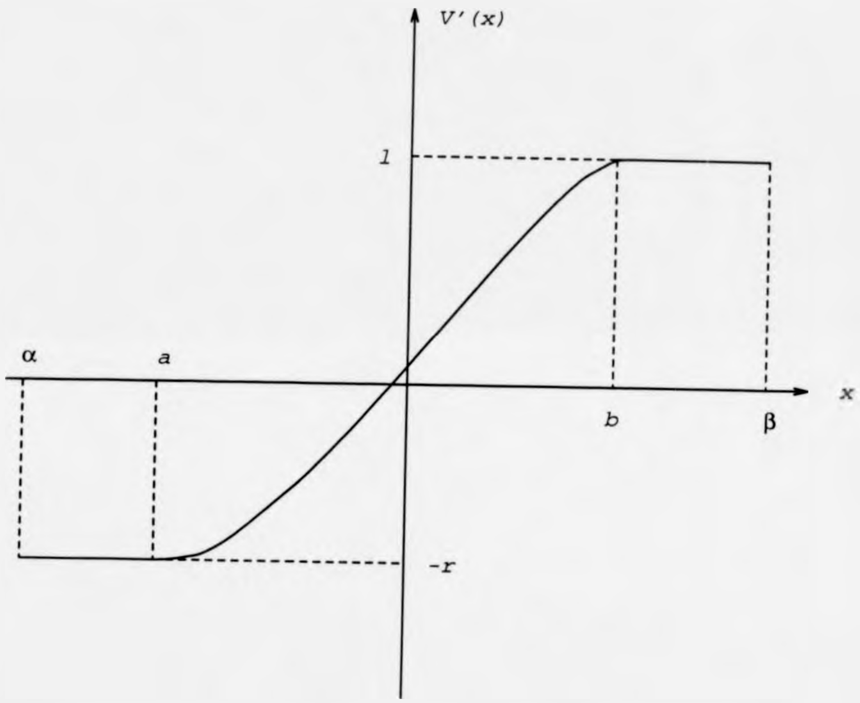


Figure 1.1: Instantaneous Control: Derivative of the Value Function.

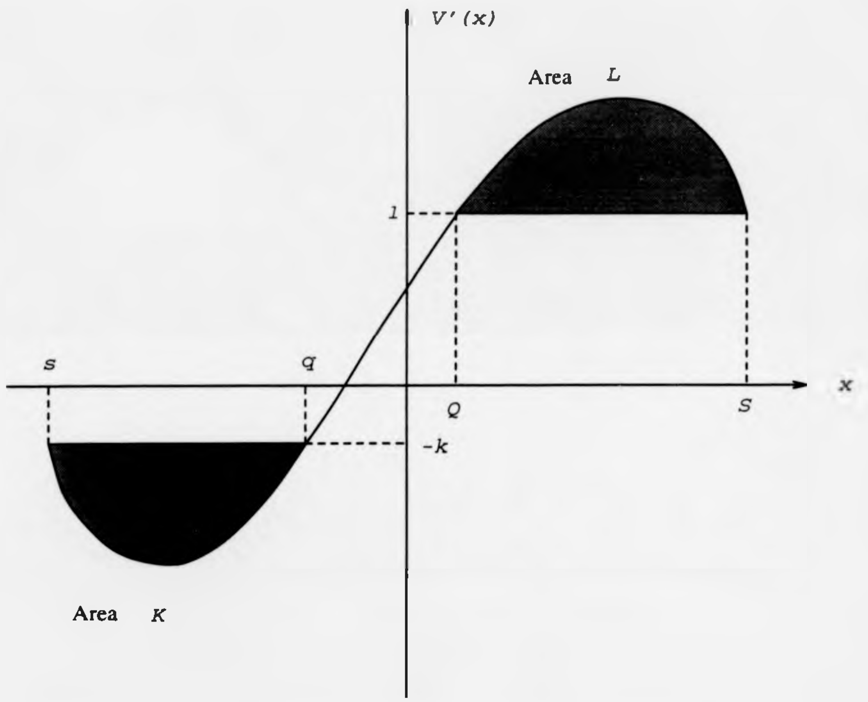


Figure 1.2: Impulse Control: Derivative of the Value Function.

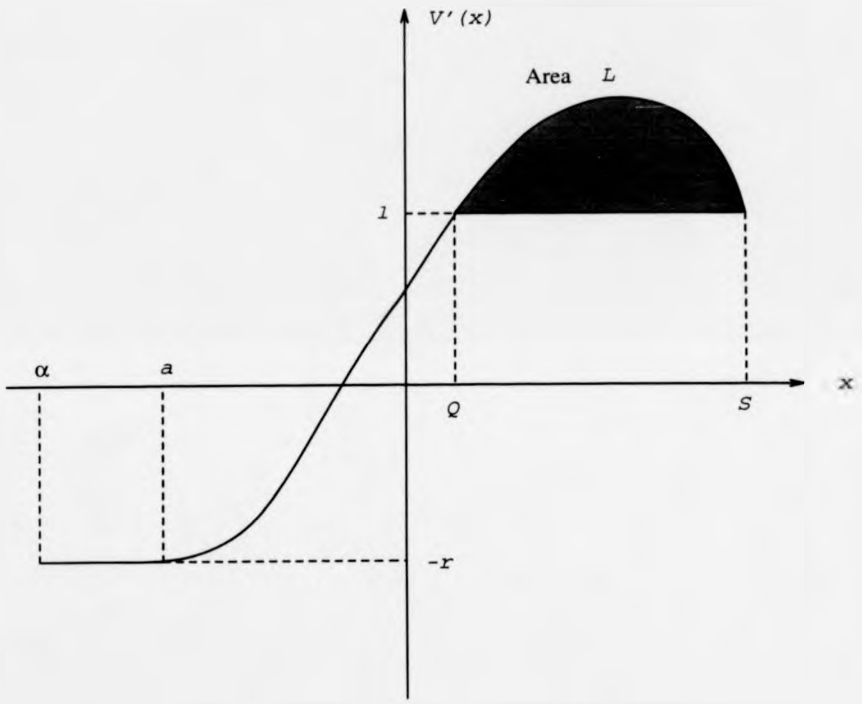


Figure 1.3: Mixed Control: Derivative of the Value Function.

# Chapter 2

# Optimal Target Zones: How an Exchange Rate Mechanism Improves upon 'Discretion'

## 2.1 Introduction

The positive analysis of target zones shows how anticipated intervention at the edges of a currency band influences the rate inside the band. But it does not explain why a policy of marginal intervention might be chosen in the first place; nor why the announced band should be credible. These are the issues examined here by applying dynamic programming techniques to the canonical target zone model of Krugman (1991).

Assume that the monetary authorities seek to stabilize the exchange rate in circumstances where there is no shortage of official reserves, but nevertheless there are costs associated with intervention. What sort of costs would yield as optimal the infinitesimal marginal intervention described by Krugman? Further, what choice of intervention points would be fully credible (in the sense that they would be incentive-compatible when fundamentals reach the anticipated intervention point)? These are the questions addressed here.

The exact solutions turn out to involve high order polynomial functions, so for expositional purposes we make use of a quadratic approximation in the value function which enable us to employ the diagrammatic approach of Harrison, Sellke and Taylor (1983) and Dixit (1991, 1991b) to illustrate our results. These diagrams also show clearly the link between the "second-order smooth-pasting" required of the value function as a condition of optimality, and the tangency of the exchange rate at the edge of the band implied by arbitrage.

Flood and Garber (1992) noted that Krugman's tangency condition only emerged in the limit as discrete intervention is reduced to infinitesimal size. In the second part of the paper, the same diagrammatic approach is used to indicate how the presence of lump sum costs makes discrete intervention optimal; and how the same limiting argument applies here too.

## 2.2 Costs of Stabilization

What sort of costs would yield as optimal a stabilization policy of intervening at the edges of a currency band? To solve this "inverse optimal" question we begin by assuming that there are explicit costs associated with intervention. This allows us to make direct use of results obtained on the control of Brownian motion processes (–subject to the time consistency requirement explained below).

Let these explicit intervention costs, be symmetrical, and consider three functional forms: strictly convex, strictly proportional or lump sum. Note first that strictly convex costs can be immediately dismissed for present purposes, as they yield *continuous* intervention as optimal policy. This is because strictly convex costs[1] approach zero faster than the size of the intervention itself, so even the smallest deviation of the exchange rate will induce some correction, see Fleming and Rishel (1975), Malliaris and Brock (1983) and Dixit (1991b). The continuous management of exchange rates associated with quadratic intervention costs is described in Svensson (1992).

However, costs vary strictly in proportion to size of intervention, no action is called for until the marginal costs of the exchange rate's deviation from target match the marginal cost of intervention. Then the optimal policy is to implement a barrier on fundamentals, using *instantaneous* control, see Harrison and Taksar (1983), Bertola and Caballero (1990) and Dixit (1991b). We use this cost structure in deriving the optimal target zone in the next section.

Finally note if the cost of changing the fundamentals includes a lump sum, then the optimal policy involves *impulse* control, and fundamentals are adjusted by a discrete amount when the barrier is reached, as in the so called $s, S$ policy of inventory control, see Harrison *et al* (1983) and Dixit (1991b). We use the diagrammatic approach of Harrison *et al* to indicate how the discrete intra-marginal intervention rules described

[1] We assume that those strictly convex costs are continuous and continuously differentiable in intervention.

on the control of Brownian motion processes (-subject to the time consistency requirement explained below).

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by Flood and Garber (1992) emerge as optimal when lump sum costs are added to the proportional costs.

To establish the optimality of barrier policies, the analysis of section 3 and 4 assume explicit costs of intervention; inside the control barrier only the deviation of the exchange rate from its target is penalised. We show finally that such "discretionary" equilibria can indeed be improved by credible precommitment to certain rules.

## 2.3 Infinitesimal Intervention

### 2.3.1 Time Consistent Barriers

Let the exchange rate be the present discounted value of future velocity-adjusted money, so

$$s_t = E_t \int_t^{\infty} k_x e^{-\beta(x-t)} dx \tag{2.3.1}$$

where $s_t$ is the logarithm of exchange rate at time $t$, $k_t$ is the velocity-adjusted money stock (in log form), and $\beta$ is the discount factor (the inverse of the semi-elasticity of demand for money).

Let fundamentals evolve as the resultant of two influences, first a random walk in velocity itself and second official interventions which include buying and selling foreign currency in exchange for domestic money[2], so

$$dk_t = \sigma dW_t + dR_t - dL_t, \qquad (2.3.2)$$

where $W_t$ is a standard Brownian motion, $\sigma$ is a parameter measuring the volatility of fundamentals, $R_t/L_t$ are two right/left continuous processes which represent marginal intervention to buy/sell foreign currency.

Assuming that the monetary authorities seek to minimise the squared deviation of the exchange rate from a (constant) target value[3] subject to intervention costs proportional to the size of intervention, we can write the value function as

$$V(k) = \min_{R,L} E_0[\int_0^\infty s_t^2 e^{-\rho t} dt + \int_0^\infty c e^{-\rho t}(dR_t + dL_t)], \qquad (2.3.3)$$

subject to (2.3.1) and (2.3.2), where $\rho$ is the discount factor and $c$ is

[2]For fundamentals driven by a Brownian motion and with drift, the following results only need slight changes. However, if fundamentals are mean-reverting, the exchange rate is given by a Kummer's function (Delgado and Dumas, 1992), so the time consistent solution can only be obtained by numerical simulation. For simplicity, we adopt the assumption that fundamentals are only driven by a Brownian motion.

[3]The reason that the monetary authorities stabilize the exchange rate is primarily due to their concern about the control of inflation. By linking its currency to that of a country with a better monetary record, the central bank can "borrow" the reputation in using exchange rate policies to achieve price stability. Another reason for stabilising the exchange rate stemmed from the widespread belief that exchange rate markets constitute a case of market failure (Krugman, 1989).

Svensson (1992) suggested that the need for the central banks to adopt managed exchange rate regimes (against completely fixed exchange rates) is because that the managed exchange rate regimes may provide some monetary independence, in the sense of ability to stabilize domestic interest rates. So the objective function of the monetary authorities he adopted includes minimising variability in both exchange rate and interest differential, which leads to the objective function being linear quadratic in both exchange rate and the fundamentals. This does alter the following solutions, however, the approach adopted here still remains applicable even to the modified objective function.

the constant unit cost of intervention in either direction.

As discussed above, the form of the cost function has been selected so as to rule out continuous intervention; i.e., it has been chosen to make reflecting barriers the optimal policy. But inside these barriers the value function will be a function of *unregulated* Brownian motion. Hence it will satisfy the Hamilton-Jacobi-Bellman equation

$$\rho V(k) = \frac{1}{2}\sigma^2 V_{kk}(k) + s^2, \tag{2.3.4}$$

which can be derived directly by differentiating (2.3.3) to obtain $E(dV) = \rho V - s^2$ and noting that if $V$ is a stationary function of $k$, then $E(dV) = \sigma^2/2V_{kk}(k)$ by Itô's lemma.

The symmetric cost structure ensures that the value function is symmetric, so we need consider only one of the reflecting barriers. The two boundary conditions needed to solve (2.3.4) are thus provided by the conditions for optimal intervention at, say, the upper barrier $\bar{k}$, specifically

$$V_k(\bar{k}) = c, \tag{2.3.5}$$

$$V_{kk}(\bar{k}) = 0. \tag{2.3.6}$$

The first of these is the condition that the marginal welfare cost must match the (constant) unit cost of intervention; the other is the "second-order smooth-pasting" required of an optimum.

To find the solution $V(k)$, we assume that the form of the policy is known to the market; therefore the exchange rate will be the hyperbolic sine function of fundamentals described by Krugman, i.e.,

$$\begin{aligned} s &= \frac{A}{2}(e^{\lambda k} - e^{-\lambda k}) + k \\ &= A\sinh(\lambda k) + k \end{aligned} \tag{2.3.7}$$

with parameters $\lambda$ and $A$ defined by, $\lambda = \sqrt{\frac{2\beta}{\sigma^2}}$ and $-\lambda A = \cosh^{-1}(\lambda \bar{k})$, where $\bar{k}, -\bar{k}$ are the reflecting barriers for the fundamental $k$, see Krugman (1991) or Svensson (1991). So the market expects fundamentals to be "regulated" at fixed barrier points, and this generates an $S$-shaped pattern for the exchange rate.[4]

Substitution of these expectations into (2.3.4) provides an explicit representation of the value function which, given the symmetry of the problem and the conditioning on $A$, can be written as

$$\begin{aligned} V(k;A) = {} & \frac{A^2}{\rho - 4\beta}\sinh^2(\lambda k) + \frac{2A}{\rho - \beta}k\sinh(\lambda k) + \frac{2A\lambda\sigma^2}{(\rho-\beta)^2}\cosh(\lambda k) \\ & + \frac{1}{\rho}k^2 + \frac{\sigma^2}{\rho}\left(\frac{\lambda^2 A^2}{\rho - 4\beta} + \frac{1}{\rho}\right) + B\cosh(\mu k), \end{aligned} \tag{2.3.8}$$

where $\lambda$ and $A$ are as defined above, and

$$\mu = \sqrt{\frac{2\rho}{\sigma^2}}.$$

Determining the optimal barrier involves differentiating the value function partially with respect to $k$, treating $A$ as predetermined, and applying the boundary conditions already described. This will define the choice of the optimal barrier *conditional* on market expectations of

[4] To derive the time-consistent optimal policy for general cases, see Cohen and Michel (1988).

intervention at upper and lower barriers. The time consistent solution itself must be a fixed-point, where market conjectures are satisfied by the optimal choice of barriers. So the conditions for the optimal barrier are found by the application of (2.3.5) and (2.3.6) to the partial derivatives of $V$

$$\begin{aligned} B\mu \sinh(\mu \bar{k}) + \frac{2}{\lambda}\Big(\frac{1}{\rho - 4\beta} - \frac{\rho + \beta}{(\rho - \beta)^2}\Big) \tanh(\lambda \bar{k}) \\ + 2\Big(\frac{1}{\rho} - \frac{1}{\rho - \beta}\Big)\bar{k} = c, \end{aligned} \tag{2.3.9}$$

$$\begin{aligned} B\mu^2 \cosh(\mu \bar{k}) + \frac{2}{\rho - 4\beta} \tanh^2(\lambda \bar{k}) - \frac{2\lambda}{\rho - \beta} \bar{k} \tanh(\lambda \bar{k}) \\ + \frac{2}{\rho} - \frac{4\rho}{(\rho - \beta)^2} + \frac{2}{\rho - 4\beta} = 0, \end{aligned} \tag{2.3.10}$$

together with the consistency requirement

$$A\lambda \cosh(\lambda \bar{k}) = -1. \tag{2.3.11}$$

Eliminating $B$ by substitution yields the desired result, an implicit fixed-point equation for the optimal time consistent barrier, as follows

$$\mu^{-1} \tanh(\mu \bar{k}) = \frac{\frac{2}{\rho - 4\beta} \frac{\tanh(\lambda \bar{k})}{\lambda} - \frac{2\bar{k}}{\rho - \beta} - \frac{2(\rho + \beta)}{(\rho - \beta)^2} \frac{\tanh(\lambda \bar{k})}{\lambda} + \frac{2\bar{k}}{\rho} - c}{\frac{2}{\rho - 4\beta} \tanh^2(\lambda \bar{k}) - \frac{2\lambda}{\rho - \beta} \bar{k} \tanh(\lambda \bar{k}) - a}, \tag{2.3.12}$$

where

$$a = \frac{4\rho}{(\rho - \beta)^2} - \frac{2}{\rho - 4\beta} - \frac{2}{\rho}$$

For given $\bar{k}$, the optimal target zone will be defined by $-\bar{s}, \bar{s}$, where $\bar{s} = \bar{k} - \lambda^{-1} \tanh(\lambda \bar{k})$.

The band width in fundamentals $\bar{K}$ from equation (2.3.12) is a Nash equilibrium (specifically, a Cournot equilibrium, see Henderson and Quandt, 1980, p202). Here we assume that public has perfect knowledge about the monetary authority's intervention policy, so it cannot manipulate public expectations. In the game played between the public and the monetary authority, the public fully anticipates central bank's intervention policy (in the marginal intervention case, it is equivalent to knowing $\bar{K}$), so it forms rational expectations (choosing $A$ in (2.3.11)) to determine the exchange rate behaviour inside the band. In this sense, equation (2.3.11) can be interpreted as the *rational expectations* constraint. Meanwhile, the monetary authority takes public expectations (parameter $A$) as given to decide optimally the intervention policy (the marginal intervention barrier $\bar{K}$)[5]. This gives arise the time consistency constraint. The interaction between the public and the monetary authority generates the time consistent solution in (2.3.12). To see the qualitative nature of the solution, we provide the following example with approximation.

[5]Substitution of (2.3.8) into (2.3.5) and (2.3.6), and eliminating $B$ yields,

$$\frac{1}{\mu}\tanh(\mu\bar{k})\{\frac{4\lambda^2 A}{\rho-4\beta}\sinh^2(\lambda\bar{k}) + \frac{2\lambda^2 A}{\rho-\beta}\bar{k}\sinh(\lambda\bar{k}) + \frac{4\lambda\rho A}{(\rho-\beta)^2}\cosh(\lambda\bar{k})$$
$$+ \frac{2\lambda^2 A}{\rho-4\beta} + \frac{2}{\rho}\} + c = \frac{2\lambda A^2}{\rho-4\beta}\sinh(\lambda\bar{k})\cosh(\lambda\bar{k})$$
$$+ \frac{2(\rho+\beta)A}{(\rho-\beta)^2}\sinh(\lambda\bar{k}) + \frac{2\lambda A}{\rho-\beta}\bar{k}\cosh(\lambda\bar{k}) + \frac{2}{\rho}\bar{k}.$$

This shows how optimal intervention barrier $\bar{k}$ depends on public expectations $A$, so this equation can be interpreted as a time consistency constraint.

### 2.3.2 A Quadratic Approximation and the Harrison Diagram

The nature of the solution and how it changes in response to changes in costs etc., is made much more apparent when the value function (2.3.8) is somewhat simplified. This can be achieved by replacing the conjecture of (2.3.3) above by

$$s = (\lambda A + 1)k \tag{2.3.13}$$

i.e., the linear approximation of Krugman's hyperbolic solution near the origin, the approximation recommended for small target zones by Delgado and Dumas (1992).

Using (2.3.13) we find the value function simplifies to

$$V(k; A) = \frac{(1 + A\lambda)^2}{\rho}(\frac{\sigma^2}{\rho} + k^2) + B\cosh(\mu k). \tag{2.3.14}$$

i.e., it is the sum of two terms, a quadratic form and a symmetric term in exponentials.

The boundary conditions specified above require that, at the upper barrier $\bar{k}$,

$$V_k(\bar{k}; A) = 2(1 + A\lambda)^2\frac{\bar{k}}{\rho} + \mu B\sinh(\mu\bar{k}) = c, \tag{2.3.15}$$

$$V_{kk}(\bar{k}; A) = \frac{2(1 + A\lambda)^2}{\rho} + \mu^2 B\cosh(\mu\bar{k}) = 0. \tag{2.3.16}$$

which can be used to determine $\bar{k}$ and $B$, conditional on $A$. Specifically, substituting for $B$ and $A$, we find

$$\mu^{-1}\tanh(\mu\bar{k}) = \bar{k} - \frac{c\rho}{2(1-\operatorname{sech}(\lambda\bar{k}))^2} \quad (2.3.17)$$

where $A\lambda$ has been replaced by $-\operatorname{sech}(\lambda\bar{k})$ according to (2.3.11) above.

To an approximation, this defines the optimal subgame-perfect target zone, given the constant marginal (and average) intervention cost $c$. The link between the optimising conditions (2.3.15) and (2.3.16) and the tangency condition derived by Krugman can be seen most easily using the diagrammatic approach employed by Harrison *et al* (1983) and Dixit (1991b).

Given the conjecture of $A$, one can plot the derivative of the value function as a function of $k$. If, for example, $B = 0$ so the (approximated) value function (2.3.9) is quadratic, then $V_k(k)$ is a linear function of $k$, $V_k = [2(1+A\lambda)^2/\rho]k$ as shown by the line UU in Figure 2.1. For $B < 0$, the expectation of future intervention lowers the cost function and generates solutions such as that labelled WW. The schedule WW has been chosen so as to satisfy the two boundary conditions that $V_k(\bar{k}) = c$ and $V_{kk}(\bar{k}) = 0$, so it is tangent to c at $\bar{k}$ (and $-\bar{k}$). How can one be sure that the intervention point identified by the optimality conditions is indeed consistent with the conjecture embedded in $A$? The answer is by noting that the tangency condition for the exchange rate given in Krugman (1991) occurs at the same value of $\bar{k}$, i.e. the solution for the exchange rate, $s = k + A\sinh(\lambda k)$, must also reach a maximum at $\bar{k}$ (minimum at $-\bar{k}$) so that at the barriers the exchange rate is tangent to the edges of the band. (This is shown in the figure where $S$ is tangent to $\bar{S}$ at $\bar{k}$.) The time consistent solution requires both these "tangency conditions" are satisfied at the same value of $k$.

Arbitrage arguments show that the exchange rate will be tangent to its band at any intervention point, no matter how it is chosen—provided it is fully credible. In the next section, indeed, we discuss how a credible intervention "rule", where the second order smooth pasting of the value function is *not* satisfied, might be preferred to the time consistent optimum just derived.

The quadratic approximation of equation (2.3.17) suggests clearly how the band will change in response to changes in costs condition and the discount factor. Raising the discount factor will tend to flatten the schedule UU–implying wider barriers for intervention and a wider band. For a given discount factor, raising $c$ will also widen the barriers and the band. While qualitative nature of these changes is clear enough, the quantitative answers provided by (2.3.16) and (2.3.17) will not of course be exact.

## 2.4 Lump Sum Costs and Discrete Intervention

The target zone model has been generalised to include the case where the policy authority may use discrete intervention. Flood and Garber (1992) show that in this case the exchange rate is still related to fundamentals as in equation (2.3.4) above, but that intervention takes place inside the currency band. They also note that in the limit as the size of intervention is reduced, the outcome approaches the infinitesimal marginal intervention described by Krugman. In this section we use the Harrison diagram to indicate graphically how lump sum costs of intervening lead to discrete intervention as the optimal policy; and how infinitesimal intervention emerges in the limit as these lumpy costs vanish.

If there are lump sum costs $C$ (in addition to the proportional costs $c$) then the value function becomes

$$V(k) = \min_{R,L} E_0[\int_0^\infty s_t^2 e^{-\rho t} dt + + \sum_{i=1}^{\infty} (C + c|\xi_i|) e^{-\rho \tau_i}], \qquad (2.4.1)$$

subject to (2.3.1) and (2.3.2) above. Here $\tau_i$ denotes "stopping time" when intervention takes place, $|\xi_i|$ denotes the size of the intervention which is equal to $D$ given bellow. Given $C > 0$, so-called $S, s$ policy will be optimal, i.e., there will be intervention at $\bar{k}$ which reduces the fundamental to $\bar{k} - D$, where $\bar{k}$ , $D$ depend on $C$, $c$. The behaviour of the exchange rate between $\bar{k}$ and $-\bar{k}$ will be as in equation (2.3.7), as Flood and Garber pointed out, so the value function will take the form given in equation (2.3.8); but the boundary conditions for an optimal policy are now

$$V_k(\bar{k}; A) = c \qquad (2.4.2)$$

$$V_k(\bar{k} - D; A) = c \qquad (2.4.3)$$

$$V(\bar{k}; A) = V(\bar{k} - D, A) + C + cD \qquad (2.4.4)$$

where $A$ is defined by

$$\bar{k} + A \sinh(\lambda \bar{k}) = \bar{k} - D + A \sinh(\lambda(\bar{k} - D)) \qquad (2.4.5)$$

Consider for simplicity the approximated value function of (2.4.1) above. The implications of applying the boundary conditions given above, subject to the consistency requirement that the barriers so selected match the Flood -Garber conjecture embedded in the value function, can be seen clearly in Figure 2.2 where the derivative of the value function and the exchange rate are shown in relation to the fundamental $k$. The optimal discrete intervention requires the derivative of the value function to match the proportional cost $c$ as at the points X and Y in the figure; in addition, the integral between $\bar{k} - D$ and $\bar{k}$ must match the total intervention costs of $C + cD$, as shown. Since the area defined by X,Y, $\bar{k}$ and $\bar{k} - D$ is $cD$, the shaded area must be equal to the lumpy cost $C$. The "no profitable arbitrage" condition implies that $s(\bar{k}) = s(\bar{k} - D)$, as Flood and Garber pointed out (and consistency requires that the conjecture embedded in the value function is the optimal policy chosen). So once again one finds a close analogy between the condition on the exchange rate implied by arbitrage and the optimality condition required of the marginal value function.

That the policy of discrete intervention within the currency band will give way to a policy of infinitesimal marginal intervention when $C$ tends to zero, is evident from the figure; as $C \to 0$, so $D \to 0$ thus, in the limit, one obtains the smooth pasting solution shown earlier in Figure 2.1. (This can, of course, be demonstrated more formally by showing how the three boundary conditions listed as (2.4.1)–(2.4.4) above tend to the two conditions (2.3.5) and (2.3.6) as $C \to 0$).

## 2.5 Rules Rather Than Discretion

The intervention policies derived above have been obtained using the techniques of dynamic programming. But in a context like this, where expectations play a crucial role, it is well known that such "discretionary" equilibria can be improved upon by adopting rules—if only some precommitment mechanism is available to enforce these rules (Kydland and Prescott 1977).

If that precommitment can be achieved by membership of an Exchange Rate Mechanism (ERM). Why should the rules enforced by an ERM offer room for improvement over the dynamic programming outcome? It is because the conjecture about the exchange rate, which was taken as predetermined when applying the optimality conditions, is in fact determined by the boundary conditions themselves. Taking explicit account of the effect of the barriers on exchange rate expectations would surely lead an ERM to select narrower barriers so as to stabilize the exchange rate.

But how narrow should these barriers be? Consider specifically the case where there are no lump sum costs. Assume also that there is to be marginal infinitesimal intervention at the ERM barriers. If so, the Krugman conjecture (2.3.7) will still apply to the exchange rate; and the derivative of the value function should still equal $c$ ($-c$) at upper (lower) barrier for the fundamental. But, at such predetermined barriers, second order smooth pasting is no longer appropriate (see Whittle 1983), as we drop the condition $V_{kk}(k;A) = 0$ that signifies the optimal choice of discretionary barriers.

What are we to put in its place? Consider for instance the "state dependent" criterion, that the barriers be chosen so as to

$$\min_{\bar{k}} \quad V(0; A(\bar{k}))$$
$$\text{subject to} \quad V_k(\bar{k}; A) = 0.$$

Together with the consistency condition (2.3.11), the rule is chosen so as to minimise discounted costs conditional on starting at the middle of the band. The narrower bands selected this way (which we denote $-\bar{k}_R$, $\bar{k}_R$) not only reduce the value function conditional on starting at $k = 0$, but the integrated costs implied by $-\bar{k}_R$, $\bar{k}_R$ are in fact lower than those associated with the discretionary barriers for all starting values of $k$ such that $0 \leq k \leq \bar{k}_R$; so the rule chosen strictly dominates the dynamic programming outcome.

The superiority of this rule over discretion is shown in Figure 2.3 (cf. Figure 2 in Constantinides and Richard, 1978). There the value function implied by the Hamilton-Jacobi-Bellman (HJB) equations of dynamic programming is labelled $V^D(k; A(\bar{k}_D))$; and as shown, it satisfies first and second order smooth pasting at $\bar{k}_D, -\bar{k}_D$ (where there are points of inflexion and slopes of $c, -c$ respectively). The narrower barriers selected by our state dependent criterion are shown as $\bar{k}_R$, $-\bar{k}_R$ and the associated value function is labelled $V^R(k; A(\bar{k}_R))$ with a slope of $c, -c$ at these barriers (but no point of inflexion). That $V^R(0; A(\bar{k}_R)) < V^D(0; A(\bar{k}_D))$ should hardly be surprising, given the assumption of costless commitment; what is remarkable is that $V^R$ lies strictly below $V^D$ for all values of $k$ between the barriers. (The proof is given in Appendix A).

That choosing the barriers $-\bar{k}_R$, $\bar{k}_R$ so as to minimise $V(0; A(\bar{k}_R))$

will indeed improve on discretion is proved in the Appendix. But, one might reasonably ask, would these barriers have been chosen if one considered the minimisation at a different point? Is the choice of $-\bar{k}_R$, $\bar{k}_R$ unconditionally optimal? It appears that it is, for (using a quadratic approximation of the value function) it can be shown that $V^R(0; A(\bar{k}_R)) < V(0; A(\bar{k}))$ for $\forall k$, $\bar{k} \in (0, \bar{k}_D)$ and $\bar{k} \neq \bar{k}_R$. This proof is sketched in the Appendix, where for this reason we refer to $-\bar{k}_R$, $\bar{k}_R$ as the optimal rule.

## 2.6 Conclusion

For a single monetary authority aiming to stabilise the exchange rate with proportional costs of intervention, the optimal policy is a target zone with a reflecting barrier on fundamentals. Assuming these fundamentals follow a Wiener process we solve for the "time consistent" optimal barrier at which marginal intervention will take place. We then use our results to answer the inverse optimal question: how big would proportional intervention costs have to be to make existing exchange rates bands optimal? The analysis, which uses the popular monetary model, shows the link between the arbitrage conditions for the exchange rates derived by Krugman and the "smooth pasting" conditions that apply to the value function.

Not all observed intervention takes place at the edge of exchange rate bands: one explanation for intra-marginal intervention is that there may be *lump sum* costs involved. We show how the time consistent optimal barriers for the fundamental are affected by such lumpy costs, and describe the various combinations of fixed and variable costs which solve the inverse optimal problem for a given band width.

Last of all we note that, because the exchange rate is assumed to discount future policy, the optimal "discretionary" equilibrium can in principle be improved by precommitment to a rule, imposed for example by an Exchange Rate Mechanism. We show narrower bands dominate the optimal "discretionary" outcome, and solve for the optimal band width for the Exchange Rate Mechanism.

## 2.7 FIGURES

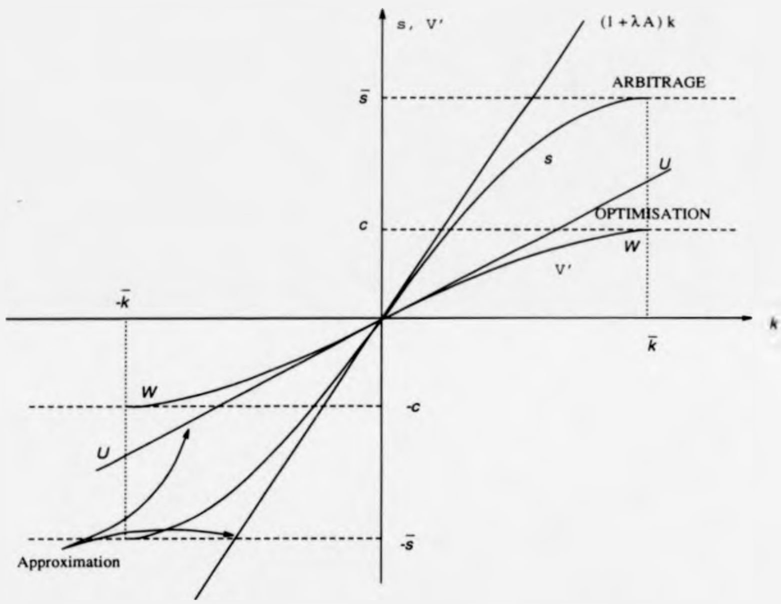



Figure 2.1: The Optimal Time Consistent Target Zone.

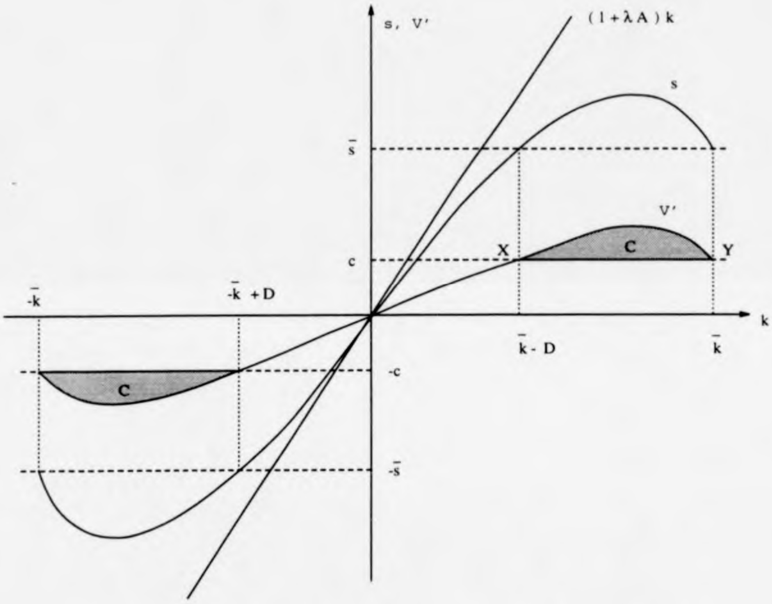



Figure 2.2: Lump Sum Costs and Discrete Intervention.

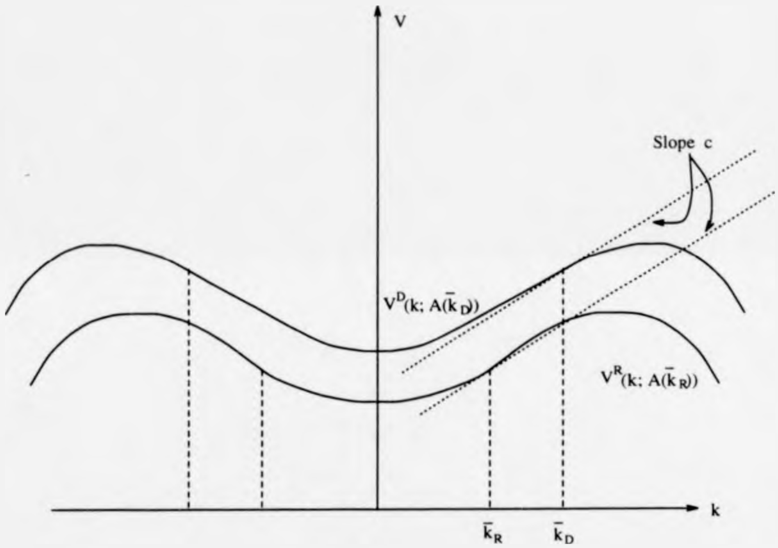



Figure 2.3: A Rule which Dominates the HJB Solution.

# Chapter 3

# Irreversibility and Oil Production

## 3.1 Introduction

Getting oil onshore involves three essential stages, those of exploration, development and extraction. They are succinctly described by Paddock *et al* (1988, p481) as follows:

> *Exploration* involves seismic and drilling activity to obtain information on the quantities of hydrocarbon reserves present in the tract, as well as the costs of bringing them out. If the exploration results are favourable, the firm may then proceed to the *development* stage, which involves putting the equipment in place to extract the oil: for example, constructing platforms and drilling production wells. ... *Extraction* involves using the installed capacity to take the hydrocarbons out of the ground.

A recent econometric analysis of exploration and extraction of oil in the UK continental shelf by Hashem Pesaran (1990) investigates the parameters describing the first and third stages using data from 1978 to 1986, a time when the international oil markets were particularly volatile. The framework used there involves optimising the rate of exploration and of extraction, given that the decision to incur development costs has already been made. In this paper, by contrast, we focus on the decision *whether to move on from exploration to extraction*, i.e., whether to pay the development costs involved in building oil platforms and the like.

With the objective of finding explicit analytical results on the factors governing the decision to develop, modelling the industry is kept as simple as possible. Thus we treat these development costs as if they were payable as an instantaneous lump sum when the decision to develop has been made; but see Adelman and Paddock (1980) for discussion and justification of this "collapsing" technique. Initially it is assumed that extraction begins immediately after the decision to develop has been taken, but later we add a stochastic "development lag" to allow for the time required to build platforms etc.. To keep things tractable, we sidestep the optimising decisions which Pesaran analyses by treating the rate of exploration and extraction as fixed functions of the price of oil and the volume of reserves respectively.

The techniques used here have already been applied to the decision whether to open (or close) a coal mine by Brennan and Schwartz (1985). Applications to offshore oil extraction include the paper by Paddock *et al* (1988) already referred to; Bjerkholt and Brekke (1988) and recent work by Brekke and Øksendal (1991,1992). Essential to all these studies is the assumption that prices of the natural resource follow a geometric Brownian motion[1], and that the lump sum costs of extraction be more or less irreversible, a combination of features which leads even risk neutral investors to raise the trigger prices for extraction above certainty equivalent levels. In the case of oil, where for geological reasons extraction rates are linked to reserve levels, trigger prices should of course reflect reserve levels as well.

Section 2 begins with a review of the way in which unique optimal *trigger prices* for extraction and close-down are determined when the level of reserves is infinite. Then in section 3 we turn to the case of finite reserves where the trigger prices turn out to depend on the level of reserves, so defining *switching boundaries*. The elegant results recently obtained by Brekke and Øksendal are described first. We note in particular that as both development and extraction costs are specified to be independent of reserve levels, one can reformulate their model in terms of the *value* of reserves. So the problem of finding switching boundaries can be reduced to that of finding unique *value triggers* (which can be determined in precisely the same way as the price triggers already described in section 2). Where costs *do* depend on the reserve levels this reduction is no longer possible. But for a specific cost structure, we are able to determine explicitly the switching boundary appropriate for an irreversible switch to production(i.e., assuming that closure is prohibitively expensive). The effect of oil price volatility on the entry price is examined for plausible parameter values: in the absence of specific tax provisions to shield the investor, the oil price volatility doubles

[1] Favero *et al* (1992) have tested the time series of Brent Crude and concluded that the random walk hypothesis cannot be rejected.

the entry entry price, i.e., *the trigger prices in the face of uncertainty is more than twice the break-even price for a constant real price.* The effect of adding a stochastic development lag is also analysed.

In section 4 of the paper, we use the switching boundaries to produce a theoretical time path for oil extraction as a function of lagged real oil prices and costs. The oil price hikes of OPEC I and OPEC II produce spikes in production; but these are smoothed by the stochastic development lag. Areas for future research are considered in conclusion.

## 3.2 Infinite Reserves and Trigger Prices

To begin with, assume that reserves are infinite, so one can ignore the need to explore. Assume also that the rate of extraction $q(t)$ can take only one of two values, $q^*$ or zero. Let $D$ be the the development cost to be paid when extraction begins and $C$ be the closure cost paid when extraction ends. (Note that $C$ can be negative if development expenditures can be recouped.) With the real price of oil $\bar{P}$ following a random walk, the decision to be made is when to remain idle ($q = 0$) and when to produce ($q = q^*$). As Brennan and Schwartz show, the solution in these circumstances is defined by two "trigger prices" which can be determined as follows.

Specifically let the prices of oil measured relative to a general price index follow a geometric Brownian motion with drift, so

$$d\bar{P}_t = \alpha \bar{P}_t dt + \sigma \bar{P}_t dW_t, \qquad (3.2.1)$$

where $\alpha$ is the rate of drift, $\sigma \bar{P}$ is the instantaneous standard deviation of $\tilde{P}$, $W_t$ is a standard Brownian motion. Let the cash flow net of tax from the oil field in production be specified as

$$(\tilde{P}(t) - \tilde{a})q^* - T,$$

where $\tilde{a}$ is the (constant) unit cost of extraction, and

$$T = \tau_1 \tilde{P} q^* + \tau_2 q^* (\tilde{P}(1 - \tau_1) - \tilde{a}),$$

with $\tau_1$ being the royalty rate and $\tau_2$ being the rate of income tax.

In what follows we use $P$ and $a$, without the tilde sign, to denote prices and costs corrected for the effects of tax, so

$$P_t = \tilde{P}_t(1 - \tau_1)(1 - \tau_2),$$
$$a = \tilde{a}(1 - \tau_2).$$

The net of tax price process is

$$dP_t = \alpha P_t dt + \sigma P_t dW_t, \tag{3.2.2}$$

and net cash flow can be written succinctly as

$$(P_t - a)q^*.$$

Let $U$ denote the value of the field when idle and $V$ the value of the field in production, where these values are linked as follows

$$U(P_0) = \sup_{\tau}\{E_0(V(P_\tau) - D)e^{-\rho\tau}\}, \tag{3.2.3}$$

$$V(P_\tau) = \sup_{\tau'}\{E_\tau[\int_\tau^{\tau'} (P_s - a)q^* e^{-\rho(s-\tau)} ds$$
$$+(U(P_{\tau'}) - C)e^{-\rho(\tau'-\tau)}]\}, \tag{3.2.4}$$

where the expectation operator is conditional on the given initial value P, $\tau$ is the first time the idle field is developed and $\tau'$ is the first time the producing oil field closes down. To solve for the functions $U, V$ and the trigger prices $P_D$ and $P_C$ we follow Dixit (1989) in using option-pricing analogy[2]. (We assume risk neutrality.)

When there is no production there is no cash flow, so the real value of the field must yield an expected capital gain equal to the real interest rate, i.e.,

$$\frac{E_0 dU}{dt} = \rho U, \tag{3.2.5}$$

where

$$\frac{E_0 dU}{dt} = \lim_{\delta \to 0} \frac{E_0[U(P(t+\delta)) - U(P(t))]}{\delta}.$$

But from Itô's lemma,

$$\frac{E_0 dU}{dt} = \frac{\sigma^2}{2} P^2 U_{PP} + \alpha P U_P. \tag{3.2.6}$$

Equating (3.2.5) and (3.2.6) yields,

$$\rho U = \frac{\sigma^2}{2} P^2 U_{PP} + \alpha P U_P, \tag{3.2.7}$$

[2]For rigourous treatments of optimal stopping see Øksendal(1985).

an ordinary differential equation with the solution

$$U(P) = A_+ P^{\lambda_+} + A_- P^{\lambda_-}, \tag{3.2.8}$$

where $\lambda_+, \lambda_-$ are the two roots of the quadratic equation

$$\frac{1}{2}\sigma^2(\lambda - 1)\lambda + \alpha\lambda - \rho = 0.$$

This represents the value of the option to extract oil from the field, conditional on paying an exercise price of $D$, the development cost. As this option will surely be worthless when $P = 0$, so $A_- = 0$.

When the field is in production, similar arguments apply. Now the value of the field will reflect the cash flow described in (3.2.4) above, and the value of option to close down at cost $C$ (when the price of oil falls sufficiently below the cost of extraction). In this case the arbitrage condition becomes,

$$\frac{E_\tau dV}{dt} + (P - a)q^* = \rho V. \tag{3.2.9}$$

From Itō's lemma it follows that,

$$\frac{E_\tau dV}{dt} = \alpha P V_P + \frac{\sigma^2}{2} P^2 V_{PP}, \tag{3.2.10}$$

so by substitution we obtain the following expression,

$$\rho V = \alpha P V_P + \frac{\sigma^2}{2} P^2 V_{PP} + (P - a)q^*, \tag{3.2.11}$$

with the solution

$$V(P) = \frac{P}{\rho - \alpha} q^* - \frac{a}{\rho} q^* + B_+ P^{\lambda_+} + B_- P^{\lambda_-}. \qquad (3.2.12)$$

The first two terms in (3.2.12) evidently measure the present discounted value of cash flows ***assuming no close-down***; the other terms give the value of the option to close, but this tends to be worthless as $P \to \infty$; so $B_+ = 0$.

As is shown in Dixit (1989), the appropriate trigger prices, denoted by $P_D$ for development and $P_C$ for closure, ( along with the undetermined coefficients $A_+$, $B_-$) can be obtained from the Value Matching and Smooth Pasting conditions implied by efficient arbitrage, i.e., Value Matching,

$$\begin{aligned} V(P_D) &= U(P_D) + D \\ U(P_C) &= V(P_C) + C, \end{aligned} \qquad (3.2.13)$$

and Smooth Pasting,

$$\begin{aligned} V_P(P_D) &= U_P(P_D) \\ V_P(P_C) &= U_P(P_C). \end{aligned} \qquad (3.2.14)$$

(Here we assume that development costs are incurred every time the abandoned field is restarted. When the field is abandoned, the development investment is lost due to quick rust of the platforms etc..)

Dixit argues that equations (3.2.6), (3.2.11), (3.2.13) and (3.2.14) are sufficient for the optimality, so the solution to the problem (3.2.1), (3.2.3) and (3.2.4) is obtained by solving equations (3.2.8), (3.2.12), (3.2.13) and (3.2.14). These trigger prices and the associated value functions are shown in Figure 3.1. The line marked NN is the Present Discounted Value of future expected net of tax cash flows in the absence of closure. The excess of $V$ above line NN gives the value of the option to close, an option that is optimally exercised (at cost C) where the net of tax price $P$ falls to $P_C$. The schedule $U$ gives the value of the option to begin extraction. This is exercised optimally, at a development cost of $D$, when the price rises to $P_D$.

Later we will work with the case where the decision to extract is irreversible, i.e., when $C \to \infty$, $P_C$ tends to zero so the value function $V$ becomes NN, the Present Discounted Value function. Raising $C$ raises the entry trigger, as one is more hesitant to enter when there is no escape. (Indeed when $C \to \infty$, the optimal entry trigger $P_D$ would still lie above the "certainty equivalent" entry trigger $P = a + D(\rho - \alpha)$ even when $D = 0$. This is shown in Figure 3.2.)

## 3.3 Finite Reserves and Switching Boundaries

Let the reserve of oil $Q(t)$ be finite and non-augmentable; and suppose that the extraction rate $q(t)$ is proportional to $Q(t)$, i.e.,

$$q(t)dt = \gamma Q(t)dt = -dQ(t). \quad (3.3.1)$$

The assumed pattern of exponential decline in extraction from a developed field reflects geological factors and is standard in the literature on petroleum extraction (see Adelman and Jacoby (1979), Adelman and Paddock(1980) and Paddock *et al* (1988)).

With lump-sum development and close-down costs, one would expect that "trigger" strategies will still apply; but the trigger prices will in general depend on the level of reserves. So the optimal policy is characterised not by two trigger prices but by two switching boundaries. In the derivation that follows we continue to use Dixit's option-pricing analogy.

### 3.3.1 Fixed Costs of Extraction (Brekke and Øksendal (1991))

We begin with the model of Brekke and Øksendal (1991,1992) where extraction costs are a fixed lump-sum and the optimal switching boundaries turn out to be two rectangular hyperbolae in $P, Q$ space. With independent variation in prices and quantities, one generally needs to solve partial differential equations(PDEs) to determine these boundaries. But as we show below this particular model can in fact be solved by ordinary differential equations(ODEs) after an appropriate transformation has been applied. Later, when we consider variable costs of extraction, the use of PDEs appears to be unavoidable.

The linked value functions for the field in both idle and production are written as

$$\begin{aligned} U(P,Q) &= \sup_{\tau}\{E_0(V(P_\tau,Q_\tau) \;-\; D)e^{-\rho\tau}\} \\ V(P_\tau,Q_\tau) &= \sup_{\tau'}\{E_\tau[\int_\tau^{\tau'}(\gamma P_s Q_s - k)e^{-\rho(s-\tau)}ds \\ &\quad +(U(P_{\tau'},Q_{\tau'}) - C)e^{-\rho(\tau'-\tau)}]\} \end{aligned}$$

where $k$ is the fixed extraction costs. Define the variable $R$ as,

$$R_t = P_t Q_t. \tag{3.3.2}$$

On differentiating $R_t$ by using Itô's lemma, we obtain,

$$dR_t = \alpha R_t dt + \sigma R_t dW_t \qquad \text{idle,} \tag{3.3.3}$$

$$dR_t = (\alpha - \gamma) R_t dt + \sigma R_t dW_t, \qquad \text{in production} \tag{3.3.4}$$

where the $\sigma$-field is not changed, so the expectation taken in the previous probability space is invariant under such transformation.

If we rewrite the linked value functions in terms of $R_t$, we have

$$U(R) = \sup_{\tau}\{E_0(V(R_\tau) - D)e^{-\rho\tau}\}, \tag{3.3.5}$$

$$V(R_\tau) = \sup_{\tau'}\{E_\tau[\int_\tau^{\tau'} (\gamma R_s - k)e^{-\rho(s-\tau)}ds + (U(R_{\tau'}) - C)e^{-\rho(\tau'-\tau)}]\}. \tag{3.3.6}$$

To obtain the ODE for $U$ and $V$ , we can proceed just as in the infinite reserve case, except of course that the state variable is now $R$, not $P$.

Thus when the field is idle, there is no cash flow, so the value of the field must yield an expected capital gain equal to the real interest rate. Using this arbitrage condition and Itô's lemma we have,

$$\rho U = \frac{\sigma^2}{2} R^2 U'' + \alpha R U'. \tag{3.3.7}$$

When the field is in production, the arbitrage condition requires that the value of the field must yield the sum of the expected capital gain and cash flows equal to the real interest,i.e., together with Itô's lemma we derive,

$$\rho V \;=\; \frac{\sigma^2}{2}R^2V'' \;+\; (\alpha-\gamma)RV' + \gamma R - k, \qquad (3.3.8)$$

Let $\lambda_+$ be the positive root of the following quadratic equation[3],

$$\frac{1}{2}\sigma^2(\lambda_+ - 1)\lambda_+ \;+\; \alpha\lambda_+ \;-\; \rho \;=\; 0, \qquad (3.3.9)$$

and $\xi_-$ be positive root of the quadratic equation,

$$\frac{1}{2}\sigma^2(\xi_- - 1)\xi_- \;+\; (\alpha-\gamma)\xi_- \;-\; \rho \;=\; 0$$

Imposing the natural boundary conditions on the options discussed in the previous section, we can write,

$$U(R) \;=\; A_+R^{\lambda_+}, \qquad (3.3.10)$$

$$V(R) \;=\; \frac{\gamma R}{\rho+\gamma-\alpha} \;-\; \frac{k}{\rho} \;+\; B_-R^{\xi_-}, \qquad (3.3.11)$$

where $U(R)$ is the value of the option to produce while the field is idle (which becomes worthless when the $R$ goes to zero). $V(R)$ contains both the expected cash flows provided no close-down occurs $(\gamma R/(\rho + \gamma - \alpha) \;-\; k/\rho)$, and another term representing the value of option to close down $(B_-R^{\xi_-})$ which becomes worthless when $R$ goes to infinity.

The Value Matching and Smooth Pasting conditions appropriate at

[3]It is straight forward to prove that for $\rho - \alpha > 0$ we have $\lambda_+ > 1$.

the point of development and of close-down, $R_D, R_C$, are

$$\begin{aligned} A_+ R_D^{\lambda_+} &= \frac{\gamma R_D}{\rho+\gamma-\alpha} - \frac{k}{\rho} + B_- R_D^{\xi_-} - D \\ A_+ R_C^{\lambda_+} &= \frac{\gamma R_C}{\rho+\gamma-\alpha} - \frac{k}{\rho} + B_- R_C^{\xi_-} + C \\ \lambda_+ A_+ R_D^{\lambda_+ -1} &= \frac{\gamma}{\rho+\gamma-\alpha} + \xi_- B_- R_D^{\xi_- -1} \\ \lambda_+ A_+ R_C^{\lambda_+ -1} &= \frac{\gamma}{\rho+\gamma-\alpha} + \xi_- B_- R_C^{\xi_- -1}. \end{aligned} \tag{3.3.12}$$

These determine the value of $R_D$,$R_C$ and imply $R_D > R_C$. The value functions and triggers are depicted in Figure 3.3.

The optimal policies of firm are completely determined by these two critical values for the revenue. Thus an idle field should start production when $R$ reaches $R_D$, while a productive field should be closed when $R$ falls below $R_C$; in all other cases current activity should remain unchanged. Then for $R_C < R < R_D$ the extraction should continue if development costs have already been paid, otherwise the field should be left idle. (This band of hysteresis is of course due to the presence of lumpy costs.) In this special case we see that a "two dimensional problem" involving the price of oil and the quantity of reserves can be transformed into a one dimensional problem for which the solution is precisely analogous to the trigger price model examined in the previous section—except that it is the "value" of reserves which acts as the trigger.

The switching boundaries in $P,Q$ space described by Brekke and Øksendal (1991) now follow immediately. Specifically the development boundary is

$$PQ = R_D, \tag{3.3.13}$$

and the boundary for closure is

$$PQ = R_C, \tag{3.3.14}$$

as shown in the upper panel of Figure 3.4.

To see what these boundaries might imply, consider the sequence of decisions that would be triggered in the case reserves stand initially at $Q_0$ and where the realisation of the price process is a simple *trend increase at the rate* $\alpha$. This is (hopefully!) not a realistic case, but it will suffice to show how switching boundaries can be reached more than once. Initially there will be no extraction until the point $A$ is reached; then prices continue rising until reserves fall to point $B$, where close-down occurs. If, as assumed, prices continue to rise, however, development will recur at point $C$, and so on. The time path of extraction is shown in the lower panel of Figure 3.4. (These switching boundaries can of course be applied to any realisations of the price processes.)

Brekke and Øksendal assume that lump sum development charge $D$ is fixed independently of the volume of reserves to be extracted. But if $D$ *does* depend on the size, the switching boundaries will be affected. In the next section we note in particular that those development expenses which depend *linearly* on reserves can be treated "as if" they were proportional extraction costs. So analysing the effect of extraction costs on the switching boundary also covers the effect of size-dependent development expenditures.

### 3.3.2 Proportional Extraction Costs with no Close-down

#### 3.3.2.1 Fixed Development Costs

Alas, it is not in general possible to transform a two dimensional problem into equivalent one-dimensional form! In this respect the Brekke and Øksendal's example is a special case. Thus when total extraction costs are not fixed but vary proportionally to the volume of production, it appears that we have to work with two state variables $(P, Q)$ and solve the associated PDEs. To keep the analysis tractable, however, we assume that the switch to extraction is irreversible(i.e., $C \to \infty$).

Under these assumptions, the linked value functions become,

$$U(P,Q) = \sup_{\tau} E_0\{(V(P_\tau, Q) - D)e^{-\rho\tau}\}, \quad (3.3.15)$$

$$V(P_\tau, Q) = E_\tau \int_\tau^\infty (P_s - a)\gamma Q_s e^{-\rho(s-\tau)} ds, \quad (3.3.16)$$

where the expectation is conditional on the initial values of $P$ and $Q$, $\tau$ is the first time the idle field is developed.

Because the switch is irreversible, so the integral in (3.3.16) is taken to infinity. Using the option arguments rehearsed above, or proceeding more directly with the Feynman-Kac formula (see Øksendal 1985), one finds that $V(P, Q)$ satisfies the following PDE,

$$\rho V = \alpha P V_P + \frac{\sigma^2}{2} P^2 V_{PP} - \gamma Q V_Q + (P - a)\gamma Q. \quad (3.3.17)$$

Under the condition $\rho + \gamma - \alpha > 0$, a particular solution for this PDE is,

$$V(P,Q) = \frac{\gamma PQ}{\rho+\gamma-\alpha} - \frac{a\gamma Q}{\rho+\gamma}, \tag{3.3.18}$$

Which is the expected discounted cash flows conditional on no closure, as we verify in Appendix B.

When there is no production, there is no cash flow so equation (3.2.5) still holds; and $Q$ remains constant so (3.2.6) also holds. So $U(P,Q)$ must satisfy the PDE given as,

$$\rho U = \alpha P U_P + \frac{\sigma^2}{2} P^2 U_{PP}. \tag{3.3.19}$$

Hence we can write the solution as

$$\begin{aligned} U(P,Q) &= A_+(Q)P^{\lambda_+} + A_-(Q)P^{\lambda_-} \\ &= A(Q)P^{\lambda_+} \end{aligned}$$

where $\lambda_+$ and $\lambda_-$ are as defined above, and the natural boundary condition for $A_-$ is applied. Not surprisingly, the value of option to enter industry must reflect the volume of reserves waiting to be extracted.

In order to solve for the development boundary we apply the switching conditions, i.e., Value Matching,

$$U = A(Q)P^{\lambda_+} = \frac{\gamma PQ}{\rho+\gamma-\alpha} - \frac{a\gamma Q}{\rho+\gamma} - D = V - D \tag{3.3.20}$$

and Smooth Pasting,

$$\lambda_+ A(Q)P^{\lambda_+ - 1} = \frac{\gamma Q}{\rho+\gamma-\alpha} \tag{3.3.21}$$

and the optimal switching boundary is

$$\frac{\gamma PQ}{\rho+\gamma-\alpha}(1-\frac{1}{\lambda_+})-\frac{a\gamma Q}{\rho+\gamma}-D=0. \tag{3.3.22}$$

This is shown as the schedule $DD$ in Figure 3.5. Note that as $Q \rightarrow \infty$ the development trigger tends to $P_{min} = a(\rho+\gamma-\alpha)/(\rho+\gamma)(1-\frac{1}{\lambda_+})$. The contrast with what is found in the previous section is due to the proportional extraction costs.

To assess the effect of the oil price volatility on the entry entry trigger price $P_S$, we compare it with the break-even under certainty denoted $P_D$. This break-even price is given by the condition that reserves match costs in present discounted value, i.e.,

$$\frac{\gamma P_D Q}{\rho+\gamma-\alpha}=\frac{a\gamma Q}{\rho+\gamma}+D$$

Here the left hand side is the present discounted value of oil revenues ($\gamma Q$ is the initial extraction rate, which is discounted by $\rho+\gamma-\alpha$, i.e., by the real interest rate *plus* the rate of depreciation of oil reserves *less* any expected rise in the real price of oil). The right hand side is the present discounted value of variable costs *plus* the lump sum cost of development.

How does the break-even price calculated in this way compare with the stochastic entry trigger price $P_S$? For a given value of $Q$ and all other parameters identical, we find

$$P_S=\frac{\lambda_+}{\lambda_+-1}P_D$$

where $\lambda_+$ is the positive root of the polynomial in equation (3.3.9). This result is a replica of the standard formula in the literature of irreversible investment (see for example Kester (1984), Dixit (1992), McDonald and Siegel (1986) and Pindyck (1988, 1991)). To get a feel for the quantitative impact of oil price volatility, let us calculate the ratio of $P_S$ to $P_D$ for plausible values of the key parameters. Specifically let there be no trend forecasted in the real price of oil ($\alpha = 0$); and let the real interest rate used to discount revenues be 5% ($\rho = 0.05$). As for the volatility of oil price we set $\sigma^2 = 0.09$ which implies that the standard deviation of oil prices 12 months away is a little below 40% of its current value. (This is broadly consistent with real oil price movements over the last few years.) These parameter value imply $\lambda_+ = 2.5$, and so, from the formula above, $P_S = 2.5P_D$. That is, **WITH PLAUSIBLE PARAMETER VALUES, THE STOCHASTIC ENTRY PRICE WILL BE MORE THAN TWICE THE BREAK-EVEN PRICE UNDER CERTAINTY**[4].

Since the volatility of oil price can effectively double the impact of lumpy development costs (relative to expected future revenues), price uncertainty would seem to play an important role in the decision to develop reserves. It is, however, important to note that the tax provisions applying in the UKCS allow such costs (plus uplift) to be written off against the future payment of Petroleum Revenue Tax. The impact of such tax provisions–under which the Government shares some of the risks involved–in *reducing* the effect of price volatility is something we plan to investigate further in future.

[4]Moreover, it is not difficult to show that if future oil price becomes more volatile, the firm requires even higher trigger price to enter.

#### 3.3.2.2 Variable Development Costs

More generally , it is possible that the development costs $D$ may depend on the size of the oil field $Q$[5]. How will this affects the shape of the switching boundary? We solve for the case that $D$ is a linear function of the size. Suppose

$$D = D_0 + \delta Q, \tag{3.3.23}$$

where $D_0$ is the lumpy cost and $\delta Q$ is the proportional cost.

Notice that equations (3.3.20)–(3.3.22) will be unchanged even if $D$ depends on $Q$, then we have the switching boundary given by

$$\frac{\gamma PQ}{\rho+\gamma-\alpha}(1-\frac{1}{\lambda_+}) = \frac{a\gamma Q}{\rho+\gamma} + D_0 + \delta Q, \tag{3.3.24}$$

where the first term on the right hand side of equation (3.3.24) is the marginal extraction cost. If we rewrite (3.3.24) and let the marginal extraction cost be $a/(\rho+\gamma)+\delta/\gamma$ then we have

$$\frac{\gamma PQ}{\rho+\gamma-\alpha}(1-\frac{1}{\lambda_+}) = (\frac{a\gamma}{\rho+\gamma} + \delta)Q + D_0. \tag{3.3.25}$$

Compare with (3.3.22), it is as if the marginal extraction cost has been increased by $\delta$. So the shape of this boundary is qualitatively like that given by (3.3.22) but the minimum price of the boundary becomes

$$P_{min} = \frac{a(\rho+\gamma-\alpha)+\delta}{(\rho+\gamma)(1-\frac{1}{\lambda_+})}. \tag{3.3.26}$$

We shall use this boundary to derive the time path of oil extraction in

[5] Where $Q$ is the initial size of a field and is given exogenously.

section 4.

#### 3.3.2.3 Adding a Stochastic Development Lag

From the data given in Table 1 of Favero, Pesaran and Sharma (1992), we observe that there is a development lag (i.e., the period from Annex B approval to the start-up of production) which varies from several months up to more than five years. So the model we developed so far, assuming instantaneous development, needs some modification. In order to capture this delay (and for the convenience of the analysis in what follows) we treat the development lag as exogenous, assuming it is a random time $T_D$ in a probability space $(\Omega^D, \mathcal{F}^D, P^D)$ which is independent of the probability space generated by the Brownian motion affecting the oil prices. Furthermore we assume that the development cost is paid as an exogenous lump sum at the beginning of the development phase; and when this investment is made, it cannot be withdrawn.

Suppose that the entry occurs at time $\tau$. For given random development lag $T_D$, the field will not start yielding cash flow until time $\tau + T_D$. The expected cash flow after the completion of development is

$$\begin{aligned} V(P_{\tau+T_D}, Q_{\tau+T_D}) &= E_{\tau+T_D} \int_{\tau+T_D}^{\infty} (P_s - a)\gamma Q_s e^{-\rho(s-\tau-T_D)} ds \\ &= \frac{\gamma P_{\tau+T_D} Q_{\tau+T_D}}{\rho + \gamma - \alpha} - \frac{\gamma a Q_{\tau+T_D}}{\rho + \gamma}, \end{aligned} \tag{3.3.27}$$

where price and reserves are those at $\tau + T_D$.

How much will this expected cash flow be equivalent, in present value terms, to that evaluated at $\tau$? Since there is no extraction during the period of development, so

$$Q_{\tau+T_D} = Q_\tau = Q_0 = Q$$

and the present expected cash flow at the beginning of the development is

$$\begin{aligned} V(P_\tau, Q_\tau) &= e^{-\rho T_D} E_\tau\left(\frac{\gamma P_{\tau+T_D} Q_{\tau+T_D}}{\rho+\gamma-\alpha} - \frac{\gamma a Q_{\tau+T_D}}{\rho+\gamma}\right) \\ &= e^{-\rho T_D}\left[\frac{\gamma E_\tau(P_{\tau+T_D}) Q_\tau}{\rho+\gamma-\alpha} - \frac{\gamma a Q_\tau}{\rho+\gamma}\right], \end{aligned} \tag{3.3.28}$$

where $E_\tau(P_{\tau+T_D})$ is the future price of oil $T_D$ periods ahead forecasted at $\tau$.

Given the time trend $\alpha$ in the price of oil, the expected price will be appreciated at the end of the development, namely

$$E_\tau(P_{\tau+T_D}) = P_\tau e^{\alpha T_D}$$

Having obtained the value function in production at the entry time $\tau$, we can write the value function for the idle field with the option to switch to development and subsequently to extraction as[6]

$$\begin{aligned} U(P,Q) &= \sup_\tau E_0 E^D\{[(\frac{\gamma P_\tau Q_\tau e^{(\alpha-\rho)T_D}}{\rho+\gamma-\alpha} - \frac{\gamma a Q_\tau e^{-\rho T_D}}{\rho+\gamma}) - D]e^{-\rho\tau}\} \\ &= \sup_\tau E_0\{[\frac{\gamma P_\tau Q_\tau}{\rho+\gamma-\alpha} E^D e^{(\alpha-\rho)T_D} \\ &\quad - \frac{\gamma a Q_\tau}{\rho+\gamma} E^D e^{-\rho T_D} - D]e^{-\rho\tau}\}, \end{aligned} \tag{3.3.29}$$

[6]We thank Avinash Dixit for pointing out the errors in the previous draft.

where $E^D$ is the expectation operator taken in the probability space $(\Omega^D, \mathcal{F}^D, P^D)$.

Applying boundary conditions, we have

$$A(Q)P^{\lambda_+} = \frac{\gamma PQ}{\rho+\gamma-\alpha} E^D e^{(\alpha-\rho)T_D} - \frac{a\gamma Q}{\rho+\gamma} E^D e^{-\rho T_D} - D$$

$$\lambda_+ A(Q)P^{\lambda_+ -1} = \frac{\gamma Q}{\rho+\gamma-\alpha} E^D e^{(\alpha-\rho)T_D}. \qquad (3.3.30)$$

So we derive the switching boundary

$$\frac{\gamma PQ}{\rho+\gamma-\alpha}\frac{\lambda_+ - 1}{\lambda_+} E^D e^{(\alpha-\rho)T_D} - \frac{a\gamma Q}{\rho+\gamma} E^D e^{-\rho T_D} - D = 0. \qquad (3.3.31)$$

How this switching boundary is affected by the development lag relative to the case where there is no delay in development? To assess this effect, we compare these two cases for given identical initial reserves and all the other parameters.

Without development lag, the optimal entry boundary is given by equation (3.3.22). Denote the oil price in this case by $P^N$, then the floor price for entry is

$$P_m^N = \frac{\lambda_+ - 1}{\lambda_+}\frac{a(\rho+\gamma-\alpha)}{\rho+\gamma}$$

The floor price for entry with development lag is

$$P_m = \frac{E^D e^{-\rho T_D}}{E^D e^{(\alpha-\rho)T_D}} P_m^N$$

It is interesting to see how the ratio of these two floor entry prices response to the changes in $\alpha$ and $T_D$ for given discount rate. From simple derivations we obtain the followings[7],

$$\frac{\partial}{\partial \alpha}\left(\frac{P_m}{P_m^N}\right) = -\frac{E^D e^{-\rho T_D} E^D (T_D e^{(\alpha-\rho)T_D})}{(E^D e^{(\alpha-\rho)T_D})^2} < 0$$

and for $\alpha \geq 0$

$$\max P_m = \lim_{\alpha \to 0} P_m = P_m^N$$
$$\frac{P_m}{P_m^N} \leq 1$$

Furthermore, we have

$$\frac{\partial}{\partial T_D}\left(\frac{P_m}{P_m^N}\right) = -\frac{\alpha E^D e^{-\rho T_D}}{E^D e^{(\alpha-\rho)T_D}} < 0$$

because the field can only yield the cash flow after the completion of development, the real oil price relevant to it is that at time $\tau + T_D$. For a given very large initial reserves and nonnegative time trend in the price of oil, the field will be developed at a lower trigger price if the trend becomes larger or the development takes longer.

If the initial reserves are relatively small, from equation (3.3.22) and (3.3.31) we obtain

$$\frac{P - P_m}{P^N - P_m} = \frac{1}{E^D e^{(\alpha-\rho)T_D}} \frac{P^N - P_m^N}{P^N - P_m}. \tag{3.3.32}$$

For $\alpha = 0$, equation (3.3.32) becomes

$$\frac{P - P_m}{P^N - P_m} = \frac{1}{E^D e^{-\rho T_D}} > 1$$

[7]We assume that differentiation and integration are interchangeable.

So the development lag shifts the switching boundary outward as the burden of development costs is effectively increased by the need to finance the delay. The schedule DD without development lag and schedule LL with development lag are drawn in Figure 3.6(a).

If $\alpha > 0$, we first look at the case where $\alpha - \rho < 0$, the left hand side of equation (3.3.32) has the following properties.

$$\lim_{P^N \to P^N_m} \frac{1}{E^D e^{(\alpha-\rho)T_D}} \frac{P^N - P^N_m}{P^N - P_m} = 0$$

$$\lim_{P^N \to \infty} \frac{1}{E^D e^{(\alpha-\rho)T_D}} \frac{P^N - P^N_m}{P^N - P_m} = \frac{1}{E^D e^{(\alpha-\rho)T_D}} > 1$$

and

$$\frac{\partial}{\partial P^N}\left(\frac{1}{E^D e^{(\alpha-\rho)T_D}} \frac{P^N - P^N_m}{P^N - P_m}\right) = -\frac{1}{E^D e^{(\alpha-\rho)T_D}} \frac{P_m - P^N_m}{(P^N - P_m)^2}$$

so $P$ is less than $P^N$ for $P^N < \bar{P}$ and greater than $P^N$ for $P^N > \bar{P}$ (the critical size related to $\bar{P}$ is $\bar{Q}$), the schedule DD and LL are shown in Figure 3.6(b).

Since the trend in the real price of oil is positive, the relatively large sized field (larger than $\bar{Q}$) will be developed at a lower trigger price (compare with the case without development lags) because it can generate sufficient expected cash flow ($T_D$ period later) to cover the lumpy development costs. But with the size smaller than $\bar{Q}$, the expected future operating profits are insufficient to cover both entry costs and the value of waiting option, so their developments will be deferred.

## 3.4 Time Path of Aggregate Oil Production

In the previous sections, we have discussed the optimal switching boundaries for a specific oil field. Any given tract of sea which has been leased for exploration will, however, contain several fields of varying size. The switching boundaries imply that, even if they all have the same development cost, fields will be developed at different times, biggest first, smaller later. So a given realisation of real oil price process will imply a sequence of development decisions and a specific time path for oil extraction.

We assume for simplicity that there is no exploration and the switching boundaries are the same across all the oil fields. In particular, we adopt the irreversible switching model without exploration given in section 3.2. In this case, it is sufficient to assume that the development costs $D$ and the exponential extraction rates $\gamma$ are the same for all the oil fields[8].

Suppose development can be achieved instantly[9] by paying a development cost $D$, and the proven reserves $Q$ are distributed in $[0, Q]$ according to a distribution density function $f(Q)$ and

$$Q_T = \int_0^Q Qf(Q)dQ, \tag{3.4.1}$$

where $Q_T$ is the total proven reserves.

[8]For the case where the development costs are given as $D = D_0 + \delta Q$, it would be necessary instead that to assume $D_0$ and $\delta$ are the same across the fields.

[9]For a given fixed development lag, the results remain qualitatively unchanged.

Imagine that if the realisation of price process is given in Figure 3.7, where it has two peaks $P_1$ and $P_2$ and $P_2 > P_1$. (These might represent the price hikes due to OPEC I and OPEC II for example.) New fields may be developed during the price increasing phase from $P_0$ to $P_1$, but no development will be triggered during the price decreasing phase after $P_1$. In the second price increasing phase, more fields will be developed only when the price surpasses $P_1$. So the price process which is effective in triggering new oil fields is the non-decreasing process $\bar{P}_t$ shown in Figure 3.7. In what follows, we shall define this "effective" oil price $\bar{P}_t$ more accurately.

The optimal switching boundary in section 3.2 is given in equation (3.3.22), we can rewrite it as follows

$$(P - P_{min})Q = D', \tag{3.4.2}$$

where $P_{min} = a(\rho+\gamma-\alpha)/(\rho+\gamma)(1-\frac{1}{\lambda_+})$ and $D' = (\rho+\gamma-\alpha)D/\gamma(1-\frac{1}{\lambda_+})$.

Consider the price process (3.2.2) in a given period $[0, T]$, $P_t$ is continuous and adapted to the $\sigma$-filtration $\{\mathcal{F}_t\}$.Dividing the time period into $N$ points and

$$0 = t_1 < t_2 < \ldots < t_N = T$$

Then we can discretise the price process $P_t$ by a sequence $\{P_{t_i}\}$, $i = 1, 2, \ldots, N$, and $\{P_{t_i}\}$ is adapted to $\{\mathcal{F}_{t_i}\}$.

Now we can define the "effective" price process $\bar{P}_{t_i}$ as

For $i = 1$

$$\hat{P}_{t_i} = P_{t_i}. \tag{3.4.3}$$

For $i > 1$

$$\hat{P}_{t_{i+1}} = \begin{cases} P_{t_{i+1}} & if \quad P_{t_{i+1}} > \hat{P}_{t_i} \\ \hat{P}_{t_i} & if \quad P_{t_{i+1}} \leq \hat{P}_{t_i} \end{cases} \tag{3.4.4}$$

$$\Delta\hat{P}_{t_i} = \hat{P}_{t_{i+1}} - \hat{P}_{t_i} \geq 0. \tag{3.4.5}$$

The process $\hat{P}_{t_i}$ so defined is obviously adapted to $\{\mathcal{F}_{t_i}\}$. Clearly it is non-decreasing, it equals the previous peak if the current real oil price is non-increasing; it equals the current real oil price if it surpasses the previous peak. One can take the limit $N \to \infty$, so $\hat{P}_t$ becomes a continuous non-decreasing process adapted to $\{\mathcal{F}_t\}$.

The reserves to be developed for a given "effective" oil price $\hat{P}_s$ are

$$\hat{Q}_s = \frac{D'}{\hat{P}_s - P_{min}}. \tag{3.4.6}$$

But if $P < P_{min}$, no fields will be triggered as development will only begin if oil prices first reach $P_{min}$. For convenience, we take $t = 0$ as the first time any field is developed. At this time, the developed reserves are

$$L_0 = \int_{\hat{Q}_0}^{\bar{Q}} Qf(Q)dQ \quad if \quad \hat{Q}_0 < \bar{Q}, \tag{3.4.7}$$

and the developed reserves at time $s > 0$ is

$$L_s = \int_{\hat{Q}_s}^{\bar{Q}} Qf(Q)dQ, \tag{3.4.8}$$

because $\hat{P}_s$ is a non-decreasing process, so $\hat{Q}_s$ is a non-increasing process and $L_s$ is obviously a non-decreasing process. For a given realisation of real oil price $P_t$, $\hat{P}_t$ and $L_s$ are shown in Figure 3.7. In order to describe the aggregate oil extraction rate at time $t$, we first look at the contribution of those oil field developed at time $t = 0$,

$$\begin{aligned} q_A(t;0) &= \gamma e^{-\gamma t} L_0 \\ &= \gamma e^{-\gamma t} \int_{Q_0}^{Q} Q f(Q) dQ. \end{aligned} \quad (3.4.9)$$

The contribution of those oil fields triggered between $[s, s + \Delta s]$ is

$$\begin{aligned} q_A(t; s, s + \Delta s) &= \gamma e^{-\gamma(t-s)} (L_{s+\Delta s} - L_s) \quad (3.4.10) \\ &= \gamma e^{-\gamma(t-s)} \Delta L_s. \end{aligned}$$

So the aggregate oil extraction rate at time $t$ is,

$$q_A(t) = \gamma e^{-\gamma t} L_0 + \int_0^t \gamma e^{-\gamma(t-s)} dL_s \quad (3.4.11)$$

For the previously specified real oil price process $P_t$, $q_A(t)$ is shown in Figure 3.8.

The real oil price series from 1970 onwards has two sharp increases in 1975 and 1979 (OPEC I and OPEC II). The extraction path implied by our model would have two big increases in 1974 and 1979 (with a declining in between). But the big increase in oil production in UKCS occurred with a substantial delay after OPEC I, and there was no declining phase before the second big increase after OPEC II. The delay of production in response to the price increase may be due to a development lag (the average development lag for the six fields with Annex

B approval at the time of OPEC I was about two and half years). A stochastic development lag will have a smoothing effect on the series for extraction (as will subsequent rounds of leasing). In what follows we consider the time path of oil production for a given lease and a known distribution of the random development lag.

As shown in section 3.2, the optimal switching boundary in this case is similar to (3.4.6) except that $P_{min}$ is replaced by $(E^D e^{-\rho T_D} / E^D e^{-(\alpha-\rho)T_D}) P_{min}$, and $D'$ replaced by $D'/E^D e^{-(\alpha-\rho)T_D}$, but $\hat{P}_t$, $\hat{Q}_t$ are as defined above. Developed reserves at time $s$ will be

$$L_s = \int_{\hat{Q}_s}^{Q} \chi(T_D \leq s) Q(T_D) f(Q(T_D)) dQ, \tag{3.4.12}$$

where

$$\chi(T_D \leq s) = \begin{cases} 1 & if \quad T_D \leq s \\ 0 & if \quad T_D > s \end{cases}$$

So the time path of oil extraction is

$$q_A(t) = \int_0^t \gamma e^{-\gamma(t-s)} dL_s, \tag{3.4.13}$$

for $L_s$ defined above.

For price series shown in Figure 3.7, the effect of first adding a fixed development lag, and then smoothing the series by randomising the lag is illustrated in Figure 3.8.

## 3.5 Conclusions and Topics for Further Research

The fact that extracting oil from the sea involves distinct and sequential stages invites at least two sorts of study. First is how to optimise the rate of activity during each of these stages; second is when to switch *between* stages. In the econometric study referred to above, for instance, Pesaran addresses the first issue and seeks in particular to characterise exploration and extraction. In this paper, however, we have focussed on the issue of when to switch between the stages. For simplicity, rates of exploration and production have been represented as linear functions of state variables; and development costs have been "collapsed" to a lump sum.

The case where production costs are also a lump sum (and there is no exploration) has already been analysed by Brekke and Øksendal (1991,1992), who find the switching boundaries are rectangular hyperbolae in $P, Q$. It appears that their results can be obtained in a much simpler fashion, namely by transforming the problem into univariate form and using Dixit's approach to find two switching point for $R = PQ$. When production costs vary with the extraction rate, however, this transformation no longer applies, and obtaining the solution involves solving PDEs, and we determine the switching boundary for irreversible entry into production in section 3.2 above. The impact of the uncertainty of future oil price on the entry boundary is assessed and compared with the certainty equivalent break-even boundary, its magnitude is significant. When plausible parameter values were used to calculate the effect of oil price volatility on entry, it was found that in the face of uncertainty the price might need to be twice as high as the certainty equivalent "break-even" price to trigger development.

In section 4 such switching boundaries have been used to generate a time path for oil extraction given a time path for oil prices and geological distribution of fields by size.

In this paper, we have assumed that development costs are not deductible against Petroleum Revenue Tax. In next chapter we aim to consider in more detail the effects of tax deductibility in mitigating the impact of price uncertainty on the decision to develop.

As additional topics for research, we aim to study the impact of risk averse behaviour by using Contingent Claims Analysis, and to compare the results with the exogenous high real rate of interest used in the *Brown Book*. We also would like to incorporate the effects of *successive leases* on the flow of oil produced. Simulation methods may well be called for here. The implications of allowing for some *reversibility* of extraction can already be studied in the Brekke and Øksendal model. It might be desirable to expand this to other cases; a better alternative might be to incorporate the option to quit as maintenance costs tend to be very high.

## 3.6 FIGURES

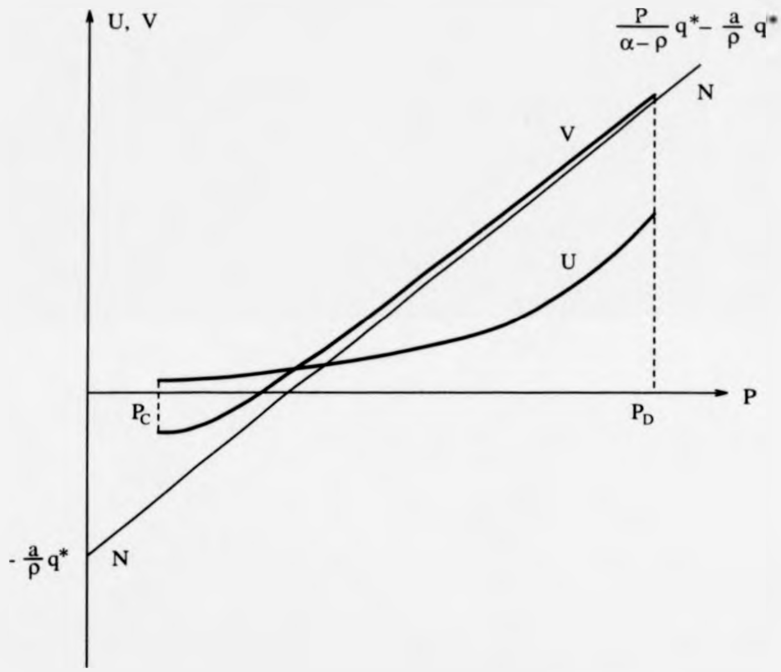



Figure 3.1: Value Functions and Trigger Prices for Infinite Reserves.

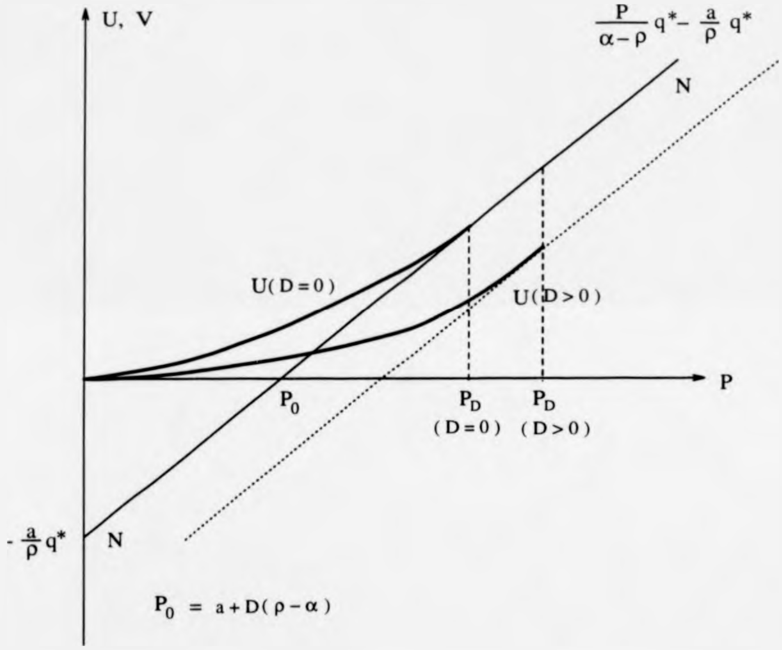



Figure 3.2: Irreversible Entry for Infinite Reserves.

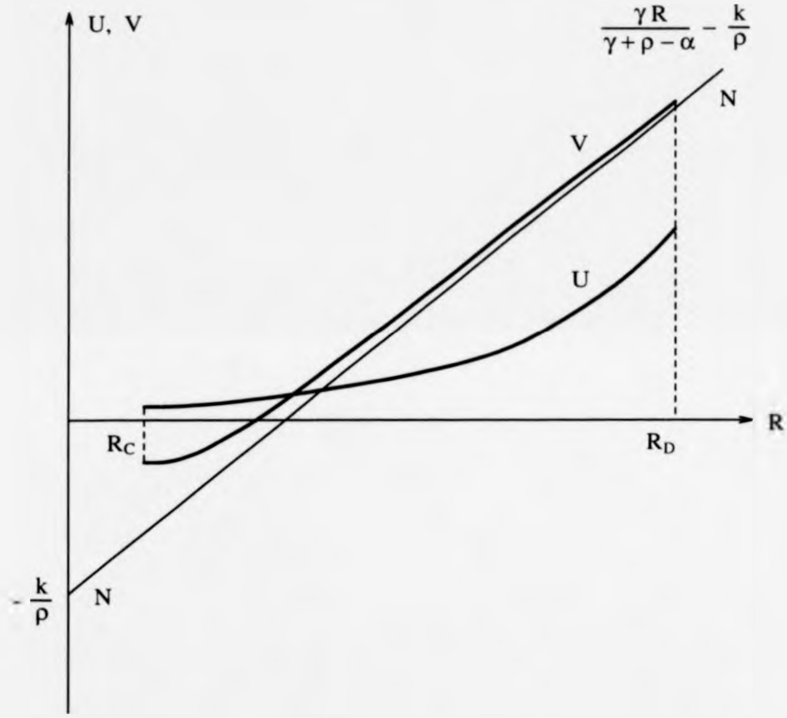
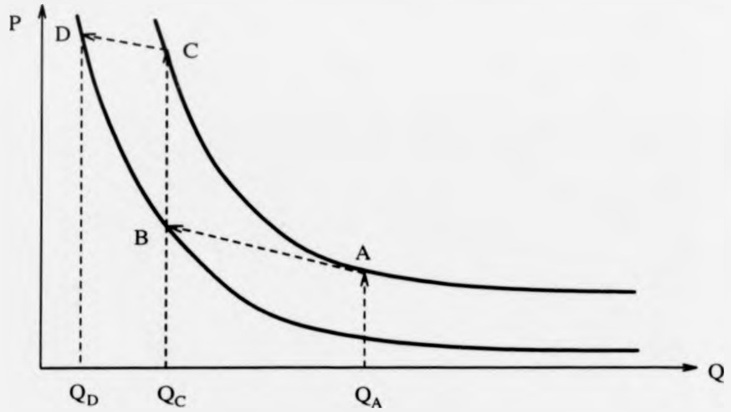
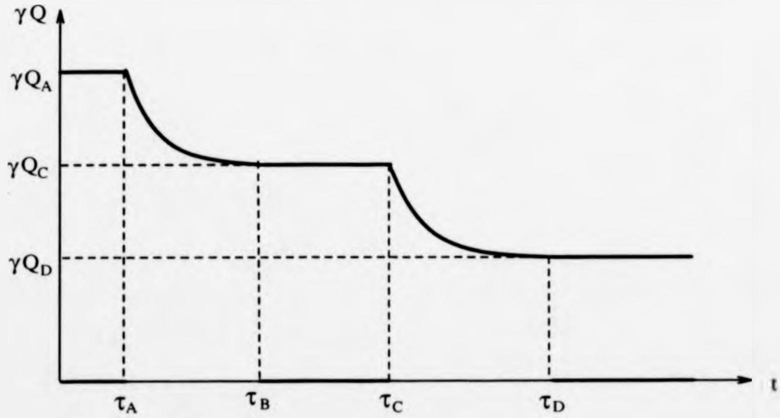



Figure 3.3: Finite Reserves: B&Ø Model and its Trigger Reserves.




4 (a) : Optimal Development Boundaries .




4 ( b ) : Time Path of Oil Extraction.

Figure 3.4: Optimal Switching Boundary and Time Path of Extraction.

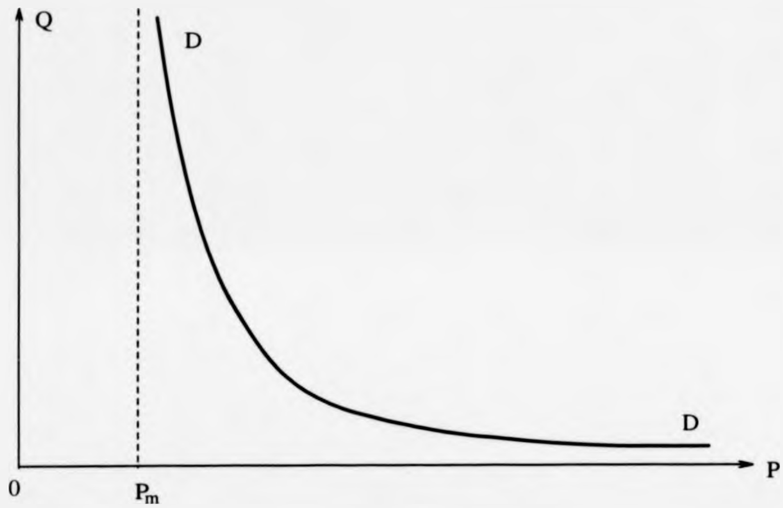
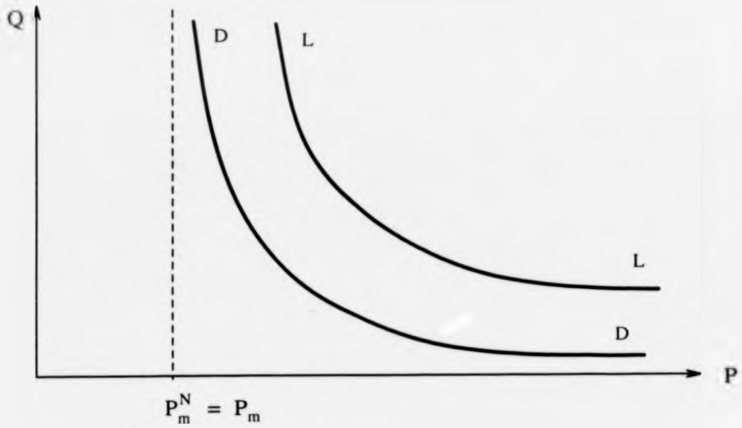
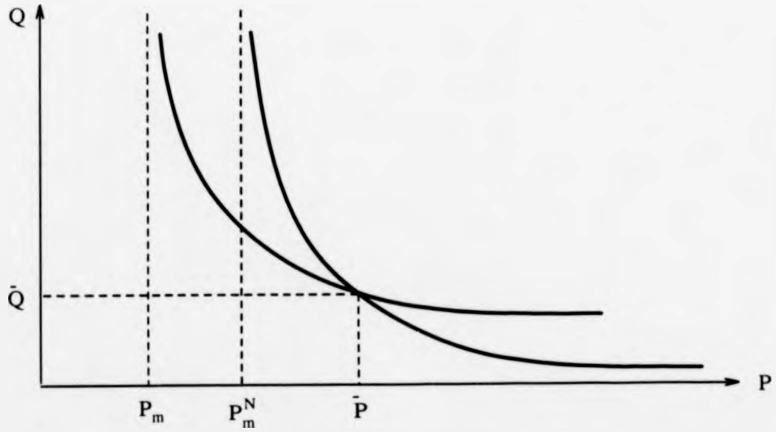



Figure 3.5: Finite Reserves: Optimal Switching Boundary with Proportional Operating Costs and no Close-down.




6(a): $\alpha = 0$




6(b): $\alpha < \rho$

Figure 3.6: Optimal Entry Boundary with Random Development Lag.

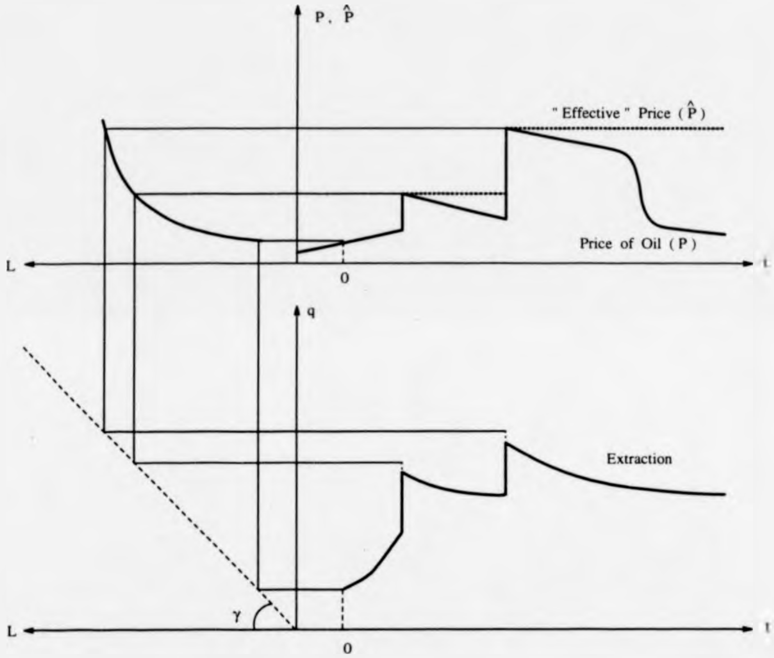



Figure 3.7: Time-path for Oil Extraction.

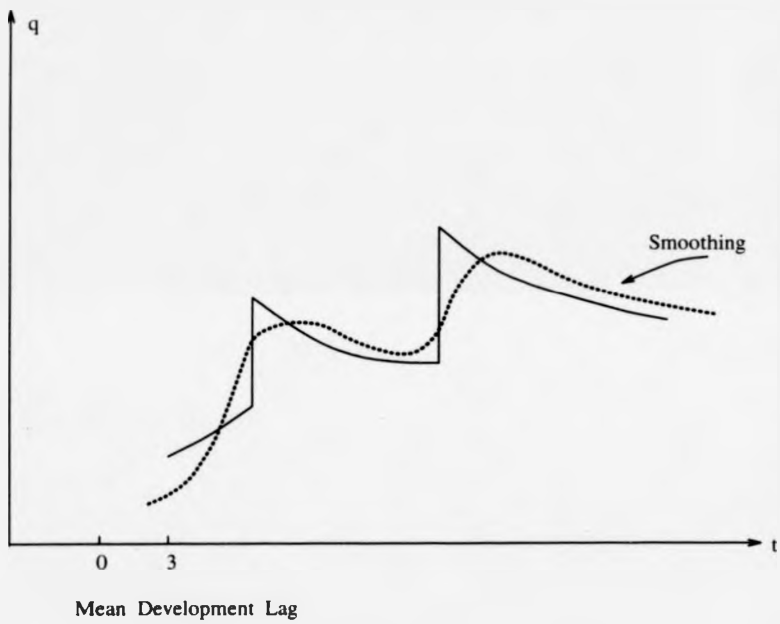



Figure 3.8: The Effect of Development Lags.

# Chapter 4

# Optimal Investment under Stochastic Capital Depreciation

## 4.1 Introduction

In his paper published in Oxford Economics Papers (1985), Mark Precious provides a deterministic model of investment, where the investor anticipates the possibility of sales constraints, as in Blanchard and Sachs (1982). The model neatly combines the cost minimisation and profit maximization approaches to investment. The former is typically used when the firm is always facing a demand constraint, so it minimises the total costs for given level of output (cf Brechling (1975), Hausman (1972)). The latter approach is typically used when the firm faces no demand constraints (cf Jorgenson (1963) and Brechling (1975)). Precious combines the two by allowing for anticipated regime switching.

Precious's model is analysed by using deterministic optimal control theory, maximising the expected present discounted value of cash flow, subject to the capital accumulation dynamics and demand constraints. This intertemporal optimisation problem yields two evolutionary equations (Euler equations), one describing the optimal evolution of the shadow price of the capital and the other describing the process of capital accumulation itself, often analysed by using a phase diagram. Given constant return to scale technology, linear dependence of capital accumulation on gross investment and capital depreciation, and no other constraints on the labour market, the intertemporal problem leads to unique equilibrium for both the demand constrained and unconstrained cases, as the problem is manifold stable (saddle point stable). The phase diagram technique helps to illustrate various responses of the firm's investment to unanticipated wage, interest rate and price shocks in each different regime, and clarifies the relationship between the policy implication and its regime. This technique also allows us to explain the firm's behaviour when making the transition from an unconstrained regime to the constrained one, or vice versa. The transition might be induced either by exogenous shocks (unanticipated) or endogenous changes of the production level (anticipated). Note that if the transition is induced by any endogenous changes, the shadow price cannot change discretely; instead, some continuity conditions must be satisfied at the transition boundary in order to guarantee that the transition occurs optimally.

Since 1985, however, the investment theory has undergone a major change as attention has shifted to cope with the investment decision under price uncertainties, with lumpy costs[1]. These developments involves the use of impulse and instantaneous control or optimal stopping, which makes the general intertemporal optimisation problem technically intractable unless strong assumptions are made. This topic of irreversible investment was dealt with in the the previous chapters.

In this chapter, attention is focussed on generalising Precious's intertemporal model by incorporating one element of uncertainty, specifically by assuming that the capital depreciation follows geometric Brownian motion. For simplicity, the irreversibility constraint on investment (considered previously) is relaxed, and the output price is assumed to be constant. The primary concern of adopting the assumption that the capital depreciation follows the geometric Brownian motion is due to technical reasons. In considering the switching case, this will normally lead to a high order ODE (ordinary differential equation) which generally can only be solved by numerical methods. However, incorporating more uncertainty factors (eg. price uncertainty) will lead to nonlinear PDEs (partial differential equations) and as the switching boundary must be determined simultaneously with the optimal investment decision. This free boundary problem is even numerically difficult to solve.

In dealing with the deterministic intertemporal optimisation problem, the maximum principle can be used to obtain the Euler equations, and phase diagrams can help to show the stability of the problem. This is because, in the deterministic case, there exists a unique mapping between state dependent optimal policies and the time dependent ones, so the phase diagram can be used not only to analyse the sta-

[1]The cases of proportional investment costs under uncertainty are considered by Bertola (1989) and Pindyck (1988).

bility of equilibrium, but also to portray state dependent trajectories leading towards that equilibrium. However, in the stochastic case, this unique mapping generally does not exist. For this reason, use is made of stochastic dynamic programming, i.e., the HJB (Hamilton-Jacobi-Bellman) equation (though phase diagram like picture will be used on occasion to illustrate state dependent trajectories).

As a benchmark, the basic points of deterministic regime switching in Precious's model are summarised in Appendix C, and the programming method is used instead to reinterpret the optimality condition in terms of value functions. The use of high order smooth pasting conditions is explained. In section 2, the stochastic version of the Precious's model is formulated as a regime switching problem with the optimality condition deduced by optimal stopping. In section 3, an approximation method is used to assess the precautionary investment behaviour (the optimal investment decreases when the variance of capital depreciation increases). And section 4 explains stochastic smoothing as a consequence of the *reversibility* of the anticipated switching and the precautionary investment is caused by the concavity of the marginal value of capital ($V_k(k)$). And finally, section 5 concludes this chapter.

## 4.2 Anticipated Stochastic Regime Switching

Although it is interesting to consider the uncertainty related to factor prices or stochastic shift of demand constraint, it is generally difficult to find the optimal solution to such regime switching problem as that will induce partial differential equations, and the optimal investment will be simultaneously determined with the free boundary of switching. Considering uncertainty on the capital depreciation is straight forward, technically it will make the problem tractable and also we shall see that the lose of the expected profits due to capital depreciation has a second order term attributed to such uncertainty.

To specify this uncertainty, we assume that the capital depreciation follows a geometric Brownian motion, namely

$$\frac{dK_d(t)}{K(t)} = -\delta dt + \sigma dZ(t), \tag{4.2.1}$$

where $K_d(t)$ is the depreciated capital, $\delta$ is the depreciation rate, $Z(t)$ is a standard Brownian motion, $|\sigma| > 0$ is its instantaneous variance. This equation indicates that the capital depreciation has a deterministic trend, but it is perturbated by a random term which is unanticipated. This Brownian motion has a normal distribution and the mean is zero, the variance is $\sigma\sqrt{dt}$.

Similar to the deterministic case the value function of the firm is the expected present profits conditional on the current information available (at time $t$), i.e.,

$$V(k_t) = E_t \int_t^{\infty} \{pQ(t) - wL(t) - [I(t) + C(I)]\} e^{-r(s-t)} ds, \tag{4.2.2}$$

where $Q(t), L(t), I(t)$ are the output, employment and investment respectively at time $t$. $p, w$ are the price of output and wage rate assumed to be constant. $C(I)$ is the adjustment costs, $r$ is the interest rate. The constraints are given as follows,

$$Q(s) = F(K, L) = K^{\alpha}L^{1-\alpha}, \quad \text{for } 0 < \alpha < 1; \tag{4.2.3}$$

$$Q(s) \leq \bar{Q}; \tag{4.2.4}$$

$$K(t) = k. \tag{4.2.5}$$

Where the first constraint describes the technological condition available to the firm, the second is a demand constraint, the third is the capital accumulation dynamics and the last one is the initial capital stock.

Using (4.2.1), we have the capital accumulation equation follows a diffusion process, namely

$$dK(s) = (I(s) - \delta K(s))ds + \sigma K(s)dZ(s), \quad s \geq t. \tag{4.2.6}$$

Because all the function are time homogeneous, applying dynamic programming, we have a time homogeneous Bellman equation (Fleming and Rishel 1975, Krylov 1981),

$$rV(k) = \max_{I,L}\{\frac{1}{2}\sigma^2k^2V_{kk}(k)+(I-\delta k)V_k(k)+pF-wL-(I+C)+\lambda(\bar{Q}-F)\}, \tag{4.2.7}$$

where $\lambda$ is the shadow cost of demand constraint and subscripts denote derivatives. Unlike that in the deterministic case, the Bellman equation here is a second order ordinary differential equation. This is because the Brownian motion term is of the order $\sqrt{ds}$, applying Itô's lemma to the value function will lead to a second order term.

The first and second order conditions can be expressed as follows,

$$V_k(k) = 1 + C'(I), \tag{4.2.8}$$

$$F_L = \frac{w}{p - \lambda}, \tag{4.2.9}$$

and the Hessian is

$$H = \begin{bmatrix} -C''(I) & 0 \\ 0 & (p-\lambda)F_{LL} \end{bmatrix} \tag{4.2.10}$$

where subscripts denote partial derivatives, and $H$ is negative definite Hessian matrix for $p - \lambda \geq 0$. As the firm can choose both capital in place (through investment) and the labour, the first order conditions show that investment continues until marginal value of the firm by installing a unit capital is equal to its marginal costs (list price *plus* the marginal adjustment cost of investment), and the adjustment of labour ceases only if the marginal product of labour is balanced by its real wage.

From (4.2.8), because adjustment cost is convex, then $C'(I)$ is invertible, then

$$I = I(V_k(k) - 1). \tag{4.2.11}$$

From Krylov (1981), we have that the solution to this problem exist and unique provided in this infinite horizon case if the transversality conditions are satisfied. Furthermore, we have that the value function is twice continuously differentiable (cf Krylov chapter 2).

In this stochastic case the deterministic equilibria will not be the equilibria here, for if $k > 0$, the Brownian motion can alway drive away the process even it reaches the equilibria.

One can also view this constrained optimisation problem in regime switching setting. For an initially unconstrained firm, it faces the anticipation of switching to constrained regime, let $\tau > t$ is the first time that the firm becomes demand constrained, the value function of firm which is initially unconstrained can be written as,

$$V^u(K_t) = E_t\{\int_t^{\tau} (pQ(t) - wL - (I+C))e^{-r(s-t)} + V^c(K_\tau)e^{-r(\tau-t)}\}, \tag{4.2.12}$$

where the first term on the right hand side of equation (4.2.12) is the expected future profits when the firm is unconstrained, the second term is the expected profits of the firm which is constrained but has the future anticipation to switch to the unconstrained regime.

Let $\tau'$ be the first time the initially constrained firm becomes unconstrained, then the value function is simply,

$$V^c(K_t) = E_t\{\int_t^{\tau'} (pQ - wL - (I+C))e^{-r(s-t)}ds + V^u(K_{\tau'})e^{-r(\tau'-t)}\}. \tag{4.2.13}$$

where superscripts denote different regimes.

Since production function, adjustment cost function are time homogeneous, $p, w$ are constant, so $V(k)$ and the switching point related to $\tau$ and $\tau'$ are time homogeneous.

The Bellman equation in two regimes are[2]

$$rV^u(k) = \max_{I,L}\{\frac{1}{2}\sigma^2k^2V^u_{kk}(k) + (I-\delta k)V^u_k(k) + pQ - wL - (I+C)\}, \tag{4.2.14}$$

[2]The optimal stopping time $\tau$, $\tau'$ are Markov time, see Friedman (1975).

where subscripts denote derivatives. And

$$rV^c(k) = \max_{I,L}\{\frac{1}{2}\sigma^2 k^2 V^u_{kk}(k) + (I - \delta k)V^c_k(k) + pQ - wL - (I + C)\}, \tag{4.2.15}$$

with the boundary conditions for given switching point at $\hat{k}$

$$V^u(\hat{k}) = V^c(\hat{k}), \tag{4.2.16}$$

and

$$V^u_k(\hat{k}) = V^c_k(\hat{k}). \tag{4.2.17}$$

(cf Krylov 1981 chapter 2).

As indicated by Whittle (1983) that value matching and smooth pasting conditions given in equations (4.2.16) and (4.2.17) are due to the facts that the state variable $k$ has no jump at boundary and that the switching occurs reversibly.

Because the first order condition of this stochastic case have the same functional forms as those in the deterministic case, then Propositions C.2 and C.3 in Appendix C are still valid. So the switching boundary $\hat{k}$ can be determined in the same way as that in the deterministic case. Since the uncertainty on capital depreciation will only add one more term, namely, the expected profit change due to the change of shadow cost, to the Bellman equation, this will not affect the state dependent behaviour of marginal productivity of labour. Therefore, we must have the same switching point as that given in the deterministic case. This leads to that non-homogeneous term of equations (4.2.14) and (4.2.15) are matched at the boundary. Notice that the smooth pasting condition given in (4.2.17) demands the joining condition for the investment at the boundary provided that the first order conditions for investment on both regimes have the same functional forms. Then we can conclude that the second order smooth pasting condition should be satisfied:

$$V^c_{kk}(\hat{k}) = V^u_{kk}(\hat{k}). \tag{4.2.18}$$

The same argument as used in proving Proposition C.5 can carry through, i.e., the discontinuity of the second order derivatives of the value function will lead to a new switching point less than or greater than $\hat{k}$, this violates the claim that the switching point selected by optimal stopping is optimal.

It is intuitive that dynamic programming for the constrained optimisation and optimal stopping for optimal regime switching yield the same solution to this investment problem, especially the adoptation of second order smooth pasting condition at the boundary. In the former case, the constrained optimisation is treated as a single problem which yields a value function that is twice continuously differentiable. However, adopting the optimal stopping explicitly divides the problem into two distinct regimes based on whether demand constraint binds or not. This approach characterises the anticipated stochastic regime switching as a boundary value problem, where, apart from the Bellman equations derived in each regime, boundary conditions are natural consequences of the global optimality.

## 4.3 The Response of Investment to the Change of $\sigma^2$.

To consider the response of the optimal investment to the change of future uncertainty with respect to the capital depreciation, for simplicity, we assume that the switching is endogenous and the demand constraint is included. The optimal behaviour of the firm will provide the optimal investment policies in both regime and of course for the anticipated switching, this induces that the value function is twice continuously differentiable in the whole regime which implies the second order smooth pasting condition given by equation (4.2.18). the adjustment costs of investment are a quadratic function of the investment. Here, we only approximate the Bellman equation in the unconstrained regime, since the Bellman equation in the constrained regime has the same homogeneous part and the same approximation scheme applies.

Before approximation, we first simplify the Bellman equation in the unconstrained regime. From equation (4.2.14), optimising over $L$ yields

$$L^* = \left[\frac{w}{(1-\alpha)p}\right]^{-\frac{1}{\alpha}} k, \tag{4.3.1}$$

$$pF(k, L^*) - wL^* = \frac{\alpha w}{1-\alpha}\left[\frac{w}{(1-\alpha)p}\right]^{-\frac{1}{\alpha}} k = \zeta k, \tag{4.3.2}$$

where

$$\zeta = \frac{\alpha w}{1-\alpha}\left[\frac{w}{(1-\alpha)p}\right]^{-\frac{1}{\alpha}}.$$

Optimising over $I$ yields

$$V_k(k) = 1 + C'(I). \tag{4.3.3}$$

If

$$C(I) = \frac{1}{2}\beta I^2, \tag{4.3.4}$$

then

$$I^* = \frac{V'(k) - 1}{\beta}. \tag{4.3.5}$$

Substituting (4.3.2), (4.3.4) and (4.3.5) into equation (4.2.14) yields

$$rV(k) = \frac{1}{2}\sigma^2 k^2 V_{kk}(k) - \delta k V_k(k) + \frac{1}{2}\frac{(V_k(k) - 1)^2}{\beta} + \zeta k. \tag{4.3.6}$$

Applying boundary conditions to (4.3.6), one can solve for $V(k)$ uniquely in the unconstrained regime. In what follows, we approximate this solution by using Taylor series expansion around $k = 0$. And by rescaling the approximation, the convergent interval can be extended to $[0, \hat{k}]$.

### 4.3.1 The Approximation Scheme

Considering switching cases where sales constraint starts binding for $k \geq \bar{k}$, we seek the solution to (4.3.6) satisfying joining conditions at $k = \bar{k}$ and the boundary condition when $k \to 0$. However, it may be easy to begin with an initial condition at $k = 0$, namely,

$$\lim_{k \to 0} V(k) = V(0) > 0. \tag{4.3.7}$$

Such property is not difficult to verify. Using (4.2.14) and letting $k \to 0$, one obtains

$$rV(0) = IV_k(0) - (I + C(I)),$$

applying (4.3.4) leads to

$$rV(0) = IC'(I) - C,$$

using (4.3.5), one finally obtains

$$rV(0) = \frac{1}{2}\beta I^2 \geq 0,$$

where $I$ will be determined by the boundary conditions at $k = \hat{k}$ and transversality condition at $k \to \infty$, and the strict inequality holds for $I > 0$.

Since $V(k)$ is bounded within $[0, \hat{k}]$, we can expand $V(k)$ by a Taylor series,

$$V(k) = \sum_{i=0}^{\infty} c_i k^i. \tag{4.3.8}$$

Then

$$\begin{aligned} V_k(k) &= \sum_{i=0}^{\infty} i c_i k^{i-1}, \\ &= \sum_{i=0}^{\infty} (i+1) c_{i+1} k^i, \end{aligned}$$

so

$$kV_k(k) = \sum_{i=0}^{\infty} i c_i k^i. \tag{4.3.9}$$

For the second order derivative, we have

$$V_{kk}(k) = \sum_{i=0}^{\infty} i(i-1)c_i k^{i-2},$$

then

$$k^2 V_{kk}(k) = \sum_{i=0}^{\infty} i(i-1)c_i k^i. \qquad (4.3.10)$$

For the non-linear term

$$\begin{aligned}(V_k(k))^2 &= \Big(\sum_{i=0}^{\infty}(i+1)c_{i+1}k^i\Big)^2 \\ &= \sum_{i=0}^{\infty}\Big(\sum_{l=0}^{i}(l+1)c_{l+1}(i-l+1)c_{i-l+1}\Big)k^i. \qquad (4.3.11)\end{aligned}$$

Substituting equations (4.3.9)–(4.3.11) into (4.3.6) yields

$$\begin{aligned} r\sum_{i=0}^{\infty} c_i k^i &= \frac{1}{2}\sigma^2 \sum_{i=0}^{\infty} i(i-1)c_i k^i - \delta \sum_{i=0}^{\infty} i c_i k^i \\ &+ \frac{1}{2\beta}\Big\{\sum_{i=0}^{\infty}\Big(\sum_{l=0}^{i}(l+1)c_{l+1}(i-l+1)c_{i-l+1}\Big)k^i \\ &- 2\sum_{i=0}^{\infty}(i+1)C_{i+1}k^i + 1\Big\} + \zeta k. \qquad (4.3.12)\end{aligned}$$

If $i = 0$, then

$$c_1 = 1 \pm \sqrt{2\beta r c_0}. \qquad (4.3.13)$$

where $c_0 > 0$ is ensured by the equation (4.3.7).

If $i = 1$, then

$$c_2 = \frac{2\beta[(r+\delta)c_1 - \zeta] - 1}{4(c_1 - 1)}. \qquad (4.3.14)$$

If $i = 2$, then

$$c_3 = \left[\frac{r+\delta+\zeta}{c_1 - 1} - \sigma^2\right]\frac{\beta c_2}{3(c_1-1)}. \tag{4.3.15}$$

For $i > 2$, we have

$$\begin{aligned} c_{i+1} = & \frac{1}{2(i+1)(c_1-1)}\{[r + i\delta - \frac{1}{2}i(i-1)\sigma^2]c_i \\ & - \frac{1}{2\beta}\sum_{l=0}^{i-2}(l+2)c_{l+2}(i-l)c_{i-l}\}. \end{aligned} \tag{4.3.16}$$

The stable manifold for the constrained case is downward sloping and asymptotically tends to $O$ when $k \to \infty$ Then the transversality condition of the constrained value function demands that the first order derivative, if $k \to \infty$, satisfies

$$\lim_{k\to\infty}\frac{dV^c(k)}{dk} = \mu_s^c(k). \tag{4.3.17}$$

By the joining conditions at $\dot{k}$, the overall first derivative of the value function is downward sloping, this demands

$$\frac{dV(k)}{dk} < 0. \tag{4.3.18}$$

By its first order approximation we have

$$c_1 < 0. \tag{4.3.19}$$

Also, the value function is locally concave, by its second order approximation, then

$$c_2 < 0. \tag{4.3.20}$$

Notice that the coefficients of the first three terms of Taylor series expansion do not depend on $\sigma^2$, the lowest order approximation of the unconstrained value function (which depends on $\sigma^2$) is the fourth term, namely

$$V^u(k,\sigma^2) \sim \left[\frac{r+\delta+\zeta}{c_1-1} - \sigma^2\right] \frac{\beta c_2}{3(c_1-1)} k^3, \tag{4.3.21}$$

and

$$\frac{d}{dk} V^u(k,\sigma^2) \sim \left[\frac{r+\delta+\zeta}{c_1-1} - \sigma^2\right] \frac{\beta c_2}{3(c_1-1)} k^2. \tag{4.3.22}$$

### 4.3.2 Response of Value Function to a Change of $\sigma^2$

Using the joining conditions, the value function for the switching case can be approximated by $V^u(k)$ at the point where $k$ is small, then

$$\frac{\partial}{\partial\sigma^2} V^u(k,\sigma^2) \simeq -\frac{\beta c_2}{3(c_1-1)} k^3 < 0, \tag{4.3.23}$$

$$\frac{\partial}{\partial\sigma^2}\frac{d}{dk} V^u(k,\sigma^2) \simeq -\frac{\beta c_2}{3(c_1-1)} k^2 < 0. \tag{4.3.24}$$

Notice that the level of investment depends on $V_k(\cdot)$, increase the variance of stochastic capital depreciation will decrease the value function and investment of the firm.

Rescaling the Taylor expansion

$$V(k) = \sum_{i=0}^{\infty} c_i \frac{k^i}{\bar{k}}, \qquad k < \bar{k}, \tag{4.3.25}$$

for very large $\bar{k}$ will ensure the convergence of the series in the interval

$[0, \hat{k}]$, but will not change the results qualitatively.

The anticipated stochastic regime switching solution can be shown in Figure 4.2. The stochastic solution $RR$ $(V_k(k))$ is lower than its deterministic counterpart. The smoothness conditions for both cases are satisfied at $\hat{k}$, and the stochastic solution $RR$ will asymptotically tend to both $A'D$ and $S^cS^c$ due to the transversality conditions.

## 4.4 Stochastic Smoothing

Apart from the property that stochastic capital depreciation reduces investment relative to the deterministic case, another main characteristic of the stochastic regime switching case is that the investment appears to be smoother than its deterministic counterpart. Specifically, in the deterministic case, the first order derivative of investment with respect to capital is continuous at the switching boundary, but its second order derivative is discontinuous. In the stochastic case, both first and second order derivatives of investment with respect to capital are continuous. In terms of value functions, the stochastic regime switching has one more higher contact condition, namely, the third order smooth pasting condition at the switching boundary.

Such behaviour is mainly caused by the reason that in the absence of uncertainty, the firm which is initially unconstrained will keep expanding its size along the optimal trajectory leading towards the equilibrium. Since the future conditions can be exactly forecasted, the switching is essentially irreversible. However, convergence in a unique direction can no longer be achieved if there is an unanticipated change in the capital in place. After the firm just enters the constrained regime, a sudden jump in the capital depreciation can pull the firm back into the unconstrained regime. So the switching is *reversible*. Such reversibility of switching generally requires a higher order contact condition which will be specified in what follows.

Since the value functions for both deterministic and stochastic cases are twice continuously differentiable, it is straight forward to derive that investment is continuous at the switching boundary for both cases (because first order smooth pasting of value function is satisfied). Furthermore, using equations (C.3.16) and (C.3.17) in Appendix C, one obtains

$$V_{kk}(k) = C''(I)I'(k). \tag{4.4.1}$$

Because $V_{kk}(k)$ and $I(k)$ are continuous at the switching boundary, so is $I'(k)$.

To see that $I''(k)$ is discontinuous at the boundary in deterministic case, we first differentiate the Bellman equation (C.2.7) in Appendix C with respect to $k$, which yields

$$(I - \delta k)V_{kk}(k) = (r + \delta)V_k(k) - \frac{\partial}{\partial k}(pQ - wL), \tag{4.4.2}$$

where $Q < \bar{Q}$ indicates the unconstrained regime and $Q = \bar{Q}$ denotes the constrained regime.

Differentiating equation (4.4.2) once more with respect to $k$ yields

$$(I - \delta k)V_{kkk}(k) = (r + 2\delta - I'(k))V_{kk}(k) - \frac{\partial^2}{\partial k^2}(pQ - wL). \tag{4.4.3}$$

In the unconstrained regime, from equation (4.3.3), one derives

$$\frac{\partial^2}{\partial k^2}(pQ - wL) = \frac{\partial^2}{\partial k^2}(\zeta k) = 0. \tag{4.4.4}$$

In the constrained regime, using Cobb-Douglas production function, we have

$$L = (k^{-\alpha}\bar{Q})^{\frac{1}{1-\alpha}}, \tag{4.4.5}$$

then

$$\frac{\partial^2}{\partial k^2}(p\bar{Q} - wL) = -\frac{\alpha}{(1-\alpha)^2}w(\bar{Q})^{\frac{1}{1-\alpha}}k^{-\frac{1}{1-\alpha}-1} < 0. \tag{4.4.6}$$

At the switching boundary, since $I$, $I'(k)$, and $V_{kk}(k)$ are continuous but not $\frac{\partial^2}{\partial k^2}(pQ - wL)$, then $V_{kkk}$ is discontinuous at $k = \hat{k}$.

Differentiating equation (4.4.1) with respect to $k$ and assuming quadratic adjustment function (equation (4.3.5)), one can show

$$V_{kkk}(k) = \beta I''(k). \tag{4.4.7}$$

Therefore, $I''(k)$ is discontinuous at the switching boundary in the deterministic case.

However, such behaviour is altered in the stochastic regime switching case. Differentiating equations (4.2.14) and (4.2.15) with respect to $k$ yields

$$\frac{1}{2}\sigma^2 V_{kkk}(k) = (r+\delta)V_k(k) - (I + \sigma^2 k - \delta k)V_{kk}(k) + \frac{\partial}{\partial k}(pQ - wL), \tag{4.4.8}$$

where $Q < \bar{Q}$ and $Q = \bar{Q}$ indicate unconstrained and constrained regimes respectively.

Adopting Cobb-Douglas production function, and using the optimality condition for $L$ leads to

$$\frac{\partial}{\partial k}(pQ - wL) = w\frac{F_K}{F_L} \tag{4.4.9}$$

where

$$F_L = \frac{w}{p - \lambda}, \quad F(K, L) = Q \tag{4.4.10}$$

and $\lambda = 0$ indicates the unconstrained regime.

Since $V_k(k)$, $V_{kk}(k)$ and $I$ are continuous at the switching boundary, and furthermore, from Proposition C.3 in Appendix C, $\lambda$ is continuous, then from equation (4.4.8), $V_{kkk}(k)$ is continuous. Using equation (4.4.7), we conclude that $I''(k)$ is continuous at the switching boundary in the stochastic case.

In the deterministic regime switching case, the optimal investment in the demand unconstrained regime is reduced relative to the case where no demand constraint will ever be in place (see Appendix C). Such reduction in the level of investment is primarily due to the anticipation of demand constraint which will eventually be in place. As the optimal investment policy in the unconstrained regime generates a unique direction of convergence to constrained regime, the value function has only to be twice continuously differentiable to satisfy the global optimality. From equation (4.4.1), the concavity of value function suggests that there would be a reduction in investment in the unconstrained regime because of rational expectations of the binding sales constraint.

In the stochastic regime switching case, as the path from unconstrained regime to the constrained does not have unique direction, the second order smooth pasting condition of the value function alone cannot achieve global optimality. The consequence of rationally anticipating the binding sales constraint leads to the third order smooth pasting in the value function, so, from (4.4.7), investment function tends to be smoother than that in the deterministic case. As from (4.3.5), the optimal investment depends on marginal value of capital ($V_k(k)$), the response of investment to a sudden change in capital depends on the third order derivative of value function with respect to $k$ (because Brownian motion has a second order effect). Using (4.3.21), we can show

$$V^u_{kkk}(k,\sigma^2) \sim \left[\frac{r+\delta+\zeta}{c_1-1} - \sigma^2\right] \frac{\beta c_2}{3(c_1-1)} < 0.$$

Therefore, the precautionary investment is related to the concavity of $V_k(k)$.

## 4.5 Conclusion

In this chapter, a stochastic version of Precious's model (1985) is developed to treat the optimal investment under sales constraints. The anticipated switch of regimes due to explicitly imposed sales constraints can be clearly characterised by using the optimal stopping technique.

The introduction of the uncertainty on the capital depreciation reduces firm's demand for investment. Such investment is reduced further if the variability of the capital depreciation increases.

Unlike in the deterministic case, the anticipated stochastic regime switching is reversible, global optimality of the problem requires a higher contact condition than that in the absence of uncertainty. Specifically, the value function smooth pasts up to the third order and optimal investment smooth pastes up to its second order at the switching boundary. As optimal investment linearly depends on the marginal value of capital ($V_k(k)$), the precautionary investment is caused by the concavity in $V_k(k)$.

To capture more realistic investment behaviour in a intertemporal setting, however, this model needs to incorporate more uncertainty factors, namely uncertainty in the output prices and demand shift. And the resolution of such model may involve the use of numerical techniques.

## 4.6 FIGURES

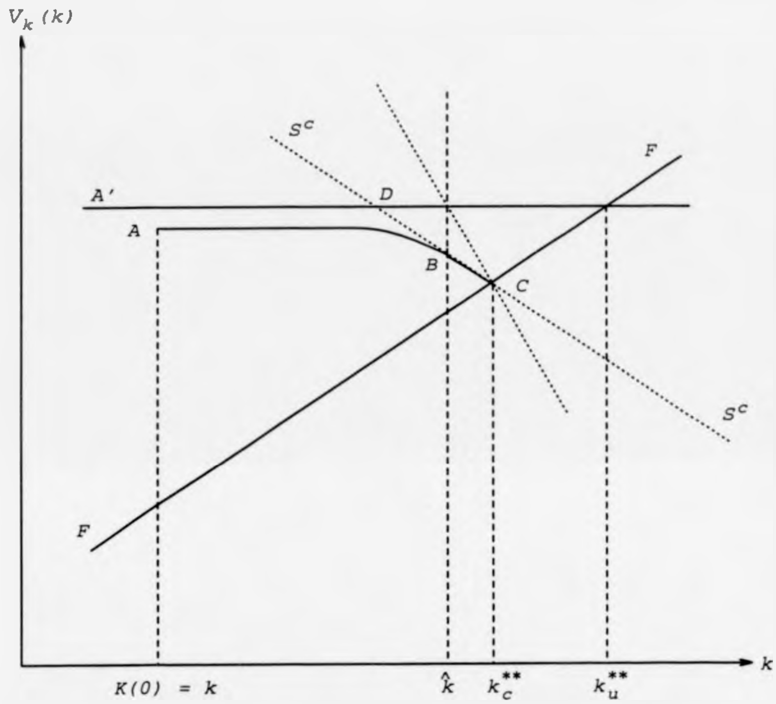



Figure 4.1: Anticipated Deterministic Regime Switching.

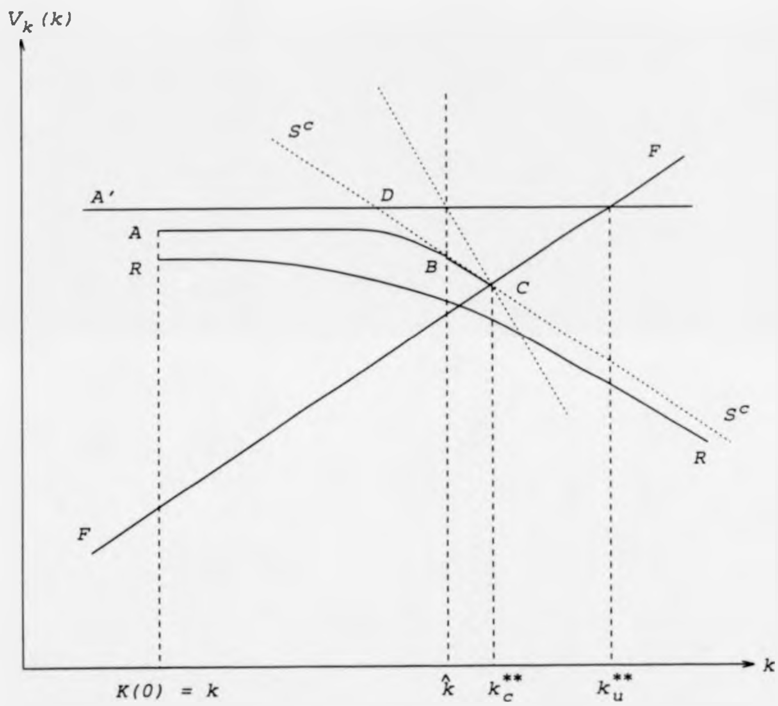



Figure 4.2: Anticipated Stochastic Regime Switching.

# Chapter 5

# Continuous Time Optimal Consumption-Portfolio Decisions with Shortselling and Borrowing Constraints (I)

## 5.1 Introduction

Modern consumption theory generally treats households' consumption and investment decisions (portfolio policies) simultaneously. Such treatment recognises the dual roles of the households in making their financial decisions. As a consumer, the household chooses how much of its income and wealth to allocate to current consumption. As an investor, the household solves the portfolio-selection problem to determine the fractional allocations of its saving among the available investment opportunities. In general, the optimal consumption-investment decisions cannot be made independently of each other.

The approach to the analysis of optimal intertemporal consumption and portfolio policies is stochastic dynamic programming, the use of which in this context was pioneered by Mossin (1968), Samuelson (1969), and Hakansson (1970) in discrete time and Merton (1969, 1971) in continuous time.

By applying continuous-time stochastic control, Merton solved the basic two-asset model assuming a risky asset with log-normally distributed return and a riskless asset with a constant interest rate. (Discrete time versions can be found earlier, see, for example, Mossin (1968) and Samuelson (1969).) He derived explicit optimal consumption and portfolio rules for households with preferences that exhibit either constant relative risk aversion or constant absolute risk aversion. The intertemporal age-dependent behavior of optimal consumption is shown to be consistent with the Modigliani-Brumberg life-cycle hypothesis (for life-cycle hypothesis, see Ando and Modigliani 1963, Modigliani and Brumberg 1954). The derived optimal portfolio rules have the same structure as those prescribed in the Markowitz-Tobin mean-variance model.

By expanding the model to the many assets case, Merton (1971, 1973) showed that if the dynamics of the asset prices satisfy a joint lognormal distribution, the derived structure of each household's optimal demands for assets is such that all optimal portfolios can be generated by simple combinations of just two portfolios—a riskless bond and a risky asset. This mutual fund theorem is identical in form with the well-known separation theorem of static mean-variance model. Closed-form solutions can be obtained for the HARA (Hyperbolic Absolute Risk Aversion) family of utility functions. It is further shown that they are the only time-additive and time-independent preference orderings that lead to optimal consumptions which are *linear* in wealth.

In Merton's models, an assumption of $U'(0) = \infty$ was made to prevent the consumption from going negative[1]. This restriction was removed by Karatzas, Lehoczky, Sethi and Shreve (1986). They provided the optimal single-agent consumption and investment policies and value functions for wealth for arbitrary, smooth, concave utility functions of consumption which are assumed only to satisfy conditions required for the finiteness of value functions. They carefully treated the consumption constraint $c \geq 0$ and addressed the possibility of bankruptcy, i.e., when the wealth becomes zero. Stock prices were again modelled by constant coefficient geometric Brownian motion processes, and explicit optimal consumption and investment formulas were obtained. But shortselling and borrowing constraints are not considered. In a paper by Lehoczky, Sethi and Shreve (1983), shortselling and borrowing constraints were introduced for a two-asset model where the mean rate of return on the stock is equal to the interest rate. The solutions shown that the borrowing constraint is only active when wealth is below some threshold.

However, this does not indicate that the use of stochastic dynamic programming is necessary in treating such type of problems. Pliska (1982, 1986), Cox and Huang (1987a, 1987b), and Pagès (1987) have adopted a

[1]Some of the utility functions in HARA family do not satisfy the condition $U'(0) = \infty$.

martingale representation technology instead of dynamic programming to study optimal intertemporal consumption and portfolio policies, while Chamberlain (1988), Duffie and Huang (1985), and Huang (1987) have used it in a general equilibrium setting.

In the martingale approach one solves the consumption and portfolio problem be separating it into two parts. First, one transforms the dynamic consumption-portfolio problem into a static variational problem and solves the static problem to find the optimal consumption bundle. Then one applies the martingale representation theorem to determine the portfolio trading strategy needed to generate the optimal consumption bundle (for a summary, see Karatzas *et al*, 1988). Using this technique, Cox and Huang (1987a) found a unique system of Arrow-Debreu state prices (or, after normalisation by the bond price, a unique equivalent martingale measure) that is consistent with the absence of arbitrage, and portfolio policies contingent on such prices and the optimal consumption bundle generated turn out to be identical to that given by dynamic programming.

The extension along this line can be found in He and Pearson (1988, 1989) who considered the intertemporal consumption and portfolio policies in continuous time economies when markets are dynamically incomplete (a market in which not all contingent claims can be created by dynamic trading in the existing securities). They found that there are infinite many Arrow-Debreu state prices (or equivalent martingale measures) that are consistent with the absence of arbitrage. The optimal portfolio policies are formed on a minimax local martingale measure such that the household's demands for contingent claims that are not marketed are zero.

Another extension to the original consumption-portfolio model can be found in Eastham and Hasting (1988) who considered the portfolio policies when transaction costs are involved every time the portfolio is re-balanced. They shown that optimal portfolio policies are piecewise constant if transaction costs have a fixed component. Similar work can also be found in Dumas and Luciano (1991), Grossman and Laroque (1990), Davies and Norman (1990) and Svensson and Werner (1993).

All the models discussed above have a common feature, that is, the income is generated solely by the returns on risky and riskless assets, or, at least, the risks associated with the future labour income can be fully diversified to (or hedged by) the risks associated with the returns on risky and riskless assets. By this formulation, the attention was somewhat focussed on the portfolio policies instead of the characterisation of the consumption behaviour.

Other types models (called pure consumption models for future references) which focus mainly on characterising consumption behaviour (but generally ignore the portfolio policies) can be found in Flavin (1981), Campbell and Mankiw (1981, 1991), Deaton (1987, 1991), Gali (1990) and Zeldes (1987a). They all assumed that risks associated with the future labour income are not diversifiable. The main results shown that if households can borrow against their future income, the consumption process tends to be less volatile than the income process (consumption smoothing as in the permanent Income Hypothesis); if the borrowing is restricted, consumption process follows more closely to that of income, i.e., it becomes more volatile. Furthermore, they shown that the precautionary motive of saving is positively related to the variability of the income process. The higher the variance in future income, the higher the precautionary saving.

How these results can be extended to the consumption behaviour generated by the consumption-portfolio model is not obvious. In the pure consumption models, income process is predetermined while that in the consumption-portfolio model it is endogenous. As introducing borrowing constraint to the consumption-portfolio model may reduce the variability of income itself, the above results cannot be made directly applicable. In the following two chapters, we adopt the standard two-asset consumption-portfolio model to investigate the effects of imposing borrowing and short-selling constraints on the consumption behaviour. This chapter formulates a two-asset model under borrowing and short-selling constraints, and provides a general framework to treat the regime switching problem induced by the borrowing restriction. Next chapter presents the optimal consumption and portfolio rules for both constant relative risk aversion and constant absolute risk aversion utility functions, and it is shown that the precautionary saving is a consequence of the imposed borrowing constraint.

The plan of this chapter is as follows: section 2 formulates the asset price and wealth processes when the risks of future labour income can be fully diversified. In section 3, borrowing and short-selling constraints are specified as the restrictions on the portfolio policies. Section 4 formally sets up a two-asset consumption-portfolio model of households, where (optimality conditions (Bellman equation and related boundary conditions) are given at the end of the section. Section 5 constructs a regime switching framework which is equivalent to the optimality conditions given in section 4, but can be used to treat the constrained consumption-portfolio decisions effectively. Section 6 concludes this chapter.

## 5.2 Asset Price and Wealth Dynamics

The introduction of the continuous type of stochastic differential equations for asset prices demands two economic assumptions, namely, that trading is continuous and that the underlying stochastic variables follow diffusion-type motion with a continuous sample path (thus excluding the Poisson-directed processes) (Merton, 1990). The first assumption can be justified by the assumption that the length of time between two revisions are short or it is short compared to the change of other factors (e.g. the change of capital stock etc.). However, the observation of the data seems to be not consistent with the assumption that the prices follow geometric Brownian motion, because it is too peaked to be consistent with the Gaussian distribution. To resolve the problem, Mandelbrot (1963a, b) and Fama (1963, 1965) maintain the independent increments and stationarity assumptions but replace the Gaussian assumption with a more general stable Pareto-Levy distribution assumption. Even though it is better able to fit the fat tail, there is little empirical evidence to support the adoption of this stable Paretian hypothesis. Furthermore, Cootner (1964) suggests that the infinite variance property of the non-Gaussian stable distribution implies that most of the statistical tools based on the finite-moment assumptions are useless. The alternative path is to consider finite-moment processes with non-stationary distribution (Cootner, 1964). This approach makes continuous-time analysis promising.

The general continuous-time framework, which requires that the underlying process be a mixture of diffusion and Poisson processes, can accommodate a wide range of specific hypotheses including the 'reflecting barrier' model (Cootner, 1964). Rosenberg (1972) pointed out that a Gaussian model with a changing variance rate appears to 'explain' the observed fat-tail characteristics of stock-market returns. A number of papers have contributed to estimating and testing the parameters of these continuous-time processes. For example, Rosenfeld (1980) has developed statistical techniques for estimating the parameters of continuous-time processes and has applied them in constructing a likelihood test for choosing between a diffusion process with a changing variance rate and a mixed diffusion and Poisson process. As discussed by Merton (1976, 1980), if the parameters are slowly varying functions of time, then it is possible to exploit the different 'time scale' of the component parts of continuous-time processes to identify and estimate these parameters.

As Merton (1990) suggests, considerably more research is required before a judgement can be made as to the success of the approach (i.e., the hypothesis that stochastic returns on assets are mixed diffusion and Poisson processes). However, the extensive mathematical literature on the distributional characteristics of these processes together with their finite-moment properties make the development of hypothesis tests considerably easier for these processes than for the stable Pareto-Levy processes.

As in Merton (1990), with the assumption of continuous trading and some very mild regularity conditions (cf Merton, p.62, 1990), the stochastic processes of asset returns can be generally divided into three types: namely, continuous-sample-path without 'rare events', continuous-sample-path with 'rare events' and discontinuous-sample-path with 'rare events' (Merton, chap.3, 1990). The first two types or the combination of the first two types can be represented by diffusion processes, and the third type is a Poisson-directed process. Therefore without consideration of the 'rare events', one can describe the asset returns using diffusion processes.

To derive the wealth process for a small investor whose behaviour cannot affect the market structure, we first specify the price processes for both bond and stock. Assuming that the interest rate of bond, which pays no coupon, is a constant. The price of bond is given as

$$\frac{dP_0(t)}{P_0(t)} = rdt, \qquad P_0(0) = p_0 \tag{5.2.1}$$

where $P_0(t)$ is the price of bond at time $t$, $p_0$ is its initial price, $r > 0$ is a constant interest rate.

Following Merton (1969, 1971, 1973), Black and Scholes (1973), the non-dividend-paying stock price is modelled by a geometric Brownian motion with constant coefficients, namely,[2]

$$\frac{dP_1(t)}{P_1(t)} = \alpha dt + \sigma dW_t, \qquad P_1(0) = p_1. \tag{5.2.2}$$

where $P_1(t)$ is the price for stock at time $t$, $\alpha$ is its mean rate of return, $\sigma$ is its instantaneous standard deviation, $p_1$ is its initial price and $W_t$ is a standard Brownian motion.

[2]For general formulation of security price processes, see Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), Pliska (1986) and Chamberlain (1988).

Let $\{c(t), t \geq 0\}$ be the consumption rate process, $\{n_i(t), t \geq 0\}$ be the processes for the number of shares held in asset $i$, $i = 0, 1$; whose price $P_i(t)$ is given by equations (5.2.1) and (5.2.2). To derive the wealth process, we first describe it in a discrete representation and then convert it to a continuous version by taking the limits. Assume all the decisions for choosing consumption rate and investment rules are made at the beginning of a particular period, say, $t - h, t$ and $t + h$, where $h$ is a very small interval. Then at the end of period $t - h$, the financial wealth of the agent is

$$x(t) = \sum_{i=0}^{1} n_i(t-h)P_i(t). \tag{5.2.3}$$

Here, we assume that all the income is generated from the return on the bond and/or the stock: the case with non-diversifiable labour income is not considered. If labour income is fully diversifiable, we can, without loss of generality, focus on the case where there is no labour income and all income is derived from traded wealth (Blanchard and Fischer (1989), chapter 6)[3]. At the beginning of period $t$, the agent decides the consumption rate and the new portfolio for the next period, namely

$$-c(t)h = \sum_{i=0}^{1} [n_i(t) - n_i(t-h)]P_i(t), \tag{5.2.4}$$

[3]The assumption that labour income is fully diversifiable is made under technical consideration, otherwise, the problem is not analytically tractable (see Zeldes 1989a). For non-diversifiable labour income cases see Flavin (1981), Campbell and Mankiw (1989, 1991), Deaton (1987, 1991) and Gali (1990).

Incrementing (5.2.3) and (5.2.4) by $h$, we obtain

$$\begin{aligned}-c(t+h)h &= \sum_{i=0}^{1}[n_i(t+h)-n_i(t)]P_i(t+h)\\ &= \sum_{i=0}^{1}[n_i(t+h)-n_i(t)][P_i(t+h)-P_i(t)]\\ &\quad+\sum_{i=0}^{1}[n_i(t+h)-n_i(t)]P_i(t).\end{aligned} \tag{5.2.5}$$

and

$$x(t+h)=\sum_{i=0}^{1}n_i(t)P_i(t+h). \tag{5.2.6}$$

Taking the limits as $h \to 0$, we have the continuous versions of equations (5.2.5) and (5.2.6),

$$-c(t)dt=\sum_{i=0}^{1}dn_i(t)dP_i(t)+\sum_{i=0}^{1}dn_i(t)P_i(t), \tag{5.2.7}$$

and

$$x(t)=\sum_{i=0}^{1}n_i(t)P_i(t). \tag{5.2.8}$$

Differentiate equation (5.2.8) using Itô's lemma,

$$dx(t)=\sum_{i=0}^{1}n_i(t)dP_i(t)+\sum_{i=0}^{1}dn_i(t)P_i(t)+\sum_{i=0}^{1}dn_i(t)dP_i(t), \tag{5.2.9}$$

substitute back into equation (5.2.7),

$$dx(t)=\sum_{i=0}^{1}n_i(t)dP_i(t)-c(t)dt. \tag{5.2.10}$$

Define the ratio of investment made in the stock to wealth to be

$\pi(t)$, namely

$$\pi(t) \equiv \frac{n_1(t)P_1(t)}{x(t)}, \tag{5.2.11}$$

so the fraction of the wealth invested in the bond is

$$\pi_0(t) = 1 - \pi(t). \tag{5.2.12}$$

Substitute equations (5.2.1), (5.2.2), (5.2.11) and (5.2.12) into (5.2.10), we obtain the wealth process

$$dx(t) = (\alpha - r)\pi(t)x(t)dt + (rx(t) - c(t))dt + \pi(t)x(t)\sigma dW_t \tag{5.2.13}$$
$$x(0) = x.$$

The interpretation of this equation is clear: the change of the wealth in an interval $dt$ is equal to the return on the bond $(1 - \pi(t))rx(t)dt$ *plus* the return on the stock $\alpha\pi(t)x(t)dt + \pi(t)x(t)\sigma dW_t$ *less* consumption $c(t)dt$.

## 5.3 Constraints

Apart from some physical implausibility conditions (i.e., non-negativity conditions on wealth and consumption), the liquidity constraints are normally the main constraints which cause concern. A number of papers have shown that liquidity constraints can have important effects on individual consumption behaviour, and on the behaviour of aggregate consumption, output and asset returns. Pissarides (1978), by assuming different liquidities of assets, suggested a different result from the standard wealth theory. Empirically, Zeldes (1989) tests the permanent income hypothesis against the alternative hypothesis that consumers optimise expected utility function subject to a well-specified sequence of borrowing constraints. The results generally support the hypothesis that an inability to borrow against future labour income affects the consumption of a significant portion of the population.

Various forms of liquidity constraints have been examined in the literature, each of which involves some price or quantity restrictions on the holding of assets. Above all, two types of quantity constraints are basically considered, the borrowing and shortselling constraints.

In Zeldes (1989), the borrowing constraint gives a lower bound which the current wealth level should lie above. This constraint prohibits an individual from consuming today the proceeds from supplying labour in the future. On the other hand, most consumers are able to borrow to purchase assets (e.g. mortgages or stock on margin). However, it seems a reasonable hypothesis that consumers cannot borrow, on net, against nontraded assets such as future labour income, in other words, that debt cannot exceed the total value of traded assets. The implications for the wealth equation (5.2.13) are that the wealth must always be nonnegative:

$$x(t) \geq 0 \tag{5.3.1}$$

and the fraction of the total wealth invested in the bond must be bounded by a finite non-positive number

$$1 - \pi(t) \geq \xi', \quad \text{where} \quad 0 \geq \xi' > -\infty; \tag{5.3.2}$$

or

$$\pi(t) \leq 1 - \xi' = \xi, \quad \text{where} \;\; \xi \geq 1. \tag{5.3.3}$$

If $\xi = 1$, the agent can invest all its wealth in the stock but cannot borrow at a riskless rate, so his investment in bonds is always nonnegative; if $1 < \xi < \infty$, the borrowing at a riskless rate is limited and the bigger the number $\xi$, the more the agent can borrow.

Likewise, the shortselling constraint can be specified as

$$\pi(t) \geq 0 \tag{5.3.4}$$

that is simply saying that the agent cannot short sell stock.

Apart from these constraints, we further demand that the consumption must be nonnegative

$$c(t) \geq 0, \quad \text{for} \;\; t \geq 0. \tag{5.3.5}$$

These restrictions will form all the constraints considered in the following chapters.

$x = 0$ is an absorbing state according to equation (5.2.13), i.e., if $x(0) = 0$ then $x(t) = 0$ for $t \geq 0$; and for $x(0) > 0$, we have $x(t) \geq 0$. Therefore, this constraint has already been embedded in equation (5.2.13) provided that $x(0) \geq 0$. In what follows, we only consider the other three constraints, namely, (5.3.3), (5.3.4) and (5.3.5).

## 5.4 A Two-asset Model

In this section, we first specify the value function of the agent given its smooth utility function, then provide the conditions for admissible optimal consumption and investment rules, and finally give the sufficient condition (Bellman equation) for the optimality of these policies.

### 5.4.1 Value function

Consider a small investor whose utility function $U(c)$ is smooth on $(0, \infty)$. Specifically, we shall choose two types of utility functions which are commonly used: the constant relative risk aversion utility $U(c) = c^{1-\eta}/(1-\eta)$, and the constant absolute risk aversion utility $U(c) = e^{-\eta c}/(-\eta)$, where $\eta > 0$. The utility function is extended to $[0, \infty)$ by defining

$$U(0) = \lim_{c \downarrow 0} U(c), \qquad U'(0) = \lim_{c \downarrow 0} U'(c),$$

where the limits can be $\pm\infty$.

We define the state where wealth reaches zero as *bankruptcy*, and assign a value $P$ to it. The first random time to reach this state is given by

$$T_b = \inf\{t \geq 0 : x(t) = 0\}, \tag{5.4.1}$$

and, if $T_b < \infty$, then the agent receives "value" $P$ at time $T_b$, and the decision problem terminates (see Karatzas *et al* 1986, Lehoczky *et al* 1983).

For the utility functions specified earlier, the investor has to choose $\{c(t), t \geq 0\}$ and $\{\pi(t), t \geq 0\}$ so as to maximise the present discounted value of expected utility up to the bankruptcy time

$$V_{c(t),\pi(t)}(x) \equiv E_x[\int_0^{T_b} e^{-\beta t}U(c(t))dt + Pe^{-\beta T_b}]. \tag{5.4.2}$$

$V_{c(.),\pi(.)}(x)$ is the value function of the agent for given consumption and investment strategies, $\beta > 0$ is a constant discount factor, $E_x$ is an expectation operator conditional on the initial wealth $x$ given in equation (5.2.13). Notice that $P = U(0)/\beta$ is the natural bankruptcy value which is equivalent to continuing the problem indefinitely after the wealth reaches zero but allowing only zero consumption (Karatzas *et al*, 1986).

Not all the numbers assigned to $P$ are meaningful. As suggested by Karatzas *et al* (1986), the model is interesting only when

$$\frac{1}{\beta}U(0) \leq P < \frac{1}{\beta}\lim_{c\to\infty} U(c). \tag{5.4.3}$$

Because if $P \geq \beta^{-1}\lim_{c\to\infty} U(c)$, one should consume to bankruptcy instantly since choosing any other policies always produces lower value function, so the value function is identical to $P$. There is no optimal policy since instantaneous bankruptcy cannot be achieved. On the other hand, if $P < U(0)/\beta$, one behaves as if $P$ were the natural "value" $U(0)/\beta$[4]. We shall stick to this parameterisation of $P$ throughout this chapter and the next.

For this specified range of bankruptcy value $P$, the optimal value

[4]It is evident that if $P < U(0)/\beta$, when wealth reaches zero, the agent is always better off by simply consuming nothing and continue staying at that state indefinitely (which yields natural value $U(0)/\beta$), rather than going bankrupt (which yields a value $P < U(0)/\beta$).

function is defined as

$$V^*(x) \equiv \sup_{c(t)\geq 0, 0\leq \pi(t)\leq \xi} V_{c(t),\pi(t)}(x), \quad x \geq 0, \tag{5.4.4}$$

where $\xi \geq 1$, and from the previous subsection, we notice consumption is nonnegative and shortselling and borrowing constraints are satisfied.

Since the case for $r > 0, \alpha = r$ has been considered in Lehoczky *et al* (1983), we shall only investigate the remaining cases where $\alpha \neq r$. Before proceeding, we define some constants for future reference. First, let

$$\gamma = \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2},$$

where $\sigma \neq 0$.

Second, we define two roots in the following quadratic equation

$$\gamma\lambda^2 - (r - \beta - \gamma)\lambda - r = 0. \tag{5.4.5}$$

For $\gamma$ defined above. it is not difficult to verify that there are two distinct roots to the equation, namely

$$\lambda_- < -1, \qquad \lambda_+ > 0;$$

and

$$\lambda_+\lambda_- = -\frac{r}{\gamma}. \tag{5.4.6}$$

Third, the convergence condition of value function is, for any value of $\lambda_- < 0$

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty, \quad \forall c > 0. \tag{5.4.7}$$

Now we have completed the formulation of the two-asset model for a small investor: the agent, facing given market structure, wishes to find consumption and investment rules to maximise its present discounted value of expected utility (equation (5.4.4)) subject to a well specified wealth process (equation (5.2.13)). In what follows, we summarise some technical requirements.

### 5.4.2 Admissible policies

In order to have a well defined value function, the suitable policies $(c(t), \pi(t)$ for $t \geq 0)$ should be chosen so that the wealth equation (5.2.13) has a unique solution. To achieve that, we let $\{W_t, \mathcal{F}_t, t \geq 0\}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t, t \geq 0\}$ is a nondecreasing, right-continuous family of $\sigma$-filtrations. An admissible consumption process $\{c(t), t \geq 0\}$ is a nonnegative process adapted to $\{\mathcal{F}_t\}$ which satisfies almost surely

$$\int_0^t c(s)ds < \infty, \quad t \geq 0. \tag{5.4.8}$$

For each $\{\mathcal{F}_t\}$-adapted random process $\{\pi(t) : 0 \leq \pi(t) \leq \xi, t \geq 0\}$, define the $\{\mathcal{F}_t\}$-stopping time

$$T(\pi) = \sup\{t \geq 0 : \int_0^t \sigma^2 \pi^2(s)ds < \infty\}. \tag{5.4.9}$$

For $c(t)$ and $\pi(t)$ described above, if $0 \leq t < T(\pi)$, equation (5.2.13) has a unique solution (Karatzas and Shreve 1985),

$$x(t) = e^{z_t}(x - \int_0^t c(s)e^{-z_s}ds), \quad 0 \leq t < T(\pi), \tag{5.4.10}$$

where $x$ is the initial condition for $x(t)$, and

$$z_t = \int_0^t ((\alpha - r)\pi(s) + r - \frac{1}{2}\sigma^2\pi^2(s))ds + \int_0^t \sigma\pi(s)dW_s, \quad 0 \leq t < T(\pi). \tag{5.4.11}$$

Since $t < T(\pi)$, the Itô integral in equation (5.4.10) is defined almost surely.

Following the definitions given in Karatzas *et al* (1986), for each $x > 0$, we call ***admissible at*** $x$ any adapted pair of consumption and investment processes $\{c(t), \pi(t) : c(t) \geq 0, 0 \leq \pi(t) \leq \xi, t \geq 0\}$ for which either $T(\pi) = \infty$ or $T_b < T(\pi)$ or $\lim_{t \uparrow T(\pi)} x(t)$ exists and is zero. The supremum in (5.4.4) is taken over all pairs $c(t), \pi(t)$ of consumption and investment processes admissible at $x$ for which $V_{c(.),\pi(.)}(x)$ is well defined by (5.4.2). As was shown in Karatzas *et al*, if $\pi(t)$ is unconstrained, $V_{c(.),\pi(.)}(x)$ is well defined whenever $c(t)$ and $\pi(t)$ are admissible at $x$; and in Lehoczky *et al* (1983), $V_{c(.),\pi(.)}(x)$ is also well defined even when $\pi(t)$ is unconstrained[5].

As a consequence of equation (5.4.3), there exists some $\bar{c} > 0$ for which $U(\bar{c}) > \beta P$. The pair $c(t) \equiv \bar{c}, \pi(t) = 0$ is admissible for any $x > 0$ and yields $V_{c(.),\pi(.)}(x) > P$, so $V^* > P$ for $x > 0$. And as a absorbing boundary for $x = 0$, we always have $V^*(0) = P$.

[5]In the unconstrained case, we may have a situation where $\lim_{x \downarrow 0} \pi(x) = +\infty$, see below.

### 5.4.3 Sufficient conditions for optimality—Bellman equation

Here we provide a sufficient condition (Bellman equation) as a theorem without proof: the relevant proof is given in Fleming and Rishel (1975) or Karatzas *et al* (1986).

**Theorem 5.1** *With $P$ defined in (5.4.3), assuming $V : (0, \infty) \to (P, \infty)$ is a $C^2$ function satisfying the Bellman equation*

$$\begin{aligned} \beta V(x) = & \max_{c \geq 0, 0 \leq \pi \leq \xi} [(\alpha - r)\pi x V'(x) + (rx - c)V'(x) \\ & + \frac{1}{2}\sigma^2 \pi^2 x^2 V''(x) + U(c)], \qquad (5.4.12) \\ & x > 0, \\ V(0) = & P \end{aligned}$$

*Then $V(x)$ is optimal in (5.4.4) for $x > 0$.*

Since the treatment for $c \geq 0$ is considered in Karatzas *et al* (1986), we consider the break-down of the Bellman equation for the case with the borrowing constraint. This specific separation of regimes will be discussed in section 5.

**Theorem 5.2** *With conditions satisfied in Theorem 5.1, if the borrowing constraint is active for $0 < x < \bar{x}$, and $V(x)$ satisfies the following set of Bellman equations:*

$$\begin{aligned} \beta V(x) = & \max_{c \geq 0, \pi \geq 0} [(\alpha - r)\pi x V'(x) + (rx - c)V'(x) \\ & + \frac{1}{2}\sigma^2 \pi^2 x^2 V''(x) + U(c)], \qquad (5.4.13) \end{aligned}$$

$$
\begin{aligned}
&\text{for} \quad x \geq \bar{x} > 0; \\
\beta V(x) = &\max_{c \geq 0}[(\alpha - r)\xi x V'(x) + (rx - c)V'(x) \\
&+\frac{1}{2}\sigma^2\xi^2 x^2 V''(x) + U(c)], \qquad (5.4.14) \\
&\text{for} \quad \bar{x} > x > 0; \\
V(0) = &P
\end{aligned}
$$

*where $\bar{x}$ is the critical wealth level below which borrowing constraint binds, and $V(x)$ is continuously differentiable up to the second order at $\bar{x}$, then $V(x)$ is optimal in (5.4.4).*

**Proof:** If a solution to (5.4.13) and (5.4.14) exists and is twice continuously differentiable (since it is twice continuously differentiable at $\bar{x}$), then this solution also satisfies (5.4.12). So $V(x)$ is optimal.

Theorems 5.1 and 5.2 are only sufficient conditions of the optimality. The existence of the solutions to equation (5.4.12) were provided in Karatzas *et al* (1986) for a very general set of utility functions. If these solutions do not violate the shortselling and borrowing constraints, they must be the optimal solutions even to the cases where constraints are explicitly imposed. Unfortunately, in many cases, those solutions will activate the shortselling or borrowing constraints. In the following section, we provide a regime switching framework in which a set of equivalent joining conditions are developed. We defer the applications to the cases of constant relative and constant absolute utility functions to next chapter.

## 5.5 Framework of Regime Switching

The sufficient conditions for the optimality of consumption-investment policies are given as variational relationship in Theorem 5.2. The reason that the borrowing constraint binds at the lower end of the wealth (or consumption) is that the portfolio policy ($\pi$) is a decreasing function of the optimal consumption, so a decreasing function of wealth (since optimal consumption is an increasing function of wealth, see next chapter for further details). Furthermore, the smoothing conditions given in Theorem 5.2 are direct consequences of the conditions of continuity of both value function and portfolio policy across the switching boundary. So in this section, we provide another version of joining conditions at the switching boundary equivalent to those in Theorem 5.2. And then, we establish an optimal stopping problem where the conditions for optimality are equivalent to those given in Theorem 5.2.

### 5.5.1 Equivalent joining conditions

The joining conditions in Theorem 5.2 may not be easy to use in order to deal with the regime switching problem. Here, we provide a set of equivalent joining conditions at the switching boundary in the following theorem.

**Theorem 5.3** *The necessary and sufficient conditions for the value function to be twice continuously differentiable at the switching boundary are that both $c$ and $\pi$ have to be continuous at the switching boundary.*

**Proof:** In what follows, we denote the values in the unconstrained regime by subscript $u$ and those in the constrained regime by subscript $c$, and let the shadow portfolio policy $\pi$ in the constrained regime be defined in the same way as that in the unconstrained regime. Optimising over $c$ in the variational equations (5.4.13) and (5.4.14) in Theorem 5.2, we derive

$$V'(x) = U'(c). \tag{5.5.1}$$

**Necessary:** If the value function is twice continuously differentiable, from the above equation

$$c_u(\bar{x}) = c_c(\bar{x}), \tag{5.5.2}$$

where $\bar{x}$ denotes the switching boundary.

From the definition of $\pi$ that

$$\pi = -\frac{(\alpha - r)}{\sigma^2}\frac{V'(x)}{xV''(x)},$$

we derive

$$\pi_u(\bar{x}) = \pi_c(\bar{x}). \tag{5.5.3}$$

**Sufficient:** If $c$ and $\pi$ are continuous at the switching boundary, then

$$U'(c_u(\bar{x})) = U'(c_c(\bar{x})). \tag{5.5.4}$$

From equation (5.5.1), it is straight forward to show that

$$V'_u(\bar{x}) = V'_c(\bar{x}). \tag{5.5.5}$$

Using the definition of $\pi$, one can derive

$$V''_u(\bar{x}) = V''_c(\bar{x}). \tag{5.5.6}$$

Finally, using Bellman equations (5.4.13) and (5.4.14), we derive

$$V_u(\bar{x}) = V_c(\bar{x}). \tag{5.5.7}$$

This set of equivalent joining conditions at the switching boundary indicates the continuity of controls. So if the borrowing constraint is specified as $\pi \leq \xi$ for $\xi \geq 1$, then the joining condition can be written as

$$\pi_u(\bar{x}) = \xi = \pi_c(\bar{x}), \tag{5.5.8}$$

and

$$c_u(\bar{x}) = c_c(\bar{x}) \tag{5.5.9}$$

where $\bar{x}$ is the switching boundary.

### 5.5.2 Regime switching as an optimal stopping problem.

The breakdown of the original variational equation (5.4.12) in Theorem 5.1 into those in Theorem 5.2 effectively creates two regimes which depend on whether or not the borrowing constraint is binding. Suppose the agent initially possesses some small amount of wealth under which the borrowing constraint is binding: if a sudden movement of the Brownian motion makes wealth go up, the agent may find itself in the situation where the optimal portfolio does not activate the borrowing constraint. Since the switching is fully anticipated, the rational behaviour of the agent is to find an optimal time to switch over. In what follows, we construct an optimal stopping problem which describes such rational behaviour and provide a set of necessary conditions for the optimality of the consumption-investment policies.

Let $x = \bar{x}$ be the switching state (which will be determined later) for $0 < x \leq \bar{x}$, $\pi = \xi$; and $x > \bar{x}$, $\pi < \xi$. Suppose the initial wealth of an agent is

$$0 < x(0) \leq \bar{x}, \tag{5.5.10}$$

and define a stopping time which is the first time that the wealth surpasses $\bar{x}$

$$T_1 = \inf\{t \geq 0, x(t) > \bar{x}\}. \tag{5.5.11}$$

So, up to time $T_1$, the agent's portfolio policy is such that the borrowing constraint is binding. The agent has to choose a set of appropriate consumption-investment policies *as well as* a suitable stopping time $T_1$ or equivalently a suitable boundary $\bar{x}$ such that the present discounted value of expected utility is optimised. The value function of such an agent is given as follows:

$$V_c(x) = \sup_{T_1, c \geq 0} E_x[\int_0^{T_1} U(c(t))e^{-\beta t}dt + e^{-\beta T_1}V_u(x(T_1))]. \tag{5.5.12}$$

The dynamics of wealth are given by equation (5.2.13) by setting $\pi = \xi$, namely,

$$dx(t) = (\alpha - r)\xi x(t)dt + (rx(t) - c(t))dt + \xi x(t)\sigma dW_t, \quad (5.5.13)$$
$$x(0) = x.$$

where $V_u(x(T_1))$ is the unconstrained value function at the switching boundary $x(T_1) = \bar{x}$.

The optimality conditions are such that

$$\beta V_c(x) = \sup_{c \geq 0}[(\alpha - r)\xi x V'_c(x) + (rx - c)V'_c(x) + \frac{1}{2}\xi^2 x^2 \sigma^2 V''_c(x) + U(c)], \quad (5.5.14)$$

$$V_c(\bar{x}) = V_u(\bar{x}), \quad (5.5.15)$$

$$V'_c(\bar{x}) = V'_u(\bar{x}). \quad (5.5.16)$$

(See Krylov (1980) and Whittle (1983).) The smooth pasting condition at the boundary is due to the fact that the switching barrier is optimally chosen.

The unconstrained value function and optimality conditions can be constructed similarly. If the initial wealth of the agent is

$$x(0) > \bar{x}, \quad (5.5.17)$$

and the stopping time which also defines the switching boundary is given by

$$T_2 = \inf\{t \geq 0 : x(t) < \bar{x}\}, \quad (5.5.18)$$

then the optimal portfolio is such that the borrowing constraint is inactive, and the value function is

$$V_u(x) = \sup_{T_1, c \geq 0, \pi(t)} E_x[\int_0^{T_2} U(c(t))e^{-\beta t}dt + e^{-\beta T_2}V_c(x(T_2))], \quad (5.5.19)$$

the corresponding wealth process is given in equation (5.2.13), where $V_c(x(T_2))$ is the constrained value function defined in (5.5.12) and $x(T_2) = \bar{x}$.

The optimality conditions are such that

$$\beta V_u(x) = \sup_{c \geq 0, \pi} [(\alpha - r)\pi x V'_u(x) + (rx - c)V'_u(x) + \frac{1}{2}\pi^2 x^2 \sigma^2 V''_u(x) + U(c)], \quad (5.5.20)$$

plus equations (5.5.15) and (5.5.16). Furthermore, because the switching is reversible, from Whittle (1983):

$$V''_c(\bar{x}) = V''_u(\bar{x}). \quad (5.5.21)$$

Equations (5.5.15), (5.5.16) and (5.5.21) indicate that the value functions are twice continuously differentiable at the switching boundary, which is due to the facts that the switching barrier is optimally chosen and the switching is reversible. Together with the variational equations (5.5.14) and (5.5.20), this set of optimality conditions is the same as those provided in Theorem 5.2, so these two problems are identical. Since the second order smooth pasting condition here is a necessary condition, if a solution exists it must be both necessary and sufficient. We summarise these results as the following theorem.

**Theorem 5.4** *The breakdown of the original two-asset model into regimes* *in Theorem 5.2 is identical to the optimal stopping problem described above, since they satisfy the same set of optimality conditions, namely, (5.5.14), (5.5.20), (5.5.15), (5.5.16), and (5.5.21). In the optimal stopping problem, the second order smooth pasting condition (5.5.21) is necessary.*

It is not difficult to understand the results above. Simply to break down the original two-asset model under a borrowing constraint into two regimes and to optimise them in each different regime may not provide the desired solution as in Theorem 5.1, which indicates a global optimality solution. Fortunately, global optimality can be achieved by further choosing an optimal switching barrier, which is the essence of the construction of this optimal stopping problem.

## 5.6 Conclusion

In this chapter, we have constructed a two-asset model to deal with consumption/portfolio decisions of households. The results under unlimited borrowing and shortselling are well-known. However, the introduction of the constraints, especially the borrowing constraint, complicates the problem a great deal, since many classical results are not valid. Furthermore, the method used in Karatzas *et al* (1986) is incomplete in solving this two-asset model under constraints.

Fortunately, the optimal stopping technique can be used to cope with such regime switching problem, where the regime is defined by whether the borrowing constraint is active or not. The breakdown of the regimes will be justified after we prove that the optimal portfolio policy ($\pi$) under no constraints is a decreasing function of wealth level for $\alpha - r > 0$ in next chapter. The optimal solution for both constant relative and constant absolute risk aversion utility functions can be well characterised under the borrowing constraint. In next chapter, we will discuss in more detail the consumption/portfolio behaviour of the households under constraints.

# Chapter 6

# Continuous Time Optimal Consumption-Portfolio Decisions with Shortselling and Borrowing Constraints (II)

## 6.1 Introduction

Using the framework developed in last chapter, we will be able to characterise the consumption/portfolio behaviour of an agent with imposed shortselling and borrowing restrictions for various combinations of parameters. For simplicity, we only consider two special cases, namely, where the utility function is either CRRA (Constant Relative Risk Aversion) or CARA (Constant Absolute Risk Aversion).

For the simple two-asset model specified in last chapter. the optimal portfolio policy depends mainly on the difference between the rate of return on the risky asset and that of the safe asset. If this difference is negative, shortselling constraint will bind and the agent will be only allowed to invest in the riskless asset. In such situation, the model degenerates into a deterministic one with solutions characterised in Lehoczky *et al* (1983). If the rate of return on the risky asset is greater than that of the riskless asset the shortselling constraint will not be violated, but whether borrowing constraint binds depends on other parameters. In this situation, characterisation of the solution becomes more complicated. In this chapter, we shall mainly focus on this situation and see how the introduction of the borrowing constraint will affect optimal solutions.

For $\alpha > r$, $P > U(0)/\beta$ and the lower bound of the optimal portfolio policies under no constraints less than borrowing limit $\xi$, imposing borrowing constraint separates the optimal solution into two parts. When initial wealth ($x$) is low, borrowing constraint binds, the agent exhibits risk taking tendency by adopting optimal consumption higher than that when no constraints are imposed. Such behaviour is linked to a convex portion of the value function close to $x = 0$. Increasing $x$ makes value function concave as the probability of $x$ being absorbed at $x_t = 0$ diminishes. When initial wealth surpasses a threshold $\bar{x}$, the borrowing constraint becomes inactive. In the unconstrained regime, the variability of optimal consumption process relative to that of income tends to be larger than that in the unconstrained case, and the saving to income ratio also becomes larger. These results are in line with those predicted by the pure consumption models (see, for examples, Deaton 1991, 1992;

Gali 1990 and Zeldes 1989a), even the income processes are generally different (exogenous in pure consumption models and endogenous in consumption-portfolio models). So the effects of introducing borrowing constraint are two-fold: when wealth level is low, it generates risk taking behaviour; while the wealth level is high, the agent becomes more risk averse.

As the unconstrained optimal solutions are necessary to construct the optimal consumption and portfolio policies under borrowing constraint, we provide them in Appendix D. The arrangement of this chapter is as follows: section 2 presents optimal consumption-portfolio rules for both CRRA and CARA utility functions without imposing constraints. Section 3 provides the constrained solutions to the case where utility function is CRRA. Section 4 gives the constrained optimal consumption-portfolio policies for CARA utility function. In both these sections, the issues of consumption smoothing and precautionary saving will be discussed. Finally, section 5 concludes this chapter.

## 6.2 Constrained Solutions to CRRA Utility Function

### 6.2.1 Non-regime-switching Solutions

The non-regime-switching solutions under shortselling and borrowing constraints have three types: shortselling constrained, borrowing constrained and unconstrained for all wealth level. In what follows, we list the results.

#### 6.2.1.1 Cases when $\alpha - r < 0$

From Proposition D.1, if $\alpha - r < 0$, the unconstrained optimal portfolio policy is negative ($\pi < 0$). So, any unconstrained solution violates the shortselling constraint, and the constrained solutions must have the portfolio policy such that $\pi = 0$. From the Bellman equation (5.4.12), the problem becomes deterministic, the solutions in Lehoczky *et al* (1983) apply here.

#### 6.2.1.2 Cases when $\alpha - r > 0$

There are two different cases when $\alpha - r > 0$. First, when $\xi \geq (\alpha - r)/(\sigma^2\eta)$ and $P = U(0)/\beta$, the unconstrained optimal portfolio policy ($\pi$) violates neither shortselling nor borrowing constraints. The unconstrained optimal solution also constitutes the optimal solution even when the constraints are explicitly imposed. So, the optimal value function is (D.3.26), the optimal portfolio policy is (D.3.25), and the optimal consumption is determined by (D.3.24). These solutions are shown in Figure 6.1 with the value function in the upper panel, consumption and portfolio policy in the lower panel. Because the value function is concave in $x$, the behaviour is essentially risk averse (Arrow 1965, Pratt 1964).

Second, when $\xi \leq (\alpha - r)/(\sigma^2\eta)$ and $P \geq U(0)/\beta$, the optimal portfolio policy is such that $\pi = \xi$. In this case, the unconstrained portfolio policy ($\pi$) has a lower bound $(\alpha - r)/(\sigma^2\eta)$, so the adoptation of such policy always violates the borrowing constraint. Thus, the optimal portfolio policy ($\pi$) under constraints binds everywhere (for $x \geq 0$), and the solution is essentially the continuous version of that in

Zeldes (1989a).

### 6.2.2 Regime-switching Solutions

The parameterisation of the model other than those described in last section is such that $\alpha - r > 0$, $P > U(0)/\beta$, and $\xi > (\alpha - r)/(\sigma^2\eta)$. In this situation, the unconstrained optimal portfolio policy has a lower bound $(\alpha - r)/(\sigma^2\eta)$ when $x \to \infty$ (see Proposition D.1), and unlimited borrowing occurs at $x = 0$. Since $\pi(c)$ is a strictly decreasing function of $c$ (and therefore of $x$, cf Proposition D.1), the part of the unconstrained solution which violates the borrowing limit appears in the region where wealth falls below a threshold. In this case, it is natural to construct the constrained optimal solution in such a way that below certain threshold ($\bar{x}$), the portfolio policy binds ($\pi = \xi$); and above it, the unconstrained solution can be utilised. The switch between regimes is clearly characterised by the optimal solution under the borrowing constraint, and the optimality is guaranteed by the joining conditions (see Theorem 5.4).

In what follows, we first sketch the general properties of such regime-switching solution, and then discuss the consumption variability and precautionary savings in the unconstrained regime.

#### 6.2.2.1 General properties

As the decomposition suggested by Theorem 5.2, the solution at the lower wealth level is borrowing constrained (see also Proposition D.1). Since the analytical solution of this non-linear ODE cannot be obtained, we start with describing the behaviour at $x = 0$. When wealth level is high, borrowing constraint no longer binds. However, instead of using the original unconstrained solution for given $P$, we have to choose another unconstrained solution with lower $P'$, which can smooth paste on to the solution in the constrained regime. This will lower the value function almost everywhere (except at $x = 0$). In the unconstrained regime, the consumption level and $\pi$ associated with this value function are also reduced.

To characterise the constrained solution, we derive the consumption level at bankruptcy and compare it with its unconstrained counterpart. The risk taking behaviour associated with convexity of the value function under certain conditions is also discussed. Here, we summarise the results in the following lemmas.

**Lemma 6.1** *In the constrained regime, if $U(0)/\beta < P < \lim_{c\uparrow\infty} U(c)/\beta$ and the utility function is CRRA, then $V(0) = P$ has a unique solution $c > 0$.*

**Proof** Using Bellman equation (5.4.14) and letting $x \to 0$, we derive

$$\beta V(0) = U(c) - cU'(c). \tag{6.2.1}$$

For the CRRA utility function specified in (D.3.2), the above equation becomes

$$\beta V(0) = \eta U(c), \tag{6.2.2}$$

and

$$\lim_{c\downarrow 0} \eta U(c) = U(0) = \begin{cases} 0 & \text{if } \; 1 - \eta > 0, \\ -\infty & \text{if } \; 1 - \eta \leq 0. \end{cases} \tag{6.2.3}$$

Furthermore, because $U(c)$ is a strictly increasing function of $c$, and $P > U(0)/\beta$, equation (6.2.1) has a unique solution $c > 0$.

Lemma 6.1 not only provides the consumption level at $x = 0$, but, more importantly, provides sufficient initial conditions for solving the non-linear ODE in the constrained regime. It is apparent that for $c > 0$, $U'(c) = V'(x)$, then apart from the initial condition for the value function, its derivative is also determined at $x = 0$. So the solution can, at least, be solved numerically.

**Lemma 6.2** *Let $a_c$ and $a_u$ be the consumption at bankruptcy for constrained and unconstrained cases respectively. For a given $P$, if $U(0)/\beta < P < \lim_{c\uparrow\infty} U(c)/\beta$, then $a_c > a_u$.*

**Proof** The values of $a_c$ and $a_u$ can be determined by equations (6.2.1) and (D.1.18) respectively. For given $a > 0$, we can rewrite equation (D.1.18) as

$$\begin{aligned} V_u(0;a) &= \frac{U(a)}{\beta} - \frac{aU'(a)}{\beta} - \frac{(U'(a))^{\rho_-}}{\beta\lambda_-}\int_a^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} - \frac{aU'(a)}{\beta\lambda_-} \\ &= V_c(0;a) - \frac{aU'(a)}{\beta\lambda_-} f(a), \qquad (6.2.4) \end{aligned}$$

where $V_u$, $V_c$ denote unconstrained and constrained value functions, and

$$f(a) = a + (U'(a))^{\lambda_-}\int_a^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} > 0.$$

Since the second term on the right hand side of equation (6.2.4) is positive, then

$$V_u(0;a) > V_c(0;a).$$

Finally, because $V_u$, $V_c$ are both strictly decreasing functions of $a$, if

$$V_u(0; a_u) = V_c(0; a_c) = P,$$

then

$$a_c > a_u.$$

A more interesting feature of the value function is that it may have a convex portion near the origin. This may appear to explain the above risk taking behaviour when wealth level is very low. The result is presented in the following lemma.

**Lemma 6.3** *If the borrowing constraint is active, and* $(\alpha - r)\xi + r > \beta$, *then the value function is convex at* $x = 0$.

**Proof** Since the value function is twice continuously differentiable, we can approximate $V(x)$ and $V'(x)$ as

$$V(x) = V(0) + V'(0)x + o(x),$$
$$V'(x) = V'(0) + V''(0)x + o(x).$$

Substituting into the Bellman equation (5.4.14), and equating the first order terms, we derive

$$V''(0) = [(\alpha - r)\xi + r - \beta][V'(0)]^{1+1/\eta} > 0. \qquad (6.2.5)$$

The continuity of the value function (up to its second order derivatives) creates a region near $x = 0$ where the value function is convex. Using equation (D.1.4), we obtain $C'(x) < 0$, namely, the consumption decreases when wealth increases.

When the initial wealth $x$ is close to zero, the borrowing constraint binds, the prospect of escaping the constrained regime through excessive borrowing is restricted (in the unconstrained case $\pi \rightarrow +\infty$ as $x \rightarrow 0$) while the danger of being absorbed at $x_t = 0$ is high. As depressing consumption generates less expected utility, the agent will consume more (than that in the unconstrained case) so accelerates the pace to bankruptcy.

#### 6.2.2.2 Optimal solutions under the borrowing constraint

In what follows, we summarise the properties of this regime switching solution in the following theorem.

**Theorem 6.1** *For the utility function specified in (D.3.2) and* $U(0)/\beta < P < \lim_{c\rightarrow\infty} U(c)/\beta$, *if* $1 < \xi < \infty$ *and* $\alpha - r > 0$, *the borrowing constraint binds for* $x \leq \bar{x}$. *The solution for* $x > \bar{x}$ *is obtained by using another unconstrained solution such that* $P' < P$, *where* $P'$ *and* $\bar{x}$ *are determined by Theorem 5.3. Specifically, the constrained consumption is higher at* $x = 0$ *and* $\bar{x}$ *is less than the wealth level at which the unconstrained solution attains* $\pi = \xi$. *Furthermore, for* $x > \bar{x}$, *the constrained consumption is everywhere lower than its unconstrained counterpart.*

**Proof** Since many features are already provided in last chapter and the lemmas above, here we only prove the last statement.

To prove

$$\bar{x} < x_u(\pi = \xi, P), \tag{6.2.6}$$

(where $x_u(\pi = \xi, P)$ is the unconstrained wealth level at which $\pi = \xi$) we only have to show that $\partial\pi/\partial P > 0$ where $\pi$ is given in (D.3.20).

From (D.3.14), differentiating $a$ with respect to $P$ yields

$$\frac{\partial a}{\partial P} = -\frac{\gamma(1+\lambda_-\eta)\rho_+}{\eta^2}a^\eta > 0, \quad 1+\lambda_-\eta < 0. \tag{6.2.7}$$

Differentiating $\pi$ in (D.3.20) with respect to $a$ yields

$$\frac{\partial\pi}{\partial a} = \frac{\alpha - r}{\sigma^2\eta}\cdot\frac{(1+\lambda_+\eta)^2}{[1-(\frac{a}{c})^{1+\lambda_+\eta}]^2}(\frac{a}{c})^{\lambda_+\eta}\frac{1}{c} > 0.$$

So

$$\frac{\partial\pi}{\partial P} = \frac{\partial\pi}{\partial a}\cdot\frac{\partial a}{\partial P} > 0.$$

Thus reducing $P$ to $P'$ also reduces the corresponding wealth level.

To prove that the consumption under borrowing restriction in the unconstrained regime ($x > \bar{x}$) is less than its unconstrained counterpart, we only have to prove for $x > \bar{x}$

$$\frac{\partial c}{\partial P}|_x > 0. \tag{6.2.8}$$

From (D.3.16), fixing $x$ and differentiating $c$ with respect to $a$ yields

$$\frac{\partial c}{\partial a}|_x = \frac{1+\lambda_+\eta}{1+\lambda_+\eta(\frac{a}{c})^{1+\lambda_+\eta}}(\frac{a}{c})^{\lambda_+\eta} > 0. \tag{6.2.9}$$

Together with (6.2.7), we have

$$\frac{\partial c}{\partial P}|_x = \frac{\partial c}{\partial a}|_x \cdot \frac{\partial a}{\partial P} < 0.$$

So we complete the proof.

With the aid of this theorem, we can draw the pictures for this regime-switching solution under the borrowing constraint. The curve $ABC$ in Figure 6.2 shows the value function under borrowing constraint, where $AB$ is the part in the constrained regime and $BC$ in the unconstrained regime. At $B$, value function is twice continuously differentiable according to Theorem 5.2. This value function (given initial condition $P$) joins another unconstrained value function $A'BC$ at $B$, which has a lower bankruptcy value $P'$. So $ABC$ is lower than the unconstrained solution $AD$. For $(\alpha - r)\xi + r > \beta$, the value function in the constrained regime is first convex and then concave.

Figure 6.3 shows the corresponding consumption under borrowing restriction. If $(\alpha - r)\xi + r > \beta$, the part $AB$ in the constrained regime has a decreasing portion and joins the unconstrained solution $A'BC$ (bankruptcy value $P'$) at $B$. The part in the unconstrained regime ($BC$) is lower than that if borrowing constraint is removed.

Finally, the optimal portfolio policy is shown in Figure 6.4. The solution $ABC$ is everywhere lower than its unconstrained counterpart $DEF$, and the switching point $B$ is to the left of $E$ which produces the same $\pi$ if not constrained. Both portfolio policies $BC$ and $EF$ asymptotically tend to $(\alpha - r)/\sigma^2\eta$.

#### 6.2.2.3 Consumption variability and precautionary savings

In the solution described above, how does the variability of consumption and the level of savings response to the introduction of the borrowing constraint in the unconstrained regime? Apparently, the introduction of the borrowing constraint reduces the returns on the assets therefore reduces the income variability. It also decreases consumption, and from (D.3.22), decreases the variability of consumption. In this case, it is sensible to look at instead the consumption variability relative to that of income.

From wealth dynamics (5.2.13), the income volatility is

$$\mathrm{Var}(dI_t) = \pi^2(t)x^2(t)\sigma^2 dt,$$

using (D.3.22), we define

$$R_{CI} = \sqrt{\frac{\mathrm{Var}(dc_t)}{\mathrm{Var}(dI_t)}} = \frac{\sqrt{2\gamma}c}{\sigma\eta\pi x} \qquad (6.2.10)$$

as a relative measure of consumption volatility to that of income.

To assess its response to borrowing constraint, we differentiate $R_{CI}$ with respect to $P$ for a fixed wealth level,

$$\frac{\partial R_{CI}}{\partial P}|_x = \frac{\sqrt{2\gamma}}{\sigma\eta\pi^2 x}(\pi\frac{\partial c}{\partial P}|_x - c\frac{\partial \pi}{\partial P}|_x). \qquad (6.2.11)$$

From (6.2.7) and (6.2.9), one obtains

$$\frac{\partial c}{\partial P}|_x > 0. \qquad (6.2.12)$$

From (D.3.18) and (D.3.19), one derives

$$\pi = \frac{\lambda_+(\alpha - r)}{\sigma^2}(\frac{f(c)}{rx} - 1), \qquad (6.2.13)$$

where $f(c)$ is given by (D.3.8). So

$$\frac{\partial \pi}{\partial P}|_x = \frac{(\alpha - r)\lambda_+\lambda_-\eta}{\sigma^2(1+\lambda_-\eta)rx}\frac{\partial c}{\partial P}|_x > 0. \qquad (6.2.14)$$

substituting (6.2.12) and (6.2.14) into (6.2.11) yields

$$\frac{\partial R_{CI}}{\partial P}|_x = -\frac{2\gamma\lambda_+}{\sigma^2\eta x\pi^2}\cdot\frac{\partial c}{\partial P}|_x < 0. \qquad (6.2.15)$$

so reducing $P$ will increase $R_{CI}$, thus consumption becomes more volatile relative to income when the borrowing constraint is imposed.

To assess the saving behaviour, we define the ratio of consumption to expected income. Since from (5.2.13) the expected income is $rx + (\alpha - r)\pi x$, the consumption income ratio can be expressed as

$$R_S = \frac{1}{x}\frac{c}{(\alpha - r)\pi + r}. \qquad (6.2.16)$$

Differentiating $R_S$ with respect to $P$ and keeping $x$ fixed yields

$$\frac{\partial R_S}{\partial P}|_x = \frac{1}{x}\frac{1}{[(\alpha - r)\pi + r]^2}\{(\alpha - r)(\pi\frac{\partial c}{\partial P}|_x - c\frac{\partial \pi}{\partial P}|_x) + r\frac{\partial c}{\partial P}|_x\}. \qquad (6.2.17)$$

Substituting (6.2.12) and (6.2.14) into (6.2.17) yields

$$\frac{\partial R_S}{\partial P}|_x = -\frac{\gamma\lambda_+\rho_-}{x[(\alpha - r)\pi + r]^2}\frac{\partial c}{\partial P}|_x > 0, \qquad (6.2.18)$$

so the saving to income ratio $1 - R_S$ is a decreasing function of $P$, i.e., the introduction of the borrowing constraint will increase savings relative to income.

Such behaviour is not difficult to explain. Since the borrowing is limited, the agent's ability to spread consumption is restricted. To maintain certain level of consumption, the consumption has to follow more closely to income than it would otherwise. So the volatility of consumption relative to income increases. The ability to borrow can be viewed as an insurance device, when the access to such insurance policy is limited and such restriction is fully anticipated, the consumers must provide it themselves. So they have the motive to accumulate their wealth, which consequently leads to precautionary savings.

## 6.3 Constrained Solutions to CARA Utility Function

### 6.3.1 Non-regime-switching Solutions

The non-regime-switching solutions under shortselling and borrowing constraints have two types: shortselling constrained and unconstrained for all wealth level. Here we list the results.

#### 6.3.1.1 Cases when $\alpha - r < 0$

Similar to the CRRA utility function case, if $\alpha - r < 0$, from (D.2.11) $\pi < 0$. So the optimal $\pi$ must be $\pi = 0$, the problem degenerates into a deterministic one with all the solutions provided in Lehoczky *et al*

(1983).

#### 6.3.1.2 Cases when $\alpha - r > 0$

If $\alpha - r > 0$, $P = U(0)/\beta$ and $-\lambda_{-}(\alpha - r)/\sigma^2 \leq \xi$, from Proposition D.2, the optimal portfolio policy in the unconstrained case does not activate the borrowing constraint. So the value function and consumption are given in (D.4.12) and (D.4.11) by setting $B = 0$. The optimal portfolio policy is given by (D.4.14). The picture of the value function is shown in the upper panel of Figure 6.5, consumption and portfolio policies are sketched in the lower panel.

### 6.3.2 Regime-switching Solutions for CARA Utility Function

There are two cases other than those described above. First is that $\alpha - r > 0$, $P = U(0)/\beta$ and $-\lambda_{-}(\alpha - r)/\sigma^2 > \xi$. The second case has the parameters $\alpha - r > 0$ and $P > U(0)/\beta$. In what follows, we discuss these two cases separately.

#### 6.3.2.1 Case when $P = U(0)/\beta$

When $x \leq \bar{x}$, the unconstrained portfolio policy violates the borrowing constraint because $-\lambda_{-}(\alpha - r)/\sigma^2 > \xi$. But when $x$ is large enough, the unconstrained portfolio policy falls below the constraint (see Proposition D.2). Notice that the value function in this case is the lowest possible one, so how should the constrained solution be formed?

Such solution can be constructed in three different regimes: in the regime $0 \leq x \leq \bar{x}_1$, $c = 0$ and $\pi = \xi$; in the regime $\bar{x}_1 < x \leq \bar{x}_2$,

$c > 0$ and $\pi = \xi$; and in the last regime $x > \bar{x}_2$, its own unconstrained solution is used. In this case, obviously we must have $\bar{x}_1 > \bar{x}$, otherwise the consumption in the second regime is higher than its unconstrained counterpart, which results in an everywhere higher value function.

Figure 6.6 shows optimal policies in the above case. For $x \leq \bar{x}_2$, the optimal portfolio policy is $\xi$ which joins the unconstrained solution at $K$. The consumption becomes positive for wealth greater than $\bar{x}_1$ and joins its unconstrained part at $J$. The consumption variability and saving level relative to those of income do not change in the unconstrained regime compared with the unconstrained solution.

#### 6.3.2.2 Cases when $P > U(0)/\beta$

From Proposition D.3 and D.4, the unconstrained portfolio policy violates the borrowing constraint at the lower wealth level. Since Lemmas 6.1–6.3 can be easily extended to the general utility functions, they must apply here, and Theorem 6.1 also carries through. Because the regime switching solution for $P \geq P^*$ are very much the same as that under CRRA utility function, we only provide the pictures of optimal policies for the case where $U(0)/\beta < P < P^*$. To avoid ambiguity, we denote $\bar{x}$ as the consumption constraint binding point and $\bar{\bar{x}}$ as the portfolio regime switching point. The optimal consumption path $ABD$ is shown in Figure 6.7 and optimal portfolio policy $ABC$ in Figure 6.8.

Now we turn to the discussion of consumption variability and precautionary saving. As in last section, we define the ratio of consumption variance to that of income, i.e.,

$$R_{CI} = \frac{\sqrt{2\gamma}}{\sigma\eta\pi x}. \tag{6.3.1}$$

First, we deal with the case where $P \geq P^*$ $(0 < y < \bar{y} \leq 1)$. Differentiating (D.4.4) with respect to $P$ yields

$$\frac{\partial \bar{y}}{\partial P} = \rho_+ \frac{r\eta}{\lambda_+}\left(\frac{1}{\lambda_-} + \ln \bar{y}\right)^{-1} < 0. \tag{6.3.2}$$

Substituting (D.4.5) into (D.4.6), fixing $x$ and differentiating $y$ with respect to $\bar{y}$ yields

$$\frac{\partial y}{\partial \bar{y}}|_x = \frac{y^{\lambda_+ - 1}}{\bar{y}}(1 - \lambda_+ y^{\lambda_+} \ln \bar{y}) > 0. \tag{6.3.3}$$

For $y \leq 1$, differentiating $f(y)$ in (D.4.13) with respect to $y$ yields

$$\frac{\partial f(y)}{\partial y} = -\frac{1}{\eta y} < 0. \tag{6.3.4}$$

Now we differentiate $R_{CI}$ in (6.3.1) with respect to $P$ and using (6.3.2)–(6.3.4) and (D.4.14), so

$$\begin{aligned}\frac{\partial R_{CI}}{\partial P}|_x &= \frac{\sqrt{2\gamma}}{\sigma\eta\pi^2 x}\frac{\partial \pi}{\partial P}|_x \\ &= \frac{2}{\sigma^2\lambda_-\eta x^2\pi^2}\frac{\partial f}{\partial y}|_x\frac{\partial y}{\partial \bar{y}}|_x\frac{\partial \bar{y}}{\partial P} < 0.\end{aligned} \tag{6.3.5}$$

For the case where $U(0)/\beta < P < P^*$ $(\bar{y} > 1)$, the problem has to be separated into two parts according to (D.4.11), but the values from (6.3.2) to (6.3.4) have the same sign, so we still obtain (6.3.5). Thus imposing borrowing constraint also increases the consumption volatility relative to that of income in the CARA case.

Similar to CRRA case, the consumption income ratio is defined by

$$R_S = \frac{1}{x}\frac{c}{(\alpha - r)\pi + r}. \tag{6.3.6}$$

In the regime $y \geq 1$, $c \equiv 0$, but from (6.3.5)

$$\frac{\partial \pi}{\partial P}|_x > 0,$$

so income increases in the regime and all the income is used as savings.

For $y < 1$ and $c > 0$, let $c = -\ln y/\eta$, differentiating (6.3.6) with respect to $P$ yields

$$\frac{\partial R_S}{\partial P}|_x = \frac{1}{x[(\alpha - r)\pi + r]^2}\{-\gamma\lambda_+\rho_- - \frac{1}{r\lambda_-\eta x}\}(-\frac{1}{\eta y})\frac{\partial y}{\partial P}|_x > 0. \tag{6.3.7}$$

Hence, imposing the borrowing constraint also induces precautionary savings.

## 6.4 Conclusion

In this chapter, using the regime switching framework developed in the last chapter, we characterise the solutions under shortselling and borrowing constraints for various parameters. When $\alpha - r < 0$, the problem under constraint degenerates into a deterministic one with all the solutions given in Lehoczky *et al* (1983). For $\alpha - r > 0$, some unconstrained solutions do not activate either constraint, so they are optimal even when the constraints are explicitly imposed. Another type of solutions in this case make the borrowing constraint always bind, so they become problems as if income is not diversifiable.

The most interesting case when $\alpha - r > 0$ are regime switching problems, where borrowing constraint binds only at lower wealth level. For $(\alpha - r)\xi + r > \beta$, and $P > U(0)/\beta$ and the initial wealth is low, the agent exhibits risk taking behaviour (when initial wealth is close to zero, the agent consumes more in the borrowing constrained case than that in the unconstrained case) which is associated with a portion of convex value function. This is because first when $P > U(0)/\beta$, going bankrupt (which generates expected utility $P$) is more attractive than consuming nothing indefinitely (which has expected utility $U(0)/\beta$). Second, when $x$ is small, borrowing constraint binds, the danger of the wealth process being absorbed at the origin outweighs the prospect of getting out of the constrained regime as the fund is insufficient to finance investment in risky assets due to restricted borrowing. The combination of these two factors makes the current consumption more attractive than future expected consumption. So when the wealth is low, the borrowing constrained agent would consume more, which accelerates the pace to bankruptcy.

However, when the wealth level is sufficiently high, borrowing constraint no longer binds, the agent becomes more risk averse than that under no constraints. The anticipation of binding borrowing constraint increases the consumption volatility relative to that of income; it also induces precautionary savings which is similar to the simulation results obtained by Deaton (1991, 1992) and to that predicted by Carroll (1991) with no liquidity constraints but assuming a voluntary abstinence from borrowing.

## 6.5 FIGURES

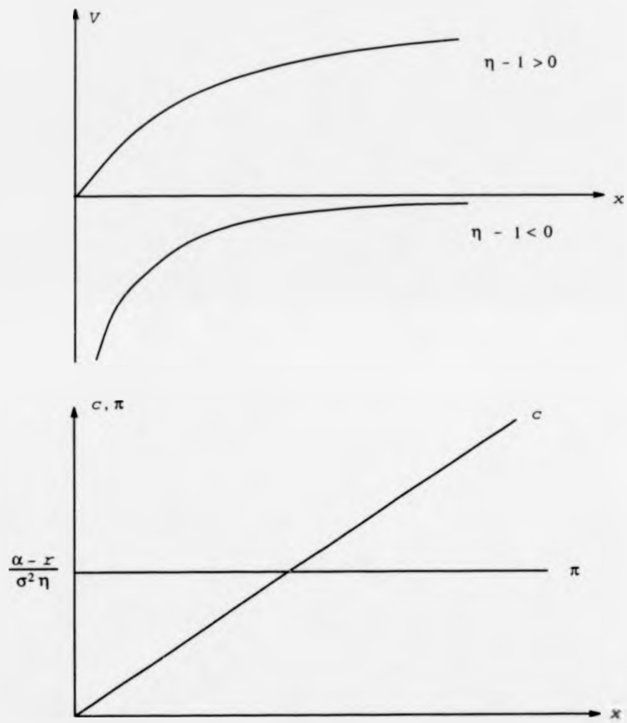



Figure 6.1: Value Function, Optimal Consumption and Portfolio Policies: $P = U(0)/\beta$, $(\alpha - r)/(\sigma^2\eta) \leq \xi$.

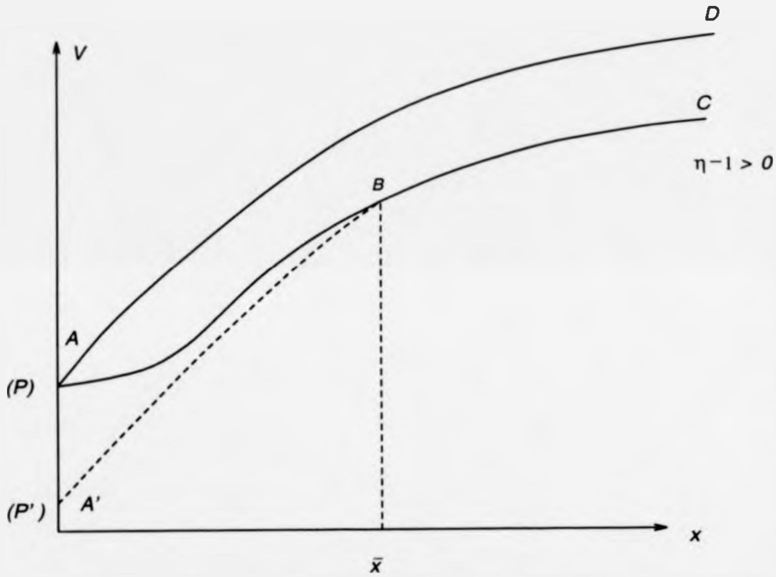



Figure 6.2: Optimal Value Function for CRRA Utility Function: $P > U(0)/\beta$, $(\alpha - r)/(\sigma^2\eta) < \xi$.

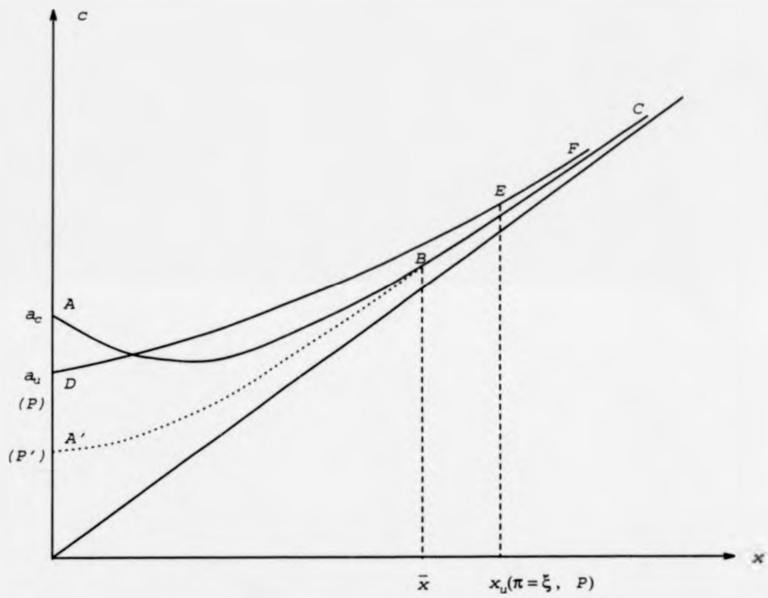



Figure 6.3: Optimal Consumption for CRRA Utility Function: $P > U(0)/\beta$, $(\alpha - r)/(\sigma^2\eta) < \xi$.

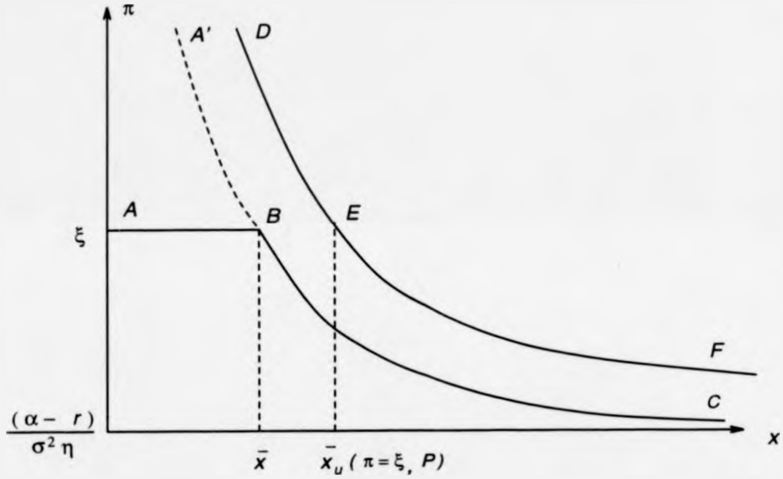



Figure 6.4: Optimal Portfolio Policy for CRRA Utility Function: $P > U(0)/\beta$, $(\alpha - r)/(\sigma^2\eta) < \xi$.

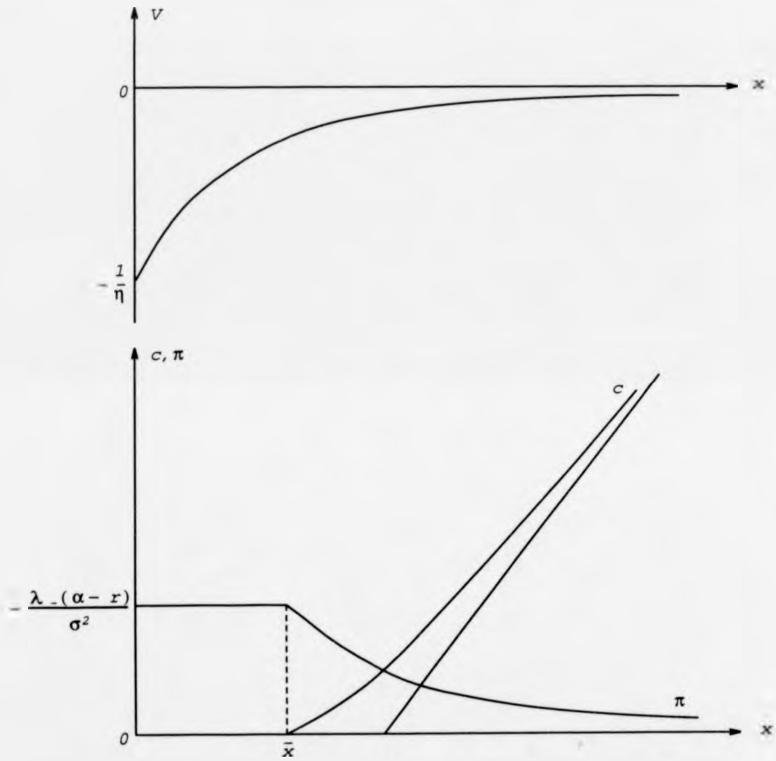



Figure 6.5: Value Function and Optimal Policies for CARA Utility Function: $P = U(0)/\beta$, $-\lambda_-(\alpha - r)/\sigma^2 \leq \xi$.

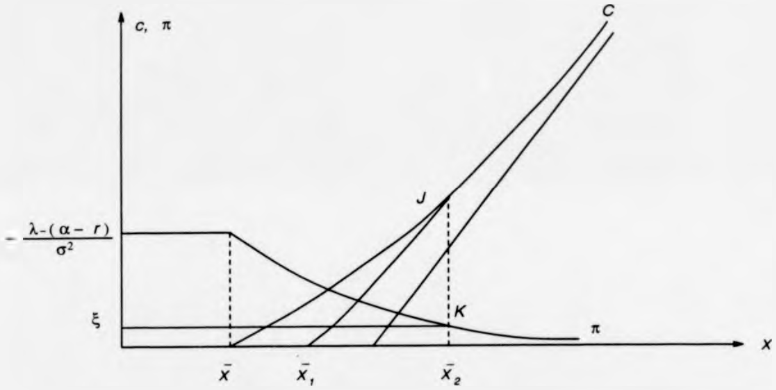



Figure 6.6: Optimal Consumption and Portfolio Policies for CARA Utility Function: $P = U(0)/\beta$, $-\lambda_-(\alpha - r)/\sigma^2 > \xi$.

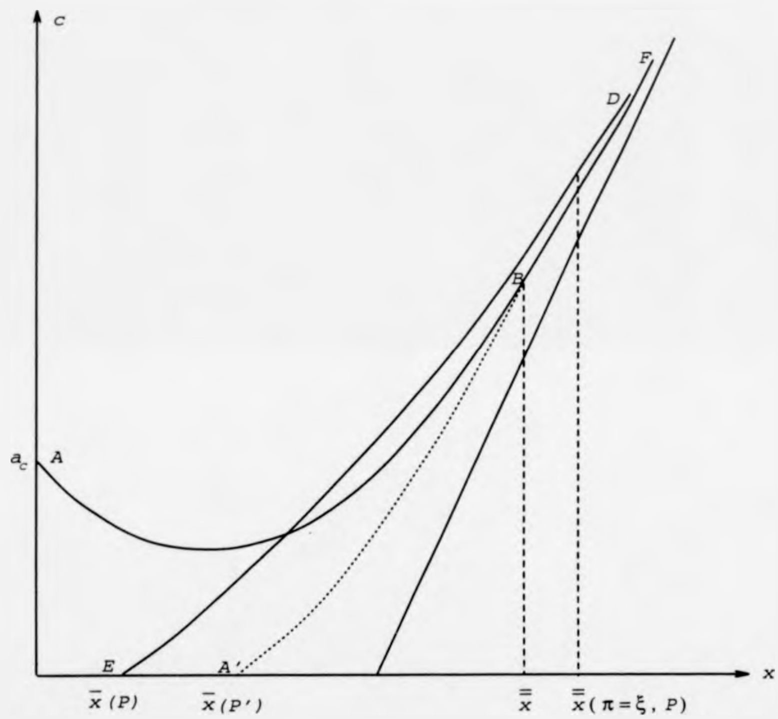



Figure 6.7: Optimal Consumption for CARA Utility Function: $P > U(0)/\beta$.

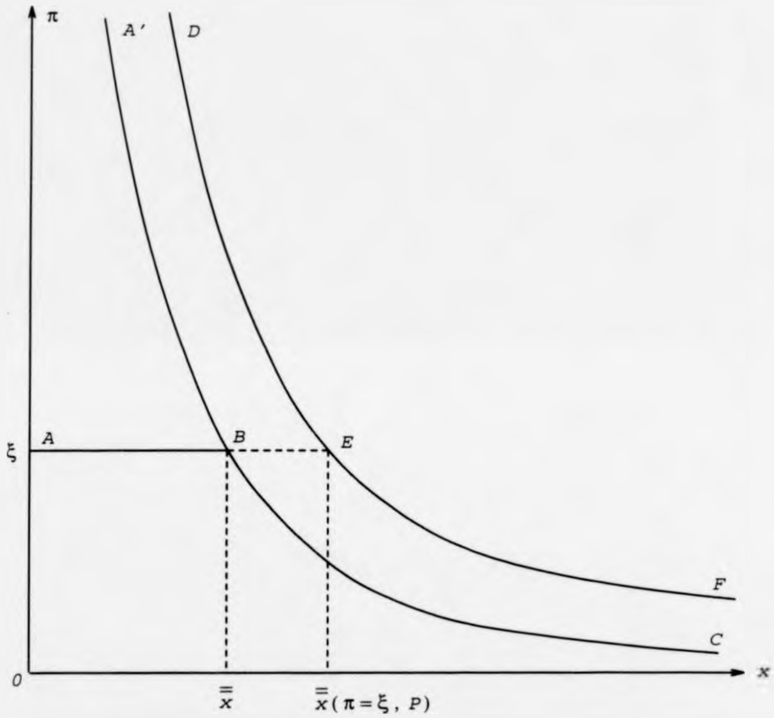



Figure 6.8: Optimal Portfolio Policy for CARA Utility Function: $P > U(0)/\beta$.

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# Appendix A

# The Optimal Rule

First we show that for $\bar{k} \in (0, \bar{k}_D)$, where $\bar{k}_D$ denotes time consistent intervention barrier, $V(0; A(\bar{k}))$ attains minimum at an interior. Then using quadratic approximation for value function (given that $\bar{k}_D << 1$), we show that, for such choice of $\bar{k}_R$, the corresponding value function indeed dominates those generated by $\bar{k}$ where $\bar{k} \in [0, \bar{k}_D]$ and $\bar{k} \neq \bar{k}_R$.

From (2.3.8) together with consistency condition, the value function at the central parity ($k = 0$) can be written as

$$V(0; A(\bar{k})) = B(\bar{k}) - \frac{2\sigma^2 \text{sech}(\lambda \bar{k})}{(\rho - \beta)^2} + \frac{\sigma^2}{\rho}\left(\frac{\text{sech}^2(\lambda \bar{k})}{\rho - 4\beta} + \frac{1}{\rho}\right), \qquad \text{(A.0.1)}$$

where $B(\bar{k})$ is rearranged using equation (2.3.9), so

$$B(\bar{k}) = \mu^{-1}\text{csch}(\mu \bar{k})\{c - \left(\frac{1}{\rho - 4\beta} - \frac{\rho + \beta}{(\rho - \beta)^2}\right)\frac{2\tanh(\lambda \bar{k})}{\lambda} + \frac{2\beta \bar{k}}{\rho(\rho - \beta)}\}. \qquad \text{(A.0.2)}$$

Differentiating $V(0; A(\bar{k}))$ with respect to $\bar{k}$ yields

$$\frac{\partial V(0; A(\bar{k}))}{\partial \bar{k}} = -\text{csch}(\mu \bar{k})\coth(\mu \bar{k})\{c - \left(\frac{1}{\rho - 4\beta} - \frac{\rho + \beta}{(\rho - \beta)^2}\right)2\text{sech}^2(\lambda \bar{k})$$

$$+\frac{2\beta}{\rho(\rho-\beta)}\}$$
$$+\mu^{-1}\text{csch}(\mu\bar{k})\{2(\frac{1}{\rho-4\beta}-\frac{\rho+\beta}{(\rho-\beta)^2})\tanh^2(\lambda\bar{k})-a\}$$
$$+2\sigma^2\lambda\text{sech}(\lambda\bar{k})\tanh(\lambda\bar{k})\{\frac{1}{(\rho-\beta)^2}-\frac{\text{sech}(\lambda\bar{k})}{\rho(\rho-4\beta)}\}. \quad \text{(A.0.3)}$$

Evaluating at time consistent barrier $\bar{k}=\bar{k}_D$, by substituting equation (2.3.12) from the text to above equation, yields

$$\lim_{\bar{k}\to\bar{k}_D}\frac{\partial V(0;A(\bar{k}))}{\partial\bar{k}}=-\mu^{-1}\text{csch}(\mu\bar{k}_D)\{-\frac{2(\rho+\beta)}{(\rho-\beta)^2}\tanh^2(\lambda\bar{k}_D)$$
$$+\frac{2\lambda}{\rho-\beta}\bar{k}_D\tanh(\lambda\bar{k}_D)\}$$
$$+2\sigma^2\lambda\text{sech}(\lambda\bar{k}_D)\tanh(\lambda\bar{k}_D)\{\frac{1}{(\rho-\beta)^2}$$
$$-\frac{\text{sech}(\lambda\bar{k}_D)}{\rho(\rho-4\beta)}\}>0,$$
$$\text{for}\ \ \bar{k}_D>0. \quad \text{(A.0.4)}$$

Evaluating at $\bar{k}=0$ yields

$$\lim_{\bar{k}\downarrow 0}\frac{\partial V(0;A(\bar{k}))}{\partial\bar{k}}=-\infty. \quad \text{(A.0.5)}$$

Since $\frac{\partial V(0;A(\bar{k}))}{\partial\bar{k}}$ is continuous in $(0,\bar{k}_D]$, it certainly attains minimum at a interior point $\bar{k}_R$.

To find everywhere dominant value function within $(0,\bar{k}_D)$, we rewrite the value function (2.3.8) incorporating (A.0.1)

$$V(k;A)=V(0;A)+B(\cosh(\mu k)-1)+\frac{2\sigma^2\lambda A}{(\rho-\beta)^2}(\cosh(\lambda k)-1)$$
$$+\frac{A^2}{\rho-4\beta}\sinh^2(\lambda k)+\frac{2A}{\rho-\beta}k\sinh(\lambda k)+\frac{k^2}{\rho}. \quad \text{(A.0.6)}$$

For $\bar{k}_D << 1$, $k \leq \bar{k}_D$, so we can approximate the above value function up to the quadratic term,

$$V(k;A) = V(0;A) + \frac{\rho}{\sigma^2}\{B + \frac{2\sigma^2\lambda A}{(\rho-\beta)^2} + \frac{\sigma^2}{\rho}(\frac{\lambda^2 A^2}{\rho-\beta} + \frac{1}{\rho})\}k^2. \quad (A.0.7)$$

Substituting the quadratic approximation of equation (A.0.1) to the equation above yields

$$V(k;A) = V(0;A)(1 + \frac{\rho}{\sigma^2}k^2). \quad (A.0.8)$$

It is obvious that for any number $k$, $k \in [0, \bar{k}]$, $V(k;A)$ attains minimum if and only if $V(0;A)$ is minimum. Thus, the choice of the barrier $\bar{k}_R$ generates an everywhere dominant value function for $\bar{k} \in (0, \bar{k}_D)$.

# Appendix B

# Proportional Extraction Costs: Value Function of Production without Closure

In the case where the switch from idle to production is irreversible and the extraction costs are proportional to the extraction rate, the value function is the expected discounted cash flow conditional on the initial oil price and reserves, i.e., it is the expected stochastic integral given as,

$$V(P, Q) = E_0 \int_0^\infty (P_t - a)\gamma Q_t e^{-\rho t} dt. \tag{B.0.9}$$

Given price and reserves evolve as,

$$dP_t = \alpha P_t dt + \sigma P_t dW_t, \quad P_0 = P, \tag{B.0.10}$$

$$dQ_t = -\gamma Q_t dt, \quad Q_0 = Q. \tag{B.0.11}$$

From (B.0.9) we have

$$V(P,Q) = \int_0^{\infty} (\gamma E_0(P_t Q_t) - a\gamma E_0(Q_t)) e^{-\rho t} dt. \tag{B.0.12}$$

Differentiating $P_t Q_t$ using Itô's lemma,

$$d(P_t Q_t) = (\alpha - \gamma) P_t Q_t + \sigma P_t Q_t dW_t. \tag{B.0.13}$$

Taking expectations on both sides of equations (B.0.11) and (B.0.13) and denoting

$$\overline{P_t Q_t} = E_0(P_t Q_t)$$
$$\bar{Q}_t = E_0(Q_t)$$

then

$$d\overline{P_t Q_t} = (\alpha - \gamma)\overline{P_t Q_t} dt, \quad \overline{P_0 Q_0} = PQ, \tag{B.0.14}$$
$$d\bar{Q}_t = -\gamma \bar{Q}_t dt, \quad \bar{Q}_0 = Q. \tag{B.0.15}$$

Solving (B.0.14) and (B.0.15) we have

$$\overline{P_t Q_t} = PQ e^{(\alpha - \gamma)t}, \tag{B.0.16}$$
$$\bar{Q}_t = Q e^{-\gamma t}. \tag{B.0.17}$$

Using convergence condition $\rho + \gamma - \alpha > 0$, and substitute (B.0.16), (B.0.17) into (B.0.12) we obtain,

$$V(P,Q) = \frac{\gamma PQ}{\rho + \gamma - \alpha} - \frac{a\gamma Q}{\rho + \gamma}. \tag{B.0.18}$$

# Appendix C

# Anticipated Deterministic Regime Switching

In order to provide a basis for comparison with the stochastic regime switching, I first give a digression on the deterministic regime switching. Some results of this section can be found in Precious, but I shall focus my attention on the higher order joining conditions at the boundary when switching occurs and some properties of value functions and shadow costs of various constraints. First part briefly shows the Hamiltonian approach to the optimal investment problem in infinite time horizon. In the second part, this problem is viewed in a different perspective, namely, in dynamic programming context. There, an optimality condition for the value function in contrast to that in the first part (Euler equations) is derived. The last part will summarise some basic properties of the anticipated deterministic regime switching, the interpretation will be given for the higher order joining conditions.

## C.1 Hamiltonian Approach

Consider a partial equilibrium optimal investment problem, where the firm is maximising the future discounted stream of profits subject to exogenous output constraint. Suppose the production function is Cobb-Douglas, the price for capital goods is unity and the costs of investment is convex with respect to the investment rate. The profit maximising firm will act to maximise the present discounted value of the cash flows which is given by

$$V(k) = \int_t^\infty \{pQ(t) - wL(t) - [I(t) + C(I)]\} e^{-r(s-t)} ds, \qquad \text{(C.1.1)}$$

where all the functions are defined in the same way as those in the text, all the constraints still apply here, except for the capital accumulation equation which is written as

$$\dot{K}(s) = I(s) - \delta K(s); \qquad \text{(C.1.2)}$$

The firm will maximise the expected present value (C.1.1) subject to constraints (4.2.3)-(4.2.5), and (C.1.2). The Hamiltonian formed for this problem is

$$H = [pF(K,L) - wL - (I + C(I)] + \mu[I - \delta K] + \lambda(\bar{Q} - F(K,L)), \quad \text{(C.1.3)}$$

where $\mu$ is the shadow costs of constraint (C.1.2), $\lambda$ is the shadow costs of constraint (4.2.4), and

$$\lambda(\bar{Q} - F(K,L)) = 0, \qquad \text{(C.1.4)}$$

or

$$\lambda = 0 \quad \text{if} \quad \bar{Q} > F(K, L) \tag{C.1.5}$$

$$\lambda > 0 \quad \text{if} \quad \bar{Q} = F(K, L) \tag{C.1.6}$$

Here, both $I$ and $L$ are control variables with $L \geq 0$. The first order condition for $I$ yields

$$\mu = 1 + C'(I), \tag{C.1.7}$$

which shows the relationship between the shadow costs of capital, the price of investment goods (which is assumed to be unity) and marginal adjustment costs.

The first order condition for $L$ yields

$$F_L = \frac{w}{p - \lambda} \tag{C.1.8}$$

so if the firm is demand constrained, the marginal productivity of labour is not greater than the real wage. When the firm is strictly unconstrained, the marginal productivity of labour is equal to the real wage.

The optimality condition for $\mu$ yields

$$\dot{\mu} = (r + \delta)\mu - (p - \lambda)F_k, \tag{C.1.9}$$

which gives a dynamic equation for shadow costs $\mu$. All the primes and subscripts here denote derivatives and partial derivatives respectively. And $\lambda = 0$ is for the case where the firm is strictly unconstrained.

From equation (C.1.7), because of the convexity of adjustment costs, we have a unique inverse for $I$, namely

$$I = I(\mu - 1). \tag{C.1.10}$$

Incorporating equations (C.1.8)–(C.1.10) and (C.1.2) yields two Euler equations for $\mu$ and $K$,

$$\dot{K}(s) = I(\mu - 1) - \delta K, \tag{C.1.11}$$

$$\dot{\mu}(s) = (r + \delta)\mu - w\frac{F_K}{F_L}. \tag{C.1.12}$$

It is advantageous to express these Euler equations in phase diagrams to see the dynamic response of $\mu$ and $K$ and the state dependent behaviour of $\mu$ on $K$. Several examples of anticipated regime switching can be found in Precious (1985), one of them will be discussed later after the description of the dynamic programming approach.

## C.2 Dynamic Programming Approach

The Hamiltonian approach to deterministic Precious model leads to Euler equations which give the dynamic response to both shadow costs and capital accumulation. The state dependent behaviour is then investigated by using phase diagrams. The state dependent optimal investment rule is simply given by the inverse of shadow costs $\mu$, and the labour input is then determined by marginal productivity of labour condition. Before preceding to the use of phase diagrams, we first look at this problem in a different perspective, which might give further implications.

Consider the optimisation problem given from the equations (C.1.1)–(4.2.5), the dynamic programming approach will lead to an optimality condition which describes the state dependent behaviour in terms of value function, namely

$$rV(k) = \max_{I,L}\{(I - \delta k)V'(k) + pQ - wL - (I + C) + \lambda(Q - F(k, L)\}. \tag{C.2.1}$$

The first order conditions for $L$ is the same as (C.1.8). Provided that investment is not restricted to positive numbers (i.e., disinvestment is allowed), the first order condition for $I$ yields

$$V'(k) = 1 + C'(I). \tag{C.2.2}$$

Comparing this with equation (C.1.7), we have

$$V'(k) = \mu(k), \tag{C.2.3}$$

so the interpretation of $\mu$ is clear, it is simply the change of the value of the firm with respect to that of capital (the same interpretation is also provided in Precious (1985), but derived by different method).

The second order condition is given by a Hessian matrix formed by the following entries:

$$\mathcal{L}_{LL} = (p - \lambda)F_{LL}, \tag{C.2.4}$$

$$\mathcal{L}_{II} = -C''(I), \tag{C.2.5}$$

$$\mathcal{L}_{IL} = \mathcal{L}_{LI} = 0, \tag{C.2.6}$$

where $\mathcal{L}$ denotes the right hand side of (C.2.1). It is easy to verify that this matrix is negative definite provided $p \geq \lambda$, which ensures that the optimal investment policy exists.

From the first order conditions, we can determine that the optimal employment and investment are $L^*, I^*$ respectively, so the Bellman equation becomes

$$rV(k) = pF(k, L^*) - wL^* + (I^* - \delta k)V'(k) - (I^* + C(I^*)). \quad \text{(C.2.7)}$$

The interpretation of this Bellman equation is transparent. The left hand side of the equation is the expected future discounted profits. The first two terms of the right hand side are the profits generated by production alone without further investment. The last two terms constitute the opportunity cost of carrying out investment. The term $(I^* - \delta k)V'(k)$ represents the payoff of doing the investment, where $(I^* - \delta k)$ contributes to the expansion or contraction of the firm. The term $I^* + C(I^*)$ represents the investment costs.

Unlike the Euler equations which give the instantaneous rules for labour input and investment (or $\mu$ equivalently) at any given time, the Bellman equation gives an marginal decision rule for the firm's intertemporal investment problem with respect to a given state $k$. So we have effectively transformed the dynamic optimal policies into the state contingent optimal policies. Notice that equation (C.2.7) is simply an arbitrage condition. It is optimal if the return on firm's assets is equal to the profits *plus* the expected gain on further investment.

If the optimisation is taken in the infinite time horizon, the transversality condition will confine the state dependent $V'(k)$ onto the stable manifold given by Euler equation without considering the regime switching possibility. In the unconstrained case, $V'(k)$ is a constant (as $\mu$), the optimal investment yields a constant rule which compensates the rate of capital depreciation as it is given when the system is in equilibrium. Therefore, when the initial capital stock is less than that of the equilibrium capital level, the firm will keep expanding its size, and the rate of expansion is decreasing over time as well as over $k$. If the firm's initial capital stock is greater than the equilibrium capital stock, the size of the firm is decreasing over time (and over $k$), until it converges to the equilibrium. We notice that the simplicity of using the phase diagrams is due to that under optimal policies there is unique mapping between states and time, or in another word, the optimal policies expressed in terms of time are equivalent to those in terms of states.

Using the first order optimality condition for $I$, differentiate it with respect to $k$, one has

$$V''(k) = C''(I^*)I'(k), \tag{C.2.8}$$

It is easy to see that in the unconstrained case, $V'' = 0, C'' > 0$, so $I'(k) = 0$.

In the constrained case, the stable manifold is downward sloping. Because $C''(I^*) > 0$, from equation (C.2.8) $V''(k) < 0$, i.e., the investment is decreasing when $k$ increases.

## C.3 Anticipated Regime Switching

In order to compare the deterministic and stochastic regime switching for given exogenous demand constraints, I shall digress the anticipated deterministic switching from demand unconstrained regime to constrained regime, which is provided in Precious (1985). Precious considers the case where there is the possibility that for falls in the interest rate and wage rate at some initial date $t$ the previously demand constrained firm finds itself facing no constraint after the sudden relaxation of sales restriction, but because of the desired gradual increase in output (the equilibrium output level of the unconstrained firm is greater than the previous constrained one) in response to such changes, the firm will, at some later date $t'$, find itself facing a demand constraint (the equilibrium output level of the unconstrained firm is also greater than that of the constrained one). Given rational expectation which in this model implies perfect foresight the firm must anticipate that it will eventually switch to the constrained regime. It is important to know how this anticipation will affect the firm's investment behaviour and what would be the implication of the higher order joining condition.

For given output, adjustment costs do not depend on time explicitly, $p, w$ remain constant, the shadow costs of capital accumulation should also be state dependent. In this case, the switching is state dependent at a given capital level $\dot{k}$. Below $\dot{k}$ the demand constraint is not binding because the output level of the unconstrained firm for given lower level of capital stock should also be lower. Above $\dot{k}$ the firm is virtually constrained.

Before drawing the phase diagram for this anticipated switching as that in Precious (1985), I shall summarise some basic properties for the value function (or $\mu$). The first two propositions are given in Precious and the second one has been proved there. The other properties are provided for giving clearer picture of this switching.

**Proposition C.1** *For given initially predetermined capital stock $k$, if it is less than $\bar{k}$, the firm will choose the unconstrained value function even it facing future demand constraints.*

In order to prove this proposition, we have to notice two facts: the optimal value function of the unconstrained firm is an upward sloping straight line and the value function of the constrained firm has the concave form. If at the point $k$ where $V_k^u = V_k^c$, the value function of the unconstrained firm is greater than that of the constrained firm, then the optimal value function of the unconstrained firm will always be greater that that of the constrained firm (separating hyperplane theorem).

**Proof:** Consider at point $k$ where $V_k^u = V_k^c$. Optimise equation (C.2.1) over $L$ for both constrained and the unconstrained cases. Subtracting them yields,

$$r(V^u - V^c) = \alpha p \left[\frac{w}{(1-\alpha)p}\right]^{\frac{\alpha-1}{\alpha}} k - (\alpha(p-\lambda)+\lambda) \left[\frac{w}{(1-\alpha)(p-\lambda)}\right]^{\frac{\alpha-1}{\alpha}} k \tag{C.3.1}$$

Because the two terms on the right hand side of equation (C.3.1) are all positive, we can consider their ratio.

Let

$$R = \frac{\alpha p\left[\frac{w}{(1-\alpha)p}\right]^{\frac{\alpha-1}{\alpha}}}{(\alpha(p-\lambda)+\lambda)\left[\frac{w}{(1-\alpha)(p-\lambda)}\right]^{\frac{\alpha-1}{\alpha}}} \frac{\alpha p}{\alpha(p-\lambda)+\lambda}\left[\frac{p-\lambda}{p}\right]^{\frac{\alpha-1}{\alpha}} \tag{C.3.2}$$

Let

$$x = \frac{\lambda}{p-\lambda}, \quad x \geq 0 \tag{C.3.3}$$

Then

$$R = \frac{\alpha}{\alpha + x}[1+x]^{\frac{1}{\alpha}} \tag{C.3.4}$$

Differentiate $R$ with respect to $x$ yields,

$$R' = \frac{(1+x)^{\frac{1}{\alpha-1}}(1-\alpha)x}{(\alpha+x)^2} \tag{C.3.5}$$

When $x = 0, R = 1$, we have $R'(x > 0) > 0$, so

$$R \geq 1 \tag{C.3.6}$$

The strict inequality holds for $x > 0$. So we complete the proof.

The reason for this is clear. For the firm to instantaneously produce a higher level of output than would a unconstrained firm will drive down the marginal productivity of labour below the real wage because the shadow costs of demand constraint is positive $\lambda > 0$. In that case, profits can be increased by shedding labour and output which could remain the marginal productivity of labour to be the real wage. Therefore, even if the firm expects to become demand constrained in the future, the output produced at the initial stage for given the capital level is lower than that of the constrained firm will be nevertheless the unconstrained one.

**Proposition C.2** *The marginal productivity of labour for both unconstrained and constrained firms are equal at the switching point.*

The proof of this proposition is in Precious (1985, p.15). It is also pointed out there that the switching point can be determined by the interception of $\dot{\mu} = 0$ loci for both unconstrained and constrained cases. This is due to the fact that no jump of the shadow costs of capital accumulation is allowed during switching for this rational expectation model. This proposition leads immediately to the following property.

**Proposition C.3** *At switching point, the shadow costs of demand constrained firm is zero.*

It is easy to verify this proposition. Given the matching condition of marginal productivity of labour provided in Proposition C.2, we have

$$F_L^u = \frac{w}{p}, \tag{C.3.7}$$

and

$$F_L^c = \frac{w}{p - \lambda}. \tag{C.3.8}$$

where superscripts denote the regimes.

If at $\mathring{k}$, $F_L^u$, $F_L^c$ are matched, then $\lambda = 0$. This suggests that the switching occurs smoothly in terms of output, below $\mathring{k}$ the unconstrained firm will naturally increase its output when $k$ increases and at $\bar{k}^-$ it reaches the level $\bar{Q}$. At $\bar{k}^+$ the demand constraint starts binding, the unconstrained firm is now being constrained. In another word, $\bar{k}$ is the highest capital level which the unconstrained firm can operate optimally.

For given switching point $\bar{k}$ provided by Proposition C.2, using Proposition C.3 and Euler equations, we have the following property.

**Proposition C.4** *At the switching point $\bar{k}$, the derivatives of the shadow costs of capital accumulation with respect to capital should be matched.*

**Proof:** The Euler equations for the unconstrained firm are,

$$\dot{\mu} = (r + \delta)\mu - pF_K \tag{C.3.9}$$

$$\dot{K} = \frac{\mu - 1}{\beta} - \delta K \tag{C.3.10}$$

$$pF_L = w \tag{C.3.11}$$

Then,

$$\frac{d\mu}{dK} = \frac{(r + \delta)\mu - \alpha p\left[\frac{w}{(1-\alpha)p}\right]^{\frac{\alpha - 1}{\alpha}}}{\frac{\mu - 1}{\beta} - \delta K} \tag{C.3.12}$$

The Euler equations for the constrained firm have the similar form as those for the unconstrained firm, the only difference results from the dynamics of $\mu$, namely,

$$\dot{\mu} = (r + \delta)\mu - (p - \lambda)F_K \tag{C.3.13}$$

where $\lambda$ is the shadow cost of sales constraint.

Then,

$$\frac{d\mu}{dK} = \frac{(r+\delta)\mu - \alpha p\left[\frac{w}{(1-\alpha)(p-\lambda)}\right]^{\frac{\alpha-1}{\alpha}}}{\frac{\mu-1}{\beta} - \delta k}. \tag{C.3.14}$$

At the switching point $k$, the shadow costs $\lambda$ are the same for the both regimes, therefore at $k$,

$$\left[\frac{d\mu}{dK}\right]^u = \left[\frac{d\mu}{dK}\right]^c \tag{C.3.15}$$

This completes the proof.

Notice that the shadow price $\mu$ is actually the derivative of the firm's optimal value function so this proposition is to ensure that the second order smooth pasting condition is satisfied.

Propositions C.2 and C.3 are the lower order smooth conditions at the switching boundary in this rational expectation model. Remember that this first order optimality condition for investment established the relationship between shadow costs of capital accumulation and optimal investment (given price for the investment goods is unity). Then no anticipated jump for $\mu$ leads to the joining condition for the investment at the boundary, and this also leads to the joining conditions for both marginal product of labour and output. All these joining conditions ensure that the value function is twice continuously differentiable in both regimes.

The higher order smooth pasting condition for $\mu'$ (or $V''$) is ensured as the lower order smooth conditions are satisfied and also dynamical optimality conditions (Euler equation) hold. From (C.2.7) we have

$$\mu'(k) = C''(I^*)I'(k), \qquad \text{(C.3.16)}$$

and

$$V''(k) = \mu'(k). \qquad \text{(C.3.17)}$$

At the switching boundary, $I$ is matched due to the first order smooth pasting condition of value functions. Then the change of investment due to the change of $k$ on the two sides of the switching point are the same.

The reason for Propositions C.4 is clear, it is simply due to the fact that the switching point is optimally chosen. Suppose $\mu'$ is not matched, and $[\mu'(k)]^u > [\mu'(k)]^c$ rewrite Bellman equation (C.2.1) here, we have

$$rV(k) = pF(k, L^*) - wL^* + (I^* - \delta k)V'(k) - (I^* + C(I^*)). \qquad \text{(C.3.18)}$$

Notice that by lower order smooth condition, every term in equation (C.3.18) on both sides of the switching point is equal. Because $\dot{k}$ is less than equilibrium capital of both constrained and unconstrained firm, the optimal investment rule demands

$$(I^* - \delta k) > 0 \quad \text{around} \quad k = \dot{k}. \qquad \text{(C.3.19)}$$

For given $[\mu'(k)]^u > [\mu'(k)]^c$ the expected change of optimal net investment gain for the unconstrained firm is greater than that of the constrained firm. So even when $k$ immediately crosses the boundary, the firm will still remain unconstrained, therefore $\hat{k}$ is not the optimal switching point, instead the optimal switching point if it exist should be greater than $\hat{k}$.

The same logic applies if we construct the case where $[\mu'(k)]^u < [\mu'(k)]^c$. This leads to the optimal switching point less than $\hat{k}$. Combine these two cases we have the smooth pasting condition for $\mu$ indeed ensure that the switching point is optimally chosen.

Using Propositions C.1–C.4, the phase diagram for this anticipated regime switching problem is shown in Figure 4.1.

The optimal switching path is given by curve $\widehat{ABC}$ , and the point $A$ is determined by initial capital stock $K(0)$, the Euler equation for the unconstrained firm gives a tangent condition at $B$. In the unconstrained regime when the firm anticipates the future switching to the constrained regime at $\hat{k}$, the investment behaviour is cautious.

In order to clarify this statement, we first develop the following proposition.

**Proposition C.5** *The value function for the unconstrained firm under anticipated switching is concave, its corresponding shadow costs $\mu$ is alway below the one without switching.*

**Proof:** The second part of Proposition C.5 is easy to verify. From Proposition C.4, we notice that the optimal solution of $\mu$ for the unconstrained firm when facing switching should be tangent to constraint stable manifold $S^cS^c$ at $B$, and this solution can not cross the unconstrained stable manifold $A'B'$, then $AB$ is alway below $A'B'$.

To prove the first part we have to use Euler equations for the unconstrained case. We note that any point below $A'B'$, the time change of $\mu$ is negative, and the solution lies above line $FC$ which gives the time change of $k$ is positive, namely for any point along $AB$ we have

$$\frac{d\mu}{dt} < 0, \tag{C.3.20}$$

and

$$\frac{dK}{dt} > 0. \tag{C.3.21}$$

Then

$$\frac{d\mu}{dK} = \frac{d\mu}{dt} / \frac{dK}{dt} < 0, \tag{C.3.22}$$

or

$$V''(K) < 0. \tag{C.3.23}$$

Here we conclude Proposition C.5.

The investment behaviour given by Proposition C.5 for the unconstrained firm facing switching is cautious, the level of investment is always less than that of totally unconstrained firm and it keeps decreasing when $k$ increases. The optimal behaviour of the firm will inevitably drive the unconstrained firm to become constrained. If the firm increases the investment over $k$ or keeps it in a higher level, the firm will more quickly become demand constrained than it could have by reducing the investment over $k$. For given value function is concave over $k$, the payoff of exercising future investment opportunity is diminishing. If the firm maintains the same level of investment as it would when the firm is totally unconstrained, the net gain of investment opportunity declines due to that the adjustment costs will remain unchanged, and the value of exercising investment opportunity decreases more quickly than it adopts decreasing investment policy. Therefore it is not surprising why the unconstrained firm will squeeze out some unnecessary investment and keep it declining when facing the anticipation of switching to the constrained regime.

# Appendix D

# Unconstrained Solutions

If shortselling and borrowing are not restricted, there is a monotonic mapping between the wealth $x(t)$ and optimal consumption $c(t)$ (denoted by $X(c)$). Under the optimally selected portfolio rule, one can derive a linear ordinary differential equation for $X(c)$ from the Bellman equation, and the general solutions obtained only have some undetermined parameters. The forms of optimal portfolio and consumption rules can thereafter be obtained. Furthermore, by some transformations, a value function which depends on the optimal consumption can be found to satisfy a similar ordinary differential equation. Verifying this value function subject to the initial conditions, we can therefore parameterise the optimal portfolio and consumption rules.

However, such method may fail if consumption starts binding below some wealth level $\bar{x} > 0$ because the mapping between $x(t)$ and $c(t)$ does not exist for $x < \bar{x}$. In this situation, we choose $y = dV^*(x)/dx$ to be the intermediate variable, where $V^*(x)$ is the optimal value function. By adopting such alternative, the construction of the optimal solution follows the same way as described above.

In what follows, we provide the solution methods to general utility functions in the first two sections. Section 1 deals with the case where consumption constraint ($c = 0$) is not binding. In this section, we first derive the monotonic mapping $X(c)$ from the Bellman equation and give the relevant forms of optimal portfolio and consumption policies. Second, we derive the corresponding value functions. And finally, we present the optimal solutions to various cases. Any proof omitted here can be found in Karatzas *et al* (1986). Section 2 deals with the case where consumption constraint ($c = 0$) binds when wealth falls below a threshold. The procedure to obtain the optimal solutions in this section is similar to that used in Section 1. The last two sections present the solutions to CRRA and CARA utility functions, where the proofs that the unconstrained portfolio policy ($\pi$) is a decreasing function of consumption (or a increasing function of $dV^*(x)/dx$) are provided.

## D.1 General Utility Function: $c = 0$ not Binding

### D.1.1 Mapping between optimal consumption and wealth.

Let $X(c)$ be the mapping from the optimal consumption $c$ to its corresponding wealth $x$, then to find the mapping $X(c)$, first, we optimise this variational equation over portfolio $\pi$, given that $V''(x) < 0$, and no constraint applying to $\pi$, then

$$\pi = -\frac{(\alpha - r)V'(x)}{\sigma^2 x V''(x)}, \tag{D.1.1}$$

the Bellman equation therefore becomes

$$\beta V(x) = \frac{\gamma(V'(x))^2}{V''(x)} + \max_{c\geq 0}[(rx - c)V'(x) + U(c)]. \tag{D.1.2}$$

Let $C(x)$ be the inverse mapping of $X(c)$. If $c = 0$ is not binding, maximising over $c$ gives

$$V'(x) = U'(C(x)), \tag{D.1.3}$$

with second order derivative of value function satisfying

$$V''(x) = U''(c(x))C'(x), \quad \text{and} \quad C'(x)X'(c) = 1. \tag{D.1.4}$$

The Bellman equation reduces to

$$\beta V(X(c)) = -\frac{\gamma(U'(c))^2 X'(c)}{U''(c)} + [rX(c) - c]U'(c) + U(c). \tag{D.1.5}$$

Differentiating with respect to $c$ yields a second-order, linear, ordinary differential equation

$$\gamma X''(c) = [(r - \beta - 2\gamma)\frac{U''(c)}{U'(c)} + \frac{\gamma U'''(c)}{U''(c)}]X'(c) + [\frac{U''(c)}{U'(c)}]^2(rX(c) - c). \tag{D.1.6}$$

It is not difficult to verify that the general solution is given by

$$X(c;a,B) \equiv B(U'(c))^{\lambda_+} + \frac{c}{r} - \frac{1}{\gamma(\lambda_+ - \lambda_-)}[\frac{(U'(c))^{\lambda_+}}{\lambda_+}\int_a^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{(U'(c))^{\lambda_-}}{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}], \quad c > a. \tag{D.1.7}$$

With

$$X'(c;a,B) > 0$$

where $a \geq 0$ is the lowest possible consumption level under which the agent will choose to exit; $B \leq 0$ is a constant which generates a non-linear relationship between consumption and wealth. These parameters will be determined by the introduction of the bankruptcy conditions.

Because $\lim_{c\to\infty} X(c;a,B) = \infty$, $X(c;a,B)$ maps consumption $[a,\infty)$ onto wealth $[X(a;a,B),\infty)$, and its inverse $C(x;a,B)$ exists, is increasing and maps $[X(a;a,B),\infty)$ onto $[a,\infty)$. With the aid of this mapping, we have established the feedback rule between optimal consumption and wealth, so the optimal policies are:

$$c(t) = C(x(t)), \tag{D.1.8}$$

$$\pi(t) = -\frac{(\alpha - r)U'(c(t))}{\sigma^2 x(t)U''(c(t))C'(x(t))}, \tag{D.1.9}$$

for initial wealth $x_0$ such that $x_0 > X(a;a,B)$. Furthermore, the optimal consumption satisfies the following stochastic differential equation

$$dc(t) = -\frac{U'(c(t))}{U''(c(t))}[r - \beta + \gamma\frac{U'(c(t))U'''(c(t))}{(U''(c(t)))^2}]dt$$

$$- \frac{\alpha - r}{\sigma} \frac{U'(c(t))}{U''(c(t))} dW_t, \tag{D.1.10}$$
$$c(0) = c_0 = C(x_0);$$

with the instantaneous variance of the consumption being

$$\frac{Var(dc(t))}{dt} = \frac{(\alpha - r)^2}{\sigma^2} \left(\frac{U'(c(t))}{U''(c(t))}\right)^2 = 2\gamma \left(\frac{U'}{U''}\right)^2. \tag{D.1.11}$$

### D.1.2 Optimal value function

For the optimal policies given above, the corresponding value function can be obtained through a simple transformation. Let $x_0 > X(a; a, B)$, and corresponding $C_0 = C(x_0)$. By the Markov property

$$V_{c(.),\pi(.)}(x_0) = H(c_0) \equiv E_{x_0}\left[\int_0^{T_{X(a,a,B)}} e^{-\beta t} U(c(t))dt + \bar{v}e^{-\beta T_{X(a,a,B)}}\right]. \tag{D.1.12}$$

where $T_{X(a;a,B)}$ is the first time that the wealth reaches zero, and $\bar{v}$ can be interpreted as the corresponding bankruptcy value.

Let $y = U'(c)$, assume that the inverse $c = I(y)$ exists and define

$$\begin{aligned} G(y_0) &= H(I(y_0)) \\ &\equiv E_{x_0}\left[\int_0^{T_{X(a,a,B)}} e^{-\beta t} U(I(y_t)dt + \bar{v}e^{-\beta T_{X(a;a,B)}}\right], \quad \text{(D.1.13)} \\ &\quad 0 < y_0 < U'(a). \end{aligned}$$

Using Feynman-Kac formula

$$\begin{aligned} \beta G(y) &= -(r - \gamma)yG'(y) + \gamma y^2 G''(y) + U(I(y)), \\ &\quad 0 < y < U'(a), \end{aligned} \tag{D.1.14}$$

$$\lim_{y \uparrow U'(a)} G(y) = \bar{v}.$$

Transforming back using $H(c) = G(U'(c))$, then

$$\beta H(c) = -\frac{U'(c)}{U''(c)}[r - \beta + \gamma \frac{U'(c)U'''(c)}{(U''(c))^2}]H'(c)$$
$$+\gamma(\frac{U'(c)}{U''(c)})^2 H''(c) + U(c), \quad c > a. \qquad \text{(D.1.15)}$$
$$\lim_{c \downarrow a} H(c) = \bar{v}.$$

The non-explosive general solution for the value function ($J(c, \cdot, \cdot)$ is a function of optimal consumption) is

$$J(c; a, A) = A(U'(c))^{\rho_+} + \frac{U(c)}{\beta} - \frac{1}{\gamma(\rho_+ - \rho_-)}[\frac{(U'(c))^{\rho_+}}{\rho_+}\int_a^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$+\frac{(U'(c)^{\rho_-}}{\rho_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}]. \qquad \text{(D.1.16)}$$

and

$$\rho_\pm = 1 + \lambda_\pm$$

(See Karatzas *et al* (1986) Proposition 8.1.)

Now we provide a theorem from Karatzas *et al* (1986) Theorem 9.1.

**Theorem D.1** *For* $a \geq 0, B \leq 0$, *the function*

$$V(x; a, B) \equiv J(c(x; a, B); a, \frac{\lambda_+}{\rho_+}B), \quad x > X(a; a, B) \qquad \text{(D.1.17)}$$

*satisfies the Bellman equation.*

The value function obtained in equation (D.1.16) is the mapping from optimal consumption to the present discounted value of expected utility.

The mapping from the wealth to the value function is achieved by substituting optimal consumption policy in (D.1.8) to (D.1.16).

### D.1.3 Selection of arbitrary constants $a$ and $B$

What left to be determined for the optimal value function and optimal consumption and investment policies are the constants $a$ and $B$ for given initial conditions. Since the value function has to satisfy the initial condition $V(0) = P$, let $x = 0$ and subsequently $c \to a$, the value function (D.1.16) becomes

$$V(0) = -\frac{1}{\rho_+}[-\frac{U'(a)^{\rho_-}}{\gamma\lambda_-\rho_-}\int_a^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} - \frac{\rho_+}{\beta}U(a) + \frac{\lambda_+}{r}aU'(a)], \quad a > 0. \tag{D.1.18}$$

For $P > U(0)/\beta$ and $U'(0) = +\infty$, $V(0) = P$ has one and only one positive solution $a$.

For given $a$, $B$ can be chosen such that $x(a; a, B) = 0$, so

$$B(U'(a))^{\lambda_+} + \frac{a}{r} - \frac{U'(a)^{\lambda_-}}{\gamma\lambda_-(\lambda_+ - \lambda_-)}\int_a^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} = 0. \tag{D.1.19}$$

## D.2 General Utility Function: $c = 0$ Binds

If $U(0)$ and $U'(0)$ are both finite, the previous method may fail to produce solutions. In what follows, we deal with this situation by letting $y = dV^*(x)/dx$ be the intermediate variable. Since the optimal value function $V^*(x)$ is strictly concave, the mapping from $x$ to $y$ is invertible. Furthermore, for $x \geq \bar{x}$ ($\bar{x}$ is the wealth level under which $c = 0$ binds), the relationship between wealth and optimal consumption is given by the first order optimality condition $dV^*(x)/dx = U'(c) = y$.

Define a function $I$ which is the inverse of $U'(c)$ such that

$$I: \quad (0, U'(0)] \to [0, \infty).$$

We can extend $I$ by setting $I \equiv 0$ on $[U'(0), \infty)$. From equation (D.1.7), let $a = 0$ and $c = I(y)$, we derive

$$\mathcal{X}(y; B) = By^{\lambda_+} + \frac{1}{r}I(y) - \frac{1}{\gamma(\lambda_+ - \lambda_-)}\{\frac{y^{\lambda_+}}{\lambda_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\lambda_-}}{\lambda_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}\}, \tag{D.2.1}$$

where $\mathcal{X}(y; \cdot)$ is a mapping from $y$ (defined as $dV(x)/dx$) to wealth and $B \leq 0$.

By analogy to equation (D.1.16), one obtains the value function as

$$\mathcal{J}(y; B) = Ay^{\rho_+} + \frac{1}{\beta}U(I(y)) - \frac{1}{\gamma(\lambda_+ - \lambda_-)}\{\frac{y^{\rho_+}}{\rho_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\rho_-}}{\rho_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}\}, \tag{D.2.2}$$

where $\mathcal{J}(y; \cdot)$ is the value function in terms of optimal $y$ (or $dV(x)/dx$) and $A \leq 0$. From Theorem D.1 we have $A = \frac{\lambda_+}{\rho_+}B$.

Define $P^*$ as

$$P^* = \frac{1}{\beta}U(0) - \frac{(U'(0))^{\rho_-}}{\beta\lambda_-}\int_0^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}, \tag{D.2.3}$$

then equation (D.1.18) has an unique root $a \geq 0$ if and only if $P \geq P^*$. For $U(0)/\beta \leq P < P^*$, (D.1.18) has no nonnegative solution because $c = 0$ binds when $x > 0$. Since the case when $P \geq P^*$ can be dealt with by using the method in previous subsection, we only consider the case when $U(0)/\beta \le P < P^*$.

Because $c = 0$ binds for $x < \bar{x}$, the consumption takes the form

$$c = \begin{cases} 0, & 0 \le x \le \bar{x}, \\ C(x; 0, B), & x \ge \bar{x}. \end{cases} \tag{D.2.4}$$

where

$$B = -\frac{\beta \bar{y}^{\lambda_-}}{\gamma(\lambda_+ - \lambda_-)\bar{y}^{\rho_+}} \cdot (P - \frac{1}{\beta}U(0)), \tag{D.2.5}$$

and $\bar{y}$ is defined as

$$\bar{y}^{\rho_-} = -\beta\lambda_-(P - \frac{1}{\beta}U(0))[\int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}]^{-1}. \tag{D.2.6}$$

So $\bar{x}$ can be determined by

$$\bar{x} = B(U'(0))^{\lambda_+} - \frac{(U'(0))^{\lambda_-}}{\gamma\lambda_-(\lambda_+ - \lambda_-)} \int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}. \tag{D.2.7}$$

Since from equation (D.2.1), $\mathcal{X}'(y; B) < 0$, the inverse $\mathcal{Y}(x; B)$ exists and is a mapping from $[0, \infty)$ to $(0, \bar{y}]$, where $\bar{y} = \infty$ if $B = 0$.

By the definition $V'(x) = \mathcal{Y}(x; B)$, and using equation (D.2.1), (D.2.2), one derives

$$V'(x) = \mathcal{Y}(x : B) = \frac{\mathcal{J}'(\mathcal{Y}(x; B); \frac{\lambda_+}{\rho_+} B)}{\mathcal{X}'(\mathcal{Y}(x; B); B)} > 0, \;\; x > 0, \tag{D.2.8}$$

$$V''(x) = \mathcal{Y}'(x; B) = \frac{1}{\mathcal{X}'(\mathcal{Y}(x; B); B)} < 0, \quad x > 0. \tag{D.2.9}$$

and the optimal policies in terms of $y_t$ are simply

$$c_t = I(V'(x_t)) = I(y_t), \tag{D.2.10}$$

$$\pi_t = -\frac{(\alpha - r)V'(x_t)}{\sigma^2 x_t V''(x_t)} = -\frac{(\alpha - r)y_t \mathcal{X}'(y_t; B)}{\sigma^2 \mathcal{X}(y_t; B)}. \qquad \text{(D.2.11)}$$

## D.3 Solutions to CRRA Utility Function

In what follows, given CRRA utility function, we compute the optimal consumption and portfolio policies of equations (D.1.8) and (D.1.9) and value function (D.1.16) for given value of $a$ and $B$. To compute (D.1.8), first we have to determine the mapping given in equation (D.1.7) and then take the inverse. After we obtain (D.1.7) and (D.1.8), it is straightforward to derive (D.1.9). Furthermore, using (D.1.16) and Theorem D.1, we can calculate value function as a function of optimal consumption, using (D.1.8) the value function which depends on the wealth can be then determined.

As a requirement for the finiteness of the value function, for CRRA utility functions, we demand specifically (see equation (5.4.7))

$$1 + \lambda_- \eta < 0. \qquad \text{(D.3.1)}$$

For the CRRA utility functions given as

$$U(c) = \frac{c^{1-\eta}}{1-\eta}, \qquad \eta > 0, \qquad \text{(D.3.2)}$$

The first order derivative is

$$U'(c) = c^{-\eta}. \qquad \text{(D.3.3)}$$

Rearranging (D.1.7), we arrive at

$$rX(c;a,B) = rB(U'(c))^{\lambda_+} - \frac{\lambda_-}{r(\lambda_+ - \lambda_-)}g(c;a) + \frac{\lambda_+}{r(\lambda_+ - \lambda_-)}f(c), \tag{D.3.4}$$

where

$$g(c;a) = c - (U'(c))^{\lambda_+}\int_a^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}, \tag{D.3.5}$$

$$f(c) = c + (U'(c))^{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}. \tag{D.3.6}$$

Substituting (D.3.2) and (D.3.3) into (D.3.5) yields

$$g(c;a) = \frac{c}{1+\lambda_+\eta}[\lambda_+\eta + (\frac{a}{c})^{1+\lambda_+\eta}]. \tag{D.3.7}$$

Substituting (D.3.2), (D.3.3) into (D.3.6) and using (D.3.1),

$$f(c) = \frac{\lambda_-\eta}{1+\lambda_-\eta}c. \tag{D.3.8}$$

So (D.3.4) becomes

$$\begin{aligned} rX(c;a,B) = rBc^{-\lambda_+\eta} &- \frac{\lambda_-}{\lambda_+ - \lambda_-}\frac{\lambda_+\eta}{1+\lambda_+\eta}(\frac{a}{c})^{1+\lambda_+\eta} \\ &+ \frac{\lambda_+\eta}{1+\lambda_+\eta}\frac{\lambda_-\eta}{1+\lambda_-\eta}c. \end{aligned} \tag{D.3.9}$$

For $a \geq 0, B \leq 0$, $rX(c;a,B)$ is a monotonic increasing function of $c$, the inverse $c = C(x)$ exists and is unique.

To determine $a \geq 0$, from (D.1.18) we define

$$F(c) = -\frac{(U'(c))^{\rho_-}}{\gamma\lambda_-\rho_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} - \frac{\rho_+}{\beta}U(c) + \frac{\lambda_+}{r}cU'(c), \tag{D.3.10}$$

where

$$\rho_\pm = 1 + \lambda_\pm, \quad \text{and} \quad \rho_+\rho_- = -\frac{\beta}{\gamma}. \tag{D.3.11}$$

Substituting (D.3.2), (D.3.3) into (D.3.10) and using (D.3.1)

$$F(c) = [(1-\eta)(\frac{1}{\gamma\lambda_-\rho_-(1+\lambda_-\eta)} + \frac{\lambda_+}{r}) - \frac{\rho_+}{\beta}]U(c). \tag{D.3.12}$$

Using (D.3.11)

$$F(c) = \frac{\eta^2}{\gamma(1+\lambda_-\eta)}\frac{c^{1-\eta}}{1-\eta}. \tag{D.3.13}$$

Equation (D.1.18) becomes

$$\frac{\eta^2}{\gamma(1+\lambda_-\eta)}\frac{a^{1-\eta}}{1-\eta} = -\rho_+P. \tag{D.3.14}$$

The left hand side of the above equation is a strictly decreasing function of $a$. It is obvious that if $U(0)/\beta < P < \lim_{c\to\infty} U(c)/\beta$, (D.3.14) has a unique solution $a > 0$.

To determine $B$, we use (D.1.19) and the solution obtained from (D.3.14), so

$$B = -\frac{a^{1+\lambda_+\eta}}{r(\lambda_+ - \lambda_-)}\frac{\lambda_-}{1+\lambda_-\eta}[1 - \eta(\lambda_+ - \lambda_-)] < 0. \tag{D.3.15}$$

To eliminate the arbitrary constants in (D.3.9), we substitute (D.3.15) and the solution to (D.3.14) into (D.3.9), then

$$rX(c;a,B) = \frac{\lambda_+\eta}{1+\lambda_+\eta}\frac{\lambda_-\eta}{1+\lambda_-\eta}c[1 - (\frac{a}{c})^{1+\lambda_+\eta}]. \tag{D.3.16}$$

Now we turn to the determination of optimal portfolio policy given in equation (D.1.9). Since (D.3.16) is a monotonic increasing function, we have

$$C'(x)X'(c;a,B) = 1. \qquad \text{(D.3.17)}$$

Substituting into (D.1.9)

$$\begin{aligned} \pi &= -\frac{(\alpha - r)}{\sigma^2}\frac{U'(c)}{U''(c)}\frac{X'(c;a,B)}{X(c;a,B)}, \\ &= -\frac{(\alpha - r)}{\sigma^2}\frac{U'(c)}{U''(c)}\frac{(rX)'}{rX}. \end{aligned} \qquad \text{(D.3.18)}$$

From (D.3.4), differentiating $rX$ with respect to $c$, we have

$$(rX(c;a,B))' = \lambda_+ \cdot \frac{U''(c)}{U'(c)}[rX(c;a,B) - f(c)], \qquad \text{(D.3.19)}$$

putting back into (D.3.18), so

$$\pi(c;a,B) = \frac{(\alpha - r)}{\sigma^2\eta}\frac{1 + \lambda_+\eta(\frac{a}{c})^{1+\lambda_+\eta}}{1 - (\frac{a}{c})^{1+\lambda_+\eta}}. \qquad \text{(D.3.20)}$$

For the specified CRRA utility function given in (D.3.2), and chosen parameters $a, B$ given in (D.3.14) and (D.3.15), the value function becomes

$$J(c;a,B) = -\frac{\eta^2}{\gamma}\frac{1}{1 + \lambda_+\eta}\frac{1}{1 + \lambda_-\eta}\frac{c^{1-\eta}}{1 - \eta}[1 - \frac{(1-\eta)\lambda_+}{\rho_+}(\frac{a}{c})^{1+\lambda_+\eta}]. \qquad \text{(D.3.21)}$$

For the utility function given in (D.3.2), the variance becomes

$$\frac{Var(dc(t))}{dt} = \frac{2\gamma}{\eta^2}c^2(t). \qquad \text{(D.3.22)}$$

### D.3.1 Solutions when $P = U(0)/\beta$

When the utility function is CRRA and $P$ is the natural bankruptcy value, the optimal consumption is a linear function of wealth and optimal portfolio rule is such that the investment made in the stock (or in the bond) is a constant proportion of the total wealth. To provide these results, we first present a theorem without proof which is a combination of Theorems 10.1 and 11.4 in Karatzas *et al* (1986).

**Theorem D.2** *Provided that the utility function is $U(c) = \frac{c^{1-\eta}}{1-\eta}$, $\eta > 0$, and $P = U(0)/\beta$; by setting $a = 0$, $B = 0$, the optimal consumption and portfolio rules are given in (D.1.8) and (D.1.9), the value function is given by (D.1.16).*

To derive the optimal consumption, we set $a = 0$ in equation (D.3.16) ($B = 0$ is a consequence of (D.3.15) by letting $a = 0$) obtaining

$$X(c; 0, 0) = \frac{\lambda_+\eta}{1 + \lambda_+\eta} \frac{\lambda_-\eta}{1 + \lambda_-\eta} \frac{c}{r}. \tag{D.3.23}$$

So the optimal consumption can be written in a feedback form

$$c = \left(\frac{\lambda_+\eta}{1 + \lambda_+\eta} \frac{\lambda_-\eta}{1 + \lambda_-\eta}\right)^{-1} rx. \tag{D.3.24}$$

Letting $a = 0$ in equation (D.3.20), we obtain the optimal portfolio policy

$$\pi = \frac{(\alpha - r)}{\sigma^2\eta}. \tag{D.3.25}$$

The optimal value function corresponding to these policies is achieved by setting $a = 0$ in equation (D.3.21)

$$J(c; 0, 0) = -\frac{\eta^2}{\gamma}\frac{1}{1+\lambda_+\eta}\frac{1}{1+\lambda_-\eta}\frac{c^{1-\eta}}{1-\eta}, \qquad \text{(D.3.26)}$$

written in a feedback form

$$V(x) = J(C(x); 0, 0) = \frac{x^{1-\eta}}{1-\eta}\left(\frac{1}{r}\frac{\lambda_+\eta}{1+\lambda_+\eta}\frac{\lambda_-\eta}{1+\lambda_-\eta}\right)^{\eta}. \qquad \text{(D.3.27)}$$

From equations (D.3.23) and (D.3.24), the optimal consumption is a linear function of wealth with a constant propensity to consume which depends on all the parameters determining the price processes as well as the parameter describing the risk aversion; the optimal portfolio rule is a constant independent of the wealth level. It is obvious that if $\alpha - r < 0$, the optimal behaviour of the agent is to short sell stock; if $(\alpha - r)/\sigma^2\eta > \xi \geq 1$, the mean return rate on stock is sufficiently large so that even the risk averse agent will activate the borrowing constraint to finance its investment in stock.

### D.3.2 Solution when $P > U(0)/\beta$

In this case, equation (D.3.14) always has a unique solution $a > 0$, so the consumption constraint is inactive. The combination of Theorems 11.2 and 12.1 in Karatzas *et al* (1986) provides the following theorem.

**Theorem D.3** *For $U(0)/\beta < P < \lim_{c\to\infty} U(c)/\beta$, let $a$ be the unique positive solution to equation (D.3.14), and let $B$ be given by (D.3.15). The optimal consumption is the inverse of equation (D.3.16), the optimal portfolio rule is given by equation (D.3.20) and the value function* *is given in (D.3.21).*

When $\alpha - r < 0$, from (D.3.20) we have $\pi(c; a, B) < 0$, the optimal behaviour of the agent is to shortsell stock to maintain certain consumption level. When $\alpha - r > 0$, then $\pi(c; a, B) > 0$ and $\lim_{c \downarrow a} \pi(c; a, B) = +\infty$, so at the lower end of consumption (or equivalently when wealth level reaches zero) this policy involves unlimited borrowing, therefore this policy cannot be considered as optimal when borrowing restriction is imposed. We summarise the properties of the portfolio policy $\pi$ in the following proposition.

**Proposition D.1** *For the optimal portfolio policy given in (D.3.20), if $a > 0$ and $\alpha - r < 0$, then*

$$\pi(c; a, B) < 0.$$

*If $\alpha - r > 0$, then*

$$\lim_{c \downarrow a} \pi(c; a, B) = +\infty, \quad \text{(D.3.28)}$$

$$\lim_{c \downarrow \infty} \pi(c; a, B) = \frac{\alpha - r}{\sigma^2 \eta}, \quad \text{(D.3.29)}$$

*and $\pi(c; a, B)$ is a strictly decreasing function of $c$.*

**Proof** Equations (D.3.28)-(D.3.29) are obvious. To prove $\pi(c; a, B)$ is a strictly decreasing function of $c$, we take the derivative of $\pi(c; a, B)$ with respect to $c$, which yields

$$\frac{\partial \pi(c; a, B)}{\partial c} = -\frac{\alpha - r}{\sigma^2 \eta} \cdot \frac{(1 + \lambda_+ \eta)^2}{[1 - (\frac{a}{c})^{1+\lambda_+\eta}]^2} \cdot \frac{1}{c} (\frac{a}{c})^{1+\lambda_+\eta} < 0. \quad \text{(D.3.30)}$$

Therefore, we complete the proof.

## D.4 Solutions to CARA Utility Function

For given CARA utility function as

$$U(c) = -\frac{e^{-\eta c}}{\eta}, \quad \eta > 0, \tag{D.4.1}$$

equation (5.4.7) is finite, so is the value function.

From (D.2.3), using the utility function above, we obtain

$$P^* = -\frac{1}{\beta\eta}(1 - \frac{1}{\lambda_-{}^2}). \tag{D.4.2}$$

Whether $c = 0$ binds or not, we use the method described in section 2.2. Notice if $c = 0$ does not binds, let $\bar{y}$ be the maximum $y$ such that $\mathcal{X}(\bar{y}; \bar{y}, B) = 0$, then $y < \bar{y} \leq U'(0) = 1$. Otherwise, we have $\bar{y} > 1$.

The difference between the cases that $c = 0$ binds or not is that $\bar{y}$ (or $a$) and $B$ are determined in different ways. If $P \geq P^*$, equation (D.1.18) has a unique root $a \geq 0$, then $a$ (or $\bar{y} = U'(a)$) and $B$ can be determined by (D.1.18) and (D.1.19), namely, substituting (D.4.1) into (D.1.18) and (D.1.19) yields

$$V(0) = -(-\frac{\lambda_- - 1}{\gamma\lambda_-{}^2\eta} + \frac{\lambda_+}{r}a)\frac{e^{-\eta a}}{\rho_+} = P.$$
$$B = -(\frac{a}{r} + \frac{1}{\gamma\lambda_-{}^2(\lambda_+ - \lambda_-)\eta})e^{\lambda_+\eta a} < 0, \; a \geq 0.$$

Let

$$y = U'(c), \bar{y} = U'(a), \tag{D.4.3}$$

then $y < \bar{y} \leq 1$ and the above two equations become

$$V(0) = (\frac{\lambda_- - 1}{\gamma \lambda_-^2 \eta} + \frac{\lambda_+}{r\eta} \ln \bar{y}) \frac{\bar{y}}{\rho_+} = P. \tag{D.4.4}$$

$$B = -(-\frac{1}{r\eta} \ln \bar{y} + \frac{1}{\gamma \lambda_-^2 (\lambda_+ - \lambda_-)\eta}) \bar{y}^{-\lambda_+} < 0. \tag{D.4.5}$$

For given $\bar{y}$ and $B$, one can determine wealth as a function of $y$ from (D.2.1), i.e.,

$$\mathcal{X}(y; \bar{y}, B) = (B + \frac{1}{\gamma \lambda_-^2 (\lambda_+ - \lambda_-)\eta \bar{y}^{\lambda_+}}) y^{\lambda_+} - \frac{\ln y}{r\eta} - \frac{1}{r\eta}(\frac{1}{\lambda_+} + \frac{1}{\lambda_-}). \tag{D.4.6}$$

Since (D.4.5) is a strictly decreasing function of $y$, the inverse $\mathcal{Y}(x; B)$ exists and can be used to obtain the optimal consumption.

Substituting (D.4.1) into (D.2.2), one obtains the value function

$$\mathcal{J}(y; B) = (\frac{\lambda_+}{\rho_+} B + \frac{1}{\gamma(\lambda_+ - \lambda_-)\lambda_+ \rho_+ \eta}) y_{\rho_+} - \frac{1}{r\eta} y, \tag{D.4.7}$$

where $B$ is given by (D.4.5).

If $U(0)/\beta \leq P < P^*$, $c = 0$ binds for $x \leq \bar{x}$, then $\bar{y}$ and $B$ have to be determined by (D.2.6) and (D.2.5), so

$$\bar{y} = [\beta \eta \lambda_-^2 (P + \frac{1}{\beta\eta})]^{1/\rho_-}, \tag{D.4.8}$$

$$B = -\frac{\beta}{\gamma(\lambda_+ - \lambda_-)} \cdot (\beta \eta \lambda_-^2)^{-\rho_+/\rho_-} (P + \frac{1}{\beta\eta})^{1-\rho_+/\rho_-}. \tag{D.4.9}$$

And $\bar{x}$ can therefore be written as

$$\begin{aligned}\bar{x} = &-\frac{\beta}{\gamma(\lambda_+ - \lambda_-)} \cdot (\beta\eta\lambda_-{}^2)^{-\rho_+/\rho_-}(P + \frac{1}{\beta\eta})^{1-\rho_+/\rho_-} \\ &+\frac{1}{\gamma(\lambda_+ - \lambda_-)\lambda_-{}^2\eta}.\end{aligned} \tag{D.4.10}$$

In this case, because $\bar{y} > U'(0) = 1$, the wealth equation is divided into two pieces, i.e.,

$$\mathcal{X}(y;\bar{y},B) = \begin{cases} By^{\lambda_+} + \frac{1}{\gamma(\lambda_+-\lambda_-)\lambda_-{}^2\eta}y^{\lambda_-}, & y \geq 1, \\ \\ (B + \frac{1}{\gamma(\lambda_+-\lambda_-)\lambda_+{}^2\eta\bar{y}^{\lambda_+}})y^{\lambda_+} - \frac{\ln y}{r\eta} - \frac{1}{r\eta}(\frac{1}{\lambda_+} + \frac{1}{\lambda_-}), & y \leq 1, \end{cases} \tag{D.4.11}$$

and $\mathcal{X}(y;\bar{y},B)$ is continuous at $y = 1$.

The value function can be broken down similarly as

$$\mathcal{J}(y;\bar{y},B) = \begin{cases} \frac{\lambda_+}{\rho_+}By^{\lambda_+} + \frac{1}{\gamma(\lambda_+-\lambda_-)\rho_-\lambda_-\eta}y^{\rho_-} - \frac{1}{\beta\eta}, & y \geq 1 \\ \\ (\frac{\lambda_+}{\rho_+}B + \frac{1}{\gamma(\lambda_+-\lambda_-)\rho_+\lambda_+\eta})y^{\rho_+} - \frac{1}{r\eta}y, & y \leq 1. \end{cases} \tag{D.4.12}$$

where $B$ is given by (D.4.9).

In both cases, the optimal portfolio policy is given in (D.2.11). Because $\mathcal{X}' < 0$, so if $\alpha - r < 0$, then $\pi < 0$. For $\alpha - r > 0$, we define the following function

$$f(y) = \begin{cases} -\frac{1}{\lambda_-\eta}y^{\lambda_-}, & y \geq 1, \\ \\ -\frac{\ln y}{\eta} - \frac{1}{\lambda_-\eta}, & y \leq 1. \end{cases} \tag{D.4.13}$$

Then the portfolio policy can be expressed as

$$\pi = \frac{\lambda_+(\alpha - r)}{\sigma^2}\{\frac{f(y)}{r\mathcal{X}} - 1\}. \tag{D.4.14}$$

In what follows, we present the solutions to three different cases, and the propositions related to the portfolio policies only consider the case where $\alpha - r > 0$ since otherwise $\pi < 0$.

### D.4.1 Solutions when $P = U(0)/\beta$

In this case, the value function and the wealth equation are obtained by setting $B = 0$ in (D.4.12) and (D.4.11). The properties of $\pi$ are given in the following proposition.

**Proposition D.2** *For the CARA utility function specified in (D.4.1) and $P = U(0)/\beta$, if $\alpha - r > 0$ and $y \geq 1$, then*

$$\pi(y;\bar{y},0) = -\frac{\lambda_-(\alpha - r)}{\sigma^2}; \tag{D.4.15}$$

*if $y < 1$, then*

$$\lim_{y\uparrow 1} \pi(y;\bar{y},0) = -\frac{\lambda_-(\alpha - r)}{\sigma^2}, \tag{D.4.16}$$

$$\lim_{y\downarrow 0} \pi(y;\bar{y},0) = 0, \tag{D.4.17}$$

*and $\pi(y;\bar{y},0)$ is a strictly increasing function of $y$ for $y < 1$.*

**Proof** By letting $B = 0$ in (D.4.11) and using (D.4.14), one can easily verify (D.4.15)–(D.4.17). Here, we only prove that $\pi(y;\bar{y},B)$ is a strictly increasing function of $y$ for $y < 1$. Differentiating (D.4.14) with respect to $y$, noticing that the sign of $\pi'$ only depends on $f'\mathcal{X} - \mathcal{X}'f$, we write

$$f'\mathcal{X} - \mathcal{X}'f = \frac{1}{\lambda_+\eta^2 y}\{\frac{1}{r}(1 - y^{\lambda_+}) + \frac{y^{\lambda_+}\ln y}{\gamma(\lambda_+ - \lambda_-)}\}.$$

Define

$$G(y) = \frac{1}{r}(1 - y^{\lambda_+}) + \frac{y^{\lambda_+}\ln y}{\gamma(\lambda_+ - \lambda_-)},$$

one immediately has

$$\lim_{y\uparrow 1} G(y) = 0,$$

and

$$\frac{\partial G(y)}{\partial y} = \frac{\lambda_+}{\gamma(\lambda_+ - \lambda_-)} y^{\lambda_+ - 1}[\ln y + \frac{1}{\lambda_-}] < 0, \quad y < 1.$$

So $G(y)$ attains minimum at $y = 1$, thus

$$f'\mathcal{X} - \mathcal{X}'f = \frac{1}{\lambda_+\eta^2 y} G(y) > 0, \quad \text{for} \quad y < 1.$$

Thus, we complete the proof.

### D.4.2 Solutions when $U(0)/\beta < P < P^*$

Similar to last case, the value function and the wealth equation are obtained from (D.4.12) and (D.4.11) but with $B < 0$. The properties of $\pi$ are summarised in the following proposition.

**Proposition D.3** *For the CARA utility function specified in (D.4.1) and $U(0)/\beta < P < P^*$, if $\alpha - r > 0$, then*

$$\lim_{y \uparrow \bar{y}} \pi(y; \bar{y}, B) = +\infty, \tag{D.4.18}$$

$$\lim_{y \downarrow 0} \pi(y; \bar{y}, B) = 0, \tag{D.4.19}$$

*and $\pi(y; \bar{y}, B)$ is a strictly increasing function of $y$.*

**Proof** Since (D.4.18) and (D.4.19) are obvious, we only prove that $\pi(y; \bar{y}, B)$ is a strictly increasing function of $y$.

In regime $y \geq 1$, from (D.4.11), (D.4.13) and (D.4.14), we derive

$$\begin{aligned}
\frac{\partial \pi(y; \bar{y}, B)}{\partial y} &= \frac{\lambda_+(\alpha - r)}{\sigma^2} \frac{f'\mathcal{X} - \mathcal{X}'f}{r\mathcal{X}^2} \\
&= \frac{\lambda_+(\alpha - r)}{\sigma^2 r \mathcal{X}^2} \frac{y^{\lambda_+ + \lambda_- - 1}}{\gamma(\lambda_+ - \lambda_-)\lambda_-{}^2 \eta^2}(1 - \frac{\lambda_+}{\lambda_-}) \\
&\quad \cdot (\beta\eta\lambda_-{}^2)^{1-\rho_+/\rho_-}(P + \frac{1}{\beta\eta})^{1-\rho_+/\rho_-} > 0.
\end{aligned}$$

In the regime $0 < y < 1$,

$$\begin{aligned}
\frac{\partial \pi(y; \bar{y}, B)}{\partial y} &= \frac{\lambda_+(\alpha - r)}{\sigma^2 r \mathcal{X}^2} \{ -\frac{\beta}{\gamma(\lambda_+ - \lambda_-)} (\beta\eta\lambda_-{}^2)^{-\rho_+/\rho_-} \\
&\quad \cdot (P + \frac{1}{\beta\eta})^{1-\rho_+/\rho_-} \cdot \frac{y^{\lambda_+ - 1}}{\eta} (\frac{\lambda_+ - \lambda_-}{\lambda_-} + \lambda_+ \ln y) \\
&\quad + \frac{1}{\lambda_+ \eta^2 y} (\frac{1}{r}(1 - y^{\lambda_+}) + \frac{y^{\lambda_+} \ln y}{\gamma(\lambda_+ - \lambda_-)}) \}.
\end{aligned}$$

The first term inside the brackets is positive, and from the proof of last proposition, the second term is also positive. So for $0 < y < 1$, $\pi'(y; \bar{y}, B) > 0$. The proof is completed.

### D.4.3 Solutions when $P \geq P^*$

In this case, because $y < \bar{y} < 1$, so the value function and wealth are given by (D.4.7) and (D.4.6). The following proposition provides the properties of $\pi$.

**Proposition D.4** *For the CARA utility function specified in (D.4.1) and $P \geq P^*$, if $\alpha - r > 0$, then*

$$\lim_{y \uparrow \bar{y}} \pi(y; \bar{y}, B) = +\infty, \tag{D.4.20}$$

$$\lim_{y \downarrow 0} \pi(y; \bar{y}, B) = 0, \tag{D.4.21}$$

*and $\pi(y; \bar{y}, B)$ is a strictly increasing function of $y$.*

**Proof** Equations (D.4.20)–(D.2.3) are obvious. To prove $\pi(y; \bar{y}, B)$ is a strictly increasing function of $y$, we differentiate $\pi(y; \bar{y}, B)$ with respect to $y$ and notice the sign of this derivative only depends on the sign of $f'\mathcal{X} - \mathcal{X}'f$. Using equation (D.4.6) and (D.4.13), we obtain the following

$$f'\mathcal{X} - \mathcal{X}'f = \frac{1}{r\eta^2 y}\{\frac{y^{\lambda_+}}{\bar{y}^{\lambda_+}}(\frac{\lambda_+ - \lambda_-}{\lambda_-} + \lambda_+ \ln y)(\ln \bar{y} + \frac{\lambda_+ + \lambda_-}{\lambda_+\lambda_-}) + \frac{1}{\lambda_+}\}.$$

Define the function in the brackets on the right hand side of the above equation as $G(y; \bar{y})$ and notice from (D.4.3) that $y < 1$, then

$$\lim_{y \uparrow \bar{y}} G(y; \bar{y}) = \frac{2\lambda_+}{\lambda_-} \ln \bar{y} + \lambda_+ (\ln \bar{y})^2 + \frac{\lambda_+}{\lambda_-{}^2} > 0, \bar{y} < 1,$$

$$\lim_{y \downarrow 0} G(y; \bar{y}) = \frac{1}{\lambda_+} > 0.$$

Differentiating $G(y; \bar{y})$ with respect to $y$ yields

$$\frac{\partial G(y; \bar{y})}{\partial y} = \frac{{\lambda_+}^2 y^{\lambda_+ - 1}}{\bar{y}^{\lambda_+}} \left(\frac{1}{\lambda_-} + \ln y\right)\left(\ln \bar{y} + \frac{\lambda_+ + \lambda_-}{\lambda_+ \lambda_-}\right).$$

Since $0 < y < \bar{y} < 1$, so

$$\frac{1}{\lambda_-} + \ln y < 0.$$

Therefore, the sign of $G'(y; \bar{y})$ only depends on $\ln \bar{y} + \frac{\lambda_+ + \lambda_-}{\lambda_+ \lambda_-}$. Notice that $G$ is positive at the boundaries and $G'$ does not change sign in between, then

$$G > 0, \quad \text{for} \quad 0 < y < \bar{y} < 1.$$

Hence

$$\frac{\partial \pi(y; \bar{y}, B)}{\partial y} > 0.$$

From the definition of $y$ in (D.4.3), because $U''(c) < 0$, so $y$ is a decreasing function of $c$. Thus from the above proposition, $\pi$ is a decreasing function of $c$. Notice that at $x = 0$, unlimited borrowing and shortselling occur, so the unconstrained solutions cannot be optimal if constraints are imposed.