On Some Twisted Kac-Moody Groups

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Contents

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This document was prepared using LaTeX.

Declaration

The material in the last three chapters of this thesis is, to the best of my knowledge, original and has not been published before.

Dedication

This work is dedicated to my grandmother,

Maria Jesús Bontempini Castillo
(21st October 1898 – 31st December 1991) .

Summary

This thesis consists of two distinct parts. The first part comprises the first three chapters and is largely of an expository nature. The second part comprises the last three chapters all of which are, to the best of our knowledge, original.

In the first part we cover the background material which we shall require in the sequel. Thus Chapter 1 deals with the theory of Kac-Moody algebras and is drawn from two main sources, namely [Kac90] and [BdK90]. Two enlightening examples are given at the end of this chapter.

Chapter 2 introduces the notion of the Kac-Moody group functor. This material is drawn largely, but not exclusively, from an extensive body of work on the topic by J. Tits. We give a presentation for Kac-Moody groups over fields and describe some of their properties.

In Chapter 3 we give an overview of some results on Kac-Moody groups. First we describe the work of J-Y. Hée generalizing the notion of twisted Chevalley groups to the Kac-Moody situation. We then give an exposition of the work of R.W. Carter and Y. Chen on the automorphisms of complex simply-connected affine Kac-Moody groups arising from extended Cartan matrices and we describe the classification of such automorphisms. In particular, we note that the family of diagonal automorphisms of such groups behave in a manner which has no analogy in the classical theory. We conclude the Chapter with an example demonstrating the limitation of Hée's results with regards to this type of automorphism.

Chapter 4 makes use of the results on Kac-Moody algebras described in §1.5 to extend the results of Hée. Suppose $\dot{A}$ is a simply-laced extended Cartan matrix and let $\mathfrak{G}_{\mathfrak{D}}^{\dot{A}}(\mathbb{K})$ be a Kac-Moody group associated to $\dot{A}$. In Chapter 4 we extend the results of Hée to the fixed point subgroup, $\mathfrak{G}_{\mathfrak{D}}^{\sigma}(\mathbb{K})$ say, of $\mathfrak{G}_{\mathfrak{D}}^{\dot{A}}(\mathbb{K})$ under a particular graph-by-diagonal automorphism. We then establish an isomorphism between the subgroup $\mathfrak{G}_{\mathfrak{D}}^{\sigma}(\mathbb{K})$ so obtained and a Kac-Moody group associated to an affine Cartan matrix of type II or III.

Thus Chapter 4 contains our main contributions for two reasons. Firstly, it provides a realization of Kac-Moody groups of types II and III in terms of those arising from extended Cartan matrices. More precisely, Propositions 4.4.3, 4.5.6, and 4.6.4 prove the following result.

Theorem 0.0.1

For each affine Cartan matrix B of type II or III there exists a simply-laced affine Cartan matrix $\dot{A}$ of higher rank with automorphisms γ, $\dot{\gamma}$, γ' and τ, $\dot{\tau}$, τ' of the group functors $\mathfrak{G}_{m}^{\dot{A}}$, $\mathfrak{G}_{ad}^{\dot{A}}$, and $\mathfrak{G}_{sc}^{\dot{A}}$ respectively, with fixed point subgroup functors $\mathfrak{G}_{m}^{\gamma,\tau}$, $\mathfrak{G}_{ad}^{\dot{\gamma},\dot{\tau}}$,

and $\mathfrak{G}_{sc}^{\gamma',\tau'}$ *respectively with group functor isomorphisms*

$$\begin{array}{lcccc} \Psi_m & : & \mathfrak{G}_m^B & \to & \mathfrak{G}_m^{\gamma,\tau} \\ \Psi_{ad} & : & \mathfrak{G}_{ad}^B & \to & \mathfrak{G}_{ad}^{\tilde{\gamma},\tilde{\tau}} \\ \Psi_{sc} & : & \mathfrak{G}_{sc}^B & \to & \mathfrak{G}_{sc}^{\gamma',\tau'} \end{array}$$

where the group functors are defined on the category of fields $\mathbb{K}$ *with* $\mathrm{char}\,\mathbb{K} \neq 2$ *satisfying the conditions that*

$$\sqrt{2} \in \mathbb{K} \textit{ if } B = {}^{\bullet}\tilde{C}_l \textit{ and}$$
$$\mathrm{char}\,\mathbb{K} \neq 3 \textit{ and } \mathbb{K} \textit{ contains a primitive cube root of unity if } B = {}^{t}\tilde{G}_2.$$

Finally, Chapter 4 contains calculations which pave the way for the last two chapters.

Chapters 5 and 6 are both straightforward consequences of Hée's theorem and calculations made in Chapter 4. In our sixth and final chapter we demonstrate how the fixed point subgroup of $\mathfrak{G}_D^{\tilde{A}}(\mathbb{K})$ under an automorphism which is the composition of a graph, diagonal and field automorphism does yield to Hée's approach. We conclude that the non-triviality of eigenspaces in the theory of automorphisms of fields plays a rôle in the applicability of Hée's theorem.

Introduction

As is the case with many research projects, the nature of this thesis has undergone various changes during the course of its production. When we began this project in October 1989, our aim was to initiate a study of the fixed point subgroups of Kac-Moody groups under certain automorphisms. In particular, we aimed to study the fixed point subgroups of Kac-Moody groups associated to affine Cartan matrices in an explicit manner such as that used for the Chevalley groups in [Ste67, §11]. In order to do this we first required a realization of the affine Kac-Moody groups of types II and III. We thus set about generalizing the results on Kac-Moody algebras in [Kac90, Chapter 8] to Kac-Moody groups.

It was at this point that we first became aware of the results of Hée announced in [H90]. These far-reaching results achieved many of the aims we had envisaged and had applications in more situations than we had anticipated. However, they did not, in their original form, apply in the situation we were studying at the time. They nevertheless gave us great insight into the methods we might use to prove some of the results we required.

Having completed the realizations of the affine Kac-Moody groups of types II and III, it seemed natural to add a field automorphism and to probe the applicability of Hée's results further. We thus decided to study the extent to which Hée's results applied to fixed point subgroups of Kac-Moody groups of extended type. Since Hée's results on Kac-Moody groups were generalizations of results from the classical theory of Chevalley groups, it seemed reasonable to consider automorphisms involving a non-trivial diagonal automorphism.

We thus turned our attention to an automorphism which was a combination of graph, field and diagonal automorphisms. Suppose that $\dot{A}$ is an extended Cartan matrix and that $\mathfrak{G}_{\mathcal{D}}^{\dot{A}}(\mathbb{K})$ is a Kac-Moody group associated to $\dot{A}$. We considered the graph automorphism, γ, of $\mathfrak{G}_{\mathcal{D}}^{\dot{A}}(\mathbb{K})$ inherited from an automorphism of the Dynkin diagram of A and as a preliminary step studied the fixed point subgroup of $\mathfrak{G}_{\mathcal{D}}^{\dot{A}}(\mathbb{K})$ obtained when combining γ with a field automorphism. We then added a diagonal automorphism and studied the result. What we discovered was that the presence of a field automorphism removed the difficulties we had previously encountered and the existence of non-trivial eigenspaces in the theory of automorphisms of fields meant that Hée's results were indeed applicable.

It remains to be seen what results are obtained when the graph automorphism is other than the one described. However, we note that only graph automorphisms induced by diagram automorphisms fixing the zeroeth vertex commute with non-trivial diagonal automorphisms. Furthermore, all such graph automorphisms have been covered in this thesis.

Chapter 1

Kac-Moody Algebras

We shall of necessity assume a familiarity with the theory of finite-dimensional semisimple Lie algebras. However, we shall not assume a great familiarity with the theory of arbitrary Lie algebras. Nevertheless, the well-established properties and notation (such as $[\cdot,\cdot]$ for the Lie bracket and $\operatorname{ad} x$ for the adjoint map of a Lie algebra $\mathfrak{g}$ associated to an element $x \in \mathfrak{g}$) will be assumed.

For the time being we shall be working over the complex field and shall denote by $\underline{n}$ and $\underline{n}_0$ the sets $\{1,\ldots,n\}$ and $\{0,1,\ldots,n\}$ respectively. Similarly, we shall denote by $\mathbb{N}$ the set of strictly positive integers and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

The first three sections of this chapter are expository. They rely heavily on [BdK90] and [Kac90]. Section 1.1 contains the general theory of Kac-Moody algebras with particular emphasis on the similarities and differences between the general theory and the finite-dimensional theory. Thus a perusal of this section should be enough to provide the necessary background in Kac-Moody algebras that I will require. For the sake of brevity I have omitted proofs wherever proofs already exist elsewhere and have given references to those sources.

Part of the material in §1.2 should also be familiar from the classical theory, but the material relating to the general theory is of sufficient importance that we deemed it necessary to highlight it by giving it a section of its own. Section 1.3 is an exposition of [Kac90, Chapter 6] and considers the structure theory of affine Kac-Moody algebras, including explicit descriptions of their root systems.

The last two sections describe realizations, by which I mean explicit constructions in terms of structures already known, of all affine Kac-Moody algebras. Section 1.4 is an exposition of [Kac90, Chapter 7] and is necessary background for §1.5. The latter is a description of the material introduced in [Kac90, Chapter 8], presented in a more explicit form. The detailed examples provided in §1.5 have not, to the best of our knowledge, appeared elsewhere. The material introduced there will be of great importance in the development of our later work.

1.1 General Theory

Basic Definitions

We start by defining the notion of a generalized Cartan matrix. Let $A = (A_{ij})_{i,j\in \underline{n}}$ be a real matrix of rank r. We call A a *generalized Cartan matrix*, or *GCM* for short, if it satisfies the following conditions;

- $A_{ii} = 2$ for all $i \in \underline{n}$,
- $A_{ij} \in \mathbf{Z}$,
- $A_{ij} \leq 0$ whenever $i \neq j$,
- $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$.

Note that we don't in general have $\det A > 0$. When all the leading minors of A are positive, then A is simply an ordinary Cartan matrix.

We use the term *Kac-Moody algebra* to describe a Lie algebra associated to a generalized Cartan matrix. Despite the fact that a deep theory can only be developed for Kac-Moody Algebras, we develop as much of the theory as is possible for Lie algebras associated to an arbitrary square matrix. To this end we define the *realization* of an arbitrary complex matrix $A = (A_{ij})_{i,j\in\underline{n}}$ to be a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that

- $\mathfrak{h}$ is a complex vector space of dimension $(2n - r)$ where $r = \operatorname{rank} A$,
- $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}$ is a set of n independent elements in $\mathfrak{h}$,
- $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of n independent elements in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$,
- $\alpha_j(\alpha_i^\vee) = \langle \alpha_j, \alpha_i^\vee \rangle = A_{ij}$ for all $i, j \in \underline{n}$ where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \mapsto \mathbb{C}$ is the dual contraction between $\mathfrak{h}$ and $\mathfrak{h}^*$.

We note that $\dim \mathfrak{h} \geq n$ and hence $\Pi^\vee$ is a basis for $\mathfrak{h}$ if and only if A is non-singular.

Two such realizations $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ are said to be *isomorphic* if there exists a vector space isomorphism

$$\phi : \mathfrak{h} \rightarrow \mathfrak{h}_1$$

such that

$$\phi(\Pi^\vee) = \Pi_1^\vee \text{ and } \phi^*(\Pi_1) = \Pi$$

where ϕ^* is the dual map of ϕ.

The following result can be found in [Kac90], though the proof we give is a corrected version of the original.

PROPOSITION 1.1.1

1. *There exists a realization for every complex $n \times n$ matrix A. Furthermore, such a realization is unique up to isomorphism.*

2. *Two complex $n \times n$ matrices A and B have isomorphic realizations if and only if B can be obtained from A by a permutation of the index set.*

Proof

We prove the existence of a realization by construction. Suppose rank $A = r$. Then, permuting the indices if necessary, we may assume that the matrix A takes the form

$$A = \begin{pmatrix} A(r) & B \\ C & D \end{pmatrix}$$

where $A(r)$ is an $r \times r$ submatrix of rank r. We now extend the matrix A to a $(2n-r) \times (2n-r)$ matrix E given by

$$E = \begin{pmatrix} A(r) & B & 0 \\ C & D & I_{n-r} \\ 0 & I_{n-r} & 0 \end{pmatrix}$$

where I_{n-r} is the $(n-r) \times (n-r)$ identity matrix. Then E is non-singular as

$$\det E = \pm \det A(r).$$

We now take $\mathfrak{h}$ to be $\mathbb{C}^{2n-r}$, $\alpha_1^\vee, \ldots, \alpha_n^\vee$ to be the first n rows of E, and $\alpha_1, \ldots, \alpha_n$ to be the first n linear coordinate functions. Then $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of the matrix A.

Now, given a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$, we can extend Π to a basis of $\mathfrak{h}^*$ by adding elements $\alpha_{n+1}, \ldots, \alpha_{2n-r} \in \mathfrak{h}^*$ so that we get

$$(\langle \alpha_j, \alpha_i^\vee \rangle) = \begin{pmatrix} A(r) & B & M \\ C & D & N \end{pmatrix}$$

for some $r \times (n-r)$ matrix M and some invertible $(n-r) \times (n-r)$ matrix N. Adding suitable linear combinations of $\alpha_1, \ldots, \alpha_r$ to the elements $\alpha_{n+1}, \ldots, \alpha_{2n-r} \in \mathfrak{h}^*$ we can make $M = 0$. Then, replacing $\alpha_{n+1}, \ldots, \alpha_{2n-r}$ by their linear combinations we get $N = I_{n-r}$.

Similarly, we extend $\Pi^\vee$ by adding elements $\alpha_{n+1}^\vee, \ldots, \alpha_{2n-r}^\vee \in \mathfrak{h}$. By analogous reasoning we get

$$(\langle \alpha_j, \alpha_i^\vee \rangle) = \begin{pmatrix} A(r) & B & 0 \\ C & D & I_{n-r} \\ 0 & I_{n-r} & 0 \end{pmatrix}$$

thus proving the uniqueness.

For part *b)* we first suppose that $A = (A_{ij})$ and $B = (B_{ij})$ have isomorphic realizations

$$(\mathfrak{h}, \Pi = \{\alpha_1, \ldots, \alpha_n\}, \Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\})$$

and

$$(\mathfrak{h}_1, \Pi_1 = \{\beta_1, \ldots, \beta_n\}, \Pi_1^\vee = \{\beta_1^\vee, \ldots, \beta_n^\vee\})$$

respectively. So there exists a vector space isomorphism

$$\phi : \mathfrak{h} \to \mathfrak{h}_1$$

such that

$$\phi(\Pi^\vee) = \Pi_1^\vee \text{ and } \phi^*(\Pi_1) = \Pi.$$

By permuting the indexing set of B if necessary, we may assume that

$$\phi(\alpha_i^\vee) = \beta_i^\vee \quad \text{and} \quad \phi^*(\beta_j) = \alpha_j.$$

However, then we have

$$\langle \beta_j, \beta_i^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle = A_{ij}.$$

So B can be obtained from A by a permutation of the indexing set. Also, from the proof of part *a)*, it is clear that if a matrix B can be obtained from a matrix A by a permutation of the indexing set then their realizations are isomorphic. □

EXAMPLE 1.1.2

1) The Cartan matrix of A_2,

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

has rank 2. Consequently we have $\mathfrak{h} = \mathbb{C}^2$ and

$$\alpha_1^\vee = (2, -1) \text{ and } \alpha_2^\vee = (-1, 2).$$

2) The generalized Cartan matrix

$$\tilde{D}_4 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

has rank 4. Hence $\mathfrak{h} = \mathbf{C}^6$ and we extend $\tilde{D}_4$ to the non-singular matrix

$$E = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

as described in the proof of Proposition 1.1.1. This gives

$$\begin{aligned} \alpha_1^\vee &= (2, 0, -1, 0, 0, 0) \\ \alpha_2^\vee &= (0, 2, -1, 0, 0, 0) \\ \alpha_3^\vee &= (-1, -1, 2, -1, -1, 0) \\ \alpha_4^\vee &= (0, 0, -1, 2, 0, 0) \\ \alpha_5^\vee &= (0, 0, -1, 0, 2, 1) \end{aligned}$$

◇

Given two matrices A_1 and A_2 with realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ respectively, we can construct a realization of the direct sum

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

of the two matrices , namely

$$(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \times \{0\} \cup \{0\} \times \Pi_2, \Pi_1^\vee \times \{0\} \cup \{0\} \times \Pi_2^\vee),$$

which is called the *direct sum* of the realizations.

A realization is said to be *decomposable* if the corresponding matrix is decomposable. Note that if we have a decomposable matrix A, then by a suitable reordering of indices we can decompose A into a direct sum of indecomposable matrices, and the corresponding realization into a direct sum of corresponding indecomposable realizations.

As in the finite-dimensional theory we use the following terminology;

- Π is called the *root basis*,
- $\Pi^\vee$ is called the *coroot basis*,
- elements from Π are called *simple roots*,
- elements from $\Pi^\vee$ are called *simple coroots*.

We also set

$$Q = \sum_{i=1}^{n} \mathbf{Z}\alpha_i, \quad Q_+ = \sum_{i=1}^{n} \mathbf{N}_0\alpha_i, \quad \text{and} \quad Q^\vee = \sum_{i=1}^{n} \mathbf{Z}\alpha_i^\vee,$$

where the Q is called the *root lattice* and $Q^\vee$ is called the *coroot lattice*.

Construction of Kac-Moody Algebras

Let $A = (A_{ij})$ be an $n \times n$ matrix over $\mathbb{C}$, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. We first introduce a Lie algebra $\tilde{\mathfrak{g}}_A(\mathbb{C})$ by means of generators and relations , namely

$$
\begin{array}{llll}
\text{Generators:} & e_i, f_i \text{ for } i \in \underline{n}, h \in \mathfrak{h}, & & \\
\text{Relations:} & [h, h'] & = 0 & \text{for } h, h' \in \mathfrak{h}, \\
 & [h, e_i] & = \langle \alpha_i, h \rangle e_i & \\
 & [h, f_i] & = -\langle \alpha_i, h \rangle f_i & \text{for } i, j \in \underline{n}. \\
 & [e_i, f_j] & = -\delta_{ij} \alpha_i^\vee &
\end{array}
\qquad (1.1.2)
$$

We note that our approach varies slightly from that taken by both [Kac90] and [BdK90] and instead we adopt for later convenience the approach taken by Jacques Tits (see for example [Tit85]), often referred to as *Tits' convention* in the literature. The difference lies largely in our choice of the generators $f_i, i \in \underline{n}$ which vary from their original choice only by a factor of -1. Thus we shall not reproduce any proofs of subsequent results where any changes that need to be made to the source quoted are simply to compensate for this fact. As far as our construction of a Kac-Moody algebra is concerned, we follow the approach originally to be found in [Moo67] and [Moo68]. However, by [Kac90, Proposition 5.12], the proofs of the results given in [Kac90] are still valid for this alternative definition.

By the uniqueness of the realization of A we note that $\tilde{\mathfrak{g}}_A(\mathbb{C})$ depends only on A.

We denote by $\tilde{\mathfrak{n}}_+$ (respectively $\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}_A(\mathbb{C})$ generated by the elements $e_i, i \in \underline{n}$ (respectively $f_i, i \in \underline{n}$).

The proof of the following structure theorem for $\tilde{\mathfrak{g}}_A(\mathbb{C})$ can be found in [Kac90, Theorem 1.2].

THEOREM 1.1.3

1. *We have a triangular decomposition of $\tilde{\mathfrak{g}}_A(\mathbb{C})$ when considered as a vector space, viz*

$$\tilde{\mathfrak{g}}_A(\mathbb{C}) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+.$$

2. *The subalgebra $\tilde{\mathfrak{n}}_+$ (respectively $\tilde{\mathfrak{n}}_-$) is freely generated by the elements $e_i, i \in \underline{n}$ (respectively $f_i, i \in \underline{n}$).*

3. *The map interchanging e_i with f_i for $i \in \underline{n}$ and mapping h to $-h$ for all elements $h \in \mathfrak{h}$, can be uniquely extended to an automorphism $\tilde{\omega}$ of the Lie algebra $\tilde{\mathfrak{g}}_A(\mathbb{C})$.*

4. *With respect to* $\mathfrak{h}$ *we get the root space decomposition:*

$$\tilde{\mathfrak{g}}_A(\mathbb{C}) = \left(\bigoplus_{\alpha\neq 0, \alpha\in Q_+} \tilde{\mathfrak{g}}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\neq 0, \alpha\in Q_+} \tilde{\mathfrak{g}}_{\alpha}\right),$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}_A(\mathbb{C}) : [h,x] = \alpha(h)x\}$. *Furthermore,*

$$\dim \tilde{\mathfrak{g}}_\alpha < \infty, \text{ and } \tilde{\mathfrak{g}}_\alpha \subset \tilde{\mathfrak{n}}_\pm \text{ for } \pm\alpha \in Q_+, \alpha \neq 0.$$

5. *The ideal of* $\tilde{\mathfrak{g}}_A(\mathbb{C})$ *defined by*

$$\mathfrak{m} = \langle \mathfrak{s}_+, \mathfrak{s}_- \rangle,$$

where

$$\mathfrak{s}_+ = \langle x = (\mathrm{ad}\, e_i)^{1-A_{ij}}(e_j) : i,j \in \underline{n} \rangle$$

and

$$\mathfrak{s}_- = \langle x = (\mathrm{ad}\, f_i)^{1-A_{ij}}(f_j) : i,j \in \underline{n} \rangle,$$

intersects $\mathfrak{h}$ *trivially and satisfies*

$$\mathfrak{m} \cap \tilde{\mathfrak{n}}_+ = \mathfrak{s}_+ \quad \text{and} \quad \mathfrak{m} \cap \tilde{\mathfrak{n}}_- = \mathfrak{s}_-.$$

Corollary 1.1.4
The natural map $\mathfrak{h} \to \tilde{\mathfrak{g}}_A(\mathbb{C})$ *is an embedding.*

Proof
This is a direct corollary of Theorem 1.1.3, part 1. □

We are now in a position to define the Lie algebra, $\mathfrak{g}_A(\mathbb{C})$, corresponding to a complex $n \times n$ matrix A.

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A, $\tilde{\mathfrak{g}}_A(\mathbb{C})$ the Lie algebra given by the presentation 1.1, and $\mathfrak{m}$ the ideal in $\tilde{\mathfrak{g}}_A(\mathbb{C})$ intersecting $\mathfrak{h}$ trivially defined in Theorem 1.1.3. Then the *Lie algebra* corresponding to A is defined to be the quotient algebra

$$\mathfrak{g}_A(\mathbb{C}) = \tilde{\mathfrak{g}}_A(\mathbb{C})/\mathfrak{m}.$$

We call n the *rank* of the Lie algebra $\mathfrak{g}_A(\mathbb{C})$ and the quadruple $(\mathfrak{g}_A(\mathbb{C}), \mathfrak{h}, \Pi, \Pi^\vee)$ the *quadruple associated to the matrix* A.

Two quadruples $(\mathfrak{g}_A(\mathbb{C}), \mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{g}_{A_1}(\mathbb{C}), \mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ are called *isomorphic* if there exists a Lie algebra isomorphism

$$\phi : \mathfrak{g}_A(\mathbb{C}) \mapsto \mathfrak{g}_{A_1}(\mathbb{C})$$

such that

$$\phi(\mathfrak{h}) = \mathfrak{h}_1, \quad \phi(\Pi^\vee) = \Pi_1^\vee, \quad \text{and} \quad \phi^*(\Pi_1) = \Pi.$$

EXAMPLE 1.1.5

Before proceeding, we recall the well-known Lie algebra corresponding to the Cartan matrix $A_1 = (2)$, namely the set of all traceless 2×2 complex matrices where the Lie product of two matrices M and N is defined to be

$$[M, N] = MN - NM.$$

We know that

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that we can take the elements

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

as generators for $\mathfrak{sl}_2(\mathbb{C})$.

We also recall that the simply connected Chevalley group corresponding to $\mathfrak{sl}_2(\mathbb{C})$ is $SL_2(\mathbb{C})$. ◇

We return now to the general theory of Kac-Moody algebras. We get the following structure theorem for $\mathfrak{g}_A(\mathbb{C})$. For the proof we refer the reader to [BdK90, theorem 10.4.3] and [Kac90, §1.3 and §3.3].

THEOREM 1.1.6

Let $\mathfrak{g}_A(\mathbb{C})$ be the Kac-Moody algebra belonging to the generalized Cartan matrix $A = (A_{ij})$. Then

1. *We have a triangular decomposition of $\mathfrak{g}_A(\mathbb{C})$ viewed as a vector space, viz*

$$\mathfrak{g}_A(\mathbb{C}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_+$ is generated by the elements e_i for $i \in \underline{n}$ and $\mathfrak{n}_-$ is generated by the elements f_i for $i \in \underline{n}$.

2. *With respect to $\mathfrak{h}$ we get the root space decomposition:*

$$\mathfrak{g}_A(\mathbb{C}) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_A(\mathbb{C}) : [h, x] = \alpha(h)x\}$$

is called the root space corresponding to α. Note that $\mathfrak{g}_0 = \mathfrak{h}$. Furthermore, $\dim \mathfrak{g}_\alpha < \infty$, and $\mathfrak{g}_\alpha \subset \mathfrak{n}_\pm$ for $\pm\alpha \in Q_+, \alpha \neq 0$.

3. *Let* $\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}\alpha_i^\vee + \mathbb{C}f_i$. *Then* $\mathfrak{g}_{(i)}$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$ *as a Lie algebra. Under this isomorphism*

$$e_i \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \alpha_i^\vee \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } f_i \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

4. *The Kac-Moody algebra* $\mathfrak{g}_A(\mathbb{C})$ *is given by the following presentation;*

$$\begin{array}{lllll} \textit{Generators:} & e_i, f_i \text{ for } i \in \underline{n}, h \in \mathfrak{h}, & & & \\ \textit{Relations:} & [h, h'] & = 0 & & \text{for } h, h' \in \mathfrak{h}, \\ & [h, e_i] & = \langle \alpha_i, h \rangle e_i & \Big\} & \\ & [h, f_i] & = -\langle \alpha_i, h \rangle f_i & & \text{for } i, j \in \underline{n}, \\ & [e_i, f_j] & = -\delta_{ij}\alpha_i^\vee & & \\ & (\mathrm{ad}\, e_i)^{1-A_{ij}} e_j & = 0 & \Big\} & \text{for } i \neq j. \\ & (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j & = 0 & & \end{array}$$

Example 1.1.7

If A is indecomposable and the determinant of each of its leading minors is positive, then the above definition gives rise to a simple Lie algebra of known type from the classical theory. ◇

The subalgebra $\mathfrak{h}$ of $\mathfrak{g}_A(\mathbb{C})$ is called the *Cartan subalgebra*. The elements

$$\{e_i, f_i : i \in \underline{n}\}$$

are called the *Chevalley generators*. In fact, the Chevalley generators generate the *derived subalgebra*

$$\mathfrak{g}'_A(\mathbb{C}) = [\mathfrak{g}_A(\mathbb{C}), \mathfrak{g}_A(\mathbb{C})].$$

Furthermore,

$$\mathfrak{g}_A(\mathbb{C}) = \mathfrak{g}'_A(\mathbb{C}) + \mathfrak{h} \qquad \text{with} \qquad \mathfrak{g}_A(\mathbb{C}) = \mathfrak{g}'_A(\mathbb{C}) \Leftrightarrow \det A \neq 0.$$

If we define

$$\mathfrak{h}' = \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee$$

then we have

$$\mathfrak{g}'_A(\mathbb{C}) \cap \mathfrak{h} = \mathfrak{h}', \qquad \text{and} \qquad \mathfrak{g}'_A(\mathbb{C}) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha \qquad \text{if} \qquad \alpha \neq 0.$$

If $\{\alpha_{n+1}^\vee, \ldots, \alpha_{2n-r}^\vee\}$ extends $\{\alpha_i^\vee\}_{i \in \underline{n}}$ to a basis of $\mathfrak{h}$, we call the set

$$\{e_i, f_i, \alpha_j^\vee \ : \ i \in \underline{n}, \ j \in \underline{2n-r}\}$$

a *Chevalley basis* of $\mathfrak{g}_A(\mathbb{C})$.

The number mult $\alpha = \dim \mathfrak{g}_\alpha$ in the root space decomposition of $\mathfrak{g}_A(\mathbb{C})$ is called the *multiplicity* of α. We call an element $\alpha \in Q$ a *root* if

$$\alpha \neq 0 \quad \text{and} \quad \text{mult}\,\alpha \neq 0.$$

For $\alpha = \sum_i k_i \alpha_i \in Q$ we call ht $\alpha = \sum_i k_i$ the *height* of α. We induce a partial ordering $\geq$ on $\mathfrak{h}^*$ by setting

$$\alpha \geq \beta \quad \text{if} \quad \alpha - \beta \in Q_+.$$

A root $\alpha > 0$ (respectively $\alpha < 0$) is called *positive* (respectively *negative*). From the root space decomposition of $\mathfrak{g}_A(\mathbb{C})$ (see theorem 1.1.3) we deduce that every root is either positive or negative. We denote by Φ, Φ_+ and Φ_- the sets of all, positive and negative roots respectively. We can then see that

$$\Phi = \Phi_+ \cup \Phi_-, \quad \text{and} \quad \Phi_+ \cap \Phi_- = \emptyset.$$

For the simple roots we have

$$\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i \quad \text{and} \quad \mathfrak{g}_{k\alpha} = 0 \text{ if } \mid k \mid > 1.$$

Since every root is either positive or negative we get

LEMMA 1.1.8
If $\beta \in \Phi_+ \setminus \{\alpha_i\}$, then $(\beta + \mathbb{Z}\alpha_i) \cap \Phi \subseteq \Phi_+$.

This result means that the ideal $\mathfrak{m}$ of $\tilde{\mathfrak{g}}_A(\mathbb{C})$ is $\tilde{\omega}$-invariant. Hence $\tilde{\omega}$ induces an involutive isomorphism ω of $\mathfrak{g}_A(\mathbb{C})$, called the *Chevalley involution* of $\mathfrak{g}_A(\mathbb{C})$. It is determined by

$$\omega(e_i) = f_i, \text{ and } \omega(f_i) = e_i \text{ for } i \in \underline{n}, \text{ and } \omega(h) = -h \text{ for } h \in \mathfrak{h}.$$

Note that since $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, we deduce that mult α = mult $(-\alpha)$ and in particular

$$\Phi_- = -\Phi_+.$$

When there is any danger of confusion, a symbol dependent on a matrix A may be followed by (A). Thus we can denote by $\Phi(A)$ and $Q(A)$ the set of roots and the root lattice associated to A respectively.

Duality of Kac-Moody Algebras

If A is a generalized Cartan matrix, then its transpose, ${}^tA$, is also a generalized Cartan matrix. Note that it follows from the definition of a realization that if

$(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of a matrix A, then $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ is a realization of ${}^tA$. So, if

$$(\mathfrak{g}_A(\mathbb{C}), \mathfrak{h}, \Pi, \Pi^\vee)$$

is the quadruple associated to A, then

$$(\mathfrak{g}_{{}^tA}(\mathbb{C}), \mathfrak{h}^*, \Pi^\vee, \Pi)$$

is the quadruple associated to ${}^tA$. The Kac-Moody algebras $\mathfrak{g}_A(\mathbb{C})$ and $\mathfrak{g}_{{}^tA}(\mathbb{C})$ are said to be *dual* to each other.

Note that the dual root lattice of $\mathfrak{g}_A(\mathbb{C})$ is the root lattice of $\mathfrak{g}_{{}^tA}(\mathbb{C})$. Denote by $\Phi^\vee \subseteq Q^\vee \subseteq \mathfrak{h}$ the root system $\Phi({}^tA)$ of $\mathfrak{g}_{{}^tA}(\mathbb{C})$. This is called the *dual root system* of $\mathfrak{g}_A(\mathbb{C})$.

In contrast to the finite-dimensional case, there is no natural bijection between Φ and $\Phi^\vee$.

Gradations of a Kac-Moody Algebra

Given an abelian group G, a decomposition

$$V = \bigoplus_{\alpha \in G} V_\alpha$$

of a vector space V into a direct sum of subspaces is called an *G-gradation* of V. Elements from V_α are called *homogeneous* of *degree* α. A subspace $U \subseteq V$ is called *graded* if

$$U = \bigoplus_{\alpha \in G} (U \cap V_\alpha).$$

EXAMPLE 1.1.9

Consider the vector space $\mathcal{L}$ of Laurent polynomials in one variable t

$$\mathcal{L} = \left\{ \sum_{i=m}^{n} c_i t^i : c_i \in \mathbb{C}, m, n \in \mathbb{Z}, n \geq m \right\}.$$

Then $\mathcal{L}$ has a $\mathbb{Z}$-gradation, namely

$$\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i \quad \text{with} \quad \mathcal{L}_i = \mathbb{C}t^k.$$

◇

A *G-gradation* of a Kac-Moody algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in G} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$ considered as a vector space into subspaces such that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

for all $\alpha, \beta \in G$.

Example 1.1.10

From Theorem 1.1.6 we have a $Q(A)$-gradation of the Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$

$$\mathfrak{g}_A(\mathbb{C}) = \bigoplus_{\alpha \in Q(A)} \mathfrak{g}_\alpha$$

where $Q(A)$ is the root lattice associated to $\mathfrak{g}_A(\mathbb{C})$. ◇

Some Ideals of a Kac-Moody Algebra

We define the *centre*, $\mathfrak{c}$, of a Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$ to be the ideal

$$\{x \in \mathfrak{g}_A(\mathbb{C}) : [x, y] = 0 \text{ for all } y \in \mathfrak{g}_A(\mathbb{C})\}.$$

We can in fact say quite a lot more about the centre of $\mathfrak{g}_A(\mathbb{C})$. To this end we have the following result.

Proposition 1.1.11

The centre $\mathfrak{c}$ of $\mathfrak{g}_A(\mathbb{C})$ satisfies the following conditions;

1. $\mathfrak{c} = \{h \in \mathfrak{h} : \langle \alpha_i, h \rangle = 0$ *for all* $i \in \underline{n}\}$
2. $\dim \mathfrak{c} = \dim \mathfrak{h} - n = n - r$ *where* $r = \operatorname{rank} A$
3. $\mathfrak{c}$ *is contained in the linear span of* $\Pi^\vee$ *and is also the centre of* $\mathfrak{g}'_A(\mathbb{C})$.

Proof

For proof see [BdK90, Lemma 11.2.1] and [Kac90, Proposition 1.6]. □

We are now in a position to give a description of the structure of ideals of $\mathfrak{g}_A(\mathbb{C})$. The proof of the following result can be found in [Kac90, Proposition 1.7].

Proposition 1.1.12

1. $\mathfrak{g}_A(\mathbb{C})$ *is simple if and only if* $\det A \neq 0$ *and for each pair of indices i and j there exist indices $k_r, r \in \underline{s}$ such that*

$$A_{ik_1} A_{k_1k_2} \ldots A_{k_sj} \neq 0.$$

2. *Provided the above condition on the indices holds, every ideal of* $\mathfrak{g}_A(\mathbb{C})$ *either contains* $\mathfrak{g}'_A(\mathbb{C})$ *or is contained in* $\mathfrak{c}$.

The Invariant Bilinear Form on a Kac-Moody Algebra

For the remainder of this section all Lie algebras will be of the form $\mathfrak{g}_A(\mathbb{C})$ where A is a generalized Cartan matrix.

We note that if we replace the Chevalley generators

$$\{e_i, f_i : i \in \underline{n}\} \quad \text{by} \quad \{e_i, \epsilon_i f_i : i \in \underline{n}\}$$

for some $\epsilon_i \in \mathbb{C}^\times$, we would have to replace the set

$$\{\alpha_i^\vee : i \in \underline{n}\} \quad \text{by} \quad \{\epsilon_i \alpha_i^\vee : i \in \underline{n}\}$$

if we wished to preserve the relations in $\mathfrak{g}_A(\mathbb{C})$. The map

$$\alpha_i^\vee \mapsto \epsilon_i \alpha_i^\vee$$

extends to an isomorphism $\phi_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$, though not necessarily uniquely since in general the set $\{\alpha_i^\vee\}_{i \in \underline{n}}$ does not span $\mathfrak{h}$. However, given such an isomorphism $\phi_{\mathfrak{h}}$, we can extend it to an isomorphism

$$\phi : \mathfrak{g}_A(\mathbb{C}) \to \mathfrak{g}_{DA}(\mathbb{C}),$$

where $D = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_n)$.

We call a matrix $A = (A_{ij})_{i,j \in \underline{n}}$ *symmetrizable* if there exists an invertible diagonal matrix

$$D = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_n)$$

with entries in $\mathbb{C}$ and a symmetric complex matrix

$$S = (S_{ij})$$

such that $A = DS$.

We then call S a *symmetrization* of A and $\mathfrak{g}_A(\mathbb{C})$ a *symmetrizable* Kac-Moody algebra.

EXAMPLE 1.1.13

1) Consider the matrix ${}^{\bullet}\tilde{A}_1 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. Then we can obtain the following two symmetrizations of ${}^{\bullet}\tilde{A}_1$:

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}.$$

2) ${}^{t}G_2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$ has a symmetrization

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 6 \end{pmatrix}.$$

◇

Note that in both the above examples all non-zero entries in the diagonal matrices are in $\mathbb{Q}_+$ and all entries in the symmetric matrices are in $\mathbb{Q}$.

LEMMA 1.1.14

1. *Given a symmetrizable matrix A, a decomposition into a product of a diagonal matrix D and a symmetric matrix S as above exists such that $\epsilon_i \in \mathbb{Q}_+$ for all $i \in \underline{n}$ and $S_{ij} \in \mathbb{Q}$ for all $i, j \in \underline{n}$.*

2. *If A is indecomposable as well as symmetrizable then the matrix D in the decomposition is uniquely determined up to a constant factor.*

Proof

See §2.3 of [Kac90] for proof. □

Now let $A = (A_{ij})_{i,j \in \underline{n}}$ be an arbitrary symmetrizable generalized Cartan matrix and fix a decomposition $A = DS$ as described above. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A and

$$\mathfrak{h}' = \sum_{i \in \underline{n}} \mathbb{C}\alpha_i^\vee \subseteq \mathfrak{h}.$$

Fix a complementary subspace $\mathfrak{h}_c$ to $\mathfrak{h}'$ in $\mathfrak{h}$. Now define a symmetric bilinear $\mathbb{C}$-valued form $(\cdot,\cdot)_\mathfrak{h}$ on $\mathfrak{h}$ by letting

$$(\alpha_i^\vee, h)_\mathfrak{h} = \epsilon_i \langle \alpha_i, h \rangle \text{ for } i \in \underline{n},\ h \in \mathfrak{h}.$$

and

$$(h, h')_\mathfrak{h} = 0 \text{ for } h, h' \in \mathfrak{h}_c.$$

REMARK 1.1.15

Note that there is no ambiguity in the definition of $(\cdot,\cdot)_\mathfrak{h}$ since $\Pi^\vee$ is a linearly independent set and

$$\begin{aligned} (\alpha_i^\vee, \alpha_j^\vee)_\mathfrak{h} &= \epsilon_i \epsilon_j S_{ji} \\ &= \epsilon_j \epsilon_i S_{ij} \\ &= (\alpha_j^\vee, \alpha_i^\vee)_\mathfrak{h} \end{aligned}$$

for all $i \in \underline{n}$. ◇

Lemma 1.1.16

With the above notation we have that

1. *the kernel of the restriction of the bilinear form* $(\cdot,\cdot)_{\mathfrak{h}}$ *to* $\mathfrak{h}'$ *coincides with* $\mathfrak{c}$, *and*

2. *the bilinear form* $(\cdot,\cdot)_{\mathfrak{h}}$ *is nondegenerate on* $\mathfrak{h}$.

Proof

We refer the reader to the proofs of lemmas [Kac90, 2.1] and [BdK90, 12.1.3].
□

Thus we can now establish a very natural isomorphism

$$\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$$

defined on $h \in \mathfrak{h}$ by

$$\langle \nu(h), h' \rangle = (h, h')_{\mathfrak{h}} \qquad \text{for} \qquad h, h' \in \mathfrak{h}.$$

Considering the special case when $h = \alpha_i^\vee$ we get

$$\langle \nu(\alpha_i^\vee), h' \rangle = (\alpha_i^\vee, h')_{\mathfrak{h}} = \epsilon_i \langle \alpha_i, h' \rangle \qquad \text{for} \qquad h' \in \mathfrak{h}$$

whence we deduce that

$$\alpha_i^\vee = \epsilon_i \nu^{-1}(\alpha_i). \tag{1.1.16}$$

We use ν to induce a bilinear form $(\cdot,\cdot)_{\mathfrak{h}^*}$ on $\mathfrak{h}^*$ by defining

$$(\alpha, \beta)_{\mathfrak{h}^*} = \left(\nu^{-1}(\alpha), \nu^{-1}(\beta)\right)_{\mathfrak{h}}$$

for $\alpha, \beta \in \mathfrak{h}^*$.

Thus for α_i and α_j in Π we have

$$\begin{aligned} (\alpha_i, \alpha_j)_{\mathfrak{h}^*} &= \left(\nu^{-1}(\alpha_i), \nu^{-1}(\alpha_j)\right)_{\mathfrak{h}} \\ &= \frac{1}{\epsilon_i \epsilon_j}(\alpha_i^\vee, \alpha_j^\vee)_{\mathfrak{h}} \\ &= \frac{1}{\epsilon_j} A_{ji} = S_{ij} = \frac{1}{\epsilon_i} A_{ij}. \end{aligned}$$

From this and equation 1.1.16, we deduce the following facts:

- $(\alpha_i, \alpha_i)_{\mathfrak{h}^*} > 0$ for $i \in \underline{n}$,

- $(\alpha_i, \alpha_j)_{\mathfrak{h}^*} \leq 0$ for $i \neq j$,
- $\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)_{\mathfrak{h}^*}}\nu^{-1}(\alpha_i)$.

Hence we obtain the usual expression for the generalized Cartan matrix, namely

$$A = \left(\frac{2(\alpha_i, \alpha_j)_{\mathfrak{h}^*}}{(\alpha_i, \alpha_i)_{\mathfrak{h}^*}}\right)_{i,j\in\underline{n}}.$$

The proof of the following fundamental result can be found in the proofs of theorems [Kac90, 2.2] and [BdK90, 12.2.1 and 12.2.2].

THEOREM 1.1.17

Let $\mathfrak{g}_A(\mathbb{C})$ be a symmetrizable Kac-Moody algebra. Fix a decomposition $A = DS$ of A as in Lemma 1.1.14. Then there exists on $\mathfrak{g}_A(\mathbb{C})$ a nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(\cdot,\cdot)$ such that:

1. *The restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ is precisely the bilinear form $(\cdot,\cdot)_{\mathfrak{h}}$ we defined earlier and is nondegenerate.*

2. *For all $x, y, z \in \mathfrak{g}_A(\mathbb{C})$*
$$([x, y], z) = (x, [y, z]),$$
i.e. $(\cdot,\cdot)$ is invariant.

3. *If $\alpha + \beta \neq 0$ then $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*

4. *For $\alpha \in \Phi$ and $\alpha \neq 0$ the restriction of $(\cdot,\cdot)$ to the subspace $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is nondegenerate.*

5. *For $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ where $\alpha \in \Phi$ we have*
$$[x, y] = -(x, y)\nu^{-1}(\alpha).$$

The Weyl Group of a Kac-Moody Algebra

Before we define the Weyl group associated to a Kac-Moody algebra it is convenient to introduce some results about $\mathfrak{g}$-modules. To this end we let V be a module over a Kac-Moody algebra $\mathfrak{g}$. Our applications will entail considering $\mathfrak{g}$ as a $\mathfrak{g}$-module and the reader may read the relevant material as if that were the only example and omit the generalities. However, they have been included in order to emphasize the fact that the particular case we will consider can be placed in context in a more general setting.

We say that an element $x \in \mathfrak{g}$ is *locally nilpotent* on V if for any $v \in V$ there exists a positive integer N such that $x^N(v) = 0$.

LEMMA 1.1.18

1. *Let $\{v_i\}_{i\in I}$, where I is some indexing set, be a system of generators of a $\mathfrak{g}_A(\mathbb{C})$-module and let $x \in \mathfrak{g}_A(\mathbb{C})$ be such that* $(\mathrm{ad}\, x)$ *is locally nilpotent on $\mathfrak{g}_A(\mathbb{C})$ and $x^{N_i}(v_i) = 0$ for some positive integers N_i, $i \in I$. Then x is locally nilpotent on V.*

2. *Let $\{g_i\}_{i\in I}$, where I is some indexing set, be a system of generators for a Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$ and let $x \in \mathfrak{g}_A(\mathbb{C})$ be such that $(\mathrm{ad}\, x)^{N_i} g_i = 0$ for some positive integers N_i, $i \in I$. Then* $\mathrm{ad}\, x$ *is locally nilpotent on $\mathfrak{g}_A(\mathbb{C})$.*

3. $\mathrm{ad}\, e_i$ *and* $\mathrm{ad}\, f_i$ *are locally nilpotent on $\mathfrak{g}_A(\mathbb{C})$.*

Proof

See Lemmas [Kac90, 3.4 and 3.5]. □

If a is a locally nilpotent operator on a vector space V we can define the *exponential* of a in the usual manner, i.e.

$$e^a = \exp a = I_V + a + \frac{1}{2!}a^2 + \cdots$$

since its action on any particular $v \in V$ will be given by a finite sum. If b is any other operator on V such that $(\mathrm{ad}\, a)^N b = 0$ for some N we can show that

$$(\exp a).b.(\exp a)^{-1} = (\exp(\mathrm{ad}\, a)).b$$

(see [Kac90, §3.8] for proof). The following result will prove useful in later calculations.

Lemma 1.1.19

Let V be a vector space over a field of characteristic 0 *and suppose a and b are operators on V such that a, b and*

$$[a, b] = ab - ba$$

are locally nilpotent and $[a, b]$ commutes with a and b. Then $a + b$ is also locally nilpotent and

$$\exp(a + b) = \exp a.\exp b.\exp\left(-\frac{1}{2}[a, b]\right).$$

Proof

This is a special case of the Campbell-Hausdorff formula and various proofs exist. In particular, the proof given in [Car72, Lemma 5.1.2] for the case when a and b are nilpotent generalizes in a natural fashion to the locally nilpotent case. □

A $\mathfrak{g}_A(\mathbb{C})$-module V is called $\mathfrak{h}$-*diagonalizable* if

$$V = \bigoplus_{\alpha\in\mathfrak{h}^*} V_\alpha$$

where

$$V_\alpha = \{v \in V : h(v) = \langle \alpha, h \rangle v \quad \text{for} \quad h \in \mathfrak{h}\}\,.$$

As usual, V_α is called the *weight space*. $\alpha \in \mathfrak{h}^*$ is called a *weight* if $V_\alpha \neq 0$, and $\dim V_\alpha$ is called the *multiplicity* of α and is denoted by $\mathrm{mult}_V\, \alpha$.

An $\mathfrak{h}$-diagonalizable module over a Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$ is called *integrable* if all e_i and f_i for $i \in \underline{n}$ are locally nilpotent on V.

REMARK 1.1.20

1) The above definitions can also be made with $\mathfrak{h}'$ replacing $\mathfrak{h}$ and $\mathfrak{g}'_A(\mathbb{C})$ replacing $\mathfrak{g}_A(\mathbb{C})$.

2) By Lemma 1.1.18, the underlying module of the adjoint representation of a Kac-Moody algebra is an integrable module.

◇

The following result can be found in [Kac90, Proposition 3.6 a)].

PROPOSITION 1.1.21
Let V be an integrable $\mathfrak{g}_A(\mathbb{C})$-module. We recall our definition

$$\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}\alpha_i^\vee + \mathbb{C}f_i\,.$$

Then, as a $\mathfrak{g}_{(i)}$-module, V decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{h}$-invariant modules.

This justifies the use of the term integrable since it essentially states that the action of $\mathfrak{g}_{(i)}$ on V can be "integrated" to the action of the special linear group $SL_2(\mathbb{C})$.

PROPOSITION 1.1.22
Let V be an integrable $\mathfrak{g}_A(\mathbb{C})$-module, $\beta \in \mathfrak{h}^*$ a weight of V and $\alpha_i \in \Pi$. Define the set of integers

$$M = \{t \in \mathbb{Z} : \beta + t\alpha_i \text{ is a weight of } V\}.$$

Then

1. *M is a closed interval of integers, in particular*

$$M = [-p, q] \cap \mathbb{Z} \qquad \text{where} \qquad p, q \in \mathbb{N}_0 \cup \infty$$

and

$$p - q = \langle \beta, \alpha_i^\vee \rangle \qquad \text{whenever} \qquad p, q \in \mathbb{N}_0.$$

2. *If $\mathrm{mult}_V\, \beta < \infty$ then p and q are both finite.*

3. *The map induced by the action of e_i on V,*

$$e_i : V_{\beta+t\alpha_i} \longrightarrow V_{\beta+(t+1)\alpha_i}$$

is injective for

$$t \in \left[-p, -\frac{1}{2}\langle\beta, \alpha_i^\vee\rangle\right).$$

4. *If we define a map $m : \mathbb{Z} \longrightarrow \mathbb{Z}$ by*

$$m(t) = \text{mult}_V\,(\beta + t\alpha_i) = \dim V_{\beta+t\alpha_i},$$

then m is a non-decreasing function on the interval

$$\left[-p, -\frac{1}{2}\langle\beta, \alpha_i^\vee\rangle\right).$$

5. *The map m is symmetric with respect to the point*

$$t_0 = -\frac{1}{2}\langle\beta, \alpha_i^\vee\rangle,$$

though $(\beta + t_0\alpha_i)$ need not be a weight of V.

6. *If both β and $\beta + \alpha_i$ are weights, then $e_i(V_\beta) \neq 0$.*

Proof

See [Kac90, Proposition 3.6]. □

We now return our attention to the construction of the Weyl group associated to a Kac-Moody algebra.

Let $\mathfrak{g}_A(\mathbb{C})$ be a Kac-Moody algebra and $\Pi = \{\alpha_i : i \in \underline{n}\}$ its system of simple roots. For every $\alpha_i \in \Pi$ we define a linear map

$$r_{\alpha_i} : \mathfrak{h}^* \to \mathfrak{h}^*$$

by

$$r_{\alpha_i}(\beta) = \beta - \langle\beta, \alpha_i^\vee\rangle\alpha_i$$

for all $\beta \in \mathfrak{h}^*$. We call r_{α_i} the *fundamental reflection* corresponding to the simple root α_i. When there is no danger of confusion we denote r_{α_i} by r_i. Note that r_i is indeed a reflection since its fixed point set is

$$H_{\alpha_i} = \{\beta \in \mathfrak{h}^* : \langle\beta, \alpha_i^\vee\rangle = 0\} \quad \text{and} \quad r_i(\alpha_i) = -\alpha_i.$$

The *Weyl group*, W, of $\mathfrak{g}_A(\mathbb{C})$ is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by all the fundamental reflections r_i for $i \in \underline{n}$.

Example 1.1.23

1) The Cartan matrix $A_1 = (2)$ has root system $\Phi = \{\alpha, -\alpha\}$ and the Weyl group $W(A_1)$ is generated by the single reflection r_α. Thus $W(A_1)$ is a cyclic group of order two.

2) Consider a Cartan matrix of type A_n, i.e. such that all the diagonal entries are 2, the superdiagonal and subdiagonal entries are all -1 and all other entries are 0. Then $W(A_n)$ is isomorphic to the full symmetric group on $n+1$ elements.

3) The Cartan matrix $G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ has Weyl group isomorphic to the dihedral group of order 12.

◇

Since $\mathfrak{h}$ and $\mathfrak{h}^*$ are dual spaces, we also have an action of the Weyl group on $\mathfrak{h}$. This is given by

$$r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee$$

for $h \in \mathfrak{h}$.

As a consequence of Proposition 1.1.22 and Lemma 1.1.18 we have the following result.

PROPOSITION 1.1.24

1. *Let V be an integrable $\mathfrak{g}_A(\mathbb{C})$-module. Then, for every $\alpha \in \mathfrak{h}^*$ and $w \in W$*

$$\operatorname{mult}_V \alpha = \operatorname{mult}_V w(\alpha).$$

In particular, the set of weights of V is W-invariant.

2. *The root system Φ of $\mathfrak{g}_A(\mathbb{C})$ is W-invariant and*

$$\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}$$

for every $\alpha \in \Phi$ and $w \in W$.

LEMMA 1.1.25

If $\alpha \in \Phi_+$ and $r_i(\alpha) < 0$, then $\alpha = \alpha_i$. That is to say, $\Phi_+ \setminus \{\alpha_i\}$ is r_i-invariant.

By considering lemma 1.1.8 we get the following result.

LEMMA 1.1.26

The bilinear form $(\cdot,\cdot)_{\mathfrak{h}^*}$ on $\mathfrak{h}^*$ is invariant under the action of the Weyl group W, so

$$(w(\alpha), w(\beta))_{\mathfrak{h}^*} = (\alpha, \beta)_{\mathfrak{h}^*}$$

for all $\alpha, \beta \in \mathfrak{h}^*$.

Proof

See [BdK90, Lemma 13.1.3] or [Kac90, Proposition 3.9]. □

In proposition 1.1.24 we showed that every fundamental reflection r_i yields a permutation of Φ and that α and $r_i(\alpha)$ have root spaces of the same dimension. Using the theory we developed earlier on $\mathfrak{g}_A(\mathbb{C})$-modules we can go further. We can show that there is an automorphism of $\mathfrak{g}_A(\mathbb{C})$ related to r_i which induces an isomorphism from $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{r_i(\alpha)}$. This is achieved in the following manner.

Lemma 1.1.27

Let π be an integrable representation of $\mathfrak{g}_A(\mathbb{C})$ on a vector space V. For $i \in \underline{n}$, set

$$r_i^\pi = (\exp e_i)(\exp f_i)(\exp e_i).$$

Then

1. $r_i^\pi(V_\alpha) = V_{r_i(\alpha)}$,
2. $r_i^{\mathrm{ad}} \in \mathrm{Aut}\, \mathfrak{g}_A(\mathbb{C})$, *and*
3. $r_i^{\mathrm{ad}}|_{\mathfrak{h}} = r_i$.

Proof

The statement we give here differs slightly from that on which it is based, namely [Kac90, Lemma 3.8]. However the difference is superficial. We note that in the original we had

$$r_i^\pi = (\exp f_i)(\exp -e_i)(\exp f_i).$$

Given that our notation is different to the original this translates to

$$r_i^\pi = (\exp -f_i)(\exp -e_i)(\exp -f_i)$$

in our notation. Finally we use the isomorphism between $\mathfrak{g}_{(i)}$ and $\mathfrak{sl}_2$ to show that

$$(\exp -f_i)(\exp -e_i)(\exp -f_i) = ((\exp e_i)(\exp f_i)(\exp e_i))^{-1}.$$

However, since r_i is an involution, we obtain the same results as the original by following the same processes. □

We continue by giving some more results about the structure of the Weyl group itself. We refer the reader to [Kac90, §§3.10–3.13] for the proofs of the remaining results in this section.

Consider an expression $w = r_{i_1} \ldots r_{i_t} \in W$. Define the *length* of w, denoted by $\ell(w)$, to be the smallest t for which such an expression exists and call the expression *reduced* in that case.

Lemma 1.1.28

Let $w = r_{i_1} \ldots r_{i_t} \in W$ be a reduced expression and let $\alpha_i \in \Pi$. Then we have

1. *$\ell(wr_i) < \ell(w)$ if and only if $w(\alpha_i) < 0$,*

2. *$w(\alpha_{i_t}) < 0$,*

3. *If $\ell(wr_i) < \ell(w)$, then there exists $s \in \underline{t}$ such that*

$$r_{i_s} r_{i_{s+1}} \dots r_{i_t} = r_{i_{s+1}} \dots r_{i_t} r_i.$$

The last part of Lemma 1.1.28 is the familiar *exchange condition.*

In order to study the geometric properties of the action of the Weyl group we introduce the notion of the fundamental chamber.

Since we are working with generalized Cartan matrices, which by definition have integer entries, we can define a realization of A over $\mathbb{R}$ by taking $\mathfrak{h}_{\mathbb{R}}$ to be a vector space of dimension $2n - r$ over $\mathbb{R}$. So if $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee)$ is a realization of A over $\mathbb{R}$, then

$$(\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee)$$

is a realization of A over $\mathbb{C}$.

Note that $\mathfrak{h}_{\mathbb{R}}$ is stable under the action of W since $Q^\vee \subseteq \mathfrak{h}_{\mathbb{R}}$. We then call the set

$$C = \{h \in \mathfrak{h}_{\mathbb{R}} : \langle \alpha_i, h \rangle \geq 0 \quad \text{for} \quad i \in \underline{n}\}$$

the *fundamental chamber.* The sets $w(C)$ for $w \in W$ are called *chambers*, and their union

$$X = \bigcup_{w \in W} w(C)$$

is called the *Tits cone.* We have corresponding dual notions of $C^\vee$ and $X^\vee$ in $\mathfrak{h}_{\mathbb{R}}^*$.

PROPOSITION 1.1.29

1. *For $h \in C$, the group*

$$W_h = \{w \in W : w(h) = h\}$$

is generated by the fundamental reflections it contains.

2. *The fundamental chamber C is a fundamental domain for the action of W on X. That is to say, any orbit $W.h$ of $h \in X$ intersects C in exactly one point. In particular, W acts simply transitively on chambers.*

3. *It can be shown that*

$$X = \{h \in \mathfrak{h}_{\mathbb{R}} : \langle \alpha, h \rangle < 0 \text{ only for a finite number of } \alpha \in \Phi_+\}.$$

So in particular, X is a convex cone.

4. *We can also show that*

$$C = \{h \in \mathfrak{h}_{\mathbb{R}} : \text{ for every } w \in W,\ h - w(h) = \sum_{i \in \underline{n}} c_i \alpha_i^\vee \text{ where } c_i \geq 0\}.$$

5. *The following conditions are equivalent:*

   (a) $| W | < \infty$,

   (b) $X = \mathfrak{h}_{\mathbb{R}}$,

   (c) $| \Phi | < \infty$,

   (d) $| \Phi^\vee | < \infty$.

6. *If $h \in X$, then $| W_h | < \infty$ if and only if h lies in the interior of X.*

Finally, we can establish that fact that the Weyl group is a Coxeter group. Recall that a Coxeter group given by a presentation of the type

$$\langle r_1, \ldots, r_n : r_i^2 = 1,\ (r_i r_j)^{m_{ij}} = 1, \quad \text{for} \quad i, j \in \underline{n} \rangle$$

where $m_{ij} \in \mathbb{N} \cup \infty$, and we use the convention that $x^\infty = 1$ for any x. So strictly speaking, the above is not really a presentation, but a presentation may be obtained by omitting all those relations $(r_i r_j)^{m_{ij}} = 1$ for which $m_{ij} = \infty$. However, it is convenient to give the presentation in this way as will become clear in the next result.

Proposition 1.1.30
The Weyl group W is a Coxeter group, where the numbers m_{ij} appearing in the presentation are given by the Cartan integers in the manner described in table 1.1.31.

$A_{ij}A_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

Table 1.1.31: Product orders in terms of Cartan integers.

1.2 Classification of Kac-Moody Algebras

In this section we investigate the classification of Kac-Moody algebras. We will see that these algebras fall into three disjoint classes which can be fully characterized by specific properties of their generalized Cartan matrices. We shall then give an explicit description of the root system Φ of a Kac-Moody algebra.

The proofs of the results in this section can be found in [Kac90] and [BdK90].

Properties of Generalized Cartan Matrices

Unless otherwise stated, we shall be dealing with a real $n \times n$ matrix $A = (A_{ij})$ satisfying the following three properties;

- A is indecomposable,
- $A_{ij} \leq 0$ for $i \neq j$,
- $A_{ij} = 0 \iff A_{ji} = 0$.

We shall call a matrix satisfying these conditions a *semi-Cartan* matrix. Note that a generalized Cartan matrix satisfies the last two conditions and we can assume the first without loss of generality.

Let ${}^t\boldsymbol{u} = (u_1, \ldots, u_n)$ be a real column vector. We write

$$\begin{array}{llll} \boldsymbol{u} > 0 & \text{if} \quad u_i > 0 & \text{for all} \quad i \in \underline{n}, & \text{and} \\ \boldsymbol{u} \geq 0 & \text{if} \quad u_i \geq 0 & \text{for all} \quad i \in \underline{n}. & \end{array}$$

We call $\boldsymbol{u}$ *positive* in the former case and *non-negative* in the latter case.

Lemma 1.2.1

Let A be a semi-Cartan matrix. Then $A\boldsymbol{u} \geq 0$ and $\boldsymbol{u} \geq 0$ together imply that either $\boldsymbol{u} > 0$ or $\boldsymbol{u} = 0$.

Lemma 1.2.2

Let $A = (A_{ij})$ be an arbitrary real $n \times n$ matrix. Then

either *there exists a vector $\boldsymbol{u} \geq 0$, $\boldsymbol{u} \neq 0$ such that ${}^tA\boldsymbol{u} \geq 0$*

or *there exists a vector $\boldsymbol{u} > 0$ such that $A\boldsymbol{u} < 0$.*

We can now present the central result of the classification theory.

Theorem 1.2.3

Let A be a semi-Cartan matrix. Then one and only one of the following three possibilities holds:

(Fin) $\det A \neq 0$,
there exists $\boldsymbol{u} > 0$ such that $A\boldsymbol{u} > 0$, and
$A\boldsymbol{v} \geq 0$ implies that either $\boldsymbol{v} > 0$ or $\boldsymbol{v} = 0$.

(Aff) corank $A = 1$,
there exists $\boldsymbol{u} > 0$ such that $A\boldsymbol{u} = 0$, and
$A\boldsymbol{v} \geq 0$ implies $A\boldsymbol{v} = 0$.

(Ind) There exists $\boldsymbol{u} > 0$ such that $A\boldsymbol{u} < 0$, and
$A\boldsymbol{v} \geq 0$ together with $\boldsymbol{v} \geq 0$ implies $\boldsymbol{v} = 0$.

EXAMPLE 1.2.4

1) Consider the matrix $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then $\det A = 3$ and

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that A_2 satisfies the first two conditions of case (Fin). To see that the last condition also holds suppose that there exists a vector $\boldsymbol{v}$ such that $A_2\boldsymbol{v} \geq 0$ but neither $\boldsymbol{v} > 0$ nor $\boldsymbol{v} = 0$ hold. So without loss of generality we may assume $\boldsymbol{v}$ is of the form ${}^t(a, 0)$ for some non-zero $a \in \mathbb{R}$. But then

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ -a \end{pmatrix},$$

contradicting the fact that $A_2\boldsymbol{v} \geq 0$. Hence A_2 satisfies all the conditions of case (Fin).

2) Consider next the matrix $\tilde{A}_1 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. Then $\det \tilde{A}_1 = 0$ and

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Also, letting $\boldsymbol{v} = {}^t(a, b)$ for arbitrary $a, b \in \mathbb{C}$, we have

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2(a-2b) \\ -(a-2b) \end{pmatrix}.$$

Thus if $a - 2b \neq 0$ we must have $\tilde{A}_1\boldsymbol{v} \not\geq 0$. Hence $\tilde{A}_1$ satisfies all the conditions for case (Aff).

3) Finally, consider $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. Then

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

and, if $\boldsymbol{v} = {}^t(a, b)$ where a and b are arbitrary elements in $\mathbb{C}$,

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a - 3b \\ -3a + 2b \end{pmatrix}.$$

In particular, suppose $A\boldsymbol{v} \geq 0$ and $\boldsymbol{v} \geq 0$ but $\boldsymbol{v} \neq 0$. Then, without loss of generality, we may assume $a > 0$. However, solving the inequalities arising from $A\boldsymbol{v} \geq 0$ then leads to the contradiction that $4a \geq 9a$. Thus we must have $\boldsymbol{v} = 0$ and A satisfies all the conditions for case (Ind).

◇

When referring to cases (Fin), (Aff), or (Ind), we shall say that A is of *finite*, *affine*, or *indefinite* type, respectively. By convention, we use the set $\underline{n}_0 = \{0, 1, \ldots, n\}$ as an indexing set for the simple roots of a Kac-Moody algebra corresponding to an matrix of affine type.

COROLLARY 1.2.5
Let A be a semi-Cartan matrix. Then

$$A \in \text{(Fin)} \Leftrightarrow \text{there exists } \mathbf{u} > 0 \text{ such that } A\mathbf{u} > 0,$$

$$A \in \text{(Aff)} \Leftrightarrow \text{there exists } \mathbf{u} > 0 \text{ such that } A\mathbf{u} = 0, \text{ and}$$

$$A \in \text{(Ind)} \Leftrightarrow \text{there exists } \mathbf{u} > 0 \text{ such that } A\mathbf{u} < 0.$$

LEMMA 1.2.6
Let A be a semi-Cartan matrix. Then both A and ${}^tA$ are of the same type.

Generalized Cartan Matrices of Finite and Affine Types

We now proceed to classify all generalized Cartan matrices of finite and affine types. We recall that a matrix of the form $(A_{ij})_{i,j \in S}$, where $S \subseteq \underline{n}$, is called a *principal submatrix* of $A = (A_{ij})_{i,j \in \underline{n}}$. We shall denote such a submatrix by A_S. The determinant of a principal submatrix shall be referred to as a *principal minor*.

LEMMA 1.2.7
If A is of finite or affine type, then any proper principal submatrix of A decomposes into a direct sum of matrices of finite type.

The following lemma ensures that Kac-Moody algebras corresponding to generalized Cartan matrices are always equipped with an invariant bilinear form such as that introduced in §1.1.

LEMMA 1.2.8
Generalized Cartan matrices of finite and affine type are symmetrizable.

It is now convenient to introduce the concept of the Dynkin diagram of a matrix. Let $A = (A_{ij})$ be a generalized Cartan matrix. We associate with A a graph $\Delta(A)$, called the *Dynkin diagram* of A as follows:

- The diagram has n vertices, each corresponding to one of the simple roots.
- If $A_{ij}A_{ji} \leq 4$ the vertices i and j are connected by

$$n_{ij} = \max(|A_{ij}|, |A_{ji}|)$$

bonds (lines). If

$$|A_{ij}| \geq |A_{ji}| \quad \text{and} \quad |A_{ij}| > 1,$$

the n_{ij} bonds are equipped with an arrow pointing from j to i.

- If $A_{ij}A_{ji} > 4$ the vertices i and j are connected by a bold-faced line equipped with an ordered pair of integers $|A_{ij}|, |A_{ji}|$.

We note in particular that A is indecomposable if and only if $\Delta(A)$ is a connected graph. Note also that that A is determined by the Dynkin diagram $\Delta(A)$ and an enumeration of its vertices. We say that $\Delta(A)$ is of finite, affine, or indefinite type, depending on the type of A. We then obtain the following result.

PROPOSITION 1.2.9
Let A be an indecomposable generalized Cartan matrix.

1. *A is of finite type $\Leftrightarrow$ all its principal minors are positive.*

2. *A is of affine type $\Leftrightarrow$ all its proper principal minors are positive and* $\det A = 0$.

3. *If A is of finite or affine type, then any proper subdiagram of $\Delta(A)$ is a union of (connected) Dynkin diagrams of finite type.*

4. *If A is of finite type, then $\Delta(A)$ contains no cycles. If A is of affine type and contains a cycle, then $\Delta(A)$ is the cycle $\tilde{A}_l$, with $l > 1$, from figure 1.2.12.*

5. *A is of affine type $\Leftrightarrow$ there exists $\delta > 0$ such that $A\delta = 0$. Such a δ is unique up to a constant factor.*

Classification of Generalized Cartan Matrices of Finite and Affine Types

We can now list all generalized Cartan matrices of finite and affine type.

THEOREM 1.2.10

1. *The Dynkin diagrams of all generalized Cartan matrices of finite type are listed in figure 1.2.11.*

2. *The Dynkin diagrams of all generalized Cartan matrices of affine type are listed in figures 1.2.12–1.2.14.*

3. *The number in brackets beside each Dynkin diagram in figure 1.2.11 is the determinant of the corresponding Cartan matrix.*

4. *The numerical labels in figures 1.2.12–1.2.14 are the coordinates of the unique vector*

$$^t\delta = (a_0, a_1, \ldots, a_n)$$

such that $A\delta = 0$ *and the* a_i *are positive relatively prime integers.*

The Dynkin diagrams given in figure 1.2.12 are often described as *extended* Dynkin diagrams.

We conclude this section with the following characterization of Kac-Moody algebras associated with generalized Cartan matrices of finite type.

PROPOSITION 1.2.15

Let A be an indecomposable generalized Cartan matrix. Then the following conditions are equivalent:

1. *A is a generalized Cartan matrix of finite type.*

2. *A is symmetrizable and the bilinear form* $(\cdot,\cdot)_{\mathfrak{h}_{\mathbb{R}}}$ *is positive definite.*

3. $|W| < \infty$.

4. $|\Phi| < \infty$.

5. $\mathfrak{g}_A(\mathbb{C})$ *is a simple finite-dimensional Lie algebra.*

6. *There exists* $\alpha \in \Phi_+$ *such that* $\alpha + \alpha_i \notin \Phi$ *for all* $i \in \underline{n}$.

Proof

See [Kac90, Proposition 4.9]. □

The root mentioned in the last part of Proposition 1.2.15 is of course the highest root of the finite root system Φ. In particular, it is unique and is given by the formula

$$\theta = \sum_{i \in \underline{n}} a_i \alpha_i$$

where the coefficients a_i are the numerical labels on the corresponding extended Dynkin diagram from figure 1.2.12.

Real and Imaginary Roots

In this section we give an explicit description of the root system Φ of a Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$ corresponding to an arbitrary matrix. Our main instrument is the notion of an *imaginary root*, which has no counterpart in the finite-dimensional theory.

Table of Dynkin Diagrams of Finite Type.

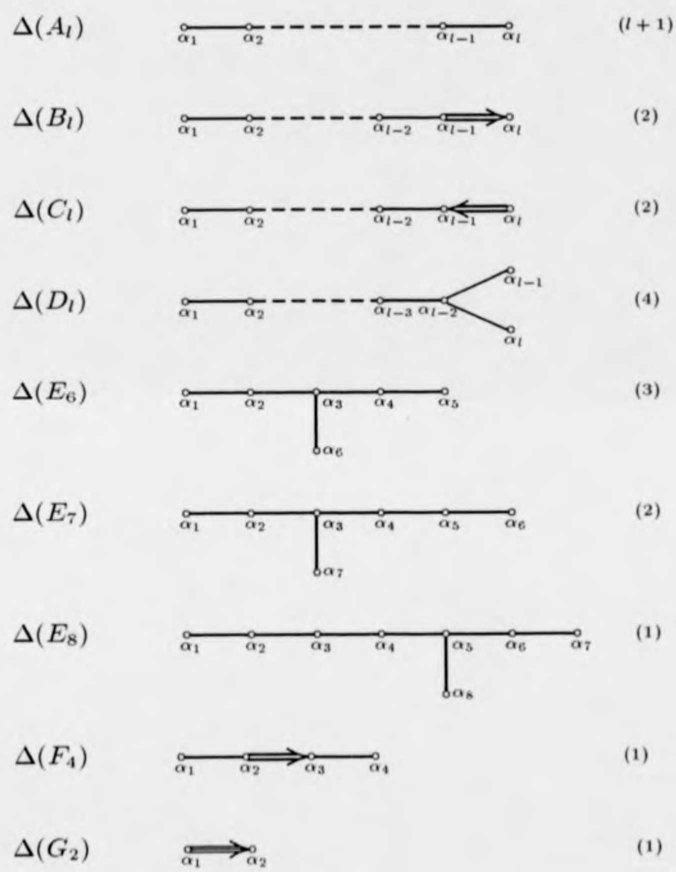


Figure 1.2.11: Table Fin

Table of Affine Dynkin Diagrams of Type I.

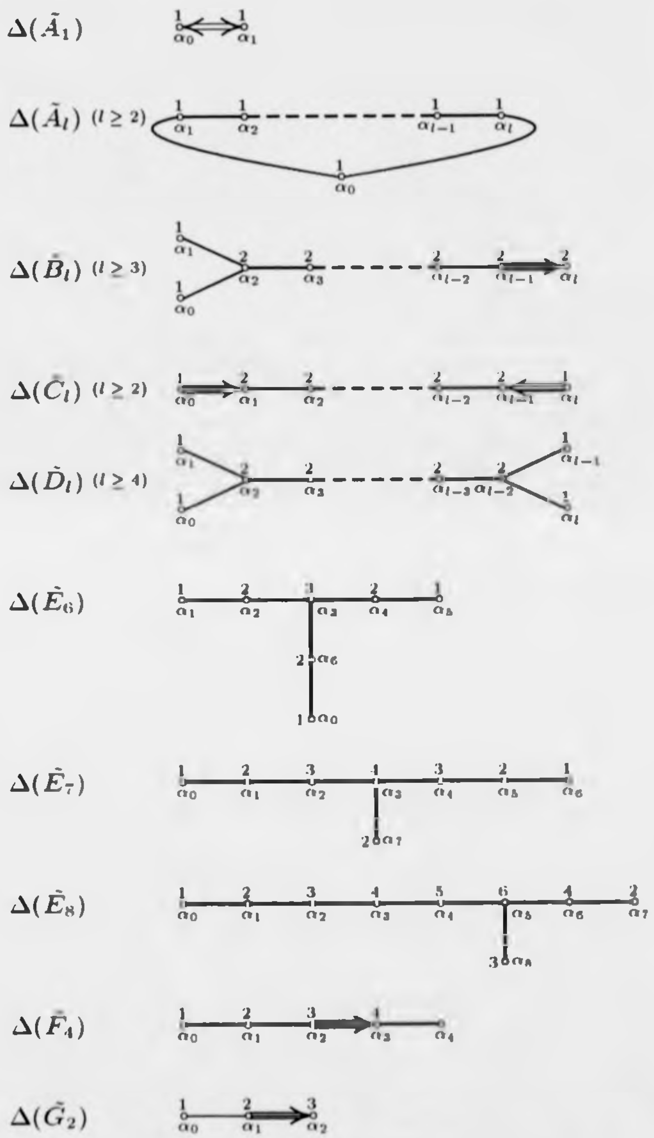


Figure 1.2.12: Table Aff I

Table of Affine Dynkin Diagrams of Type II.

$\Delta({}^{\prime\prime}\tilde{A}_1)$

$\Delta({}^{\prime}\tilde{B}_l)\,(l \geq 3)$

$\Delta({}^{\prime}\tilde{C}_l)\,(l \geq 2)$

$\Delta({}^{\prime\prime}\tilde{C}_l)\,(l \geq 2)$

$\Delta({}^{\prime}\tilde{F}_4)$

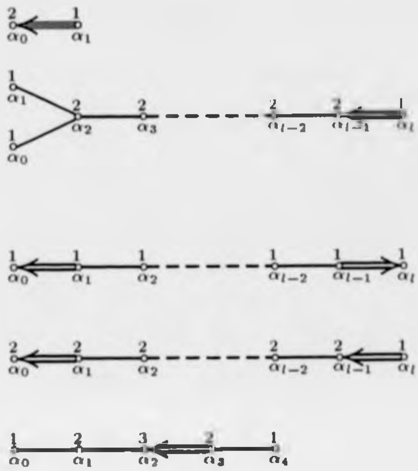


Figure 1.2.13: Table Aff II

Table of Affine Dynkin Diagrams of Type III.

$\Delta({}^{\prime}\tilde{G}_2)$

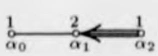


Figure 1.2.14: Table Aff III

A root $\alpha \in \Phi$ is called *real* if there exists an element $w \in W$ and a simple root $\alpha_i \in \Pi$ such that $\alpha = w(\alpha_i)$. Denote by

$$\Phi^{re} \quad \text{and} \quad \Phi^{re}_{+}$$

the sets of all real and positive real roots respectively. Note that Φ^{re} is invariant under the action of the Weyl group W.

Let α be a real root such that $\alpha = w(\alpha_i)$ for some $w \in W$ and $\alpha_i \in \Pi$. We define the *coroot* or *dual (real) root* $\alpha^\vee$ by

$$\alpha^\vee = w(\alpha_i^\vee).$$

This is independent of the choice of the presentation $\alpha = w(\alpha_i)$ (see [Kac90, §5.1] for details).

Thus we obtain a canonical W-invariant bijection

$$\Phi^{re} \to \Phi^{\vee re}.$$

We can also show that

$$\alpha > 0 \quad \Leftrightarrow \quad \alpha^\vee > 0.$$

Let α be a real root and $\alpha^\vee$ be the corresponding coroot. We define a reflection r_α by

$$r_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

for $\beta \in \mathfrak{h}^*$. Note that the reflecting hyperplane of this reflection is

$$H_\alpha = \{\beta \in \mathfrak{h}^* : \langle \beta, \alpha^\vee \rangle = 0\}.$$

The relationship between α and α_i is reflected in the following relationship between r_α and r_i.

Lemma 1.2.16

If $\alpha \in \Phi^{re}$ and $\alpha = w(\alpha_i)$ then

$$r_\alpha = w r_i w^{-1}.$$

The following proposition shows that real roots have all the properties we associate with roots in the classical theory of Lie algebras.

Proposition 1.2.17

Let α be a real root of a Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$. Then

1. mult $\alpha = 1$.

2. *$k\alpha$ is a root if and only if $k = \pm 1$.*

3. If $\beta \in \Phi$ then there exist non-negative integers p and q related by the equation

$$p - q = \langle \beta, \alpha^\vee \rangle,$$

such that

$$\beta + k\alpha \in \Phi \cup \{0\} \quad \Leftrightarrow \quad -p \leq k \leq q, \ \text{and } k \in \mathbb{Z}.$$

4. Provided that $\pm\alpha \notin \Pi$, there exists $i \in \underline{n}$ such that

$$|\operatorname{ht} r_i(\alpha)| < |\operatorname{ht} \alpha|.$$

5. If A is symmetrizable and $(\cdot,\cdot)_{\mathfrak{h}^*}$ is the bilinear form on $\mathfrak{h}^*$ induced by a standard bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}_A(\mathbb{C})$, then

(a) $(\alpha,\alpha)_{\mathfrak{h}^*} > 0$,

(b) $\alpha^\vee = \frac{2}{(\alpha,\alpha)_{\mathfrak{h}^*}}\nu^{-1}(\alpha)$, and

(c) if $\alpha = \sum_{i \in \underline{n}} k_i\alpha_i$, then $k_i(\alpha_i,\alpha_i)_{\mathfrak{h}^*} \in \mathbb{Z}(\alpha,\alpha)_{\mathfrak{h}^*}$ for $i \in \underline{n}$.

Let A be a generalized Cartan matrix, and let $(\cdot,\cdot)_{\mathfrak{h}^*}$ be the bilinear form on $\mathfrak{h}^*$ induced by a standard invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}_A(\mathbb{C})$. Then, given a real root α we have

$$(\alpha,\alpha)_{\mathfrak{h}^*} = (\alpha_i,\alpha_i)_{\mathfrak{h}^*}$$

for some simple root α_i. We call α a *short*, or *long*, real root if

$$\begin{aligned}(\alpha,\alpha)_{\mathfrak{h}^*} &= \min_{i \in \underline{n}}(\alpha_i,\alpha_i)_{\mathfrak{h}^*}, \text{ or}\\ (\alpha,\alpha)_{\mathfrak{h}^*} &= \max_{i \in \underline{n}}(\alpha_i,\alpha_i)_{\mathfrak{h}^*},\end{aligned}$$

respectively. These are independent of the choice of a standard form.

Note that if A is symmetric then all simple roots, and hence all roots, have the same square length. Such algebras will be said to be *simply-laced*. If A is not symmetric and $\Delta(A)$ is equipped with m arrows pointing in the same direction, then there are simple roots of exactly $m+1$ different square lengths.

It follows that if A is non-symmetric of finite type then every root is either short or long. Furthermore, if A is non-symmetric of affine type but not of type $\tilde{C}_l$ with $l > 1$, then every real root is either short or long. For the type $\tilde{C}_l$ with $l > 1$ there are real roots of three different lengths.

We now turn our attention to roots which do not fall into the category of real roots. A root α which is not real is called an *imaginary root*. We denote by

$$\Phi^{im} \quad \text{and} \quad \Phi^{im}_+$$

the sets of imaginary and positive imaginary roots respectively. By definition we have the following disjoint union:

$$\Phi = \Phi^{re} \cup \Phi^{im}.$$

We can also decompose Φ^{im} itself into its positive and negative constituents so that we get

$$\Phi^{im} = \Phi_+^{im} \cup \Phi_-^{im}$$

where

$$\Phi_-^{im} = -\Phi_+^{im}.$$

Although imaginary roots do not behave in the same way as real roots do, they nevertheless satisfy some useful properties.

PROPOSITION 1.2.18

1. *The set Φ_+^{im} is W-invariant.*

2. *For each $\alpha \in \Phi_+^{im}$ there exists a unique root $\beta \in -C^\vee$ which is W-equivalent to α. In particular*

$$\langle \beta, \alpha_i^\vee \rangle \leq 0$$

for all $i \in \underline{n}$. If A is symmetrizable, this is equivalent to saying

$$(\beta, \alpha_i)_{\mathfrak{h}^*} \leq 0$$

where $(\cdot\,,\cdot)_{\mathfrak{h}^*}$ is the bilinear form on $\mathfrak{h}^*$ induced by a standard invariant bilinear form on $\mathfrak{g}_A(\mathbb{C})$.

3. *If A is symmetrizable and $(\cdot\,,\cdot)_{\mathfrak{h}^*}$ is as above, then a root α is imaginary if and only if*

$$(\alpha, \alpha)_{\mathfrak{h}^*} \leq 0.$$

However, imaginary roots have properties which differ drastically from those of real roots, as the following result testifies.

PROPOSITION 1.2.19

If $\alpha \in \Phi_+^{im}$ and $r \in \mathbb{Q} \setminus \{0\}$ is such that $r\alpha \in Q$, then $r\alpha \in \Phi^{im}$. In particular,

$$n\alpha \in \Phi^{im} \quad \textit{for all} \quad n \in \mathbb{Z} \setminus \{0\}.$$

The next result is an existence proof for imaginary roots.

THEOREM 1.2.20

Let A be an indecomposable generalized Cartan matrix.

1. *If A is of finite type, then $\Phi^{\text{im}} = \emptyset$.*
2. *If A is of affine type, then*

$$\Phi_+^{\text{im}} = \{n\delta : n \in \mathbb{N}\},$$

where

$$\delta = \sum_{i \in \underline{n}} a_i \alpha_i$$

with the coefficients a_i being the numerical labels on the corresponding Dynkin diagram in figures 1.2.12–1.2.14. In this case we shall refer to δ as the fundamental *imaginary root.*

3. *If A is of indefinite type, then there exists a positive imaginary root*

$$\alpha = \sum_{i \in \underline{n}} k_i \alpha_i$$

such that, for all $i \in \underline{n}$

$$k_i > 0 \quad \text{and} \quad \langle \alpha, \alpha_i^\vee \rangle < 0.$$

A linearly independent set of roots $\Pi' = \{\alpha_i'\}_{i \in I}$, for some indexing set I, is called a *root basis* of Φ if each root α can be written in the form

$$\alpha = \pm \sum_{i \in I} k_i \alpha_i' \quad \text{where} \quad k_i \in \mathbb{Z}_+.$$

PROPOSITION 1.2.21

Let A be an indecomposable generalized Cartan matrix. Then any root basis Π' of Φ is W-conjugate to Π or $-\Pi$.

Π *is W-conjugate to $-\Pi$ $\Leftrightarrow$ A is of finite type.*

Proof

See [Kac90, Proposition and Remark 5.9]. □

A generalized Cartan matrix A is said to be of *hyperbolic* type if it is indecomposable of indefinite type and if every connected subdiagram of $\Delta(A)$ is of finite or affine type. We do not mention all the results available on the topic of hyperbolic Kac-Moody algebras since they are not relevant in the sequel. However, there are some results shared by generalized Cartan matrices of finite, affine and hyperbolic type so they have been included for the sake of interest and completeness.

Note that if A is symmetrizable, then a standard invariant bilinear form $(\cdot,\cdot)$ can be normalized so that (α_i, α_j) are integers. Since the bilinear form is positive definite on the real roots we see that

$$a = \min_{\alpha \in Q : |\alpha|^2 > 0} |\alpha|^2$$

exists and is a positive integer.

Lemma 1.2.22

Let A be a generalized Cartan matrix of finite, affine or hyperbolic type, and let

$$\alpha = \sum_{i \in \underline{n}} k_i \alpha_i \in Q.$$

1. *If* $|\alpha|^2 \leq a$, *then either* $\alpha \in Q_+$ *or* $-\alpha \in Q_+$.
2. *If* α *satisfies*

$$k_i(\alpha_i, \alpha_i)_{\mathfrak{h}^*} \in (\alpha, \alpha)_{\mathfrak{h}^*}\mathbb{Z} \quad \text{for all} \quad i \in \underline{n},$$

then either $\alpha \in Q_+$ *or* $-\alpha \in Q_+$.

Proof

See [Kac90, Lemma 5.10]. □

We can use this result to obtain a description of real and imaginary roots in terms of the square of their lengths.

Proposition 1.2.23

Let A be a generalized Cartan matrix of finite, affine, or hyperbolic type. Then

1. *the set of all real short roots is*

$$\{\alpha \in Q \; : |\alpha|^2 = a = \min_{i \in \underline{n}} |\alpha_i|^2 \}.$$

2. *the set of all real roots is*

$$\left\{ \alpha = \sum_{i \in \underline{n}} k_i \alpha_i \in Q \; : |\alpha|^2 > 0 \text{ and } \frac{k_i \, |\alpha_i|^2}{|\alpha|^2} \in \mathbb{Z} \text{ for all } i \right\}.$$

3. *the set of all imaginary roots is*

$$\{\alpha \in Q \setminus \{0\} \; : |\alpha|^2 \leq 0\}.$$

4. *if A is affine, then there exist roots of intermediate squared length m if and only if* $A = {}^{\sim}C_l$ *with* $l > 1$.

Proof

See [Kac90, Proposition 5.10]. □

Suppose γ is an automorphism of the Dynkin diagram $\Delta(A)$. Then γ induces an automorphism γ of the root lattice Q by

$$\gamma(\alpha_i) = \alpha_{\gamma(i)}.$$

Denote the group of all such automorphisms by Aut(A). The Weyl group W is another subgroup of Aut Q. Note that

$$\gamma r_i \gamma^{-1} = r_{\gamma(i)}$$

and that

$$W \cap \mathrm{Aut}(A) = 1$$

by the second part of lemma 1.1.28. Thus

$$\mathrm{Aut}\, Q \supseteq \mathrm{Aut}(A) \ltimes W.$$

The following result is a corollary of Lemma 1.2.22 and Proposition 1.2.23.

COROLLARY 1.2.24

1. *If A is indecomposable, then the group of all automorphisms of Q preserving Φ is $\pm\mathrm{Aut}(A) \ltimes W$.*

2. *If A is a symmetric matrix of finite, affine, or hyperbolic type, then the group of all automorphisms of Q preserving $(\cdot,\cdot)$ is $\pm\mathrm{Aut}(A) \ltimes W$.*

Proof
See [Kac90, Corollary 5.10]. □

1.3 Structure Theory of Affine Kac-Moody Algebras

In this section we shall describe in detail the standard bilinear form, the root system, and the Weyl group of an affine algebra $\mathfrak{g}$ associated to an indecomposable generalized Cartan matrix A of affine type in terms of the "underlying" simple finite-dimensional Lie algebra $\mathring{\mathfrak{g}}$. The proofs of all details in this section can be found in [Kac90, Chapter 6], whereas [BdK90] only cover the case when A is affine of type I.

Let $A = (A_{ij})_{i,j\in\underline{n}_0}$ be a generalized Cartan matrix of affine type. So, in particular rank $A = n$. Let $\Delta(A)$ be its Dynkin diagram. Let $\{a_i\}_{i\in\underline{n}_0}$ be the numerical labels of $\Delta(A)$. Then $a_0 = 1$ unless A is of type $A = {}^{2}C_l$ with $l > 1$ or ${}^{2}A_1$, in which case $a_0 = 2$.

We denote by $\{a_i^\vee\}_{i\in\underline{n}_0}$ the labels of the Dynkin diagram $\Delta({}^tA)$ of the dual algebra which is obtained from $\Delta(A)$ by reversing the directions of all the arrows and keeping the same enumeration of the vertices. Note that in all cases

$$a_0^\vee = 1.$$

We call the numbers

$$h = \sum_{i \in \underline{n}_0} a_i \qquad \text{and} \qquad h^\vee = \sum_{i \in \underline{n}_0} a_i^\vee$$

the *Coxeter number* and the *dual Coxeter number* of the matrix A, respectively. Recall that generalized Cartan matrices are symmetrizable (see Lemma 1.2.8). Suppose $A = DS$ where D is diagonal and S is symmetric. Let

$${}^t\delta = (a_0, \ldots, a_n) \qquad \text{and} \qquad {}^t\delta^\vee = (a_0^\vee, \ldots, a_n^\vee).$$

We know $A\delta = 0$ and ${}^tA\delta^\vee = 0$. Thus, since D is non-singular, we have $B\delta = 0$ and

$${}^t(DB)\delta^\vee = B(D\delta^\vee) = 0.$$

Since both A and B have corank 1 we conclude

$$D\delta^\vee = \lambda\delta^\vee$$

for some non-zero $\lambda \in \mathbb{R}$. If we suppose $\lambda = 1$ we obtain the following expression for D;

$$D = \operatorname{diag}\left(\frac{a_0}{a_0^\vee}, \ldots, \frac{a_n}{a_n^\vee}\right).$$

Let $\mathfrak{g} = \mathfrak{g}_A(\mathbb{C})$ be the Kac-Moody algebra associated to an affine generalized Cartan matrix $A = (A_{ij})_{i,j \in \underline{n}_0}$. Let $\mathfrak{h}$ be its Cartan subalgebra. By definition $\dim \mathfrak{h} = n + 2$. Recall that the set of simple roots

$$\Pi = \{\alpha_0, \ldots, \alpha_n\} \subset \mathfrak{h}^*$$

is linearly independent in $\mathfrak{h}^*$, and the set of simple coroots

$$\Pi^\vee = \{\alpha_0^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$$

is linearly independent in $\mathfrak{h}$.

It follows from Proposition 1.1.11 that the centre, $\mathfrak{c}$, of $\mathfrak{g}_A(\mathbb{C})$ is a one-dimensional subalgebra of $\mathfrak{h}$ and that its elements satisfy $\langle \alpha_i, h \rangle = 0$ for all $i \in \underline{n}_0$. Thus $\mathfrak{c}$ is spanned by

$$c = \sum_{i \in \underline{n}_0} a_i^\vee \alpha_i^\vee$$

which we shall refer to as the *canonical central element*.

Since $\Pi^\vee$ is a linearly independent set in $\mathfrak{h}$, we can extend it to a basis of $\mathfrak{h}$. We do this by selecting one more element, called the *scaling element*, denoted by d. We fix an element $d \in \mathfrak{h}$ such that

$$\langle \alpha_j, d \rangle = 0 \;\text{ for } j \in \underline{n}, \quad \text{and} \quad \langle \alpha_0, d \rangle = 1.$$

Note that such a d is unique up to a summand proportional to c. From the definition of d and consideration of its effect on the element

$$\delta = \sum_{i \in \underline{n}_0} a_i \alpha_i \in Q$$

we can show that $\{\alpha_0, \ldots, \alpha_n, d\}$ is a basis for $\mathfrak{h}$. It is worth noting that

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}d.$$

We define a non-degenerate symmetric bilinear $\mathbb{C}$-valued form $(\cdot,\cdot)_{\mathfrak{h}}$ on $\mathfrak{h}$ by setting

$$\begin{array}{lll} (\alpha_i^\vee, \alpha_j^\vee) = \frac{a_j}{a_j^\vee} A_{ij} & \text{for} & i,j \in \underline{n}_0, \\ (\alpha_i^\vee, d) = 0 & \text{for} & i \in \underline{n}, \\ (\alpha_0^\vee, d) = a_0, & & \text{and} \\ (d,d) = 0. & & \end{array}$$

By Theorem 1.1.17, this form can be uniquely extended to a bilinear form $(\cdot,\cdot)$ on the whole Kac-Moody algebra $\mathfrak{g}$ such that all the properties described in the theorem hold. This is a standard form and will be called the *normalized invariant form*. Henceforth we fix this form on $\mathfrak{g}$.

We now use these results we have just obtained on $\mathfrak{h}$ to determine a basis of $\mathfrak{h}^*$ and to induce a bilinear form on $\mathfrak{h}^*$. We extend Π to a basis of $\mathfrak{h}^*$ by introducing an element $\Lambda_0 \in \mathfrak{h}^*$. We fix this element by demanding that

$$\begin{array}{lll} \langle \Lambda_0, \alpha_j \rangle = 0 & \text{for} & j \in \underline{n}, \\ \langle \Lambda_0, \alpha_0 \rangle = 1, & & \text{and} \\ \langle \Lambda_0, d \rangle = 0. & & \end{array}$$

Then $\{\alpha_0, \ldots, \alpha_l, \Lambda_0\}$ is a basis of $\mathfrak{h}^*$ and we have

$$\begin{array}{lll} (\alpha_i, \alpha_j)_{\mathfrak{h}^*} = \frac{a_j^\vee}{a_j} A_{ij} & \text{for} & i,j \in \underline{n}_0, \\ (\alpha_i, \Lambda_0)_{\mathfrak{h}^*} = 0 & \text{for} & i \in \underline{n}, \\ (\alpha_0, \Lambda_0)_{\mathfrak{h}^*} = a_0, & & \text{and} \\ (\Lambda_0, \Lambda_0)_{\mathfrak{h}^*} = 0. & & \end{array}$$

The map $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by

$$\langle \nu(h), h' \rangle = (h, h') \quad \text{for} \quad h, h' \in \mathfrak{h}$$

satisfies

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee} \alpha_i, \quad \nu(c) = \delta, \quad \text{and} \quad \nu(d) = a_0 \Lambda_0.$$

It is also useful to note the behaviour of c and δ under the normalized invariant form. This is given by the following formulae;

$$(c, \alpha_i^\vee) = 0 \text{ for } i \in \underline{n}_0, \quad (c, c) = 0, \quad (c, d) = a_0,$$
$$(\delta, \alpha_i)_{\mathfrak{h}^*} = 0 \text{ for } i \in \underline{n}_0, \quad (\delta, \delta)_{\mathfrak{h}^*} = 0, \quad (\delta, \Lambda_0)_{\mathfrak{h}^*} = 1.$$

Denote by $\breve{\mathfrak{g}}$ the subalgebra of $\mathfrak{g}$ generated by the set of elements

$$\{e_i, f_i \,:\, i \in \underline{n}\}.$$

Then $\breve{\mathfrak{g}}$ is a Kac-Moody algebra associated to the matrix $\breve{A}$ obtained from A by deleting the zeroeth row and column. Furthermore, by Proposition 1.2.15, $\breve{\mathfrak{g}} = \mathfrak{g}(\breve{A})$ is a simple finite-dimensional Kac-Moody algebra whose Dynkin diagram $\Delta(\breve{A})$ is obtained from $\Delta(A)$ by removing the zeroeth vertex. We give the following results for the structure of $\breve{\mathfrak{g}}$:

- The elements $\{e_i, f_i \,:\, i \in \underline{n}\}$ are the Chevalley generators of $\breve{\mathfrak{g}}$.
- $\breve{\mathfrak{h}} = \breve{\mathfrak{g}} \cap \mathfrak{h}$ is the Cartan subalgebra of $\breve{\mathfrak{g}}$.
- $\breve{\Pi} = \{\alpha_i\}_{i \in \underline{n}}$ is the root basis of $\breve{\mathfrak{g}}$.
- $\breve{\Pi}^\vee = \{\alpha_i^\vee\}_{i \in \underline{n}}$ is the coroot basis of $\breve{\mathfrak{g}}$.
- $\breve{\Phi} = \Phi \cap \breve{\mathfrak{h}}^*$ is the root system of $\breve{\mathfrak{g}}$. This root system is finite and consists of real roots, the set $\breve{\Phi}_+ = \breve{\Phi} \cap \Phi_+$ being the set of positive roots.

Denote by $\breve{\Phi}_s$ and $\breve{\Phi}_l$ the sets of short and long roots in $\breve{\Phi}$, respectively. Then

$$\breve{\Phi} = \breve{\Phi}_s \cup \breve{\Phi}_l.$$

Put $\breve{Q} = \mathbb{Z}\breve{\Phi}$ and let $\breve{W}$ be the Weyl group of $\breve{\Phi}$.

Recall that the set of imaginary and positive imaginary roots of $\mathfrak{g}$ are

$$\Phi^{im} = \{\pm\delta, \pm 2\delta, \ldots\} \qquad \text{and} \qquad \Phi_+^{im} = \{\delta, 2\delta, \ldots\},$$

respectively. The following proposition describes the set of real roots Φ^{re} and positive real roots Φ_+^{re} in terms of $\breve{\Phi}$ and δ.

Proposition 1.3.1

Let A be an affine matrix and $k = 1, 2,$ or 3, depending on whether A is of type I, II, or III, respectively.

1. *If A is affine of type I then*

$$\Phi^{re} = \{\alpha + n\delta \,:\, \alpha \in \breve{\Phi},\ n \in \mathbb{Z}\}.$$

2. *If A is of type II or III, but not of types ${}^*\check{C}_l$ or ${}^*\check{A}_1$ then*

$$\Phi^{re} = \{\alpha + n\delta \ : \ \alpha \in \check{\Phi}_s, \ n \in \mathbf{Z}\} \cup \{\alpha + nk\delta \ : \ \alpha \in \check{\Phi}_l, \ n \in \mathbf{Z}\}.$$

3. *If A is of types ${}^*\check{C}_l$ or ${}^*\check{A}_1$ then*

$$\begin{aligned}\Phi^{re} \ &= \ \left\{\frac{1}{2}(\alpha + (2n-1)\delta) \ : \ \alpha \in \check{\Phi}_l, \ n \in \mathbf{Z}\right\} \cup \{\alpha + n\delta \ : \ \alpha \in \check{\Phi}_s, \ n \in \mathbf{Z}\} \\ &\qquad \cup \{\alpha + 2n\delta \ : \ \alpha \in \check{\Phi}_l, \ n \in \mathbf{Z}\}.\end{aligned}$$

4. *In all cases,*

$$\Phi^{re} + k\delta = \Phi^{re}$$

and

$$\Phi^{re}_+ = \{\alpha \in \Phi^{re} \ \textit{with} \ n > 0\} \cup \check{\Phi}_+.$$

We can also define

$$\check{\Phi}^\vee = \Phi^\vee \cap \check{\mathfrak{h}} \qquad \text{and} \qquad \check{Q}^\vee = \mathbf{Z}\check{\Phi}^\vee.$$

Since $a_0^\vee = 1$ in all cases, we have an orthogonal direct sum, namely

$$Q^\vee = \check{Q}^\vee \oplus \mathbf{Z}c.$$

Also, by the definition of the forms $(\cdot\,,\cdot)_{\mathfrak{h}}$ and $(\cdot\,,\cdot)_{\mathfrak{h}^*}$,

$$Q^\vee(A) \cong Q({}^tA)$$

is an isomorphism of lattices equipped with bilinear forms.

REMARK 1.3.2

For a subset S of $\mathfrak{h}^*$, denote by $\text{proj}(S)$ the orthogonal projection of S onto $\check{\mathfrak{h}}^*$. Then

- $\check{\Phi} = \text{proj}(\Phi) \setminus \{0\}$ in all cases except when A is of types ${}^*\check{C}_l$ or ${}^*\check{A}_1$.
- If A is of type ${}^*\check{C}_l$ or ${}^*\check{A}_1$ then $\text{proj}(\Phi) \setminus \{0\}$ is a non-reduced system and $\check{\Phi}$ is the associated root system.

◇

We now introduce the element

$$\theta = \delta - a_0\alpha_0 = \sum_{i \in n} a_i\alpha_i \in \check{Q}$$

which will prove to be of great importance in the sequel.

By using the formulae we know the normalized invariant bilinear form to satisfy we have

$$\begin{aligned} |\theta|^2 &= (\delta - a_0\alpha_0, \delta - a_0\alpha_0)_{\mathfrak{h}^*} \\ &= (a_0\alpha_0, a_0\alpha_0)_{\mathfrak{h}^*} \\ &= a_0^2 a_0^\vee a_0^{-1} A_{00} \\ &= 2a_0. \end{aligned}$$

Hence $|\theta|^2$ is equal to the square length of a long root if A is affine of types I, ${}^*\tilde{C}_l$ or ${}^*\tilde{A}_1$, and is equal to the square length of a short root otherwise. Thus, by Proposition 1.2.23,

$$\theta \in \check{\Phi}_+$$

in all cases. Hence we have

$$\theta^\vee = \frac{2\nu^{-1}(\theta)}{(\theta,\theta)_{\mathfrak{h}^*}} = \frac{\nu^{-1}(\delta - a_0\alpha_0)}{a_0} = \frac{c - a_0^\vee\alpha_0^\vee}{a_0},$$

from which we can easily deduce the following three relations:

$$\theta = a_0\nu(\theta^\vee), \quad |\theta^\vee|^2 = 2a_0^{-1}, \quad \text{and} \quad \alpha_0^\vee = c - a_0\theta^\vee.$$

Furthermore, we have

PROPOSITION 1.3.3

1. *If A is affine of types I, ${}^*\tilde{C}_l$ or ${}^*\tilde{A}_1$, then $\theta \in (\check{\Phi}_+)_l$ and θ is the unique root in $\check{\Phi}$ of maximal height. This height is given by $h - a_0$.*

2. *If A is affine of types II or III but not of types ${}^*\tilde{C}_l$ or ${}^*\tilde{A}_1$, then $\theta \in (\check{\Phi}_+)_s$ and is the unique root in $\check{\Phi}_s$ of maximal height. This height is given by $h - 1$.*

Unless otherwise stated, in the case of a matrix A of finite type, we shall normalize the standard invariant form $(\cdot,\cdot)$ on $\mathfrak{g}_A(\mathbb{C})$ by the condition

$$(\alpha,\alpha)_{\mathfrak{h}^*} = 2 \quad \text{if} \quad \alpha \in \Phi_l.$$

and shall call it the *normalized invariant form*. We can now deduce the following:

COROLLARY 1.3.4

Let $\mathfrak{g}$ be an affine algebra of type k. Then the ratio of the normalized invariant form of $\mathfrak{g}$ restricted to $\check{\mathfrak{g}}$ to the normalized invariant form of $\check{\mathfrak{g}}$ is equal to k.

Note that we have the following description of Π and $\Pi^\vee$:

$$\begin{aligned} \Pi &= \left\{\alpha_0 = \frac{\delta - \theta}{a_0}, \alpha_1, \ldots, \alpha_n\right\}, \\ \Pi^\vee &= \{\alpha_0^\vee = c - a_0\theta^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee\}. \end{aligned}$$

We now turn our attention to the description of the Weyl group of the affine algebra $\mathfrak{g}$. Recall that W is a Coxeter group generated by fundamental reflections $\{r_i\}_{i \in \underline{n}_0}$ which act on $\mathfrak{h}^*$ by

$$r_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i, \qquad \text{for} \qquad \beta \in \mathfrak{h}^*.$$

Since $\langle \delta, \alpha_i^\vee \rangle = 0$ for all $i \in \underline{n}_0$, we see that

$$w(\delta) = \delta \qquad \text{for all} \qquad w \in W.$$

We recall also that the invariant standard form is W-invariant.

Denote by $\breve{W}$ the subgroup of W generated by the set $\{r_i\}_{i \in \underline{n}}$. Since

$$r_i(\Lambda_0) = \Lambda_0 \qquad \text{for} \qquad i \in \underline{n},$$

we deduce that $\breve{W}$ operates trivially on $\mathbb{C}\Lambda_0 + \mathbb{C}\delta$. It also follows that $\breve{\mathfrak{h}}^*$ is $\breve{W}$-invariant.

We conclude that $\breve{W}$ operates faithfully on $\breve{\mathfrak{h}}^*$, and we can identify $\breve{W}$ with the Weyl group of the Kac-Moody algebra $\breve{\mathfrak{g}}$, operating on $\breve{\mathfrak{h}}^*$. Hence the group $\breve{W}$ is finite.

Following [Hum90, Chapter 4] we construct the so-called *affine Weyl group* $\breve{W}_{aff}$ corresponding to $\breve{W}$. To each $\alpha \in \breve{\mathfrak{h}}^*$ we associate the translation t_α which sends $\beta \in \breve{\mathfrak{h}}^*$ to $\alpha + \beta$. Then for any $w \in \breve{W}$ and $\alpha \in \breve{\mathfrak{h}}^*$

$$w t_\alpha w^{-1} = t_{w(\alpha)}$$

showing that the group of translations is normalized by $\breve{W}$. We define the affine group, $\mathrm{Aff}(\breve{\mathfrak{h}}^*)$, to be the semidirect product of $\breve{W}$ and the group of translations by elements of $\breve{\mathfrak{h}}^*$.

For each $\alpha \in \Phi$ and each $n \in \mathbb{Z}$, define an affine hyperplane

$$H_{\alpha,n} = \{\beta \in \breve{\mathfrak{h}}^* \ : \ (\alpha, \beta)_{\breve{\mathfrak{h}}^*} = n\}.$$

Note that $H_{\alpha,n} = H_{-\alpha,-n}$ and that $H_{\alpha,0}$ coincides with the projection onto $\breve{\mathfrak{h}}^*$ of the reflecting hyperplane

$$H_\alpha \ : \ \{\beta \in \breve{\mathfrak{h}}^* \ : \ \langle \beta, \alpha^\vee \rangle = 0\} \ = \ \{\beta \in \breve{\mathfrak{h}}^* \ : \ (\alpha, \beta)_{\breve{\mathfrak{h}}^*} = 0\}$$

Note also that $H_{\alpha,n}$ can be obtained by translating H_α by $\frac{n}{2}\alpha^\vee$. Define the corresponding affine reflection as follows:

$$s_{\alpha,n}(\beta) = \beta - \Big((\alpha, \beta)_{\breve{\mathfrak{h}}^*} - n \Big)\, \alpha^\vee.$$

We can also write $s_{\alpha,n}$ as $t_{(n\alpha^\vee)} s_\alpha$. In particular $s_{\alpha,0} = s_\alpha$. Denote by $\mathcal{H}$ the collection of all hyperplanes $H_{\alpha,n}$ for $\alpha \in \Phi$ and $n \in \mathbb{Z}$. The following result shows that the elements of $\mathcal{H}$ are permuted in a natural way by $\breve{W}$ as well as by certain translations in $\mathrm{Aff}(\breve{\mathfrak{h}}^*)$.

PROPOSITION 1.3.5

1. *If* $w \in \check{W}$, *then*

$$wH_{\alpha,n} = H_{w(\alpha),n} \quad \text{and} \quad ws_{\alpha,n}w^{-1} = s_{w(\alpha),n}.$$

2. *If* $\beta \in \mathfrak{h}^*$ *satisfies* $(\alpha, \beta) \in \mathbb{Z}$ *for all* $\alpha \in \check{\Phi}$, *then*

$$t_\beta H_{\alpha,n} = H_{\alpha,n+(\alpha,\beta)} \quad \text{and} \quad t_\beta s_{\alpha,n} t_\beta^{-1} = s_{\alpha,n+(\alpha,\beta)}.$$

We define the *affine Weyl group*, $\check{W}_{aff}$ to be the subgroup of $\mathrm{Aff}(\mathfrak{h}^*)$ generated by all affine reflections $s_{\alpha,n}$ where $\alpha \in \check{\Phi}$ and $n \in \mathbb{Z}$. We can describe the structure of $\check{W}_{aff}$ in terms of structures already familiar to us.

Proposition 1.3.6

$\check{W}_{aff}$ is the semidirect product of $\check{W}$ and the translation group corresponding to the coroot lattice $Q^\vee(\check{A})$.

Finally we can give a description of the Weyl group W associated to a generalized Cartan matrix of affine type.

Proposition 1.3.7

Let A be a generalized Cartan matrix of affine type and W its Weyl group. Then, in the above notation

$$W \cong \check{W}_{aff}$$

Proof

See [Kac90, §§6.5–6.6]. □

Example 1.3.8

Let $\bar{A}_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then, $\check{A}_1 = (2) = A_1$. As we saw in Example 1.1.23, $\check{W} = W(A_1) \cong C_2$. Thus $W \cong \check{W}_{aff} \cong D_\infty$, where D_∞ denotes the infinite dihedral group. ◇

1.4 Affine Kac-Moody Algebras of Type I

In this section we describe a realization of all the affine algebras of type I. Recall that such algebras have a one-dimensional centre, $\mathfrak{c}$. It transpires that such an algebra $\mathfrak{g}$ can be realized entirely in terms of an "underlying" simple finite-dimensional Lie algebra $\check{\mathfrak{g}}$. In particular, its derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is the universal central extension of the Kac-Moody algebra of polynomial maps from $\mathbb{C}^\times$ into $\check{\mathfrak{g}}$.

Laurent Polynomials and Residues

Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t. Recall that the *residue*, Res P, of a Laurent polynomial

$$P = \sum_{i \in \mathbb{Z}} m_i t^i,$$

where all but a finite number of m_i are zero, is defined by

$$\operatorname{Res} P = m_{-1}.$$

This is a linear functional on $\mathcal{L}$ defined by the properties

$$\operatorname{Res} t^{-1} = 1 \qquad \text{and} \qquad \operatorname{Res} \frac{dP}{dt} = 0.$$

Define a bilinear $\mathbb{C}$-valued function φ on $\mathcal{L}$ by

$$\varphi(P, Q) = -\operatorname{Res} \frac{dP}{dt} Q.$$

This satisfies the following two properties:

$$\varphi(P, Q) = -\varphi(Q, P), \qquad \text{and}$$
$$\varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0,$$

for $P, Q, R \in \mathcal{L}$.

Realization of the Extended Kac-Moody Algebras

We first note that the generalized Cartan matrix A giving rise to an affine Kac-Moody algebra of type I is none other than the so-called *extended Cartan matrix* of the simple finite-dimensional Lie algebra $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A})$, whose Cartan matrix $\mathring{A}$ is a matrix of finite type obtained from A by removing the zeroeth row and column.

Consider the *loop algebra*

$$\mathcal{L}(\mathring{\mathfrak{g}}) = \mathcal{L} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}.$$

This is an infinite-dimensional complex Lie algebra with the Lie bracket $[\cdot, \cdot]_0$ defined by

$$[P \otimes x, Q \otimes y]_0 = PQ \otimes [x, y]$$

for $P, Q \in \mathcal{L}$ and $x, y \in \mathring{\mathfrak{g}}$. It can be identified with the algebra of regular rational maps

$$\mathbb{C}^\times \to \mathring{\mathfrak{g}}.$$

so that the element

$$\sum_{i\in\mathbb{Z}} t^i \otimes x_i$$

corresponds to the mapping

$$z \mapsto \sum_i z^i x_i.$$

Fix a nondegenerate invariant symmetric bilinear $\mathbb{C}$-valued form $(\cdot,\cdot)$ on $\mathring{\mathfrak{g}}$. Such a form exists and is unique up to a constant multiple. We extend this form by linearity to an $\mathcal{L}$-valued bilinear form $(\cdot,\cdot)_t$ on $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$(P\otimes x, Q\otimes y)_t = PQ(x,y).$$

We also extend every derivation D of the algebra $\mathcal{L}$ to a derivation of the Lie algebra $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$D(P\otimes x) = D(P)\otimes x.$$

Now we can define a $\mathbb{C}$-valued 2-cocycle on $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$\psi(a,b) = -Res\left(\frac{da}{dt}, b\right)_t$$

for all $a,b\in\mathcal{L}(\mathring{\mathfrak{g}})$. Recall that a $\mathbb{C}$-valued 2-cocycle on a Lie algebra $\mathfrak{g}$ is a bilinear function ψ satisfying the following two conditions:

$$\psi(a,b) = -\psi(b,a), \qquad \text{and}$$
$$\psi([a,b],c) + \psi([b,c],a) + \psi([c,a],b) = 0$$

for $a,b,c\in\mathfrak{g}$. It is sufficient to check these conditions for $a = P\otimes x$, $b = Q\otimes y$, and $c = R\otimes z$, where $P,Q,R\in\mathcal{L}$ and $x,y,z\in\mathring{\mathfrak{g}}$. We have

$$\psi(a,b) = (x,y)\varphi(P,Q).$$

Hence ψ can be shown to satisfy the necessary conditions by using the properties of φ and the symmetry and invariance of $(\cdot,\cdot)$.

Denote by $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ the extension of $\mathcal{L}(\mathring{\mathfrak{g}})$ by a 1-dimensional centre associated to the cocycle ψ. In other words, as a direct sum of vector spaces,

$$\hat{\mathcal{L}}(\mathring{\mathfrak{g}}) = \mathcal{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}c$$

and the Lie bracket on $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ is given by

$$[a+\lambda c, b+\mu c] = [a,b]_0 + \psi(a,b)c$$

for $a,b\in\mathcal{L}(\mathring{\mathfrak{g}})$ and $\lambda,\mu\in\mathbb{C}$.

Finally, denote by $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ the Kac-Moody algebra that is obtained by adjoining to $\tilde{\mathcal{L}}(\check{\mathfrak{g}})$ a derivation d which acts on $\mathcal{L}(\check{\mathfrak{g}})$ as $t\frac{d}{dt}$ and which kills c. More explicitly, $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ is a complex vector space

$$\hat{\mathcal{L}}(\check{\mathfrak{g}}) = \mathcal{L}(\check{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with the Lie bracket defined by

$$\begin{aligned}&[t^m \otimes x + \lambda_1 c + \mu_1 d, t^n \otimes y + \lambda_2 c + \mu_2 d] = \\ &\quad (t^{m+n} \otimes [x,y] + \mu_1 n t^n \otimes y - \mu_2 m t^m \otimes x) - m\delta_{m,-n}(x,y)c \qquad (1.4.0)\end{aligned}$$

where $x, y \in \check{\mathfrak{g}}$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ and $\delta_{m,-n}$ is the Kronecker delta.

We show that $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ is an affine Kac-Moody algebra associated to the affine Cartan matrix A.

Let $\check{\Phi} \subset \check{\mathfrak{h}}^*$ be the root system of the Kac-Moody algebra $\check{\mathfrak{g}}$. Let $\{\alpha_i\}_{i\in \underline{n}}$ be the root basis, $\{H_i\}_{i\in\underline{n}}$ the coroot basis, and $\{E_i, F_i\}_{i\in\underline{n}}$ the Chevalley generators of $\check{\mathfrak{g}}$. Let θ be the highest root of the finite system $\check{\Phi}$ and

$$\check{\mathfrak{g}} = \bigoplus_{\alpha\in\check{\Phi}\cup 0} \check{\mathfrak{g}}_\alpha$$

be the root space decomposition of $\check{\mathfrak{g}}$. Recall that

$$(\alpha,\alpha)_{\check{\mathfrak{h}}^*} \neq 0 \qquad \text{and} \qquad \dim \check{\mathfrak{g}}_\alpha = 1$$

for $\alpha \in \check{\Phi}$. Let $\check{\omega}$ be the Chevalley involution of $\check{\mathfrak{g}}$. We choose $F_0 \in \check{\mathfrak{g}}_\theta$ such that

$$(F_0, \check{\omega}(F_0)) = \frac{2}{(\theta,\theta)_{\check{\mathfrak{h}}^*}}, \qquad (1.4.0)$$

and set $E_0 = \check{\omega}(F_0)$. Then

$$[E_0, F_0] = \theta^\vee \qquad (1.4.0)$$

by the last part of Theorem 1.1.17. Also, by [Kac90, §7.4], the elements $\{E_i\}_{i\in\underline{n}_0}$ generate the Lie algebra $\check{\mathfrak{g}}$.

We now return our attention to $\hat{\mathcal{L}}(\check{\mathfrak{g}})$. Since $\check{\mathfrak{g}}$ is simple, we see that $\mathbb{C}c$ is the centre of the Kac-Moody algebra $\hat{\mathcal{L}}(\check{\mathfrak{g}})$, and that the centralizer of d in $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ is a direct sum of Lie algebras, namely

$$(1 \otimes \check{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

In particular, $1 \otimes \check{\mathfrak{g}}$ is a subalgebra of $\hat{\mathcal{L}}(\check{\mathfrak{g}})$, and we identify $\check{\mathfrak{g}}$ with this subalgebra in the natural fashion.

Furthermore,

$$\mathfrak{h} = \check{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is an $(n+2)$-dimensional commutative subalgebra in $\hat{\mathcal{L}}(\mathfrak{g})$. We extend $\beta \in \check{\mathfrak{h}}^*$ to a linear function on $\mathfrak{h}$ by setting

$$\langle \beta, c\rangle = \langle \beta, d\rangle = 0,$$

so that $\check{\mathfrak{h}}^*$ is identified with a subspace in $\mathfrak{h}^*$. We denote by δ the linear function on $\mathfrak{h}$ defined by

$$\delta\,|_{\check{\mathfrak{h}}+\mathbb{C}c} = 0, \qquad \text{and} \qquad \langle \delta, d\rangle = 1.$$

Set

$$\begin{aligned} e_0 &= t \otimes E_0, & f_0 &= t^{-1} \otimes F_0, & & \\ e_i &= 1 \otimes E_i, \quad \text{and} & f_i &= 1 \otimes F_i, & \text{for} \quad & i \in \underline{n}. \end{aligned}$$

We deduce from equations (1.4.0), (1.4.0) and (1.4.0) that

$$[e_0, f_0] = -\left(\frac{2}{(\theta,\theta)_{\check{\mathfrak{h}}^*}} c - \theta^\vee\right).$$

The root system and root space decomposition of $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ with respect to $\mathfrak{h}$ are then

$$\Phi = \left\{ j\delta + \beta, \quad \text{where } j \in \mathbb{Z},\ \beta \in \check{\Phi} \right\} \cup \{ j\delta, \quad \text{where } j \in \mathbb{Z} \setminus \{0\}\},$$

and

$$\hat{\mathcal{L}}(\check{\mathfrak{g}}) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathcal{L}(\check{\mathfrak{g}})_\alpha \right),$$

where

$$\mathcal{L}(\check{\mathfrak{g}})_{j\delta+\beta} = t^j \otimes \check{\mathfrak{g}}_\beta, \qquad \text{and} \qquad \mathcal{L}(\check{\mathfrak{g}})_{j\delta} = t^j \otimes \check{\mathfrak{h}}.$$

We set

$$\begin{aligned} \Pi &= \{\alpha_0 = \delta - \theta, \alpha_1, \ldots, \alpha_n\}, \qquad \text{and} \\ \Pi^\vee &= \left\{ \alpha_0^\vee = \tfrac{2}{(\theta,\theta)_{\check{\mathfrak{h}}^*}} c - \theta^\vee, \alpha_1^\vee = 1 \otimes H_1, \ldots, \alpha_n^\vee = 1 \otimes H_n \right\}. \end{aligned}$$

Note that the θ used here is none other than that introduced in §1.2, i.e.

$$\theta = \sum_{i \in \underline{n}} a_i \alpha_i \in \check{Q},$$

where the a_i are the labels on the Dynkin diagram of the affine matrix A we started with (recall that $a_0 = 1$ in the cases currently under consideration). Thus

$$\langle \theta, \alpha_i^\vee \rangle = -\langle \alpha_0, \alpha_i^\vee \rangle$$

for all $i \in \underline{n}$ and so

$$A = (\langle \alpha_j, \alpha_i^\vee \rangle)_{i,j \in \underline{n}_0}.$$

In other words $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of the matrix A, since Π and $\Pi^\vee$ are linearly independent and $\mathfrak{h}$ is of the required dimension. This construction is heavily used in the proof of the following result.

THEOREM 1.4.1

Let $\mathring{\mathfrak{g}}$ be a complex finite-dimensional simple Lie algebra, and let A be its extended Cartan matrix. Then $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ is the affine Kac-Moody algebra associated to the affine matrix A, $\mathfrak{h}$ is its Cartan subalgebra, Π and $\Pi^\vee$ the root and coroot basis, and $\{e_i, f_i\}_{i \in \underline{n}_0}$ the Chevalley generators. In other words,

$$(\hat{\mathcal{L}}(\mathfrak{g}), \mathfrak{h}, \Pi, \Pi^\vee)$$

is the quadruple associated to A.

Proof

See [Kac90, Theorem 7.4] □

As an immediate consequence of this construction we have the following Corollary.

COROLLARY 1.4.2

Let $\mathfrak{g}_A(\mathbb{C})$ be an affine Kac-Moody algebra of type I and rank $n+1$. Then the multiplicity of every imaginary root of $\mathfrak{g}_A(\mathbb{C})$ is n.

We can also give an explicit description of the rest of the notions introduced in the previous sections.

The normalized invariant form $(\cdot,\cdot)$ on $\mathfrak{g}$ can be described as follows. Take the the normalized invariant form $(\cdot,\cdot)$ on $\mathring{\mathfrak{g}}$ and extend it to the whole of $\hat{\mathcal{L}}(\mathfrak{g})$ by defining

$$\begin{aligned}
&(P \otimes x, Q \otimes y) = (\operatorname{Res} t^{-1} PQ)(x, y), && \text{for } x, y \in \mathring{\mathfrak{g}}, P, Q \in \mathcal{L},\\
&(\mathbb{C}c + \mathbb{C}d, \mathcal{L}(\mathring{\mathfrak{g}})) = 0,\\
&(c, c) = (d, d) = 0, && \text{and}\\
&(c, d) = 1.
\end{aligned}$$

The result is an invariant non-degenerate symmetric bilinear form, whose restriction to $\mathfrak{h}$ coincides with the form constructed in §1.3. We refer the reader to [Kac90, §7.5] for the verification of these claims.

Note also that the element $c \in \hat{\mathcal{L}}(\mathfrak{g})$ is then the canonical central element and that the element d is the scaling element.

Let

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$$

be the triangular decomposition of the Lie algebra $\mathring{\mathfrak{g}}$. Then the triangular decomposition of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ can be expressed as follows:

$$\hat{\mathcal{L}}(\mathring{\mathfrak{g}}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where

$$\mathfrak{n}_- = \left(t^{-1}\mathbb{C}[t^{-1}] \otimes (\check{\mathfrak{n}}_+ + \check{\mathfrak{h}})\right) + \mathbb{C}[t^{-1}] \otimes \check{\mathfrak{n}}_-,$$

and

$$\mathfrak{n}_+ = \left(t\mathbb{C}[t] \otimes (\check{\mathfrak{n}}_- + \check{\mathfrak{h}})\right) + \mathbb{C}[t] \otimes \check{\mathfrak{n}}_+.$$

We can also express the Chevalley involution ω of $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ in terms of the Chevalley involution $\check{\omega}$ of $\check{\mathfrak{g}}$. We define

$$\omega\left(P(t) \otimes x + \lambda c + \mu d\right) = P(t^{-1}) \otimes \check{\omega}(x) + \lambda c + \mu d,$$

where $P(t) \in \mathcal{L}$, $x \in \check{\mathfrak{g}}$, and $\lambda, \mu \in \mathbb{C}$. Then $\omega(e_i) = f_i$ and $\omega(f_i) = e_i$ for all $i \in \underline{n}$ and ω fixes every element of $\mathfrak{h}$. Furthermore,

$$\begin{aligned} \omega(e_0) &= \omega(t \otimes E_0) \\ &= t^{-1} \otimes \check{\omega}(E_0) \\ &= t^{-1} \otimes F_0 = f_0 \end{aligned}$$

and similarly $\omega(f_0) = e_0$.

Structure Constants of Affine Kac-Moody Algebras of Type I

Given a root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi \cup 0} \mathfrak{g}_\alpha$$

of a Kac-Moody algebra $\mathfrak{g}$ with respect to some Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$, we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

whenever $\alpha, \beta, \alpha + \beta \in \Phi$. In particular, when $\alpha, \beta, \alpha + \beta \in \Phi^{re}$ their corresponding root spaces are one-dimensional, generated by single elements e_α, e_β, and $e_{\alpha+\beta}$ say, and the above relation implies that

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta},$$

for some $N_{\alpha,\beta} \in \mathbb{C}$. It is convenient to assume that $N_{\alpha,\beta} = 0$ if $\alpha, \beta \in \Phi^{re}$ but $\alpha + \beta \notin \Phi^{re}$.

The elements e_α for $\alpha \in \Phi$ are called *root vectors* and the elements $N_{\alpha,\beta}$ for $\alpha, \beta \in \Phi^{re}$ are called the *structure constants* of the Kac-Moody algebra $\mathfrak{g}$.

We consider the structure constants of the affine Kac-Moody algebra $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ constructed in §1.4. By virtue of the root system and root space decomposition of $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ given earlier, all real roots of $\hat{\mathcal{L}}(\check{\mathfrak{g}})$ can be expressed in the form $\alpha + i\delta$ for some

$\alpha \in \Phi(\mathfrak{g})$ and $i \in \mathbb{Z}$. Suppose we have established root vectors e_α and structure constants $N_{\alpha,\beta}$ for all $\alpha, \beta \in \Phi(\mathring{\mathfrak{g}})$. Then, given $\alpha \in \Phi(\mathfrak{g})$ and $i \in \mathbb{Z}$, a root vector for $\mathcal{L}(\mathring{\mathfrak{g}})_{\alpha+i\delta}$ can be taken to be $t^i \otimes e_\alpha$.

Suppose now that we have two real roots $\alpha + i\delta$ and $\beta + j\delta$ of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ where $\alpha, \beta \in \Phi(\mathring{\mathfrak{g}})$, $i, j \in \mathbb{Z}$ and $\beta + j\delta \neq \pm\alpha + i\delta$. Then, using equation (1.4.0), we obtain

$$\begin{aligned}
[t^i \otimes e_\alpha, t^j \otimes e_\beta] &= t^{i+j} \otimes [e_\alpha, e_\beta] - i\delta_{i,-j}(e_\alpha, e_\beta)c \\
&= t^{i+j} \otimes N_{\alpha,\beta} e_{\alpha+\beta} \\
&= N_{\alpha,\beta}\left(t^{i+j} \otimes e_{\alpha+\beta}\right).
\end{aligned}$$

Hence we have proved the following result.

LEMMA 1.4.3
For all real roots $\alpha + i\delta$ and $\beta + j\delta$ of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$,

$$N_{\alpha+i\delta,\beta+j\delta} = N_{\alpha,\beta}.$$

Thus the structure constants of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ depend only on the structure constants of $\mathring{\mathfrak{g}}$.

Structure Constants of Finite-Dimensional Kac-Moody Algebras

In this section we develop a systematic method for determining the structure constants of a simply-laced Kac-Moody algebra of finite type. We begin by giving some properties satisfied by the structure constants of an arbitrary simple finite-dimensional Lie algebra.

THEOREM 1.4.4
The structure constants of a simple finite-dimensional Lie algebra $\mathfrak{g}_A(\mathbb{C})$ satisfy the following relations for all $\alpha, \beta, \gamma \in \Phi(A)$:

1. $N_{\beta,\alpha} = -N_{\alpha,\beta}$.

2. *If $-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$ is the α-chain through β, then*

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = (p+1)^2.$$

3. *If $\alpha + \beta + \gamma = 0$ then*

$$\frac{N_{\alpha,\beta}}{(\gamma,\gamma)} = \frac{N_{\beta,\gamma}}{(\alpha,\alpha)} = \frac{N_{\gamma,\alpha}}{(\beta,\beta)}.$$

4. Whenever $\alpha+\beta+\gamma+\vartheta=0$ *and no pair are opposite, we have*

$$\frac{N_{\alpha,\beta}N_{\gamma,\vartheta}}{(\alpha+\beta,\alpha+\beta)}+\frac{N_{\beta,\gamma}N_{\alpha,\vartheta}}{(\beta+\gamma,\beta+\gamma)}+\frac{N_{\gamma,\alpha}N_{\beta,\vartheta}}{(\gamma+\alpha,\gamma+\alpha)}=0.$$

Proof

We refer the reader to the proof of [Car72, Theorem 4.1.2], bearing in mind that the proof of part 2, and consequently the result itself, must be altered to take into account the fact that we have adopted Tits' convention for the generators. □

We now proceed to develop a method for the systematic calculation of the structure constants in a simply-laced Kac-Moody algebra of finite type. This work is a variation on similar calculations found in [Kac90, §7.8].

Since we are only considering simply-laced algebras we can write the property in part 4 of Theorem 1.4.4 as

$$N_{\alpha,\beta}N_{\gamma,\vartheta}+N_{\beta,\gamma}N_{\alpha,\vartheta}=0$$

if

$$\alpha,\beta,\gamma,\vartheta,\alpha+\beta,\alpha+\vartheta\in\Phi \qquad \text{and} \qquad \alpha+\beta+\gamma+\vartheta=0.$$

Using the fact that $\gamma+\vartheta+(\alpha+\beta)=0$, $\alpha+\vartheta+(\beta+\gamma)=0$ and part 3 of Theorem 1.4.4, we can further reformulate this as

$$N_{\alpha,\beta}N_{\alpha+\beta,\gamma}+N_{\beta,\gamma}N_{\beta+\gamma,\alpha}=0.$$

We now define a map

$$\begin{array}{rccc}\varepsilon: & Q\times Q & \longrightarrow & \{\pm1\}\\ & (\alpha,\beta) & \mapsto & \varepsilon(\alpha,\beta)\end{array}$$

satisfying $\varepsilon(0,\beta)=1=\varepsilon(\alpha,0)$ and such that ε is bimultiplicative, i.e.

$$\begin{aligned}\varepsilon(\alpha_1+\alpha_2,\beta) &= \varepsilon(\alpha_1,\beta)\varepsilon(\alpha_2,\beta) \text{ and}\\ \varepsilon(\alpha,\beta_1+\beta_2) &= \varepsilon(\alpha,\beta_1)\varepsilon(\alpha,\beta_2).\end{aligned}$$

Thus ε is determined by its action on the fundamental roots, in particular by the elements

$$\varepsilon_{ij}=\varepsilon(\alpha_i,\alpha_j).$$

We show that taking $N_{\alpha,\beta}=\varepsilon(\alpha,\beta)$ gives rise to a consistent set of structure constants for $\mathfrak{g}$. This entails showing that ε satisfies the following conditions;

$$\begin{aligned}&\varepsilon(\alpha,\alpha)=-1,\\ &\varepsilon(\beta,\alpha)=-\varepsilon(\alpha,\beta),\\ &\varepsilon(-\alpha,-\beta)=\varepsilon(\alpha,\beta),\\ &\varepsilon(\alpha,\beta)=\varepsilon(\beta,-\alpha-\beta)=\varepsilon(-\alpha-\beta,\alpha), \qquad \text{and}\\ &\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma)=\varepsilon(\beta,\gamma)\varepsilon(\alpha,\beta+\gamma)\end{aligned}$$

whenever $\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma \in \Phi$.

We introduce the notation $i \leftrightarrow j$ to indicate that the nodes corresponding to α_i and α_j in $\Delta(A)$ are connected by a bond. Similarly $i \not\leftrightarrow j$ denotes that the corresponding nodes are not connected in $\Delta(A)$. We now choose ε_{ij} in the following manner:

$$\begin{array}{rcll} \varepsilon_{ii} & = & -1 & , \\ \varepsilon_{ij} & = & 1 & \text{if} \quad i<j, \\ \varepsilon_{ji} & = & -1 & \text{if} \quad i<j,\ i \leftrightarrow j, \text{ and} \\ \varepsilon_{ji} & = & 1 & \text{if} \quad i<j,\ i \not\leftrightarrow j. \end{array}$$

We claim that choosing the elements $\{\varepsilon_{ij}\}_{i,j\in\underline{n}}$ in this manner ensures that ε satisfies the necessary conditions.

We begin by verifying that ε satisfies the condition

$$\varepsilon(\beta,\alpha) = -\varepsilon(\alpha,\beta).$$

Suppose $\alpha, \beta \in \Phi$ with

$$\alpha = \sum_{i\in\underline{n}} m_i\alpha_i, \qquad \beta = \sum_{j\in\underline{n}} n_j\alpha_j$$

and $\alpha+\beta \in \Phi$. Then

$$\begin{aligned} \varepsilon(\alpha,\beta) &= \varepsilon\left(\sum_{i\in\underline{n}} m_i\alpha_i, \sum_{j\in\underline{n}} n_j\alpha_j\right) = \prod_{i,j\in\underline{n}} \varepsilon_{ij}^{m_in_j} \\ &= \prod_{i\in\underline{n}}(-1)^{m_in_i} \prod_{i>j,\, i\leftrightarrow j} (-1)^{m_in_j} \\ &= (-1)^{\sum_i m_in_i + \sum_{i>j,\, i\leftrightarrow j} m_in_j}. \end{aligned}$$

Similarly, we have

$$\varepsilon(\beta,\alpha) = (-1)^{\sum_i m_in_i + \sum_{i<j,\, i\leftrightarrow j} m_in_j}.$$

Thus in order to verify that $\varepsilon(\beta,\alpha) = -\varepsilon(\alpha,\beta)$ it is sufficient to show that

$$\sum_{i\leftrightarrow j} m_in_j \quad \text{is odd}$$

since $i \not\leftrightarrow i$. Note that

$$(\alpha,\beta) = -1 \quad \text{and} \quad (\alpha,\alpha) = 2 = (\beta,\beta)$$

where $(\cdot,\cdot)$ denotes the normalized invariant form. Thus

$$\begin{aligned} -1 &= \left(\sum_{i\in\underline{n}} m_i\alpha_i, \sum_{j\in\underline{n}} n_j\alpha_j\right) \\ &= \sum_{i,j\in\underline{n}} m_in_j(\alpha_i,\alpha_j) \\ &= 2\sum_{i\in\underline{n}} m_in_i - \sum_{i\leftrightarrow j} m_in_j \end{aligned}$$

Hence

$$\sum_{i \to j} m_i n_j = 2 \sum_{i \in \underline{n}} m_i n_i + 1$$

and so is odd as required.

From the above calculation of $\varepsilon(\alpha, \beta)$ it is also clear that

$$\varepsilon(-\alpha, -\beta) = \varepsilon(\alpha, \beta).$$

We next consider the condition $\varepsilon(\alpha, \alpha) = -1$ for all $\alpha \in \Phi$. It is sufficient to show this for $\alpha \in \Phi^+$ and so we proceed by induction on ht α. If $\alpha = \alpha_i \in \Pi$ then $\varepsilon(\alpha, \alpha) = -1$ by our choice of the ε_{ii}.

Now suppose $\alpha \in \Phi^+ \setminus \Pi$ with $\alpha = \beta + \gamma$ and the usual induction hypotheses. Thus

$$\varepsilon(\beta, \beta) = -1 = \varepsilon(\gamma, \gamma),$$

whence

$$\begin{aligned} \varepsilon(\alpha, \alpha) &= \varepsilon(\beta + \gamma, \beta + \gamma) \\ &= \varepsilon(\beta, \beta)\varepsilon(\gamma, \gamma)\varepsilon(\beta, \gamma)\varepsilon(\gamma, \beta) \\ &= -\varepsilon(\beta, \gamma)^2 = -1 \end{aligned}$$

as required.

Note that we can now exploit the fact that $\varepsilon(\alpha, 0) = 1$ and the bimultiplicativity of ε to obtain the fact that $\varepsilon(\alpha, -\alpha) = -1$. Using the bimultiplicativity of ε again and the fact that ε satisfies

$$\varepsilon(\alpha, \alpha) = -1 = \varepsilon(\alpha, -\alpha)$$

we can easily verify the remaining two conditions required of the map.

Thus we have shown that given any

$$\alpha = \sum_{i \in \underline{n}} m_i \alpha_i\,, \qquad \beta = \sum_{i \in \underline{n}} n_i \alpha_i \in \Phi$$

such that $\alpha + \beta \in \Phi$, defining

$$N_{\alpha, \beta} = (-1)^{\sum_i m_i n_i + \sum_{i > j, i \to j} m_i n_j}$$

gives rise to a consistent set of structure constants satisfying the required conditions for compatibility with Tits' convention.

1.5 Affine Kac-Moody Algebras of Types II and III

We now describe a realization of the remaining "twisted" affine Kac-Moody algebras. These turn out to be closely related to the algebra of equivariant polynomial maps from $\mathbb{C}^\times$ to a simple finite-dimensional Lie algebra with the action of a finite cyclic group.

Unless otherwise specified, proofs of the results in this section can be found in [Kac90, Chapter 8].

Construction for a General Automorphism

Let $\mathfrak{g}_A(\mathbb{C})$ be a simple finite-dimensional Lie algebra and let σ be an automorphism of $\mathfrak{g}_A(\mathbb{C})$ satisfying $\sigma^m = 1$ for a positive integer m. Set

$$\epsilon = e^{\frac{2\pi i}{m}}.$$

Then each eigenvalue of σ has the form ϵ^j for some $j \in \mathbb{Z}/m\mathbb{Z}$ and, since σ is diagonalizable, we have the decomposition

$$\mathfrak{g}_A(\mathbb{C}) = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j,$$

where, $\mathfrak{g}_j$ is the eigenspace of σ for the eigenvalue ϵ^j. This decomposition is a $\mathbb{Z}/m\mathbb{Z}$-gradation of $\mathfrak{g}_A(\mathbb{C})$. Conversely, if a $\mathbb{Z}/m\mathbb{Z}$-gradation as above is given, the linear transformation of $\mathfrak{g}_A(\mathbb{C})$ given by multiplying the vectors of $\mathfrak{g}_j$ by ϵ^j is an automorphism σ of $\mathfrak{g}_A(\mathbb{C})$ which satisfies $\sigma^m = 1$. We shall use the notation $\bar{s} \in \mathbb{Z}/m\mathbb{Z}$ to denote the residue of s modulo m.

Let $\mathfrak{h}_0$ be a maximal ad-diagonalizable subalgebra of the Lie algebra $\mathfrak{g}_0$.

LEMMA 1.5.1

1. *Let $(\cdot, \cdot)$ be a non-degenerate invariant bilinear form on $\mathfrak{g}_A(\mathbb{C})$. Then*

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \quad \text{if} \quad i + j \not\equiv 0 \bmod m,$$

and $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are nondegenerately paired if $i + j \equiv 0 \bmod m$.

2. *The centralizer $\mathfrak{z}$ of $\mathfrak{h}_0$ in $\mathfrak{g}_A(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{g}_A(\mathbb{C})$.*

3. *$\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}_A(\mathbb{C})$.*

It follows that $\mathfrak{h}_{\bar{0}}$ contains a regular element of $\mathfrak{g}_A(\mathbb{C})$, say x. Hence the centralizer $\mathfrak{h}'$ of x in $\mathfrak{g}_A(\mathbb{C})$ is a σ-invariant Cartan subalgebra. Let

$$\mathfrak{g}_A(\mathbb{C}) = \left(\bigoplus_{\alpha\in\Phi_+} \mathbb{C}F'_\alpha\right) \oplus \mathfrak{h}' \oplus \left(\bigoplus_{\alpha\in\Phi_+} \mathbb{C}E'_\alpha\right)$$

be the root space decomposition of $\mathfrak{g}_A(\mathbb{C})$ with respect to $\mathfrak{h}'$ and some root vectors E'_α and $F'_\alpha = E'_{-\alpha}$. Let $\bar{\gamma}$ be an automorphism of $Q(A)$. Then $\bar{\gamma}$ induces an automorphism γ of $\mathfrak{g}_A(\mathbb{C})$ defined by

$$\gamma(\alpha_i^\vee) = (\bar{\gamma}(\alpha_i))^\vee \qquad \text{and} \qquad \gamma(E'_\alpha) = E'_{\bar{\gamma}(\alpha)}$$

for $i \in \underline{n}$ and $\alpha \in \Phi$.

This enables us to give the following description of σ.

PROPOSITION 1.5.2

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra and let $\Pi' = \{\alpha'_i\}_{i\in\underline{n}}$ be a set of simple roots. Let $\sigma \in \mathrm{Aut}\,\mathfrak{g}$ be such that $\sigma^m = 1$. Then σ is conjugate to an automorphism of $\mathfrak{g}$ of the form

$$\gamma\exp(\mathrm{ad}\,\frac{2\pi i}{m}h), \qquad h \in \mathfrak{h}_{\bar{0}},$$

where γ is a diagram automorphism preserving $\mathfrak{h}$ and Π', $\mathfrak{h}_{\bar{0}}$ is the fixed point set of γ in $\mathfrak{h}$, and $\langle\alpha'_i, h\rangle \in \mathbb{Z}$ for all $i \in \underline{n}$.

Define $\mathcal{L}(\mathfrak{g})$ as described in §1.4. We associate a subalgebra $\mathcal{L}(\mathfrak{g}, \sigma, m)$ of $\mathcal{L}(\mathfrak{g})$ to the automorphism σ of $\mathfrak{g}$ as follows:

$$\mathcal{L}(\mathfrak{g}, \sigma, m) = \bigoplus_{j\in\mathbb{Z}} \mathcal{L}(\mathfrak{g}, \sigma, m)_j \tag{1.5.2}$$

where

$$\mathcal{L}(\mathfrak{g}, \sigma, m)_j = t^j \otimes \mathfrak{g}_j.$$

This decomposition is a $\mathbb{Z}$-gradation of $\mathcal{L}(\mathfrak{g}, \sigma, m)$.

Constructed in this way, $\mathcal{L}(\mathfrak{g}, \sigma, m)$ is the fixed point set of the automorphism $\hat{\sigma}$ of $\mathcal{L}(\mathfrak{g})$ defined by

$$\hat{\sigma}(t^j \otimes x) = (\epsilon^{-j}t^j) \otimes \sigma(x),$$

for $j \in \mathbb{Z}$ and $x \in \mathfrak{g}$. Hence $\mathcal{L}(\mathfrak{g}, \sigma, m)$ may be identified with the algebra of equivariant maps (with respect to the action of $\mathbb{Z}/m\mathbb{Z}$):

$$(\mathbb{C}^\times;\ \text{multiplication by } \epsilon^{-1}) \to (\mathfrak{g};\ \text{action of } \sigma).$$

Define $\hat{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c' \oplus \mathbb{C}d'$ as described in §1.4. Set

$$\hat{\mathcal{L}}(\mathfrak{g}, \sigma, m) = \mathcal{L}(\mathfrak{g}, \sigma, m) \oplus \mathbb{C}c' \oplus \mathbb{C}d'.$$

This is a subalgebra of $\hat{\mathcal{L}}(\mathfrak{g})$. $\tilde{\mathcal{L}}(\mathfrak{g},\sigma,m)$ is the fixed point set of an automorphism $\tilde{\sigma}$ of $\tilde{\mathcal{L}}(\mathfrak{g})$ defined by

$$\tilde{\sigma}\,|_{\mathcal{L}(\mathfrak{g},\sigma,m)} = \hat{\sigma}, \quad \tilde{\sigma}(c') = c', \quad \text{and} \quad \tilde{\sigma}(d') = d'.$$

The derived subalgebra of $\tilde{\mathcal{L}}(\mathfrak{g},\sigma,m)$ is

$$\hat{\mathcal{L}}(\mathfrak{g},\sigma,m) = \mathcal{L}(\mathfrak{g},\sigma,m) \oplus \mathbb{C}c'.$$

Note also that

$$\mathcal{L}(\mathfrak{g},1,1) = \mathcal{L}(g), \quad \hat{\mathcal{L}}(\mathfrak{g},1,1) = \hat{\mathcal{L}}(g), \quad \text{and} \quad \tilde{\mathcal{L}}(\mathfrak{g},1,1) = \tilde{\mathcal{L}}(g).$$

Setting $\deg c' = 0 = \deg d'$ together with the decomposition described in equation (1.5.2) defines a $\mathbb{Z}$-gradation of $\tilde{\mathcal{L}}(\mathfrak{g},\sigma,m)$, namely

$$\tilde{\mathcal{L}}(\mathfrak{g},\sigma,m) = \bigoplus_{j\in\mathbb{Z}} \tilde{\mathcal{L}}(\mathfrak{g},\sigma,m)_j.$$

Construction of the Twisted Affine Kac-Moody Algebras

Following Kac, we begin with a simple finite-dimensional Lie algebra $\mathfrak{g}$, generated by elements E_i', F_i', and H_i' for $i \in \underline{n}$, and equipped with a non-trivial diagram automorphism γ of order k. We do a case by case analysis to establish generators E_j, F_j, H_j of $\mathfrak{g}_j$ in terms of the generators of $\mathfrak{g}$ and give a result describing some specific properties these elements satisfy. Finally, we use the information gained to construct an affine Kac-Moody algebra of type II or III, depending on k, culminating in the main theorem of this section. We then give two enlightening examples of the calculations involved.

We begin by letting

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in\Phi_+} \mathbb{C}F_\alpha'\right) \oplus \mathfrak{h}' \oplus \left(\bigoplus_{\alpha\in\Phi_+} \mathbb{C}E_\alpha'\right)$$

be a simple finite-dimensional Lie algebra of type X_n admitting a non-trivial diagram automorphism, with E_α', F_α' being the Chevalley generators of $\mathfrak{g}$, and with $(\cdot,\cdot)$ denoting the normalized invariant form on $\mathfrak{g}$. Let $\bar{\gamma}$ be an automorphism of $\Delta(X_n)$ of order k $(= 2$, or $3)$, and let γ be the corresponding diagram automorphism of $\mathfrak{g}$.

Thus the pair (X_n, k) will be one of $(A_l, 2)$, $(D_l, 2)$, $(D_4, 3)$, or $(E_6, 2)$. Note that we shall consider the case when $(X_n, k) = (D_{l+1}, 2)$ rather than $(X_n, k) = (D_l, 2)$ for reasons that will become clear as we progress. We have the corresponding $\mathbb{Z}/k\mathbb{Z}$-gradations

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$$

if $k = 2$, and

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}}, \qquad \text{and} \qquad \mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}} \oplus \mathfrak{h}_{\bar{2}}$$

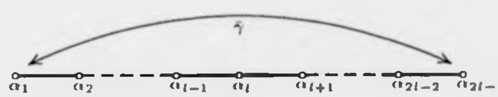
if $k = 3$. Let $\Pi' = \{\alpha_i\}_{i \in \underline{n}} \subset \mathfrak{h}'$ be the set of simple roots of $\mathfrak{g}$, enumerated as in figure 1.2.11. Let $E'_i = E'_{\alpha_i}$, $F'_i = F'_{\alpha_i}$ and $H'_i = (\alpha'_i)^\vee$ be its Chevalley basis. Introduce the following elements $\theta^0 \in \mathfrak{h}'$ and $E_i, F_i, H_i \in \mathfrak{g}$ for $i \in \underline{n}_0$.

Case 1: $(X_n, k) = (A_{2l}, 2)$. In this case the diagram automorphism is induced by the automorphism

of $\Delta(A_{2l})$, and the elements mentioned earlier are given by

$$\begin{array}{llll}
\theta^0 = \alpha_1 + \cdots + \alpha_{2l}, & & & \\
H_{l-i} = H'_i + H'_{2l-i+1} \text{ for } i \in \underline{l-1}, & H_0 = 2(H'_l + H'_{l+1}), & H_l = -(\theta^0)^\vee, \\
E_{l-i} = E'_i + E'_{2l-i+1} \text{ for } i \in \underline{l-1}, & E_0 = \sqrt{2}(E'_l + E_{l+1}), & E_l = F'_{\theta^0}, \\
F_{l-i} = F'_i + F'_{2l-i+1} \text{ for } i \in \underline{l-1}, & F_0 = \sqrt{2}(F'_l + F_{l+1}), & F_0 = E'_{\theta^0}.
\end{array}$$

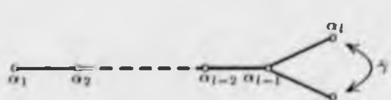
Case 2: $(X_n, k) = (A_{2l-1}, 2)$. In this case the diagram automorphism is induced by the automorphism



of $\Delta(A_{2l-1})$, and the elements in question are given by

$$\begin{array}{lll}
\theta^0 = \alpha_1 + \cdots + \alpha_{2l-2}, & & \\
H_i = H'_i + H'_{2l-i} \text{ for } i \in \underline{l-1}, & H_l = H'_l, & H_0 = -(\theta^0)^\vee - (\bar{\gamma}(\theta^0))^\vee, \\
E_i = E'_i + E'_{2l-i} \text{ for } i \in \underline{l-1}, & E_l = E'_l, & E_0 = F'_{\theta^0} + F'_{\bar{\gamma}(\theta^0)}, \\
F_i = F'_i + F'_{2l-i} \text{ for } i \in \underline{l-1}, & F_l = F'_l, & F_0 = E'_{\theta^0} + E'_{\bar{\gamma}(\theta^0)}.
\end{array}$$

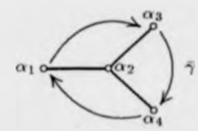
Case 3: $(X_n, k) = (D_{l+1}, 2)$. In this case the diagram automorphism is induced by the automorphism



of $\Delta(D_{l+1})$, and the elements in question are given by

$$\begin{aligned}
&\theta^0 = \alpha_1 + \cdots + \alpha_l, \\
&H_i = H_i' \text{ for } i \in \underline{l-1}, \quad H_l = H_l' + H_{l+1}', \quad H_0 = -(\theta^0)^\vee - (\bar{\gamma}(\theta^0))^\vee, \\
&E_i = E_i' \text{ for } i \in \underline{l-1}, \quad E_l = E_l' + E_{l+1}', \quad E_0 = F_{\theta^0}' - F_{\bar{\gamma}(\theta^0)}', \\
&F_i = F_i' \text{ for } i \in \underline{l-1}, \quad F_l = F_l' + F_{l+1}', \quad F_0 = E_{\theta^0}' - E_{\bar{\gamma}(\theta^0)}'.
\end{aligned}$$

Case 4: $(X_n, k) = (D_4, 3)$. In this case the diagram automorphism is induced by the automorphism

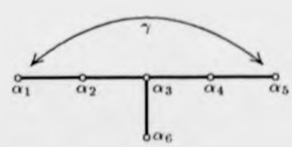


of $\Delta(D_4)$, and the elements in question are given by

$$\begin{aligned}
&\theta^0 = \alpha_1 + \alpha_2 + \alpha_3, \\
&H_1 = H_1' + H_3' + H_4' \quad H_2 = H_2', \quad H_0 = -(\theta^0)^\vee - (\bar{\gamma}(\theta^0))^\vee - (\bar{\gamma}^2(\theta^0))^\vee, \\
&E_1 = E_1' + E_3' + E_4' \quad E_2 = E_2', \quad E_0 = F_{\theta^0}' + \epsilon^2 F_{\bar{\gamma}(\theta^0)}' + \epsilon F_{\bar{\gamma}^2(\theta^0)}', \\
&F_1 = F_1' + F_3' + F_4' \quad F_2 = F_2', \quad F_0 = E_{\theta^0}' + \epsilon E_{\bar{\gamma}(\theta^0)}' + \epsilon^2 E_{\bar{\gamma}^2(\theta^0)}',
\end{aligned}$$

where $\epsilon = e^{\frac{2\pi i}{3}}$.

Case 5: $(X_n, k) = (E_6, 2)$. In this case the diagram automorphism is induced by the automorphism



of $\Delta(E_6)$, and the elements in question are given by

$$\begin{aligned}
&\theta^0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\
&H_1 = H_1' + H_5' \quad H_2 = H_2' + H_4', \quad H_3 = H_3', \quad H_4 = H_6', \quad H_0 = -(\theta^0)^\vee - (\bar{\gamma}(\theta^0))^\vee, \\
&E_1 = E_1' + E_5' \quad E_2 = E_2' + E_4', \quad E_3 = E_3', \quad E_4 = E_6', \quad E_0 = F_{\theta^0}' + F_{\bar{\gamma}(\theta^0)}', \\
&F_1 = F_1' + F_5' \quad F_2 = F_2' + F_4', \quad F_3 = F_3', \quad F_4 = F_6', \quad F_0 = E_{\theta^0}' + E_{\bar{\gamma}(\theta^0)}'.
\end{aligned}$$

We now define $\theta_0 = \theta^0$ in case 1 and

$$\theta_0 = \frac{1}{k}(\bar{\gamma}(\theta^0) + \cdots + \bar{\gamma}^k(\theta^0))$$

in the remaining cases. Let

$$\varrho = \begin{cases} l & \text{in case 1,} \\ 0 & \text{in cases 2–5,} \end{cases}$$

and let $I = \underline{l}_0 \setminus \{\varrho\}$.

PROPOSITION 1.5.3

1. *The elements $\{E_i\}_{i\in \underline{l}_0}$ generate the Lie algebra $\mathfrak{g}$.*
2. *The elements $\{E_i, F_i\}_{i\in I}$ are Chevalley generators of the Lie algebras $\mathfrak{g}_0$, with the elements*
$$\beta_i = \frac{2H_i}{(H_i, H_i)},\ i \in I$$
being the simple roots. The types of $\mathfrak{g}_0$ and the decompositions
$$\theta_0 = \sum_{i\in I} b_i\beta_i$$
are listed in table 1.5.4.
3. $[E_\varrho, F_\varrho] = -H_\varrho$, $(E_\varrho, F_\varrho) = k/b_0$ *and* $(\theta_0, \theta_0) = 2b_0/k$, *where*
$$b_0 = \begin{cases} 2 & \textit{in case 1,} \\ 1 & \textit{in cases 2–5.} \end{cases}$$
4. *The representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{\bar{1}}$ is irreducible and is equivalent to the representation on $\mathfrak{g}_{-1}$.*
5. *F_0 is the highest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ with weight θ_0. Similarly, E_0 is the lowest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ with weight $-\theta_0$.*

Note that by letting $\beta_\varrho = -\theta_0$ and $b_\varrho = 1$, we can write

$$\sum_{i\in \underline{l}_0} b_i\beta_i = 0,$$

where the b_i are the labels in figures 1.2.13–1.2.14.

The restriction of $(\cdot\,,\cdot)$ to $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ is non-degenerate, and hence defines an isomorphism

$$\nu : \mathfrak{h}_0 \to \mathfrak{h}_0^*.$$

For each $m \in \underline{k-1}_0$, let Φ_m be the set of non-zero weights of $\mathfrak{h}_0$ on $\mathfrak{g}_m$. Let

$$\mathfrak{g}_m = \bigoplus_{\beta\in\Phi_m\cup\{0\}} \mathfrak{g}_{m,\beta}$$

k	$\mathfrak{g}$	$\mathfrak{g}_{\bar 0}$	b_i
2	A_2	A_1	
2	A_{2l}, $(l > 1)$	B_l	
2	A_{2l-1}, $(l > 2)$	C_l	
2	D_{l+1}, $(l > 1)$	B_l	
3	D_4	G_2	
2	E_6	F_4	

Table 1.5.4: Types of $\mathfrak{g}_{\bar 0}$ and values of b_i

be the weight space decomposition of $\mathfrak{g}_m$ with respect to $\mathfrak{h}_{\bar 0}$. Proposition 1.5.3 implies that

$$(\beta, \beta) \neq 0, \quad \dim \mathfrak{g}_{m,\beta} = 1, \quad \text{and} \quad [\mathfrak{g}_{\bar m,\beta}, \mathfrak{g}_{-\bar m,-\beta}] = \mathbb{C}\nu^{-1}(\beta)$$

if $\beta \in \Phi_m$.

We now turn to the algebra $\tilde{\mathcal{L}}(\mathfrak{g}, \gamma, k)$. Set

$$\mathfrak{h} = \mathfrak{h}_{\bar 0} + \mathbb{C}c' + \mathbb{C}d'$$

and define $\delta \in \mathfrak{h}^*$ by

$$\delta\,|_{\mathfrak{h}_{\bar 0} + \mathbb{C}c'} = 0, \quad \langle \delta, d' \rangle = 1.$$

Set

$$e_\varrho = t \otimes E_\varrho, \quad f_\varrho = t^{-1} \otimes F_\varrho, \quad e_i = 1 \otimes E_i, \quad \text{and} \quad f_i = 1 \otimes F_i,$$

for $i \in I$. Then we have

$$[e_i, f_i] = -1 \otimes H_i \quad \text{for} \quad i \in I, \quad \text{and} \quad [e_\varrho, f_\varrho] = -\left(\frac{k}{b_0}c' + 1 \otimes H_\varrho\right).$$

The root system and the root space decomposition of $\tilde{\mathcal{L}}(\mathfrak{g}, \gamma, k)$ with respect to $\mathfrak{h}$ are then

$$\Phi = \{j\delta + \beta, \ \text{where } j \in \mathbb{Z},\ \beta \in \Phi_m,\ j \equiv m \bmod k,\ m \in \underline{k-1}\}$$
$$\cup \{j\delta, \ \text{where } j \in \mathbb{Z},\ j \neq 0\}$$

and

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,k) = \mathfrak{h} \oplus \left(\bigoplus_{\beta\in\Phi} \tilde{\mathcal{L}}(\mathfrak{g},\gamma,k)_\beta \right),$$

where

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,k)_{j\delta+\beta} = t^j \otimes \mathfrak{g}_{\bar{j},\beta}, \quad \text{and} \quad \tilde{\mathcal{L}}(\mathfrak{g},\gamma,k)_{j\delta} = t^j \otimes \mathfrak{g}_{\bar{j},0}.$$

We set

$$\Pi = \{\beta_o = \delta - \theta_0,\ \beta_i (i \in I)\},$$

and

$$\Pi^\vee = \left\{ \beta_o^\vee = \frac{k}{b_0} c' + 1 \otimes H_o,\ \beta_i^\vee = 1 \otimes H_i (i \in I) \right\}.$$

Using Proposition 1.5.3, we see that if $\mathfrak{g}$ is of type X_n and k ($= 2$ or 3) is the order of γ, then the matrix

$$A = (\langle \beta_j, \beta_i^\vee \rangle)_{i,j\in I_0}$$

is of type $X_n^{(k)}$, given by table 1.5.5, and the integers $b_0, \ldots, b_l$ are the labels on the diagram of this matrix in figures 1.2.13–1.2.14.

X_n	A_2	A_{2l}, $(l > 1)$	A_{2l-1}, $(l > 2)$	D_{l+1}, $(l > 1)$	D_4	E_6
$X_n^{(k)}$	${}^*\tilde{A}_1$	${}^*\tilde{C}_l$	${}^t\tilde{B}_l$	${}^t\tilde{C}_l$	${}^t\tilde{G}_2$	${}^t\tilde{F}_4$

Table 1.5.5: Matrices $X_n^{(k)}$ in terms of those used in the classification.

Finally, we can state the second of the realization theorems.

THEOREM 1.5.6

Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra of type $X_n = A_{2l}$, A_{2l-1}, D_{l+1}, D_4, or E_6 and let $k = 2, 2, 2, 3$, or 2, respectively. Let γ be a diagram automorphism of $\mathfrak{g}$ of order k. Note that for $k = 3$ there are two such automorphisms which are equivalent: We choose one of them. Then

- *the Lie algebra*

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,k)$$

  is an affine Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$ associated to the affine matrix A of type $X_n^{(k)}$ given by table 1.5.5 and figures 1.2.13–1.2.14,

- *$\mathfrak{h}$ is its Cartan subalgebra,*

- Φ *the root system,*
- Π *and* $\Pi^\vee$ *the root basis and the coroot basis respectively, and*
- $\{e_i, f_i\}_{i\in I_0}$ *are the Chevalley generators.*

In other words

$$\left(\hat{\mathcal{L}}(\mathfrak{g},\gamma,k), \mathfrak{h}, \Pi, \Pi^\vee\right)$$

is the quadruple associated to A.

From this result we can deduce the following informative corollary.

Corollary 1.5.7

Let $\mathfrak{g}_A(\mathbb{C})$ *be an affine algebra of rank* $l+1$ *and let* A *be of type* $X_n^{(k)}$. *Then for all imaginary roots* $j\delta$ *where* $j \in \mathbb{Z}$, $j \neq 0$

$$\operatorname{mult} j\delta = \begin{cases} l & \text{if } j \equiv 0 \bmod k, \\ \frac{(n-l)}{(k-1)} & \text{if } j \not\equiv 0 \bmod k. \end{cases}$$

The Chevalley involution ω and the triangular decomposition of the Kac-Moody algebra $\hat{\mathcal{L}}(\mathfrak{g},\gamma,k) \subset \hat{\mathcal{L}}(\mathfrak{g})$ are induced by those from $\hat{\mathcal{L}}(\mathfrak{g})$. The normalized invariant form $(\cdot\,,\cdot)$ on $\hat{\mathcal{L}}(\mathfrak{g},\gamma,k)$ is given by

$$\begin{aligned} &(P\otimes x, Q\otimes y) = k^{-1}\operatorname{Res}(t^{-1}PQ)(x,y)_{\mathfrak{g}} \quad \text{for } x,y\in\mathfrak{g},\ P,Q\in\mathcal{L}, \\ &\left(\mathbb{C}c' \oplus \mathbb{C}d', \mathcal{L}(\mathfrak{g},\gamma,k)\right) = 0, \\ &(c',c') = (d',d') = 0 \\ &(c',d') = 1, \end{aligned}$$

where $(\cdot\,,\cdot)_{\mathfrak{g}}$ is the normalized invariant form on $\mathfrak{g}$.

The canonical central element and the scaling element of $\hat{\mathcal{L}}(\mathfrak{g},\gamma,k)$ are given in terms of those of $\hat{\mathcal{L}}(\mathfrak{g})$ by

$$c = kc', \qquad \text{and} \qquad d = \frac{b_0}{k}d',$$

respectively.

Remark 1.5.8

The algebra $\mathfrak{g}_0$ is isomorphic to the algebra $\mathring{\mathfrak{g}}$ introduced in § 1.3 in all cases except for when $X_n^{(k)} = A_{2l}^{(2)}$, in which case $\mathring{\mathfrak{g}}$ is of type C_l whereas $\mathfrak{g}_0$ is of type B_l. ◇

We note that since $\hat{\mathcal{L}}(\mathfrak{g},\gamma,k)$ is the fixed point subalgebra of $\hat{\mathcal{L}}(\mathfrak{g})$ under the automorphism $\hat{\gamma}$ defined by

$$\hat{\gamma}(t^j\otimes x + \lambda c' + \mu d') = \epsilon^{-j}t^j \otimes \gamma(x) + \lambda c' + \mu d'$$

for $j \in \mathbb{Z}$, $x \in \mathfrak{g}$, and $\lambda, \mu \in \mathbb{C}$, Theorem 1.5.6 provides us with a canonical isomorphism between affine Kac-Moody algebras of types II and III and fixed point subalgebras of corresponding affine Kac-Moody algebras of type I. This justifies the use of the term *twisted* when referring to affine Kac-Moody algebras of types II and III. We give two detailed examples of the constructions described in this section.

Examples

EXAMPLE 1.5.9

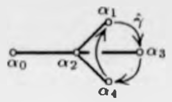
Let $\tilde{\mathfrak{g}}(D_4)$ be an affine algebra of type $\tilde{D}_4$ with Chevalley basis

$$\begin{array}{llllll} e_{\alpha_0}, & e_{\alpha_1}, & e_{\alpha_2}, & e_{\alpha_3}, & e_{\alpha_4}, & \\ f_{\alpha_0}, & f_{\alpha_1}, & f_{\alpha_2}, & f_{\alpha_3}, & f_{\alpha_4}, & \\ \alpha_0^\vee, & \alpha_1^\vee, & \alpha_2^\vee, & \alpha_3^\vee, & \alpha_4^\vee, & d' \end{array}$$

where, for each $i \in \underline{4}_0$,

$$[e_{\alpha_i}, f_{\alpha_i}] = -\alpha_i^\vee,$$

and d' is the scaling element. Let $\tilde{\gamma}$ be the diagram automorphism of $\tilde{\mathfrak{g}}(D_4)$ induced by the automorphism



of $\Delta(\tilde{D}_4)$. Denote by $\tilde{\mathfrak{g}}^{\tilde{\gamma}}$ the fixed point subalgebra of $\tilde{\mathfrak{g}}(D_4)$ with respect to $\tilde{\gamma}$.

Let $\mathfrak{g}({}^t\tilde{G}_2)$ be a twisted affine Kac-Moody algebra of type ${}^t\tilde{G}_2$ with Dynkin diagram

and with root system as described in §1.3. We shall construct an explicit isomorphism between $\mathfrak{g}({}^t\tilde{G}_2)$ and $\tilde{\mathfrak{g}}^{\tilde{\gamma}}$.

We begin by taking $\mathfrak{g} = \mathfrak{g}(D_4)$ to be a Kac-Moody algebra of type D_4 with Chevalley basis

$$\begin{array}{llll} E'_{\alpha_1}, & E'_{\alpha_2}, & E'_{\alpha_3}, & E'_{\alpha_4}, \\ F'_{\alpha_1}, & F'_{\alpha_2}, & F'_{\alpha_3}, & F'_{\alpha_4}, \\ H'_{\alpha_1}, & H'_{\alpha_2}, & H'_{\alpha_3}, & H'_{\alpha_4}, \end{array}$$

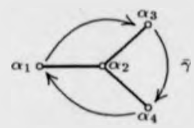
where, for each $i \in \underline{4}$,

$$\left[E'_{\alpha_i}, F'_{\alpha_i}\right] = -H'_{\alpha_i}.$$

Note that since $\mathfrak{g}$ is a simply-laced algebra we have that for all roots $\alpha_i, \alpha_j \in \Phi(\mathfrak{g})$

$$(\alpha_i + \alpha_j)^\vee = \alpha_i^\vee + \alpha_j^\vee$$

whenever $\alpha_i + \alpha_j$ is a root. Let γ be the diagram automorphism of $\mathfrak{g}$ induced by the automorphism $\bar{\gamma}$



of $\Delta(D_4)$. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}}$$

of $\mathfrak{g}$ into eigenspaces where $\mathfrak{g}_{\bar{m}}$ is the eigenspace corresponding to the eigenvalue ϵ^m for $\epsilon = e^{\frac{2\pi i}{3}}$. By Proposition 1.5.3 we have that

$$\mathfrak{g}(G_2) \overset{\psi}{\cong} \mathfrak{g}_{\bar{0}}$$

where $\mathfrak{g}(G_2)$ is a Kac-Moody algebra of type G_2 with Dynkin diagram

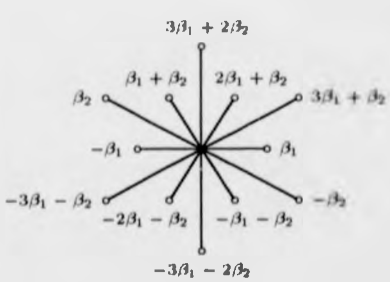
and with Chevalley basis $\{E_{\beta_i}, F_{\beta_i}, H_{\beta_i}\}_{i\in\mathbf{2}}$ such that

$$\begin{array}{ll} \psi(H_{\beta_1}) = H'_{\alpha_1} + H'_{\alpha_3} + H'_{\alpha_4} & \psi(H_{\beta_2}) = H'_{\alpha_2}, \\ \psi(E_{\beta_1}) = E'_{\alpha_1} + E'_{\alpha_3} + E'_{\alpha_4} & \psi(E_{\beta_2}) = E'_{\alpha_2}, \\ \psi(F_{\beta_1}) = F'_{\alpha_1} + F'_{\alpha_3} + F'_{\alpha_4} & \psi(F_{\beta_2}) = F'_{\alpha_2}. \end{array}$$

This construction is simply a variation on the classical construction of $\mathfrak{g}(G_2)$ from $\mathfrak{g}(D_4)$. We refer the reader to [Bou68], [Kac90, §7.9], and [Ste67, §11] for details of the original construction. We point out that we have chosen our labeling of $\Delta(G_2)$ so that the set $\Pi = \{\beta_1, \beta_2\}$ of fundamental roots of $\mathfrak{g}_{\bar{0}}$ is given by

$$\beta_1 = \frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4), \qquad \text{and} \qquad \beta_2 = \alpha_2.$$

Thus the diagram



shows the roots of $\mathfrak{g}(G_2)$ as integral combinations of fundamental roots.

The γ-stable subalgebra

$$\mathfrak{h}_{\bar{0}} = \mathbb{C}(H'_{\alpha_1} + H'_{\alpha_3} + H'_{\alpha_4}) \oplus \mathbb{C}H'_{\alpha_2}$$

is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, with respect to which $\mathfrak{g}_{\bar{0}}$ has the following Cartan decomposition,

$$\begin{aligned} \mathfrak{g}_{\bar{0}} \;=\; & \mathfrak{h}_{\bar{0}} \oplus \mathbb{C}E'_{\alpha_2} \oplus \mathbb{C}(E'_{\alpha_1} + E'_{\alpha_3} + E'_{\alpha_4}) \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2} - E'_{\alpha_2+\alpha_3} - E'_{\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2+\alpha_3} + E'_{\alpha_2+\alpha_3+\alpha_4} + E'_{\alpha_1+\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathbb{C}E'_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\ & \oplus \mathbb{C}F'_{\alpha_2} \oplus \mathbb{C}(F'_{\alpha_1} + F'_{\alpha_3} + F'_{\alpha_4}) \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2} - F'_{\alpha_2+\alpha_3} - F'_{\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2+\alpha_3} + F'_{\alpha_2+\alpha_3+\alpha_4} + F'_{\alpha_1+\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathbb{C}F'_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}. \end{aligned}$$

The eigenspaces $\mathfrak{g}_{\bar{1}}$ and $\mathfrak{g}_{\bar{2}}$ both have weight space decompositions with respect to $\mathfrak{h}_{\bar{0}}$, namely

$$\begin{aligned} \mathfrak{g}_{\bar{1}} \;=\; & \mathbb{C}(H'_{\alpha_1} + \epsilon H'_{\alpha_3} + \epsilon^2 H'_{\alpha_4}) \\ & \oplus \mathbb{C}(E'_{\alpha_1} + \epsilon E'_{\alpha_3} + \epsilon^2 E'_{\alpha_4}) \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2} - \epsilon E'_{\alpha_2+\alpha_3} - \epsilon^2 E'_{\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon E'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^2 E'_{\alpha_1+\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1} + \epsilon F'_{\alpha_3} + \epsilon^2 F'_{\alpha_4}) \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2} - \epsilon F'_{\alpha_2+\alpha_3} - \epsilon^2 F'_{\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon F'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^2 F'_{\alpha_1+\alpha_2+\alpha_4}), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{g}_{\bar{2}} \;=\; & \mathbb{C}(H'_{\alpha_1} + \epsilon^2 H'_{\alpha_3} + \epsilon H'_{\alpha_4}) \\ & \oplus \mathbb{C}(E'_{\alpha_1} + \epsilon^2 E'_{\alpha_3} + \epsilon E'_{\alpha_4}) \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2} - \epsilon^2 E'_{\alpha_2+\alpha_3} - \epsilon E'_{\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^2 E'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon E'_{\alpha_1+\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1} + \epsilon^2 F'_{\alpha_3} + \epsilon F'_{\alpha_4}) \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2} - \epsilon^2 F'_{\alpha_2+\alpha_3} - \epsilon F'_{\alpha_2+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^2 F'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon F'_{\alpha_1+\alpha_2+\alpha_4}). \end{aligned}$$

Consider

$$G_2 = (G_{ij})_{i,j\in\underline{2}} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

This is the Cartan matrix corresponding to the root system of type G_2 of $\mathfrak{g}_{\bar{0}}$. The fundamental roots of $\mathfrak{g}_{\bar{0}}$ then satisfy

$$\beta_i = \sum_{j\in\underline{2}} G_{ji}\varpi_j$$

where ϖ_j denotes the fundamental weight defined by

$$\varpi_j(H_{\beta_i}) = \delta_{ji},$$

with δ_{ji} denoting the Kronecker delta. Thus the weight β_i is given by the corresponding column of the Cartan matrix G_2.

We note also that

$$D_4 = (D_{ij})_{i,j\in\underline{4}} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

is the Cartan matrix of the Lie algebra $\mathfrak{g} = \mathfrak{g}(D_4)$.

LEMMA 1.5.10

For $m = 0, 1, 2$, the weights associated to

$$\begin{gathered} E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}, \\ E'_{\alpha_1+\alpha_2} - \epsilon^m E'_{\alpha_2+\alpha_3} - \epsilon^{2m} E'_{\alpha_2+\alpha_4}, \\ E'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^m E'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^{2m} E'_{\alpha_1+\alpha_2+\alpha_4}, \end{gathered}$$

are β_1, $\beta_1 + \beta_2$, and $2\beta_1 + \beta_2$, respectively.

Proof

We consider the case of $E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}$. Note that

$$\begin{aligned} &\left[H'_{\alpha_1} + H'_{\alpha_3} + H'_{\alpha_4}, E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}\right] \\ &\qquad = \left[H'_{\alpha_1}, E'_{\alpha_1}\right] + \left[H'_{\alpha_3}, \epsilon^m E'_{\alpha_3}\right] + \left[H'_{\alpha_4}, \epsilon^{2m} E'_{\alpha_4}\right] \\ &\qquad = \langle\alpha_1, \alpha_1^\vee\rangle E'_{\alpha_1} + \langle\alpha_3, \alpha_3^\vee\rangle \epsilon^m E'_{\alpha_3} + \langle\alpha_4, \alpha_4^\vee\rangle \epsilon^{2m} E'_{\alpha_4} \\ &\qquad = 2\left(E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}\right) \end{aligned}$$

and

$$\begin{aligned} \left[H'_{\alpha_2}, E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}\right] &= \langle\alpha_1, \alpha_2^\vee\rangle E'_{\alpha_1} + \langle\alpha_3, \alpha_2^\vee\rangle \epsilon^m E'_{\alpha_3} + \langle\alpha_4, \alpha_2^\vee\rangle \epsilon^{2m} E'_{\alpha_4} \\ &= -1\left(E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}\right). \end{aligned}$$

This demonstrates that $E'_{\alpha_1} + \epsilon^m E'_{\alpha_3} + \epsilon^{2m} E'_{\alpha_4}$ has weight β_1.

The other parts of the result follow from similar calculations. □

Thus the highest weight appearing is $2\beta_1 + \beta_2$. Note that when expressed in terms of the roots of $\mathfrak{g}$ this is precisely

$$\theta_0 = \frac{1}{3}((\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_2 + \alpha_3 + \alpha_4) + (\alpha_1 + \alpha_2 + \alpha_4)).$$

Hence we take

$$E_{\beta_0} = F'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon F'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^2 F'_{\alpha_1+\alpha_2+\alpha_4}, \quad \text{and}$$
$$F_{\beta_0} = E'_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^2 E'_{\alpha_2+\alpha_3+\alpha_4} + \epsilon E'_{\alpha_1+\alpha_2+\alpha_4}.$$

We introduce the symbols $E_{m,\beta}$ to denote the weight vector in the given weight space decomposition of $\mathfrak{g}_m$ with weight $\beta \in \Phi(G_2)$. Each $E_{m,\beta}$ will be a linear combination of root vectors of $\mathfrak{g}$ over $\mathbb{Z}(\epsilon)$.

We also note that, in the notation introduced earlier in the section,

$$\Phi_m = \begin{cases} \Phi(G_2) & \text{if } m \equiv 0 \bmod 3 \\ \Phi_s(G_2) & \text{otherwise} \end{cases}$$

where, $\Phi_s(G_2)$ denotes the set of short roots in the root system of type G_2.

We now turn our attention to the Kac-Moody subalgebra $\hat{\mathcal{L}}(\mathfrak{g}, \gamma, 3)$ of $\hat{\mathcal{L}}(\mathfrak{g})$.We note that $\hat{\mathcal{L}}(\mathfrak{g})$ is a realization of the Kac-Moody algebra $\hat{\mathfrak{g}}(D_4)$ constructed earlier and that, under the isomorphism so induced, $\hat{\mathcal{L}}(\mathfrak{g}, \gamma, 3)$ corresponds to $\hat{\mathfrak{g}}^{\hat{\gamma}}$.

We set

$$\begin{aligned} \mathfrak{h}' &= \mathfrak{h}_0 + \mathbb{C}c' + \mathbb{C}d' \\ &= \mathbb{C}(H'_{\alpha_1} + H'_{\alpha_3} + H'_{\alpha_4}) \oplus \mathbb{C}H'_{\alpha_2} + \mathbb{C}c' + \mathbb{C}d' \end{aligned}$$

and define $\delta' \in \mathfrak{h}'^*$ by

$$\delta' \mid_{\mathfrak{h}_0 + \mathbb{C}c'} = 0, \ \langle \delta', d' \rangle = 1.$$

We define

$$\begin{aligned} e_{\beta_0} &= t \otimes E_{\beta_0}, & e_{\beta_1} &= 1 \otimes E_{\beta_1}, & & & e_{\beta_2} &= 1 \otimes E_{\beta_2}, \\ f_{\beta_0} &= t^{-1} \otimes F_{\beta_0}, & f_{\beta_1} &= 1 \otimes F_{\beta_1}, & &\text{and} & f_{\beta_2} &= 1 \otimes F_{\beta_2}. \end{aligned}$$

Thus straightforward calculation using the definition of the Lie product on $\hat{\mathcal{L}}(\mathfrak{g})$ given by equation (1.4.0) in §1.4 leads to the equalities

$$[e_{\beta_2}, f_{\beta_2}] = -1 \otimes H_{\beta_2}, \qquad \text{and} \qquad [e_{\beta_1}, f_{\beta_1}] = -1 \otimes H_{\beta_1}.$$

We proceed to verify that

$$[e_{\beta_0}, f_{\beta_0}] = -(3c' + 1 \otimes H_{\beta_0}),$$

where

$$H_{\beta_0} = -(\alpha_1 + \alpha_2 + \alpha_3)^\vee - (\alpha_2 + \alpha_3 + \alpha_4)^\vee - (\alpha_1 + \alpha_2 + \alpha_4)^\vee.$$

First we note that

$$\begin{aligned} [e_{\beta_0}, f_{\beta_0}] &= \left[t \otimes F'_{\alpha_1+\alpha_2+\alpha_3}, t^{-1} \otimes E'_{\alpha_1+\alpha_2+\alpha_3}\right] + \left[t \otimes \epsilon F'_{\alpha_2+\alpha_3+\alpha_4}, t^{-1} \otimes \epsilon^2 E'_{\alpha_2+\alpha_3+\alpha_4}\right] \\ &\quad + \left[t \otimes \epsilon^2 F'_{\alpha_1+\alpha_2+\alpha_4}, t^{-1} \otimes \epsilon E'_{\alpha_1+\alpha_2+\alpha_4}\right] \\ &= 1 \otimes \left[F'_{\alpha_1+\alpha_2+\alpha_3}, E'_{\alpha_1+\alpha_2+\alpha_3}\right] - \left(F'_{\alpha_1+\alpha_2+\alpha_3}, E'_{\alpha_1+\alpha_2+\alpha_3}\right) c' \\ &\quad + 1 \otimes \left[\epsilon F'_{\alpha_2+\alpha_3+\alpha_4}, \epsilon^2 E'_{\alpha_2+\alpha_3+\alpha_4}\right] - \left(F'_{\alpha_2+\alpha_3+\alpha_4}, E'_{\alpha_2+\alpha_3+\alpha_4}\right) c' \\ &\quad + 1 \otimes \left[\epsilon^2 F'_{\alpha_1+\alpha_2+\alpha_4}, \epsilon E'_{\alpha_1+\alpha_2+\alpha_4}\right] - \left(F'_{\alpha_1+\alpha_2+\alpha_4}, E'_{\alpha_1+\alpha_2+\alpha_4}\right) c'. \end{aligned}$$

However,

$$\begin{aligned}\left(F'_{\alpha_1+\alpha_2+\alpha_3}, E'_{\alpha_1+\alpha_2+\alpha_3}\right) &= \left(\left[F'_{\alpha_1}, F'_{\alpha_2+\alpha_3}\right], E'_{\alpha_1+\alpha_2+\alpha_3}\right)\\ &= \left(F'_{\alpha_1}, \left[F'_{\alpha_2+\alpha_3}, E'_{\alpha_1+\alpha_2+\alpha_3}\right]\right)\\ &= \left(F'_{\alpha_1}, E'_{\alpha_1}\right)\\ &= 1\end{aligned}$$

by the last part of Theorem 1.1.17. Thus

$$\begin{aligned}[e_{\beta_0}, f_{\beta_0}] &= -3c' + 1\otimes((\alpha_1+\alpha_2+\alpha_3)^\vee + (\alpha_2+\alpha_3+\alpha_4)^\vee + (\alpha_1+\alpha_2+\alpha_4)^\vee)\\ &= -(3c' + 1\otimes H_{\beta_0})\end{aligned}$$

as required.

Thus the root system of $\tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)$ with respect to $\mathfrak{h}'$ is given by

$$\begin{aligned}\Phi = \{\beta + m\delta' : \beta\in\Phi_s(G_2),\ m\in\mathbb{Z}\} &\cup \{\beta + 3m\delta' : \beta\in\Phi_l(G_2),\ m\in\mathbb{Z}\}\\ &\cup \{m\delta' : m\in\mathbb{Z}\setminus\{0\}\}\end{aligned}$$

and the root space decomposition of $\tilde{\mathfrak{g}}^{\hat{\gamma}}$ with respect to $\mathfrak{h}'$ is

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,3) = \mathfrak{h}' \oplus \left(\bigoplus_{\vartheta\in\Phi} \tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)_\vartheta\right),$$

where

$$\begin{aligned}\tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)_{\beta+m\delta'} &= \mathbb{C}(t^m\otimes E_{m,\beta}) \quad \text{for } \beta\in\Phi_s(G_2),\ m\in\mathbb{Z},\\ \tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)_{\beta+3m\delta'} &= \mathbb{C}(t^{3m}\otimes E_{\bar{m},\beta}) \quad \text{for } \beta\in\Phi_l(G_2),\ m\in\mathbb{Z},\end{aligned}$$

and

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)_{m\delta'} = \begin{cases} t^m\otimes\mathfrak{h}_0 & \text{for } m\in\mathbb{Z}\setminus\{0\},\ m\equiv 0 \bmod 3\\ \mathbb{C}\left(t^m\otimes(H'_{\alpha_1}+\epsilon H'_{\alpha_3}+\epsilon^2 H'_{\alpha_4})\right) & \text{for } m\in\mathbb{Z}\setminus\{0\},\ m\equiv 1 \bmod 3\\ \mathbb{C}\left(t^m\otimes(H'_{\alpha_1}+\epsilon^2 H'_{\alpha_3}+\epsilon H'_{\alpha_4})\right) & \text{for } m\in\mathbb{Z}\setminus\{0\},\ m\equiv 2 \bmod 3.\end{cases}$$

The set

$$\Pi' = \{\beta_0 = \delta' - 2\beta_1 - \beta_2\ ,\ \beta_1\ ,\ \beta_2\}$$

is a set of fundamental roots for $\tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)$ and

$$\Pi'^\vee = \{\beta_0^\vee = 3c' + 1\otimes H_{\beta_0}\ ,\ \beta_1^\vee = 1\otimes H_{\beta_1}\ ,\ \beta_2^\vee = 1\otimes H_{\beta_2}\}$$

is the corresponding set of fundamental coroots of $\tilde{\mathcal{L}}(\mathfrak{g},\gamma,3)$.

By Theorem 1.5.6, $\left(\hat{\mathcal{L}}(\mathfrak{g},\gamma,3),\mathfrak{h}',\Pi',\Pi'^{\vee}\right) = \left(\hat{\mathfrak{g}}^{\gamma},\mathfrak{h}',\Pi',\Pi'^{\vee}\right)$ is a quadruple associated to $'\tilde{G}_2$.

Recall the Kac-Moody algebra $\mathfrak{g}('\tilde{G}_2)$ introduced earlier. The set

$$\Pi = \{\beta_0, \beta_1, \beta_2\}$$

is its fundamental root system and it has Chevalley basis

$$\begin{array}{llll} e_{\beta_0}, & e_{\beta_1}, & e_{\beta_2}, & \\ f_{\beta_0}, & f_{\beta_1}, & f_{\beta_2}, & \\ \beta_0^{\vee}, & \beta_1^{\vee}, & \beta_2^{\vee}, & d \end{array}$$

where, for each $i \in \underline{2}_0$,

$$[e_{\beta_i}, f_{\beta_i}] = -\beta_i^{\vee},$$

and d is the scaling element. Denoting by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}('\tilde{G}_2)$ generated by the fundamental coroots and d we have that

$$(\mathfrak{g}('\tilde{G}_2), \mathfrak{h}, \Pi, \Pi^{\vee})$$

is also a quadruple associated to $'\tilde{G}_2$. We proceed to describe an explicit isomorphism ϕ between these two quadruples.

We first note that from the results of §1.3 we can describe the root and coroot system of $\mathfrak{g}('\tilde{G}_2)$ entirely in terms of the underlying finite root system of type G_2 and the fundamental imaginary root, δ, and the canonical central element, c, of $\mathfrak{g}('\tilde{G}_2)$. By virtue of this description we have

$$\beta_0 = \delta - (2\beta_1 + \beta_2) \quad \text{and} \quad \beta_0^{\vee} = c - (2\beta_1 + \beta_2)^{\vee}.$$

Now,

$$\begin{aligned} (2\beta_1 + \beta_2)^{\vee} &= \frac{2(2\beta_1 + \beta_2)}{(2\beta_1 + \beta_2, 2\beta_1 + \beta_2)_{\mathfrak{h}^*}} \\ &= \frac{4\beta_1}{(\beta_1, \beta_1)_{\mathfrak{h}^*}} + \frac{2\beta_2}{1/3(\beta_2, \beta_2)_{\mathfrak{h}^*}} \\ &= 2\beta_1^{\vee} + 3\beta_2^{\vee}, \end{aligned}$$

whence we obtain.

$$\beta_0^{\vee} = c - (2\beta_1^{\vee} + 3\beta_2^{\vee}).$$

We now define a vector space isomorphism

$$\phi : \mathfrak{g}('\tilde{G}_2) \to \hat{\mathfrak{g}}^{\gamma}$$

by taking the action of ϕ on the root vectors of $\mathfrak{g}({}'\hat{G}_2)$ to be

$$
\begin{array}{lcrcl}
e_{\beta_1} & \mapsto & 1 & \otimes & (e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4}) \\
e_{\beta_2} & \mapsto & 1 & \otimes & e_{\alpha_2} \\
e_{\beta_1+\beta_2} & \mapsto & 1 & \otimes & (e_{\alpha_1+\alpha_2} - e_{\alpha_2+\alpha_3} - e_{\alpha_2+\alpha_4}) \\
e_{2\beta_1+\beta_2} & \mapsto & 1 & \otimes & (e_{\alpha_1+\alpha_2+\alpha_3} + e_{\alpha_2+\alpha_3+\alpha_4} + e_{\alpha_1+\alpha_2+\alpha_4}) \\
e_{3\beta_1+\beta_2} & \mapsto & 1 & \otimes & e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
e_{3\beta_1+2\beta_2} & \mapsto & 1 & \otimes & e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
e_{\beta_1+n\delta} & \mapsto & t^n & \otimes & (e_{\alpha_1} + \epsilon^n e_{\alpha_3} + \epsilon^{2n} e_{\alpha_4}) \\
e_{\beta_2+3n\delta} & \mapsto & t^{3n} & \otimes & e_{\alpha_2} \\
e_{\beta_1+\beta_2+n\delta} & \mapsto & t^n & \otimes & (e_{\alpha_1+\alpha_2} - \epsilon^n e_{\alpha_2+\alpha_3} - \epsilon^{2n} e_{\alpha_2+\alpha_4}) \\
e_{2\beta_1+\beta_2+n\delta} & \mapsto & t^n & \otimes & (e_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^n e_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^{2n} e_{\alpha_1+\alpha_2+\alpha_4}) \\
e_{3\beta_1+\beta_2+3n\delta} & \mapsto & t^{3n} & \otimes & e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
e_{3\beta_1+2\beta_2+3n\delta} & \mapsto & t^{3n} & \otimes & e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
e_{-\beta_1+n\delta} & \mapsto & t^n & \otimes & (f_{\alpha_1} + \epsilon^n f_{\alpha_3} + \epsilon^{2n} f_{\alpha_4}) \\
e_{-\beta_2+3n\delta} & \mapsto & t^{3n} & \otimes & f_{\alpha_2} \\
e_{-\beta_1-\beta_2+n\delta} & \mapsto & t^n & \otimes & (f_{\alpha_1+\alpha_2} - \epsilon^n f_{\alpha_2+\alpha_3} - \epsilon^{2n} f_{\alpha_2+\alpha_4}) \\
e_{-2\beta_1-\beta_2+n\delta} & \mapsto & t^n & \otimes & (f_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^n f_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^{2n} f_{\alpha_1+\alpha_2+\alpha_4}) \\
e_{-3\beta_1-\beta_2+3n\delta} & \mapsto & t^{3n} & \otimes & f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
e_{-3\beta_1-2\beta_2+3n\delta} & \mapsto & t^{3n} & \otimes & f_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
f_{\beta_1} & \mapsto & 1 & \otimes & (f_{\alpha_1} + f_{\alpha_3} + f_{\alpha_4}) \\
f_{\beta_2} & \mapsto & 1 & \otimes & f_{\alpha_2} \\
f_{\beta_1+\beta_2} & \mapsto & 1 & \otimes & (f_{\alpha_1+\alpha_2} - f_{\alpha_2+\alpha_3} - f_{\alpha_2+\alpha_4}) \\
f_{2\beta_1+\beta_2} & \mapsto & 1 & \otimes & (f_{\alpha_1+\alpha_2+\alpha_3} + f_{\alpha_2+\alpha_3+\alpha_4} + f_{\alpha_1+\alpha_2+\alpha_4}) \\
f_{3\beta_1+\beta_2} & \mapsto & 1 & \otimes & f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
f_{3\beta_1+2\beta_2} & \mapsto & 1 & \otimes & f_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
f_{\beta_1+n\delta} & \mapsto & t^{-n} & \otimes & (f_{\alpha_1} + \epsilon^n f_{\alpha_3} + \epsilon^{2n} f_{\alpha_4}) \\
f_{\beta_2+3n\delta} & \mapsto & t^{-3n} & \otimes & f_{\alpha_2} \\
f_{\beta_1+\beta_2+n\delta} & \mapsto & t^{-n} & \otimes & (f_{\alpha_1+\alpha_2} - \epsilon^n f_{\alpha_2+\alpha_3} - \epsilon^{2n} f_{\alpha_2+\alpha_4}) \\
f_{2\beta_1+\beta_2+n\delta} & \mapsto & t^{-n} & \otimes & (f_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^n f_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^{2n} f_{\alpha_1+\alpha_2+\alpha_4}) \\
f_{3\beta_1+\beta_2+3n\delta} & \mapsto & t^{-3n} & \otimes & f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
f_{3\beta_1+2\beta_2+3n\delta} & \mapsto & t^{-3n} & \otimes & f_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
f_{-\beta_1+n\delta} & \mapsto & t^{-n} & \otimes & (e_{\alpha_1} + \epsilon^n e_{\alpha_3} + \epsilon^{2n} e_{\alpha_4}) \\
f_{-\beta_2+3n\delta} & \mapsto & t^{-3n} & \otimes & e_{\alpha_2} \\
f_{-\beta_1-\beta_2+n\delta} & \mapsto & t^{-n} & \otimes & (e_{\alpha_1+\alpha_2} - \epsilon^n e_{\alpha_2+\alpha_3} - \epsilon^{2n} e_{\alpha_2+\alpha_4}) \\
f_{-2\beta_1-\beta_2+n\delta} & \mapsto & t^{-n} & \otimes & (e_{\alpha_1+\alpha_2+\alpha_3} + \epsilon^n e_{\alpha_2+\alpha_3+\alpha_4} + \epsilon^{2n} e_{\alpha_1+\alpha_2+\alpha_4}) \\
f_{-3\beta_1-\beta_2+3n\delta} & \mapsto & t^{-3n} & \otimes & e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
f_{-3\beta_1-2\beta_2+3n\delta} & \mapsto & t^{-3n} & \otimes & e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}
\end{array}
$$

for $n \in \mathbb{N}$, and by letting the action of ϕ on $\mathfrak{h}$ be given by

$$\begin{array}{lcl} \beta_2^\vee & \mapsto & \alpha_2^\vee \\ \beta_1^\vee & \mapsto & \alpha_1^\vee + \alpha_3^\vee + \alpha_4^\vee \\ \beta_0^\vee & \mapsto & 3c' - 1 \otimes (2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee) \\ d & \mapsto & d'/3. \end{array}$$

Note that we have chosen our notation for the root vectors of $\mathfrak{g}({}^t\tilde{G}_2)$ so that the elements e_ϑ for $\vartheta \in \Phi({}^t\tilde{G}_2)$ generate $\mathfrak{n}_+({}^t\tilde{G}_2)$ (see Proposition 1.3.1) and $f_\vartheta = e_{-\vartheta}$. Also note that the image of d under ϕ is forced by the image of $\beta_0^\vee$ and the fact that

$$\beta_0^\vee = c - (2\beta_1^\vee + 3\beta_2^\vee) \qquad \text{and} \qquad (c', d')_{\tilde{\mathfrak{g}}(D_4)} = 1 = (c, d)_{\mathfrak{g}({}^t\tilde{G}_2)},$$

where $(\cdot\,,\cdot)_{\tilde{\mathfrak{g}}(D_4)}$ and $(\cdot\,,\cdot)_{\mathfrak{g}({}^t\tilde{G}_2)}$ denote the normalized invariant forms on $\tilde{\mathfrak{g}}(D_4)$ and $\mathfrak{g}({}^t\tilde{G}_2)$, respectively.

Straightforward calculations show ϕ to be a Lie algebra isomorphism. Since

$$\phi(\mathfrak{h}) = \mathfrak{h}', \ \phi(\Pi^\vee) = \Pi'^\vee \ \text{ and } \ \phi^*(\Pi') = \Pi$$

ϕ satisfies all the conditions for an isomorphism between quadruples.

◇

Example 1.5.11

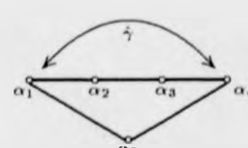
Let $\tilde{\mathfrak{g}}(A_4)$ be an affine algebra of type $\tilde{A}_4$ with Chevalley basis

$$\begin{array}{cccccc} e_{\alpha_0}, & e_{\alpha_1}, & e_{\alpha_2}, & e_{\alpha_3}, & e_{\alpha_4}, & \\ f_{\alpha_0}, & f_{\alpha_1}, & f_{\alpha_2}, & f_{\alpha_3}, & f_{\alpha_4}, & \\ \alpha_0^\vee, & \alpha_1^\vee, & \alpha_2^\vee, & \alpha_3^\vee, & \alpha_4^\vee, & d' \end{array}$$

where, for each $i \in \underline{4}_0$,

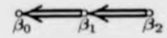
$$[e_{\alpha_i}, f_{\alpha_i}] = -\alpha_i^\vee,$$

and d' is the scaling element. Let $\tilde{\gamma}$ be the diagram automorphism of $\tilde{\mathfrak{g}}(A_4)$ induced by the automorphism



of $\Delta(\tilde{A}_4)$. Denote by $\tilde{\mathfrak{g}}^{\tilde{\gamma}}$ the fixed point subalgebra of $\tilde{\mathfrak{g}}(A_4)$ with respect to $\tilde{\gamma}$.

Let $\mathfrak{g}({}^*\tilde{C}_2)$ be a twisted affine Kac-Moody algebra of type ${}^*\tilde{C}_2$ with Dynkin diagram



and with root system as described in §1.3. We shall construct an explicit isomorphism between $\mathfrak{g}({}^*\tilde{C}_2)$ and $\hat{\mathfrak{g}}^{\bar{\gamma}}$.

We begin by taking $\mathfrak{g} = \mathfrak{g}(A_4)$ to be a Kac-Moody algebra of type A_4 with Chevalley basis

$$\begin{matrix} E'_{\alpha_1}, & E'_{\alpha_2}, & E'_{\alpha_3}, & E'_{\alpha_4}, \\ F'_{\alpha_1}, & F'_{\alpha_2}, & F'_{\alpha_3}, & F'_{\alpha_4}, \\ H'_{\alpha_1}, & H'_{\alpha_2}, & H'_{\alpha_3}, & H'_{\alpha_4}, \end{matrix}$$

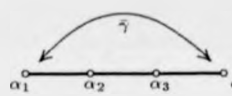
where, for each $i \in \underline{4}$,

$$\left[E'_{\alpha_i}, F'_{\alpha_i}\right] = -H'_{\alpha_i}.$$

Note that since $\mathfrak{g}$ is a simply-laced algebra we have that for all roots $\alpha_i, \alpha_j \in \Phi(\mathfrak{g})$

$$(\alpha_i + \alpha_j)^\vee = \alpha_i^\vee + \alpha_j^\vee$$

whenever $\alpha_i + \alpha_j$ is a root. Let γ be the diagram automorphism of $\mathfrak{g}$ induced by the automorphism $\bar{\gamma}$



of $\Delta(A_4)$. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

of $\mathfrak{g}$ into eigenspaces where $\mathfrak{g}_{\bar{m}}$ is the eigenspace corresponding to the eigenvalue $(-1)^{\bar{m}}$. By Proposition 1.5.3 we have an isomorphism

$$\mathfrak{g}(B_2) \overset{\psi}{\cong} \mathfrak{g}_{\bar{0}}$$

where $\mathfrak{g}(B_2)$ is a Kac-Moody algebra of type B_2 with Dynkin diagram

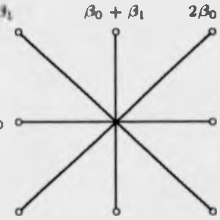
and with Chevalley basis $\{E_{\beta_i}, F_{\beta_i}, H_{\beta_i}\}_{i \in \underline{1}_0}$ such that

$$\begin{aligned} \psi(H_{\beta_0}) &= 2(H'_{\alpha_2} + H'_{\alpha_3}) & \psi(H_{\beta_1}) &= H'_{\alpha_1} + H'_{\alpha_4} \\ \psi(E_{\beta_0}) &= \sqrt{2}(E'_{\alpha_2} + E'_{\alpha_3}) & \psi(E_{\beta_1}) &= E'_{\alpha_1} + E'_{\alpha_4} \\ \psi(F_{\beta_0}) &= \sqrt{2}(F'_{\alpha_2} + F'_{\alpha_3}) & \psi(F_{\beta_1}) &= F'_{\alpha_1} + F'_{\alpha_4}. \end{aligned}$$

This construction is simply a variation on the known construction of a root system of type BC_2 from one of type A_4. We refer the reader to [Bou68], [Kac90, §7.10], and [Ste67, §11] for details of the original construction. We point out that we have chosen our labeling of $\Delta(B_2)$ so that our set $\Pi = \{\beta_0, \beta_1\}$ of fundamental roots of $\mathfrak{g}_{\bar 0}$ is given by

$$\beta_0 = \frac{1}{2}(\alpha_2 + \alpha_3), \qquad \text{and} \qquad \beta_1 = \frac{1}{2}(\alpha_1 + \alpha_4).$$

Thus the diagram



shows the roots of $\mathfrak{g}(B_2)$ as integral combinations of fundamental roots.

The γ-stable subalgebra

$$\mathfrak{h}_{\bar 0} = \mathbb{C}(H'_{\alpha_1} + H'_{\alpha_4}) \oplus \mathbb{C}(H'_{\alpha_2} + H'_{\alpha_3})$$

is a Cartan subalgebra of $\mathfrak{g}_{\bar 0}$, with respect to which $\mathfrak{g}_{\bar 0}$ has the following Cartan decomposition,

$$\begin{aligned} \mathfrak{g}_{\bar 0} \;=\;& \mathfrak{h}_{\bar 0} \oplus \mathbb{C}(E'_{\alpha_1} + E'_{\alpha_4}) \oplus \mathbb{C}(E'_{\alpha_2} + E'_{\alpha_3}) \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2} - E'_{\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2+\alpha_3} + E'_{\alpha_2+\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1} + F'_{\alpha_4}) \oplus \mathbb{C}(F'_{\alpha_2} + F'_{\alpha_3}) \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2} - F'_{\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2+\alpha_3} + F'_{\alpha_2+\alpha_3+\alpha_4}). \end{aligned}$$

The eigenspace $\mathfrak{g}_{\bar 1}$ has a weight space decomposition with respect to $\mathfrak{h}_{\bar 0}$, namely

$$\begin{aligned} \mathfrak{g}_{\bar 1} \;=\;& \mathfrak{h}_{\bar 1} \oplus \mathbb{C}(E'_{\alpha_1} - E'_{\alpha_4}) \oplus \mathbb{C}(E'_{\alpha_2} - E'_{\alpha_3}) \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2} + E'_{\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}E'_{\alpha_2+\alpha_3} \oplus \mathbb{C}(E'_{\alpha_1+\alpha_2+\alpha_3} - E'_{\alpha_2+\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\ & \oplus \mathbb{C}(F'_{\alpha_1} - F'_{\alpha_4}) \oplus \mathbb{C}(F'_{\alpha_2} - F'_{\alpha_3}) \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2} + F'_{\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}F'_{\alpha_2+\alpha_3} \oplus \mathbb{C}(F'_{\alpha_1+\alpha_2+\alpha_3} + F'_{\alpha_2+\alpha_3+\alpha_4}) \\ & \oplus \mathbb{C}F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \end{aligned}$$

where

$$\mathfrak{h}_{\bar 1} = \mathbb{C}(H'_{\alpha_1} - H'_{\alpha_4}) \oplus \mathbb{C}(H'_{\alpha_2} - H'_{\alpha_3}).$$

Consider

$$B_2 = (B_{ij})_{i,j\in \underline{1}_0} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

This is the Cartan matrix corresponding to the root system of type B_2 of $\mathfrak{g}_{\bar{0}}$. The fundamental roots of $\mathfrak{g}_{\bar{0}}$ then satisfy

$$\beta_i = \sum_{j\in \underline{1}_0} B_{ji}\varpi_j$$

where ϖ_j denotes the fundamental weight defined by

$$\varpi_j(H_{\beta_i}) = \delta_{ji},$$

with δ_{ji} denoting the Kronecker delta. Thus the weight β_i is given by the corresponding column of the Cartan matrix B_2.

We note also that

$$A_4 = (A_{ij})_{i,j\in \underline{4}} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

is the Cartan matrix of the Lie algebra $\mathfrak{g} = \mathfrak{g}(A_4)$.

LEMMA 1.5.12

For $m = 0, 1$, the weights associated to the vectors given in the above decompositions are given in table 1.5.13.

vector	associated weight
$E'_{\alpha_1} + (-1)^m E'_{\alpha_4}$	β_1
$E'_{\alpha_2} + (-1)^m E'_{\alpha_3}$	β_0
$E'_{\alpha_1+\alpha_2} - (-1)^m E'_{\alpha_3+\alpha_4}$	$\beta_0 + \beta_1$
$E'_{\alpha_2+\alpha_3}$	$2\beta_0$
$E'_{\alpha_1+\alpha_2+\alpha_3} + (-1)^m E'_{\alpha_2+\alpha_3+\alpha_4}$	$2\beta_0 + \beta_1$
$E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	$2\beta_0 + 2\beta_1$

Table 1.5.13: Weights associated to certain vectors

Proof

We consider the case of $E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$. Note that

$$\begin{aligned}\left[H'_{\alpha_1} + H'_{\alpha_4}\ ,\ E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right] & \\ &= \left[H'_{\alpha_1}, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right] + \left[H'_{\alpha_4}, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right] \\ &= (\langle \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1^\vee\rangle + \langle\langle \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_4^\vee\rangle)\, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\ &= 2E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\end{aligned}$$

and

$$\begin{aligned}\left[H'_{\alpha_2} + H'_{\alpha_3}\ ,\ E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right] & \\ &= (\langle \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2^\vee\rangle + \langle\langle \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_3^\vee\rangle)\, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\ &= 0E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\end{aligned}$$

This demonstrates that $E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$ has weight $2\beta_0 + 2\beta_1$.

The other parts of the result follow from similar calculations. □

Thus the highest weight appearing is $2\beta_0 + 2\beta_1$. Note that when expressed in terms of the roots of $\mathfrak{g}$ this is precisely

$$\theta_0 = \left(\frac{2}{2}(\alpha_1 + \alpha_4) + \frac{2}{2}(\alpha_2 + \alpha_3)\right).$$

Hence we take

$$\begin{aligned}E_{\beta_2} &= F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \quad \text{and} \\ F_{\beta_2} &= E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}.\end{aligned}$$

Denote by $\Phi(E)$ the set $\{\pm 2\beta_0, \pm(2\beta_0 + 2\beta_1)\}$. We note that

$$\Phi_m = \begin{cases}\Phi(B_2) & \text{if } m \equiv 0 \bmod 2 \\ \Phi(B_2) \cup \Phi(E) & \text{if } m \equiv 1 \bmod 2.\end{cases}$$

We introduce the the symbols $E_{m,\beta}$ to denote the weight vector in the given weight space decomposition of $\mathfrak{g}_m$ with weight $\beta \in \Phi(B_2) \cup \Phi(E)$.

We now turn our attention to the Kac-Moody subalgebra $\hat{\mathcal{L}}(\mathfrak{g}, \gamma, 2)$ of $\hat{\mathcal{L}}(\mathfrak{g})$. We note that $\hat{\mathcal{L}}(\mathfrak{g})$ is a realization of the Kac-Moody algebra $\hat{\mathfrak{g}}(A_4)$ constructed earlier and that, under the isomorphism so induced, $\hat{\mathcal{L}}(\mathfrak{g}, \gamma, 2)$ corresponds to $\hat{\mathfrak{g}}^\gamma$.

We set

$$\begin{aligned}\mathfrak{h}' &= \mathfrak{h}_0 + \mathbb{C}c' + \mathbb{C}d' \\ &= \mathbb{C}(H'_{\alpha_1} + H'_{\alpha_4}) \oplus \mathbb{C}(H'_{\alpha_2} + H'_{\alpha_3}) + \mathbb{C}c' + \mathbb{C}d'\end{aligned}$$

and define $\delta' \in \mathfrak{h}'^*$ by

$$\delta'\,|_{\mathfrak{h}_0 + \mathbb{C}c'} = 0, \ \langle \delta', d'\rangle = 1.$$

We define

$$\begin{aligned} e_{\beta_0} &= 1 \otimes E_{\beta_0}, & e_{\beta_1} &= 1 \otimes E_{\beta_1}, & & e_{\beta_2} = t \otimes E_{\beta_2}, \\ f_{\beta_0} &= 1 \otimes F_{\beta_0}, & f_{\beta_1} &= 1 \otimes F_{\beta_1}, & \text{and} \quad & f_{\beta_2} = t^{-1} \otimes F_{\beta_2}. \end{aligned}$$

Thus straightforward calculation using the definition of the Lie product on $\hat{\mathcal{L}}(\mathfrak{g})$ given by equation (1.4.0) in §1.4 leads to the equalities

$$[e_{\beta_0}, f_{\beta_0}] = -1 \otimes H_{\beta_0}, \quad \text{and} \quad [e_{\beta_1}, f_{\beta_1}] = -1 \otimes H_{\beta_1}.$$

We proceed to verify that

$$[e_{\beta_2}, f_{\beta_2}] = -(c' + 1 \otimes H_{\beta_2}),$$

where

$$H_{\beta_2} = -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^\vee.$$

First we note that

$$\begin{aligned} [e_{\beta_2}, f_{\beta_2}] &= \left[t \otimes F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, t^{-1} \otimes E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right] \\ &= 1 \otimes \left[F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right] - \left(F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right) c' \end{aligned}$$

However,

$$\begin{aligned} \left(F'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right) &= \left(\left[F'_{\alpha_1}, F'_{\alpha_2+\alpha_3+\alpha_4}\right], E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right) \\ &= \left(F'_{\alpha_1}, \left[F'_{\alpha_2+\alpha_3+\alpha_4}, E'_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right]\right) \\ &= \left(F'_{\alpha_1}, E'_{\alpha_1}\right) \\ &= 1 \end{aligned}$$

by the last part of Theorem 1.1.17. Thus

$$\begin{aligned} [e_{\beta_2}, f_{\beta_2}] &= -c' + 1 \otimes (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^\vee \\ &= -(3c' + 1 \otimes H_{\beta_2}) \end{aligned}$$

as required.

Thus the root system of $\hat{\mathcal{L}}(\mathfrak{g}, \gamma, 2)$ with respect to $\mathfrak{h}'$ is given by

$$\begin{aligned} \Phi = \ & \{\beta + m\delta' \ : \ \beta \in \Phi(B_2),\ m \in \mathbb{Z}\} \cup \{\beta + (2m-1)\delta' \ : \ \beta \in \Phi(E),\ m \in \mathbb{Z}\} \\ & \cup \{m\delta' \ : \ m \in \mathbb{Z} \setminus \{0\}\} \end{aligned}$$

and the root space decomposition of $\hat{\mathfrak{g}}^\gamma$ with respect to $\mathfrak{h}'$ is

$$\hat{\mathcal{L}}(\mathfrak{g}, \gamma, 2) = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha \in \Phi} \hat{\mathcal{L}}(\mathfrak{g}, \gamma, 2)_\alpha\right),$$

where

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,2)_{\beta+m\delta'} = \mathbb{C}(t^m \otimes E_{\bar{m},\beta}) \quad \text{for } \beta \in \Phi(B_2),\ m \in \mathbb{Z},$$
$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,2)_{\beta+(2m-1)\delta'} = \mathbb{C}(t^{2m-1} \otimes E_{\bar{1},\beta}) \quad \text{for } \beta \in \Phi(E),\ m \in \mathbb{Z},$$

and

$$\tilde{\mathcal{L}}(\mathfrak{g},\gamma,2)_{m\delta'} = \begin{cases} t^m \otimes \mathfrak{h}_0 & \text{for } m \in \mathbb{Z}\setminus\{0\},\ m \equiv 0 \bmod 2 \\ t^m \otimes \mathfrak{h}_1 & \text{for } m \in \mathbb{Z}\setminus\{0\},\ m \equiv 1 \bmod 2. \end{cases}$$

The set

$$\Pi' = \{\beta_0\ ,\ \beta_1\ ,\ \beta_2 = \delta' - 2\beta_0 - 2\beta_1\}$$

is a set of fundamental roots for $\tilde{\mathcal{L}}(\mathfrak{g},\gamma,2)$ and

$$\Pi'^{\vee} = \{\beta_0^{\vee} = 1 \otimes H_{\beta_0}\ ,\ \beta_1^{\vee} = 1 \otimes H_{\beta_1}\ ,\ \beta_2^{\vee} = c' + 1 \otimes H_{\beta_2}\}$$

is the corresponding set of fundamental coroots of $\tilde{\mathcal{L}}(\mathfrak{g},\gamma,2)$.

By Theorem 1.5.6, $\left(\tilde{\mathcal{L}}(\mathfrak{g},\gamma,2), \mathfrak{h}', \Pi'\ , \Pi'^{\vee}\right) = \left(\tilde{\mathfrak{g}}^{\gamma}, \mathfrak{h}', \Pi'\ , \Pi'^{\vee}\right)$ is a quadruple associated to ${}^*\tilde{C}_2$.

Recall the Kac-Moody algebra $\mathfrak{g}({}^*\tilde{C}_2)$ introduced earlier. The set

$$\Pi = \{\beta_0, \beta_1, \beta_2\}$$

is its fundamental root system and it has Chevalley basis

$$\begin{array}{llll} e_{\beta_0}, & e_{\beta_1}, & e_{\beta_2}, & \\ f_{\beta_0}, & f_{\beta_1}, & f_{\beta_2}, & \\ \beta_0^{\vee}, & \beta_1^{\vee}, & \beta_2^{\vee}, & d \end{array}$$

where, for each $i \in \underline{2}_0$,

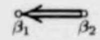
$$[e_{\beta_i}, f_{\beta_i}] = -\beta_i^{\vee},$$

and d is the scaling element. Denoting by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}({}^*\tilde{C}_2)$ generated by the fundamental coroots and d we see that

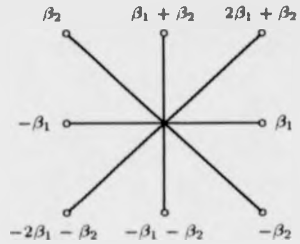
$$(\mathfrak{g}({}^*\tilde{C}_2), \mathfrak{h}, \Pi, \Pi^{\vee})$$

is also a quadruple associated to ${}^*\tilde{C}_2$. We proceed to describe an explicit isomorphism ϕ between these two quadruples.

We first note that from the results of §1.3 we can describe the root and coroot system of $\mathfrak{g}({}^*\tilde{C}_2)$ entirely in terms of the underlying finite root system $\mathring{\mathfrak{g}}({}^*\tilde{C}_2)$ of type C_2, obtained by omitting the zeroeth vertex from $\Delta({}^*\tilde{C}_2)$, and the fundamental imaginary root, δ, and the canonical central element, c, of $\mathfrak{g}({}^*\tilde{C}_2)$. Note that $\mathring{\mathfrak{g}}({}^*\tilde{C}_2)$ has Dynkin diagram



and root system



which we shall denote by $\Phi(C_2)$.

By virtue of the aforementioned description in §1.3 we have

$$\beta_0 = \frac{1}{2}(\delta - (2\beta_1 + \beta_2)) \qquad \text{and} \qquad \beta_0^\vee = c - 2(2\beta_1 + \beta_2)^\vee .$$

Now,

$$\begin{aligned}(2\beta_1 + \beta_2)^\vee &= \frac{2(2\beta_1 + \beta_2)}{(2\beta_1 + \beta_2, 2\beta_1 + \beta_2)_{\mathfrak{h}^*}} \\ &= \frac{4\beta_1}{2(\beta_1, \beta_1)_{\mathfrak{h}^*}} + \frac{2\beta_2}{(\beta_2, \beta_2)_{\mathfrak{h}^*}} \\ &= \beta_1^\vee + \beta_2^\vee ,\end{aligned}$$

whence we obtain,

$$\beta_0^\vee = c - (2\beta_1^\vee + 2\beta_2^\vee).$$

As another consequence of the results in §1.3, we have the following expression for the root system of $\mathfrak{g}({}^*C_2)$ in terms of $\Phi(C_2)$ and δ;

$$\begin{aligned}\Phi \;=\; & \left\{\frac{1}{2}(\beta + (2m-1)\delta') \;:\; \beta \in \Phi_l(C_2),\ m \in \mathbf{Z}\right\} \cup \{(\beta + m\delta' \;:\; \beta \in \Phi_s(C_2),\ m \in \mathbf{Z}\} \\ & \cup \{\beta + 2m\delta' \;:\; \beta \in \Phi_l(C_2),\ m \in \mathbf{Z}\} \cup \{m\delta' \;:\; m \in \mathbf{Z} \setminus \{0\}\} .\end{aligned}$$

We note that we can re-arrange our expression of β_0 in terms of β_1, β_2, and δ to obtain

$$\beta_2 = \delta - (2\beta_0 + 2\beta_1).$$

Using this substitution for β_2, we obtain

$$\Phi_l(C_2) = \{-2\beta_0 - 2\beta_1 + \delta\,,\ -2\beta_0 + \delta\,,\ 2\beta_0 + 2\beta_1 - \delta\,,\ 2\beta_0 - \delta\}$$

and

$$\Phi_s(C_2) = \{\beta_1\,,\ -\beta_1 - 2\beta_0 + \delta\,,\ -\beta_1\,,\ \beta_1 + 2\beta_0 - \delta\}$$

as alternative expressions for the set of long and short roots of C_2, respectively.

We now define a vector space isomorphism

$$\phi\ :\ \mathfrak{g}({}^{*}\tilde{C}_2) \rightarrow \tilde{\mathfrak{g}}^{\tau}$$

by taking the action of ϕ on the root vectors of $\mathfrak{g}({}^{*}\tilde{C}_2)$ to be

$$\begin{array}{lcrcl}
e_{\beta_1} & \mapsto & 1 & \otimes & (e_{\alpha_1} + e_{\alpha_4}) \\
e_{\beta_2} & \mapsto & t & \otimes & f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
e_{\beta_1+\beta_2} & \mapsto & t & \otimes & (f_{\alpha_1+\alpha_2+\alpha_3} - f_{\alpha_2+\alpha_3+\alpha_4}) \\
e_{2\beta_1+\beta_2} & \mapsto & t & \otimes & f_{\alpha_2+\alpha_3} \\
e_{1/2(\beta_2+(2n-1)\delta)} & \mapsto & t^n & \otimes & \sqrt{2}\,(f_{\alpha_1+\alpha_2} - (-1)^n f_{\alpha_3+\alpha_4}) \\
e_{1/2(2\beta_1+\beta_2+(2n-1)\delta)} & \mapsto & t^n & \otimes & \sqrt{2}\,(f_{\alpha_2} + (-1)^n f_{\alpha_3}) \\
e_{1/2(-\beta_2+(2n-1)\delta)} & \mapsto & t^{n-1} & \otimes & \sqrt{2}\,(e_{\alpha_1+\alpha_2} - (-1)^{n-1} e_{\alpha_3+\alpha_4}) \\
e_{1/2(-2\beta_1-\beta_2+(2n-1)\delta)} & \mapsto & t^{n-1} & \otimes & \sqrt{2}\,(e_{\alpha_2} + (-1)^{n-1} e_{\alpha_3}) \\
e_{\beta_1+n\delta} & \mapsto & t^n & \otimes & (e_{\alpha_1} + (-1)^n e_{\alpha_4}) \\
e_{\beta_1+\beta_2+n\delta} & \mapsto & t^{n+1} & \otimes & (f_{\alpha_1+\alpha_2+\alpha_3} + (-1)^{n+1} f_{\alpha_2+\alpha_3+\alpha_4}) \\
e_{-\beta_1+n\delta} & \mapsto & t^n & \otimes & (f_{\alpha_1} + (-1)^n f_{\alpha_4}) \\
e_{-\beta_1-\beta_2+n\delta} & \mapsto & t^{n-1} & \otimes & (e_{\alpha_1+\alpha_2+\alpha_3} + (-1)^{n-1} e_{\alpha_2+\alpha_3+\alpha_4}) \\
e_{\beta_2+2n\delta} & \mapsto & t^{2n+1} & \otimes & f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
e_{2\beta_1+\beta_2+2n\delta} & \mapsto & t^{2n+1} & \otimes & f_{\alpha_2+\alpha_3} \\
e_{-\beta_2+2n\delta} & \mapsto & t^{2n-1} & \otimes & e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
e_{-2\beta_1-\beta_2+2n\delta} & \mapsto & t^{2n-1} & \otimes & e_{\alpha_2+\alpha_3} \\
f_{\beta_1} & \mapsto & 1 & \otimes & (f_{\alpha_1} + f_{\alpha_4}) \\
f_{\beta_2} & \mapsto & t^{-1} & \otimes & e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
f_{\beta_1+\beta_2} & \mapsto & t^{-1} & \otimes & (e_{\alpha_1+\alpha_2+\alpha_3} - e_{\alpha_2+\alpha_3+\alpha_4}) \\
f_{2\beta_1+\beta_2} & \mapsto & t^{-1} & \otimes & e_{\alpha_2+\alpha_3} \\
f_{1/2(\beta_2+(2n-1)\delta)} & \mapsto & t^{-n} & \otimes & \sqrt{2}\,(e_{\alpha_1+\alpha_2} - (-1)^n e_{\alpha_3+\alpha_4}) \\
f_{1/2(2\beta_1+\beta_2(2n-1)\delta)} & \mapsto & t^{-n} & \otimes & \sqrt{2}\,(e_{\alpha_2} + (-1)^n e_{\alpha_3}) \\
f_{1/2(-\beta_2+(2n-1)\delta)} & \mapsto & t^{1-n} & \otimes & \sqrt{2}\,(f_{\alpha_1+\alpha_2} - (-1)^{1-n} f_{\alpha_3+\alpha_4}) \\
f_{1/2(-2\beta_1-\beta_2+(2n-1)\delta)} & \mapsto & t^{1-n} & \otimes & \sqrt{2}\,(f_{\alpha_2} + (-1)^{1-n} f_{\alpha_3}) \\
f_{\beta_1+n\delta} & \mapsto & t^{-n} & \otimes & (f_{\alpha_1} + (-1)^n f_{\alpha_4}) \\
f_{\beta_1+\beta_2+n\delta} & \mapsto & t^{-(n+1)} & \otimes & (e_{\alpha_1+\alpha_2+\alpha_3} + (-1)^{n+1} e_{\alpha_2+\alpha_3+\alpha_4}) \\
f_{-\beta_1+n\delta} & \mapsto & t^{-n} & \otimes & (e_{\alpha_1} + (-1)^n e_{\alpha_4}) \\
f_{-\beta_1-\beta_2+n\delta} & \mapsto & t^{1-n} & \otimes & (f_{\alpha_1+\alpha_2+\alpha_3} + (-1)^{1-n} f_{\alpha_2+\alpha_3+\alpha_4}) \\
f_{\beta_2+2n\delta} & \mapsto & t^{-(2n+1)} & \otimes & e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
f_{2\beta_1+\beta_2+2n\delta} & \mapsto & t^{-(2n+1)} & \otimes & e_{\alpha_2+\alpha_3} \\
f_{-\beta_2+2n\delta} & \mapsto & t^{1-2n} & \otimes & f_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
f_{-2\beta_1-\beta_2+2n\delta} & \mapsto & t^{1-2n} & \otimes & f_{\alpha_2+\alpha_3}
\end{array}$$

for $n \in \mathbb{N}$, and by letting the action of ϕ on $\mathfrak{h}$ be given by

$$\begin{array}{lcl} \beta_0^\vee & \mapsto & 2(\alpha_2^\vee + \alpha_3^\vee) \\ \beta_1^\vee & \mapsto & \alpha_1^\vee + \alpha_4^\vee \\ \beta_0^\vee & \mapsto & c' - 1 \otimes (\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee) \\ d & \mapsto & d'. \end{array}$$

Further lengthy but straightforward calculations show ϕ to be a Lie algebra isomorphism. Since

$$\phi(\mathfrak{h}) = \mathfrak{h}', \quad \phi(\Pi^\vee) = \Pi'^\vee \quad \text{and} \quad \phi^*(\Pi') = \Pi,$$

ϕ satisfies all the conditions for an isomorphism between quadruples.

◇

Chapter 2

Kac-Moody Groups

In this chapter we consider the construction of certain groups associated to Kac-Moody algebras. We shall be drawing our material almost entirely from an extensive body of work on the topic by Jacques Tits, consisting largely of [Tit81], [Tit82], [Tit85], [Tit87b], and [Tit87a].

We recall that whenever A is a Cartan matrix Chevalley and Demazure associated to each root datum $\mathcal{D} = \left(\Lambda, \{\tilde{\alpha}_i\}_{i\in\underline{n}}, \{\alpha_i^\vee\}_{i\in\underline{n}}\right)$ with Cartan matrix A a split reductive group scheme $\mathfrak{G}_\mathcal{D}$ such that Λ is the character group of a maximal split torus, $\{\tilde{\alpha}_i\}_{i\in\underline{n}}$ is a basis of the root system with respect to this torus and $\alpha_i^\vee$ is the coroot associated to $\tilde{\alpha}_i$. The importance of these group schemes lies in the fact that if $\mathbb{K}$ is an algebraically closed field, the correspondence $\mathcal{D} \leftrightarrow \mathfrak{G}_\mathcal{D}(\mathbb{K})$ is a bijection between root data of the types envisaged and connected reductive groups over $\mathbb{K}$. The groups defined in this chapter are a generalization of this construction to the case when A is a generalized Cartan matrix.

We begin by covering some background material which will be relevant in the development of the theory of Kac-Moody groups. Using the theory of Kac-Moody algebras developed in Chapter 1, we introduce the notion of prenilpotent sets of roots and define a $\mathbb{Z}$-form for the universal enveloping algebra of a Kac-Moody algebra. This $\mathbb{Z}$-form will be fundamental to the developments in §2.3 and, although it is not *a priori* clear that different $\mathbb{Z}$-forms of the universal enveloping algebra would not lead to different groups, it has been shown that this is not the case for generalized Cartan matrices of finite and affine types (see [Gar80], [Tit82] and [Tit85] for the affine case). We also introduce the concepts of a group scheme and of a group with a (B, N)-pair in §2.1.

The notion of a root datum is introduced in §2.2. In this section we also discuss how to construct the Weyl group and real root system corresponding to a generalized Cartan matrix A without having to refer to $\mathfrak{g}_A(\mathbb{C})$.

The construction of the Kac-Moody groups takes place in §2.4, though §2.3 paves the way with the introduction of the root group schemes and the toral functor. Finally, in §2.5, we give an explicit system of generators and relations for Kac-Moody groups over fields and we describe some of their properties. In particular, we give a description of a homomorphism from the simply-connected Kac-Moody group associated to a particular generalized Cartan matrix to the minimal adjoint Kac-moody group associated to the same matrix. In [Tit87a], Tits mentions that the notion of an isogeny has been generalized to Kac-Moody groups by Hée. However, to the best of our knowledge it remains unpublished and we have not seen details of the work. Nevertheless, we have avoided the temptation to extended the notion of an isogeny in its full generality and present instead only the particular cases we will require. We note that a generalized isogeny need not necessarily be surjective or have a finite kernel.

Throughout we shall assume $A = (A_{ij})_{i,j \in \underline{n}}$ to be a generalized Cartan matrix, and we shall draw on notation introduced in Chapter 1 whenever necessary.

2.1 General Background

In this section we introduce some well-known concepts and results which we will make use of during the course of this chapter.

Prenilpotent Sets of Roots

The material in this subsection can be found in [Tit87b]. Let Φ be the full root system of $\mathfrak{g}_A(\mathbb{C})$. We say that a set of roots $\Psi \subseteq \Phi$ is *closed* if

$$\alpha, \beta \in \Psi \quad \text{and} \quad \alpha + \beta \in \Phi \quad \Rightarrow \quad \alpha + \beta \in \Psi.$$

Recall that Φ^{re} is the set of real roots of $\mathfrak{g}_A(\mathbb{C})$. We say that a set of roots $\Psi \subset \Phi^{re}$ is *prenilpotent* if there exist $w, w' \in W(A)$ such that

$$w(\Psi) \subseteq \Phi^{re}_{+} \quad \text{and} \quad w'(\Psi) \subseteq \Phi^{re}_{-}.$$

Note that if $\Psi = \{\alpha, \beta\} \subset \Phi^{re}$ then Ψ is prenilpotent if and only if for all $i, j \in \mathbf{N}_0$

$$i\alpha + j\beta \in \Phi \quad \Rightarrow \quad i\alpha + j\beta \in \Phi^{re}.$$

Whenever $\{\alpha, \beta\}$ is a prenilpotent pair of roots, define

$$\theta[\alpha, \beta] = (\mathbf{N}_0\alpha + \mathbf{N}_0\beta) \cap \Phi^{re},$$

and let

$$\theta(\alpha, \beta) = \theta[\alpha, \beta] \setminus \{\alpha, \beta\}.$$

These will be finite sets.

The Universal Enveloping Algebra of a Kac-Moody Algebra

Denote by $\mathcal{U}(l)$ the universal enveloping algebra of a Lie algebra l. Unless otherwise stated, details of results quoted on the enveloping algebra of a Kac-Moody algebra can be found in [Tit87b, §4].

Recall that by part 1 of Theorem 1.1.6 the Kac-Moody algebra $\mathfrak{g}_A(\mathbb{C})$ has a triangular decomposition with respect to a Cartan subalgebra $\mathfrak{h}$, namely

$$\mathfrak{g}_A(\mathbb{C}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_+$ is generated by the elements e_i for $i \in \underline{n}$ and $\mathfrak{n}_-$ is generated by the elements f_i for $i \in \underline{n}$. It follows from this that the product mapping

$$\mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+) \to \mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$$

is bijective.

For any $u \in \mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$ and any $m \in \mathbb{N}$ we set

$$u^{[m]} = \frac{u^m}{m!} \quad \text{and} \quad \binom{u}{m} = \frac{u(u-1)\cdots(u-m+1)}{m!}.$$

For $i \in \underline{n}$ let $\mathcal{U}_{\mathbb{Z},i}$ and $\mathcal{U}_{\mathbb{Z},-i}$ be the subrings

$$\sum_m \mathbb{Z} e_i^{[m]} \quad \text{and} \quad \sum_m \mathbb{Z} f_i^{[m]}$$

of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$ respectively, let $\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})$ be the subring of $\mathcal{U}(\mathfrak{h})$ generated by all $\binom{\alpha^\vee}{m}$ for $\alpha^\vee \in \mathfrak{h}$ and $m \in \mathbb{N}$, and let $\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})_i$ be the subring

$$\sum_m \mathbb{Z} \binom{\alpha_i^\vee}{m}$$

of $\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})$. We have

$$\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})\mathcal{U}_{\pm i} = \mathcal{U}_{\pm i}\mathcal{U}_{\mathbb{Z}}(\mathfrak{h}).$$

From this and [Bou75, §12.5 Lemme 4], we deduce

$$\mathcal{U}_{\mathbb{Z},-i}\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})\mathcal{U}_{\mathbb{Z},i} = \mathcal{U}_{\mathbb{Z},i}\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})\mathcal{U}_{\mathbb{Z},-i}.$$

Let $\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-)$ and $\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)$ be the subrings of $\mathcal{U}(\mathfrak{n}_-)$ and $\mathcal{U}(\mathfrak{n}_+)$ generated by all $\mathcal{U}_{\mathbb{Z},-i}$ and all $\mathcal{U}_{\mathbb{Z},i}$ respectively. Denote by $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$ the subring of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$ generated by $\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-)$, $\mathcal{U}_{\mathbb{Z}}(\mathfrak{h})$ and $\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)$.

PROPOSITION 2.1.1

The product map

$$\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-) \otimes \mathcal{U}_{\mathbb{Z}}(\mathfrak{h}) \otimes \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+) \to \mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$$

is bijective.

We say that a subring $\mathcal{X}_{\mathbb{Z}}$ of a $\mathbb{C}$-algebra $\mathcal{X}_{\mathbb{C}}$ is a $\mathbb{Z}$*-form* of $\mathcal{X}_{\mathbb{C}}$ if the canonical map

$$\mathcal{X}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathcal{X}_{\mathbb{C}}$$

is bijective. $\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-)$ and $\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)$ are $\mathbb{Z}$-forms of $\mathcal{U}(\mathfrak{n}_-)$ and $\mathcal{U}(\mathfrak{n}_+)$ respectively. Hence, by Proposition 2.1.1, $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$ is a $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$. We note that in the classical case $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$ is called the *Kostant* $\mathbb{Z}$*-form* of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$.

Recall that $\mathfrak{g}_A(\mathbb{C})$ has a $(\oplus_i \mathbb{Z}\alpha_i)$-grading by virtue of its root space decomposition. For $i \in \underline{n}$ and $w \in W(A)$, the free $\mathbb{Z}$-module $w(\mathbb{Z}e_i)$ depends only on the root $\alpha = w(\alpha_i)$. We denote this module by $\mathfrak{g}_\alpha(\mathbb{Z})$ and let $\mathcal{U}_\alpha$ be the subring

$$\sum_m \mathbb{Z} e_\alpha^{[m]}$$

of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$ for $\alpha \in \Phi^{re}$. Recall also that $\operatorname{ad} e_i$ is a locally nilpotent on $\mathfrak{g}_A(\mathbb{C})$ (by the last part of Lemma 1.1.18) and hence also on $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$. Thus, by [Bou75, VIII.12.5 Lemme 2], it follows that $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$ is stable under the action of

$$\frac{(\operatorname{ad} e_i)^m}{m!}$$

for all $m \in \mathbb{N}$, hence is also stable under the action of $\exp \operatorname{ad} e_i$. Similarly, $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$ is stable under the action of $\exp \operatorname{ad} f_i$, whence we deduce that $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$ is stable under the action of the element

$$\begin{aligned} r_i^{\mathrm{ad}} &= \exp \operatorname{ad} e_i . \exp \operatorname{ad} f_i . \exp \operatorname{ad} e_i \\ &= \exp \operatorname{ad} f_i . \exp \operatorname{ad} e_i . \exp \operatorname{ad} f_i \end{aligned}$$

introduced in Lemma 1.1.27. By a slight abuse of notation, denote by s_i the automorphisms r_i^{ad} of $\mathfrak{g}_A(\mathbb{C})$ and of $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_A(\mathbb{C}))$. Let W^{ad} be the subgroup of $\operatorname{Aut} \mathfrak{g}_A(\mathbb{C})$ generated by the elements $\{s_i\}_{i \in \underline{n}}$. The map $s_i \mapsto r_i$ extends to a homomorphism

$$\chi : W^{ad} \to W.$$

Let $w = r_{i_1} \cdot \dots \cdot r_{i_l}$ be a reduced expression of $w \in W$. Set

$$\beta_k = (r_{i_1} \cdots r_{i_{k-1}})(\alpha_{i_k})$$

for each $k \in \underline{l}$. By Lemma 1.1.25, the elements $\beta_1, \dots, \beta_l$ are precisely those elements of Φ_+ which w^{-1} transforms into negative ones. Set

$$w^{\mathrm{ad}} = s_{i_1} \cdot \dots \cdot s_{i_l} \in W^{ad}.$$

This is an automorphism of $\mathfrak{g}_A(\mathbb{C})$, and, by abuse of notation, of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$, depending only on w and not on the chosen reduced expression. Set

$$\mathcal{U}_{\mathbb{Z},w} = \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+) \cap w^{\mathrm{ad}}(\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-)), \quad \mathcal{U}'_{\mathbb{Z},w} = \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+) \cap w^{\mathrm{ad}}(\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)),$$
$$\text{and} \quad \mathfrak{n}_w = \mathfrak{n}_+ \cap w^{\mathrm{ad}}(\mathfrak{n}_-).$$

For each $i \in \underline{n}$, set

$$\mathcal{U}'_{\mathbb{Z},i} = \mathcal{U}'_{r_i} = \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+) \cap s_i(\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)), \quad \text{and} \quad \mathfrak{n}'_i = \mathfrak{n}_+ \cap s_i(\mathfrak{n}_+).$$

Denote by $\mathcal{U}_w(\mathbb{C})$, $\mathcal{U}_\beta(\mathbb{C})$ and $\mathcal{U}'_i(\mathbb{C})$ the universal enveloping algebras of $\mathfrak{n}_w$, $\mathfrak{g}_\beta$ and $\mathfrak{n}'_i$ for $i \in \underline{n}$ and $\beta \in \Phi^{re}$. By the remarks just made about α_j and β_k we have

$$\mathfrak{n}_+ = \mathbb{C}e_i \oplus \mathfrak{n}'_i \quad \text{and} \quad \mathfrak{n}_w = \bigoplus_{k \in \underline{l}} \mathfrak{g}_{\beta_k}$$

as vector spaces, from which it follows that the product maps

$$\mathcal{U}_{\mathbb{Z},i}(\mathbb{C}) \otimes \mathcal{U}'_i(\mathbb{C}) \to \mathcal{U}(\mathfrak{n}_+)$$

and

$$\mathcal{U}_{\beta_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{U}_{\beta_l}(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C})$$

are bijective.

LEMMA 2.1.2

1. *For $i \in \underline{n}$, the product maps*

$$\mathcal{U}_{\mathbb{Z},i} \otimes \mathcal{U}'_{\mathbb{Z},i} \to \mathcal{U}_+ \quad \textit{and} \quad \mathcal{U}'_{\mathbb{Z},i} \otimes \mathcal{U}_{\mathbb{Z},i} \to \mathcal{U}_+$$

are bijective.

2. *The product maps*

$$\mathcal{U}_{\mathbb{Z},\beta_1} \otimes \cdots \otimes \mathcal{U}_{\mathbb{Z},\beta_l} \to \mathcal{U}_{\mathbb{Z},w}$$

and

$$\mathcal{U}_{\mathbb{Z},\beta_l} \otimes \cdots \otimes \mathcal{U}_{\mathbb{Z},\beta_1} \to \mathcal{U}_{\mathbb{Z},w}$$

are bijective.

Proof

See proof of [Tit87b, Lemma 1]. □

Group Schemes

We say that $\mathfrak{G}$ is a *group functor* on the category $\mathcal{C}$ if $\mathfrak{G}$ is a functor from the category $\mathcal{C}$ to the category of groups, i.e. if

- for every $\mathcal{R} \in \mathrm{Ob}(\mathcal{C})$, $\mathfrak{G}(\mathcal{R})$ is a group,
- every morphism $\phi : \mathcal{R} \to \mathcal{S}$ for $\mathcal{R}, \mathcal{S} \in \mathrm{Ob}(\mathcal{C})$ induces a group homomorphism $\mathfrak{G}(\phi) : \mathfrak{G}(\mathcal{R}) \to \mathfrak{G}(\mathcal{S})$,
- given $\mathcal{T} \in \mathrm{Ob}(\mathcal{C})$ and a morphism $\psi : \mathcal{S} \to \mathcal{T}$ then

$$\mathfrak{G}(\psi \circ \phi) = \mathfrak{G}(\psi) \circ \mathfrak{G}(\phi),$$

  and finally

- the identity map $\imath : \mathcal{R} \to \mathcal{R}$ induces the identity map $\mathfrak{G}(\imath) : \mathfrak{G}(\mathcal{R}) \to \mathfrak{G}(\mathcal{R})$ for every $\mathcal{R} \in \mathrm{Ob}(\mathcal{C})$.

Example 2.1.3

For each $\mathcal{K}$-algebra $\mathcal{A}$ the functor $\mathrm{Hom}_{\mathcal{K}}(\mathcal{A}, \cdot)$ is a group functor on the category of $\mathcal{K}$-algebras. ◇

A functor which is naturally isomorphic to one of the type described in Example 2.1.3 is said to be *representable*.

A representable functor from $\mathcal{K}$-algebras to groups is called a *group scheme over* $\mathcal{K}$. Unless otherwise specified, we shall be considering group schemes over $\mathbf{Z}$.

For more details of the following examples we refer the reader to [Wat79].

Example 2.1.4

The most straightforward example of a group scheme is $\mathfrak{Add}$. This is often referred to in the classical literature as G_a, a notation which we have not adopted so as to avoid confusion later. For any ring $\mathcal{R}$, $\mathfrak{Add}(\mathcal{R})$ is simply $\mathcal{R}$ itself considered as a group under addition. If ϕ is a ring homomorphism between two rings, $\mathfrak{Add}(\phi)$ is simply ϕ considered as a group homomorphism. ◇

Example 2.1.5

For the same reasons as in the previous example we use the notation $\mathfrak{Mult}$, instead of the more standard G_m, to denote the group scheme defined by taking $\mathfrak{Mult}(\mathcal{R})$ to be the invertible elements of $\mathcal{R}$ under multiplication. The action of $\mathfrak{Mult}$ on a ring homomorphism ϕ is simply the restriction of ϕ to a group homomorphism defined on the invertible elements of its domain. ◇

Example 2.1.6

The *special linear* group scheme of degree 2, $\mathfrak{SL}_2$, is defined by taking

$$\mathfrak{SL}_2(\mathcal{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a,b,c,d \in \mathcal{R} \ , \ ad - bc = 1 \right\}$$

and by defining

$$\begin{array}{rccc} \mathfrak{SL}_2(\phi) & : & \mathfrak{SL}_2(\mathcal{R}) & \rightarrow & \mathfrak{SL}_2(\mathcal{S}) \\ & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix} \end{array}$$

whenever $\phi : \mathcal{R} \rightarrow \mathcal{S}$ is a ring homomorphism. ◇

Groups with a (B, N)-pair

Proofs of results quoted in this subsection can be found in [Car72, §8.1–§8.3], although alternative expositions of the material can also be found in [Hum75] and [Tit74]. We note that in some sources, for example [Hum75], (B, N)-pairs may be referred to as *Tits systems*.

A pair of subgroups $\{B, N\}$ of a group G is called a *(B, N)-pair* of G if the following axioms are satisfied;

BN1 $G = \langle B, N \rangle$.

BN2 $B \cap N \lhd N$.

BN3 The group $W = \dfrac{N}{B \cap N}$ is generated by a set of elements $R = \{r_i\}_{i \in I}$, for some indexing set I, such that $r_i^2 = 1$.

BN4 If $n_i \in N$ maps to r_i under the natural homomorphism of N into W then

$$n_i B n_i \neq B.$$

BN5 If n_i is as above and $n \in N$, then

$$B n_i B . B n B \subseteq B n_i n B \cup B n B.$$

We say that a (B, N)-pair is *saturated* if

$$B \cap N = \bigcap_{n \in N} n B n^{-1}.$$

The group W is called the *Weyl group* of the (B, N)-pair. The generating set R is entirely determined by the (B, N)-pair, and the elements r_i are called the *distinguished generators* of W. For each subset $J \subseteq I$, let W_J be the subgroup of

W generated by the elements $\{r_i\}_{i\in J}$ and let N_J be the subgroup of N mapping to W_J under the natural homomorphism $\pi : N \to W$.

PROPOSITION 2.1.7
Let G be a group with a (B, N)-pair. Then

1. *For all $J \subseteq I$, $P_J = BN_JB$ is a subgroup of G.*
2. *$G = BNB$.*
3. *Let $n, n' \in N$. Then*

$$BnB = Bn'B \Leftrightarrow \pi(n) = \pi(n').$$

Thus there exists a natural one-to-one correspondence between double cosets of B in G and elements of W.

The decomposition $G = BNB$ is called the *Bruhat decomposition* of G. We introduce the notation $\mathcal{N}_G(H)$ to denote the normalizer in G of a subgroup H of G.

THEOREM 2.1.8
Let G be a group with a (B, N)-pair. Then

1. *the subgroups P_J are the only subgroups of G containing B,*
2. *$P_J = \mathcal{N}_G(P_J)$, and*
3. *distinct subgroups P_J and P_K can't be conjugate in G.*

This leads to the following result.

THEOREM 2.1.9
Let G be a group with a (B, N)-pair. Then the subgroups P_J for distinct subsets $J \subseteq I$ are all distinct and

$$P_J \cap P_K = P_{J\cap K}.$$

Thus the subgroups P_J form a lattice isomorphic to the lattice of subsets of I.

EXAMPLE 2.1.10
The group $\mathfrak{SL}_2(\mathbb{K})$, where $\mathbb{K}$ is a field, has a (B, N)-pair, where B is the subgroup of $\mathfrak{SL}_2(\mathbb{K})$ of upper triangular matrices and N is the subgroup of $\mathfrak{SL}_2(\mathbb{K})$ of monomial matrices. The subgroup $B \cap N$ is then the subgroup of diagonal matrices and W is isomorphic to the symmetric group on two elements. ◇

2.2 Root Data

Definition of a Root Datum

We define a *root datum*, $\mathcal{D}$, associated to A to be a system

$$\mathcal{D} = \left(\Lambda, \{\tilde{\alpha}_i\}_{i\in\underline{n}}, \{\alpha_i^\vee\}_{i\in\underline{n}}\right)$$

consisting of a free abelian group Λ, whose $\mathbb{Z}$-dual shall be denoted by Λ^*, and two sets of elements $\tilde{\alpha}_i \in \Lambda$ and $\alpha_i^\vee \in \Lambda^*$ indexed by $\underline{n}$, such that

$$\tilde{\alpha}_j(\alpha_i^\vee) = \langle \tilde{\alpha}_j, \alpha_i^\vee \rangle = A_{ij}$$

for all $i, j \in \underline{n}$, where $\langle\cdot,\cdot\rangle$ is the dual contraction between Λ and Λ^*.

We note that the realization of A mentioned in §1.1 is an example of a root datum associated to A. However, unlike realizations of A, there may be numerous, essentially different, root data associated to a given generalized Cartan matrix A. We mention some standard examples.

EXAMPLE 2.2.1
The first example is the *simply-connected root datum*, denoted $\mathcal{D}_{sc}$. This is characterized by the relation

$$\Lambda^* = \bigoplus_{i\in\underline{n}} \mathbb{Z}\alpha_i^\vee.$$

◇

EXAMPLE 2.2.2
The next example is the *adjoint root datum*, denoted $\mathcal{D}_{ad}$. This is obtained by taking , for each $i \in \underline{n}$,

$$\tilde{\alpha}_i = (A_{ij})_{j\in\underline{n}} \in \mathbb{Z}^n$$

and Λ to be the subgroup of $\mathbb{Z}^n$ generated by the $\tilde{\alpha}_i$ for $i \in \underline{n}$. The group Λ^* and the elements $\alpha_i^\vee$ for $i \in \underline{n}$ are then determined by the general properties of root data. Note that if A is invertible then the vectors $\tilde{\alpha}_j$ are linearly independent in $\mathbb{Z}^n$ and so

$$\Lambda = \bigoplus_{j\in\underline{n}} \mathbb{Z}\tilde{\alpha}_j.$$

◇

EXAMPLE 2.2.3
Related to the adjoint root datum is the *minimal adjoint root datum*, denoted $\mathcal{D}_m$. This is characterised by

$$\Lambda = \bigoplus_{j\in\underline{n}} \mathbb{Z}\tilde{\alpha}_j.$$

The group Λ^* and the elements $\alpha_i^\vee$ for $i \in \underline{n}$ are then determined by the general properties of root data. If A is invertible then $\mathcal{D}_{ad}$ and $\mathcal{D}_{m}$ are equivalent. ◇

EXAMPLE 2.2.4

The final example I shall give is that of the *universal root datum*, denoted $\mathcal{D}_{un}$. This is obtained by taking Λ and Λ^* to be copies of $\mathbb{Z}^{2n}$, with canonical bases denoted by

$$(u_j, v_j)_{j\in\underline{n}} \quad \text{and} \quad (u_i', v_i')_{i\in\underline{n}}$$

respectively. Letting $u_j = \tilde{\alpha}_j$ and $v_i' = \alpha_i^\vee$, we put Λ and Λ^* in $\mathbb{Z}$-duality by means of the bilinear form represented by the matrix

$$\begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix}.$$

◇

Construction of the Weyl Group and the Real Root System from Root Data

We emphasize the fact, demonstrated by example 2.2.2 in the case when A is an affine Cartan matrix, that the elements $\{\tilde{\alpha}_i\}_{i\in\underline{n}}$ need not be linearly independent. Thus, if we were to define a reflection group associated to a datum $\mathcal{D}$ in a manner analogous to that described in §1.1, we would not necessarily obtain the Weyl group associated to the generalized Cartan matrix A. In order to construct the Weyl group W and the real root system Φ^{re} associated to A from an arbitrary root datum $\mathcal{D}(A)$, we must follow the procedure described below.

We first introduce symbols α_i in one-to-one correspondence with the $\tilde{\alpha}_i$ in $\mathcal{D}$. Let $\bar{\Lambda}$ be the free abelian group generated on these symbols and let $\Pi = \{\alpha_i\}_{i\in\underline{n}}$. Thus we have a group isomorphism

$$\zeta : \mathbb{Z}^n \to \bar{\Lambda}$$

under which the canonical basis $\{b_i\}_{i\in\underline{n}}$ of $\mathbb{Z}^n$ is identified with $\{\alpha_i\}_{i\in\underline{n}}$.

Define fundamental reflections

$$r_i : \bar{\Lambda} \to \bar{\Lambda}$$

by defining

$$r_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$$

and let W be the group of transformations of $\bar{\Lambda}$ generated by $\{r_i\}_{i\in\underline{n}}$. Then W is isomorphic to the Weyl group of A and $\Phi^{re}(A) = W(\Pi)$ is the real root system associated to A. Define

$$\Phi_+^{re} = \Phi^{re} \cap \zeta\left(\sum_{i\in\underline{n}} \mathbb{N} b_i\right)$$

and let $\Phi^{re}_- = -\Phi^{re}_+$. These are the sets of positive and negative real roots, respectively.

We note that the Weyl group, W, corresponding to A will nevertheless act on Λ by

$$r_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\tilde{\alpha}_i$$

for $i \in \underline{n}$. For any root $\alpha = w(\alpha_i) \in \Phi^{re}$, where $i \in \underline{n}$ and $w \in W$, set $\tilde{\alpha} = w(\tilde{\alpha}_i)$. This is an element of Λ depending only on α and not on the particular choice of i and w.

2.3 The Root Group Schemes and the Toral Functor.

For each $\alpha \in \Phi^{re}$, we denote by $\mathfrak{U}_\alpha$ the group scheme isomorphic to $\mathfrak{Add}$ whose Lie algebra is $\mathfrak{g}_\alpha(\mathbb{Z})$. This characterizes $\mathfrak{U}_\alpha$ up to unique isomorphism. We note that given a ring $\mathcal{R}$, the group $\mathfrak{U}_\alpha(\mathcal{R})$ can be identified with the additive group of

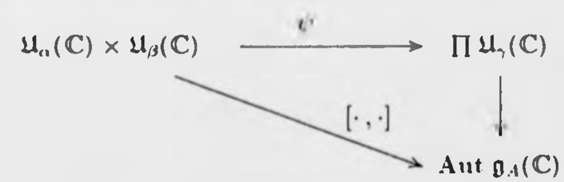
$$\mathfrak{g}_\alpha(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{R}.$$

Thus the group $\mathfrak{U}_\alpha(\mathbb{C})$ can be identified in a canonical manner with its Lie algebra, $\mathfrak{g}_\alpha$, which is a one-dimensional subalgebra of $\mathfrak{g}_A(\mathbb{C})$. We can immerse $\mathfrak{g}_\alpha$ in $\operatorname{Aut} \mathfrak{g}_A(\mathbb{C})$ by the application of $\exp \operatorname{ad}$. In this way, the groups $\mathfrak{U}_\alpha(\mathbb{C})$ for $\alpha \in \Phi^{re}$ can be viewed as subgroups of $\operatorname{Aut} \mathfrak{g}_A(\mathbb{C})$ which are permuted amongst themselves by the action of W^{ad}.

Let $\{\alpha, \beta\} \subseteq \Phi^{re}$ be a prenilpotent pair of roots. Choose a total order on the finite set $\theta(\alpha, \beta)$. Then there exists a unique $\mathbb{Z}$-morphism

$$\psi : \mathfrak{U}_\alpha \times \mathfrak{U}_\beta \to \prod \mathfrak{U}_\gamma$$

where γ runs over $\theta(\alpha, \beta)$ in the prescribed order, and such that the diagram



commutes, where $[\cdot,\cdot]$ denotes the commutator map. This is an extension to Kac-Moody algebras of the fundamental theorem of Chevalley on the integrality of the coefficients appearing in the commutator relations.

However, we can go even further. Let Ψ be a finite nilpotent subset of Φ^{re}. Thus

$$\mathfrak{g}_\Psi = \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$$

is a nilpotent Lie algebra. Let U_Ψ denote the unipotent (hence simply-connected by [Wat79, §8.5]) complex algebraic group whose Lie algebra is $\mathfrak{g}_\Psi$. This characterizes U_Ψ uniquely. The proof of the following result relies heavily on the $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$ introduced in §2.1.

PROPOSITION 2.3.1

There exists a uniquely defined group scheme $\mathfrak{U}_\Psi$ containing all $\mathfrak{U}_\alpha$ for $\alpha \in \Psi$ such that $\mathfrak{U}_\Psi(\mathbb{C}) \cong U_\Psi$ and, for any total order put on Ψ, the product morphism

$$\prod_{\alpha \in \Psi} \mathfrak{U}_\alpha \to \mathfrak{U}_\Psi$$

is an isomorphism of the underlying group schemes.

We define the *toral group functor*, $\mathfrak{T}_\mathcal{D}$, to be $\mathrm{Hom}_\mathbb{Z}(\Lambda, \mathfrak{Mult})$. Thus, for any ring $\mathcal{R}$,

$$\mathfrak{T}_\mathcal{D}(\mathcal{R}) = \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathcal{R}^\times)$$

where $\mathcal{R}^\times$ denotes the multiplicative group of invertible elements of $\mathcal{R}$. For every $r \in \mathcal{R}$, we denote by $r^{\alpha_i^\vee}$ the element of $\mathfrak{T}_\mathcal{D}(\mathcal{R})$ defined by

$$\lambda \mapsto r^{\alpha_i^\vee(\lambda)}$$

for all $\lambda \in \Lambda$.

Note that since we have an action of W on Λ we have an induced action of W on $\mathfrak{T}_\mathcal{D}(\mathcal{R})$. By a certain abuse of notation, denote by w the automorphism of $\mathfrak{T}_\mathcal{D}(\mathcal{R})$ induced by an element $w \in W$.

2.4 The Kac-Moody Group Functor

In his paper [Tit87b], Tits shows that, provided we define the Kac-Moody group functor in such a way that it satisfies a few reasonable axioms described in detail below, $\mathfrak{G}_\mathcal{D}(\mathbb{K})$ is uniquely determined up to isomorphism whenever $\mathbb{K}$ is a field. Furthermore, he gives a set of generators and defining relations for $\mathfrak{G}_\mathcal{D}(\mathbb{K})$.

The Definition of the Kac-Moody Group Functor

We shall be interested in *Kac-Moody group systems*,

$$\mathfrak{F} = \left(\mathfrak{G}_\mathcal{D}, \mathfrak{U}_+, \mathfrak{U}_-, (\varphi_i)_{i \in \underline{n}}, \eta\right),$$

consisting of a group functor $\mathfrak{G}_{\mathcal{D}}$, two subgroup functors $\mathfrak{U}_{\pm}$, homomorphisms

$$\varphi_i : \mathfrak{SL}_2 \to \mathfrak{G}_{\mathcal{D}}$$

and a homomorphism

$$\eta : \mathfrak{T}_{\mathcal{D}} \to \mathfrak{G}_{\mathcal{D}}$$

such that $\mathfrak{F}$ satisfies the following conditions;

KMG1 For every ring $\mathcal{R}$, the homomorphism

$$\eta(\mathcal{R}) : \mathfrak{T}_{\mathcal{D}}(\mathcal{R}) \to \mathfrak{G}_{\mathcal{D}}(\mathcal{R})$$

is injective.

KMG2 For all $i \in \underline{n}$ and $r \in \mathcal{R}$,

$$\varphi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta(r^{\alpha_i^\vee})$$

where $r^{\alpha_i^\vee}$ denotes the element $\lambda \mapsto r^{\alpha_i^\vee(\lambda)}$ of $\mathfrak{T}_{\mathcal{D}}(\mathcal{R})$.

KMG3 If ι is an injection of a ring $\mathcal{R}$ into a field $\mathbb{K}$, then

$$\mathfrak{G}_{\mathcal{D}}(\iota) : \mathfrak{G}_{\mathcal{D}}(\mathcal{R}) \to \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$$

is injective. Furthermore, the groups $\mathfrak{U}_+(\mathcal{R})$ and $\mathfrak{U}_-(\mathcal{R})$ are the inverse images under $\mathfrak{G}_{\mathcal{D}}(\iota)$ of $\mathfrak{U}_+(\mathbb{K})$ and $\mathfrak{U}_-(\mathbb{K})$, respectively.

KMG4 For every field $\mathbb{K}$,

- the group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is generated by the images of $\eta(\mathbb{K})$ and $\varphi_i(\mathbb{K})$ for $i \in \underline{n}$,
- the groups $\mathfrak{U}_{\pm}(\mathbb{K})$ are pronilpotent, and
- the kernel of
$$\varphi_i(\mathbb{K}) : \mathfrak{SL}_2(\mathbb{K}) \to \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$$
is central in $\mathfrak{SL}_2(\mathbb{K})$ for all $i \in \underline{n}$.

KMG5 There is a homomorphism

$$\mathrm{Ad} : \mathfrak{G}_{\mathcal{D}}(\mathbb{C}) \to \mathrm{Aut}\, \mathfrak{g}_A(\mathbb{C})$$

whose kernel is contained in $\eta(\mathfrak{T}_{\mathcal{D}}(\mathbb{C}))$, such that for $z \in \mathbb{C}$,

$$\mathrm{Ad}\left(\varphi_i \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\, z e_i,$$

$$\mathrm{Ad}\left(\varphi_i\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\right) = \exp\mathrm{ad}\,(-zf_i),$$

and, for $t \in \mathfrak{T}_{\mathcal{D}}(\mathbb{C})$

$$\mathrm{Ad}\,(\eta(t))\,(e_i) = t(\tilde{\alpha}_i)e_i,$$

and

$$\mathrm{Ad}\,(\eta(t))\,(f_i) = -t(\tilde{\alpha}_i)f_i.$$

Thus Ad endows $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$ with an action on $\mathfrak{g}_A(\mathbb{C})$. We further assume that under this action the groups $\mathfrak{U}_+(\mathbb{C})$ and $\mathfrak{U}_-(\mathbb{C})$ are the derived groups of the stabilizers in $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$ of $\mathfrak{n}_+$ and $\mathfrak{n}_-$, respectively.

The following result will be useful in calculations involving the homomorphism Ad described in condition KMG5.

Lemma 2.4.1

Let $\mathfrak{g}_A(\mathbb{C})$ be a Kac-Moody algebra and x an element in $\mathfrak{g}_A(\mathbb{C})$ such that ad x *is locally nilpotent. Then for all $\theta \in$* Aut $\mathfrak{g}_A(\mathbb{C})$,

$$\theta\exp(\mathrm{ad}\,x)\theta^{-1} = \exp(\mathrm{ad}\,\theta x).$$

Proof

This is a direct extension of [Car72, Lemma 5.1.1] with the condition of local nilpotency replacing that of nilpotency. The original proof extends in the usual way. □

In order to investigate such Kac-Moody group systems, we introduce the *Steinberg group functor* $\mathfrak{St} = \mathfrak{St}(A)$, which depends only on the generalized Cartan matrix A, and not on the full root datum $\mathcal{D}$.

The Steinberg Group Functor

We define $\mathfrak{St}$ as the inductive limit of the functors $\mathfrak{U}_\alpha$ and $\mathfrak{U}_{\theta[\beta,\gamma]}$, where $\alpha \in \Phi^{re}$ and $\{\beta,\gamma\}$ runs over all prenilpotent pairs of roots, relative to all canonical injections

$$\mathfrak{U}_\alpha \to \mathfrak{U}_{\theta[\beta,\gamma]}$$

for $\alpha \in \theta[\beta,\gamma]$.

An alternative way of describing $\mathfrak{St}$ is as follows. For each root $\alpha \in \Phi^{re}_+$, let e_α and f_α be the root vectors corresponding to α and $-\alpha$, respectively. Associated to these root vectors are well-defined isomorphisms

$$x_\alpha : \mathfrak{Add} \to \mathfrak{U}_\alpha \quad \text{and} \quad x_{-\alpha} : \mathfrak{Add} \to \mathfrak{U}_{-\alpha}.$$

For every prenilpotent pair of roots $\{\alpha,\beta\}$, choose a total ordering on $\theta(\alpha,\beta)$. Then there are well-defined integers $C_{\alpha\beta\gamma}$ such that, for any ring $\mathcal{R}$ and any $r_1, r_2 \in \mathcal{R}$ the commutation relation

$$[x_\alpha(r_1), x_\beta(r_2)] = \prod_{\gamma=i\alpha+j\beta} x_\gamma\left(C_{\alpha\beta\gamma} r_1^i r_2^j\right)$$

holds inside $\mathfrak{U}_{\theta[\alpha,\beta]}(\mathcal{R})$, where γ runs over $\theta(\alpha,\beta)$ in the prescribed order. We then define $\mathfrak{St}(\mathcal{R})$ to be the quotient of the free product of the groups $\mathfrak{U}_\alpha(\mathcal{R})$ for $\alpha \in \Phi^{re}$ by the smallest normal subgroup containing all the elements

$$x_\beta(r_2)x_\alpha(r_1)x_\beta(-r_2)x_\alpha(-r_1).\prod_{\gamma=i\alpha+j\beta} x_\gamma\left(C_{\alpha\beta\gamma} r_1^i r_2^j\right), \qquad (2.4.1)$$

for $r_1, r_2 \in \mathcal{R}$, $\{\alpha,\beta\}$ a prenilpotent pair and γ as above.

For every $i \in \underline{n}$ the automorphism s_i of $\mathfrak{g}_A(\mathbb{C})$ introduced in §2.1 induces an automorphism of the functor $\mathfrak{St}$, which we shall again denote by s_i by a slight abuse of notation.

An Approximation to the Desired Group Functor

Set $\mathfrak{U}_i = \mathfrak{U}_{\alpha_i}$ and $\mathfrak{U}_{-i} = \mathfrak{U}_{-\alpha_i}$ for each $i \in \underline{n}$ and let

$$x_i : \mathfrak{Add} \to \mathfrak{U}_i \quad \text{and} \quad x_{-i} : \mathfrak{Add} \to \mathfrak{U}_{-i}$$

be the isomorphisms associated to e_i and f_i respectively. Denote by $n_i(r)$ the canonical image of the product

$$x_i(r)x_{-i}(r^{-1})x_i(r)$$

in $\mathfrak{St}(\mathcal{R})$ for $r \in \mathcal{R}^\times$ and $i \in \underline{n}$ and set $n_i = n_i(1)$.

We define a group functor $\tilde{\mathfrak{G}}_{\mathcal{D}}$ by taking $\tilde{\mathfrak{G}}_{\mathcal{D}}(\mathcal{R})$ to be the quotient of the free product $\mathfrak{St}(\mathcal{R}) * \mathfrak{T}_{\mathcal{D}}(\mathcal{R})$ of $\mathfrak{St}(\mathcal{R})$ and $\mathfrak{T}_{\mathcal{D}}(\mathcal{R})$ by the smallest normal subgroup containing the canonical images of the following elements, where $i \in \underline{n}$, $r \in \mathcal{R}$, $t \in \mathfrak{T}_{\mathcal{D}}(\mathcal{R})$, and $r^{\alpha_i^\vee}$ is defined as in §2.3:

$$t x_i(r) t^{-1} x_i(t(\alpha_i)r)^{-1}, \qquad (2.4.1)$$

$$n_i t n_i^{-1}(n_i(t))^{-1}, \qquad (2.4.1)$$

$$n_i(r)^{-1} n_i r^{\alpha_i^\vee} \qquad \text{for } r \neq 0, \qquad (2.4.1)$$

$$n_i u n_i^{-1} s_i(u)^{-1} \qquad \text{for } u \in \mathfrak{U}_\alpha(\mathcal{R}),\ \alpha \in \Phi^{re}. \qquad (2.4.1)$$

The canonical homomorphisms

$$\mathfrak{U}_\alpha(\mathcal{R}) \to \mathfrak{St}(\mathcal{R})$$

can be shown to be injective and we shall thus identify the groups $\mathfrak{U}_\alpha(\mathcal{R})$ with their canonical images in $\mathfrak{St}(\mathcal{R})$ and thence in $\tilde{\mathfrak{G}}_{\mathcal{D}}(\mathcal{R})$. Similarly, the image of n_i in $\mathfrak{St}(\mathcal{R})$ will also be denoted by n_i.

Let $i \in \underline{n}$, and $\alpha \in \Phi^{re}$. Suppose

$$e_\alpha \in \mathfrak{g}_\alpha \quad \text{and} \quad e_{r_i(\alpha)} \in \mathfrak{g}_{r_i(\alpha)}$$

are chosen so that $e_{r_i(\alpha)} = s_i(e_\alpha)$ and let

$$x_\alpha : \mathfrak{Add} \to \mathfrak{U}_\alpha(\mathcal{R}) \quad \text{and} \quad x_{r_i(\alpha)} : \mathfrak{Add} \to \mathfrak{U}_{r_i(\alpha)}(\mathcal{R})$$

be the corresponding isomorphisms. Using equations (2.4.1)–(2.4.1) we can deduce that the element

$$n_i(r)x_\alpha(r')n_i(r)^{-1}x_{r_i(\alpha)}\left(r^{\alpha_i^\vee(\hat{\alpha})}r'\right)^{-1} \quad \text{for } r, r' \in \mathcal{R} \tag{2.4.1}$$

of $\mathfrak{St}(\mathcal{R})$ is an element of the kernel of the canonical map

$$\mathfrak{St}(\mathcal{R}) \to \tilde{\mathfrak{G}}_{\mathcal{D}}(\mathcal{R})$$

and so is a relation in $\tilde{\mathfrak{G}}_{\mathcal{D}}(\mathcal{R})$.

Let Θ_i denote the set of all roots α such that the elements of the form (2.4.1) are already equal to the identity in $\mathfrak{St}(\mathcal{R})$. That is to say, Θ_i consists of all roots α for which the relations (2.4.1), hence in particular the relations (2.4.1), are consequences of the relations (2.4.1). The following result, whose proof can be found in [Tit87b, §3.7], highlights the relevance of the set Θ_i.

LEMMA 2.4.2

Suppose $\Theta_i = \Phi^{re}$ for all $i \in \underline{n}$. Then the kernel of the canonical homomorphism

$$\mathfrak{St}(\mathcal{R}) \to \tilde{\mathfrak{G}}_{\mathcal{D}}(\mathcal{R})$$

is central. Furthermore, if $\mathcal{D}$ is a simply-connected root datum then the above homomorphism can be shown to be surjective and so we can deduce that $\mathfrak{St}(\mathcal{R})$ is a central extension of $\tilde{\mathfrak{G}}_{\mathcal{D}_{sc}}$.

Thus it would be interesting to know when $\Theta_i = \Phi^{re}$ for all $i \in \underline{n}$. The following result, from the same source as above, provides a partial answer to this question.

LEMMA 2.4.3

Suppose that all m_{ij}, defined in Table 1.1.31, are finite. Then, for any given i,

$$\Theta_i = \begin{cases} \Phi^{re} \setminus \{\pm\alpha_i\} & \text{if } m_{ij} = 2 \text{ for all } j \neq i, \\ \Phi^{re} & \text{otherwise.} \end{cases}$$

The Main Theorem on Kac-Moody Group Functors

Let $\tilde{\mathfrak{U}}_+(\mathcal{R})$ (respectively $\tilde{\mathfrak{U}}_-(\mathcal{R})$) denote the subgroup of $\tilde{\mathfrak{G}}_\mathcal{D}(\mathcal{R})$ generated by all $\mathfrak{U}_\alpha(\mathcal{R})$ for $\alpha \in \Phi_+^{re}$ (respectively $\alpha \in \Phi_-^{re}$), and let x_+ and x_- be the homomorphisms

$$r \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad r \mapsto \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

of $\mathfrak{Add}$ in $\mathfrak{SL}_2$, respectively.

THEOREM 2.4.4
Let $\mathfrak{F} = \left(\mathfrak{G}_\mathcal{D}, \mathfrak{U}_+, \mathfrak{U}_-, (\varphi_i)_{i\in\underline{n}}, \eta\right)$ be a Kac-Moody group system corresponding to some root datum $\mathcal{D} = \mathcal{D}(A)$, as defined at the beginning of §2.4.

1. *There exists a unique homomorphism of group functors*

$$\pi : \tilde{\mathfrak{G}}_\mathcal{D} \to \mathfrak{G}_\mathcal{D}$$

such that

- *the canonical map*

$$\mathfrak{T} \to \tilde{\mathfrak{G}}_\mathcal{D}$$

followed by π coincides with η,

- *the composed map*

$$\mathfrak{Add} \to \mathfrak{U}_{\pm i} \to \tilde{\mathfrak{G}}_\mathcal{D} \to \mathfrak{G}_\mathcal{D}$$

is the same as $\varphi_i \circ x_\pm$, and

- *$\pi(\tilde{\mathfrak{U}}_\pm(\mathcal{R})) \subseteq \mathfrak{U}_\pm(\mathcal{R})$ for all rings $\mathcal{R}$, with equality holding whenever $\mathcal{R}$ is a field.*

2. *If $\mathbb{K}$ is a field, then*

$$\pi(\mathbb{K}) : \tilde{\mathfrak{G}}_\mathcal{D}(\mathbb{K}) \to \mathfrak{G}_\mathcal{D}(\mathbb{K})$$

is an isomorphism.

Using highest weight modules and the $\mathbb{Z}$-form $\mathcal{U}_\mathbb{Z}$ of $\mathcal{U}(\mathfrak{g}_A(\mathbb{C}))$ described in §2.1, we can show that there exists a Kac-Moody group system $\mathfrak{F}$ for all root data $\mathcal{D} = \mathcal{D}(A)$.

2.5 Kac-Moody Groups over Fields

In the preceding sections, we have shown that a Kac-Moody Group $\mathfrak{G}_\mathcal{D}(\mathbb{K})$ over a field $\mathbb{K}$ can be constructed by means of generators and relations. We summarize this construction with more familiar notation and state a few properties satisfied by the groups $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. Unless otherwise stated, the proofs of these properties can be found in [Tit87b].

Given a generalized Cartan matrix $A = (A_{ij})_{i,j\in\underline{n}}$, with real root system Φ^{re}, fundamental root system Π and Weyl group W, and a root datum

$$\mathcal{D}\left(\Lambda, \{\tilde{\alpha}_i\}_{i\in\underline{n}}, \{\alpha_i^\vee\}_{i\in\underline{n}}\right)$$

associated to A we make the following definitions;

$$\begin{array}{lcl} H & := & \Lambda^* \otimes_{\mathbb{Z}} \mathbb{K}^\times \\ h_i(\xi) & := & \alpha_i^\vee \otimes \xi \\ x_i(\mu) & := & x_{\alpha_i}(\mu) \\ x_{-i}(\mu) & := & x_{-\alpha_i}(\mu) \\ n_i(\xi) & := & x_i(\xi)x_{-i}(\xi^{-1})x_i(\xi) \\ n_i & := & n_i(1) \end{array}$$

where $\mathbb{K}^\times$ is the multiplicative group of $\mathbb{K}$, $\xi \in \mathbb{K}^\times$, $\alpha_i^\vee \in \mathcal{D}$, $\alpha_i \in \Pi$, and $\mu \in \mathbb{K}$. Note that H is generated by elements of the form

$$\lambda \otimes \xi$$

where $\lambda \in \Lambda^*$ and $\xi \in \mathbb{K}^\times$. We define an action of W on H by

$$r_i(\lambda \otimes \xi) := r_i(\lambda) \otimes \xi$$

for $\lambda \in \Lambda^*$ and $\xi \in \mathbb{K}^*$. We can also define a map $\alpha_i : H \to \mathbb{K}^\times$ for $i \in \underline{n}$ by

$$\alpha_i(\lambda \otimes \xi) := \xi^{\lambda(\tilde{\alpha}_i)}$$

for $\lambda \in \Lambda^*$ and $\xi \in \mathbb{K}^\times$. Now, since every element $\alpha \in \Phi^{re}$ has an expression $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in \underline{n}$, we may define $\alpha : H \to \mathbb{K}^\times$ by

$$\alpha(h) = \alpha_i(w^{-1}h)$$

for $h \in H$. This definition is unambiguous.

We can now say that the Kac-Moody group $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is given by the following generators and relations;

$$\begin{array}{lllll} \text{Generators :} & h \in H \quad , \quad x_\alpha(\mu) \;\; \alpha \in \Phi^{re},\; \mu \in \mathbb{K} & & & \\ \text{Relations :} & \text{Relations in } H & & & \\ & x_\alpha(\mu)x_\alpha(\nu) & = & x_\alpha(\mu+\nu) & \\ & hx_i(\mu)h^{-1} & = & x_i(\alpha_i(h)\mu) & \\ & h_i(\xi) & = & n_i(\xi)n_i^{-1} & \\ & [x_\alpha(\mu), x_\beta(\nu)] & = & \prod\limits_{\substack{\gamma = i\alpha + j\beta \\ i,j\in\mathbb{N}}} x_\gamma(C_{\alpha\beta\gamma}\mu^i\nu^j), & \gamma\in\Phi^{re},\ \alpha\neq\pm\beta,\ \{\alpha,\beta\} \text{ prenilpotent} \\ & n_i h n_i^{-1} & = & r_i(h) & \\ & n_i x_\alpha(\mu) n_i^{-1} & = & x_{r_i(\alpha)}(\eta_{i,\alpha}\mu) & \end{array}$$

where throughout $\xi \in \mathbb{K}^\times$, $\mu, \eta \in \mathbb{K}$, and $\alpha, \beta \in \Phi^{re}$. The coefficients $C_{\alpha\beta\gamma}$ are the integers defined earlier depending on α, β, and the order in which the x_γ are taken, but not on μ or η, and $C_{\alpha,\beta,\alpha+\beta} = N_{\alpha\beta}$ where $[e_\alpha, e_\beta] = N_{\alpha\beta}e_{\alpha+\beta}$. The coefficients $\eta_{i,\alpha}$ are the integers such that

$$s_i(e_\alpha) = \eta_{i,\alpha}e_{r_i(\alpha)}$$

where $s_i = r_i^{ad}$ is the automorphism of $\mathfrak{g}_A(\mathbb{C})$ introduced in Lemma 1.1.27.

LEMMA 2.5.1

With the above notation, the relations

$$hx_\alpha(\mu)h^{-1} = x_\alpha(\alpha(h)\mu) \quad \textit{and} \quad n_i(\xi)x_\alpha(\mu)n_i(\xi)^{-1} = x_{r_i(\alpha)}\left(\eta_{i,\alpha}\xi^{-\langle\alpha,\alpha_i^\vee\rangle}\mu\right)$$

for $\alpha \in \Phi^{re}$, $\mu \in \mathbb{K}$ *and* $\xi \in \mathbb{K}^\times$ *are consequences of those already given above.*

Proof

We first prove the result for the first relation. Since $\Phi^{re} = W(\Pi)$, it is sufficient to check that

$$hx_{r_i(\alpha)}(\mu)h^{-1} = x_{r_i(\alpha)}(r_i(\alpha)(h)\mu) \text{ given that } hx_\alpha(\mu)h^{-1} = x_\alpha(\alpha(h)\mu).$$

But we have,

$$\begin{aligned} hx_{r_i(\alpha)}(\mu)h^{-1} &= hn_ix_\alpha(\mu)n_i^{-1}h^{-1} \\ &= n_i(n_i^{-1}hn_i)x_\alpha(\mu)(n_i^{-1}h^{-1}n_i)n_i^{-1} \\ &= n_ir_i(h)x_\alpha(\mu)r_i(h)^{-1}n_i^{-1} \\ &= n_ix_\alpha(\alpha_i(r_i(h))\mu)n_i^{-1} \\ &= x_{r_i(\alpha)}(\alpha_i(r_i(h))\mu). \end{aligned}$$

So the problem is reduced to showing that $\alpha(r_i(h)) = r_i(\alpha)(h)$. It is sufficient to do this in the case $h = \lambda \otimes \xi$, so we get

$$\begin{aligned} \alpha(r_i(\lambda \otimes \xi)) &= \alpha(r_i(\lambda) \otimes \xi) \\ &= \xi^{\langle\alpha, r_i(\lambda)\rangle}, \end{aligned}$$

and

$$r_i(\alpha)(\lambda \otimes \xi) = \xi^{\langle r_i(\alpha), \lambda\rangle}.$$

So the problem is further reduced to showing that $\langle\alpha, r_i(\lambda)\rangle = \langle r_i(\alpha), \lambda\rangle$, but

$$\begin{aligned} \langle\alpha, r_i(\lambda)\rangle &= \langle\alpha, \lambda - \langle\alpha_i, \lambda\rangle\alpha_i^\vee\rangle \\ &= \langle\alpha, \lambda\rangle - \langle\alpha_i, \lambda\rangle\langle\alpha, \alpha_i^\vee\rangle \\ &= \langle\alpha - \langle\alpha, \alpha_i^\vee\rangle\alpha_i, \lambda\rangle \\ &= \langle r_i(\alpha), \lambda\rangle. \end{aligned}$$

Thus $hx_\alpha(\mu)h^{-1} = x_\alpha(\alpha(h)\mu)$ is a consequence of the relations in the presentation of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$.

For the second part of the result we note that

$$\begin{aligned} n_i(\xi)x_\alpha(\mu)n_i(\xi)^{-1} &= h_i(\xi)n_i x_\alpha(\mu)n_i^{-1}h_i(\xi)^{-1} \\ &= h_i(\xi)x_{r_i(\alpha)}(\eta_{i,\alpha}\mu)h_i(\xi)^{-1} \\ &= x_{r_i(\alpha)}(\eta_{i,\alpha}r_i(\alpha)(h_i(\xi))\mu) \\ &= x_{r_i(\alpha)}(\eta_{i,\alpha}\xi^{\langle r_i(\alpha),\alpha_i^\vee\rangle}\mu) \\ &= x_{r_i(\alpha)}(\eta_{i,\alpha}\xi^{-\langle\alpha,\alpha_i^\vee\rangle}\mu). \end{aligned}$$

□

Note that the elements of the form $x_\alpha(\mu)$ for a given $\alpha \in \Phi^{re}$ and any $\mu \in \mathbb{K}$ lie in a subgroup

$$X_\alpha = \{x_\alpha(\mu) : \mu \in \mathbb{K}\}$$

of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. Furthermore, there is a homomorphism

$$\varphi : \mathfrak{SL}_2(\mathbb{K}) \to \langle X_\alpha , X_{-\alpha}\rangle$$

under which

$$\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \to x_\alpha(\mu) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix} \to x_{-\alpha}(\mu).$$

Consider the group $B_\alpha = \langle X_\alpha, H\rangle$. We have

$$H \subseteq N_{\mathfrak{G}_{\mathcal{D}}(\mathbb{K})}(X_\alpha)$$

for each $\alpha \in \Phi^{re}$, whence

$$B_\alpha = X_\alpha \ltimes H.$$

Let $N = \langle H , n_i : i \in \underline{n}\rangle$.

Lemma 2.5.2

1. *For any $w \in W$, there exists an element $n \in N$ such that*

$$nB_\alpha n^{-1} = B_{w(\alpha)}$$

   for all $\alpha \in \Phi^{re}$.

2. *If α and α' are two distinct roots, $B_\alpha \neq B_{\alpha'}$.*

3. *There exists a unique homomorphism $\omega : N \to W$ such that for $n \in N$ and $\alpha \in \Phi^{re}$,*

$$nB_\alpha n^{-1} = B_{\omega(n)(\alpha)}.$$

   Furthermore, the kernel of ω is H.

We define

$$U_+ = \langle X_\alpha : \alpha \in \Phi^{\mathrm{re}}_+ \rangle \qquad \text{and} \qquad U_- = \langle X_\alpha : \alpha \in \Phi^{\mathrm{re}}_- \rangle.$$

These are subgroups of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ and $U_+ \cap U_- = \{1\}$.

For each $\alpha \in \Phi$, $H \subseteq \mathcal{N}_{\mathfrak{G}_{\mathcal{D}}(\mathbb{K})}(X_\alpha)$ and so

$$H \subseteq \mathcal{N}_{\mathfrak{G}_{\mathcal{D}}(\mathbb{K})}(U_+) \qquad \text{and} \qquad H \subseteq \mathcal{N}_{\mathfrak{G}_{\mathcal{D}}(\mathbb{K})}(U_-).$$

Consequently we have subgroups $B_+ = U_+ \ltimes H$ and $B_- = U_- \ltimes H$ of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ with $B_+ \cap B_- = H$.

Theorem 2.5.3

The pairs (B_+, N) and (B_-, N) are saturated (B, N)-pairs of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. Both (B, N)-pairs satisfy

$$B_+ \cap N = H = B_- \cap N,$$

whence we see that they share the same Weyl group, $W \cong N/H$. Furthermore

$$B_+ \cap B_- = H.$$

When W is finite there is a unique $w_0 \in W$ whose length is maximal. An element $n \in N$ such that $\pi(n) = w_0$ will conjugate B_+ into B_- and vice versa. However, when W is infinite no such word of maximal length exists and in this case no conjugacy exists between the two (B, N)-pairs.

Thus, by Proposition 2.1.7,

$$B_+ N B_+ = \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) = B_- N B_-$$

are two Bruhat decompositions of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. However, since $B_\pm = U_\pm \ltimes H$ and $H \subseteq N$, we can further refine this decomposition to obtain

$$U_+ N U_+ = \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) = U_- N U_-.$$

Proposition 2.5.4

The map $w \mapsto B_- w B_+$ is a bijection of W onto $B_- \backslash \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) / B_+$.

This gives rise to a *Birkhoff decomposition*

$$\mathfrak{G}_{\mathcal{D}}(\mathbb{K}) = B_- N B_+$$

of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$, which we can again further refine to

$$\mathfrak{G}_{\mathcal{D}}(\mathbb{K}) = U_- N U_+.$$

We now consider some of the consequences of condition KMG5. We thus consider the group $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$ and the homomorphism

$$\mathrm{Ad} : \mathfrak{G}_{\mathcal{D}}(\mathbb{C}) \to \mathrm{Aut}\, \mathfrak{g}_A(\mathbb{C}).$$

Recall that

$$\mathrm{Ad}\left(x_{\alpha_j}(z)\right) = \exp \mathrm{ad}\, z e_{\alpha_j} \qquad \text{and} \qquad \mathrm{Ad}\left(x_{-\alpha_j}(z)\right) = \exp \mathrm{ad}\, z f_{\alpha_j}$$

for all $\alpha_j \in \Pi$. We note that this implies that $\mathrm{Ad}(n_i) = s_i$ for all $i \in \underline{n}$.

Lemma 2.5.5

With the above notation

$$\mathrm{Ad}\,(x_\alpha(z)) = \exp \mathrm{ad}\, z e_\alpha$$

and

$$\mathrm{Ad}\,(x_{-\alpha}(z)) = \exp \mathrm{ad}\, z f_\alpha$$

for all $\alpha \in \Phi_+^{re}$ *and* $z \in \mathbb{C}$.

Proof

Recall that each $\alpha \in \Phi^{re}$ is the image of at least one fundamental root under some element of the Weyl group W and that W is generated by the elements r_i. Suppose $\alpha = w(\alpha_j)$ for some $j \in \underline{n}$ and some $w \in W$. We prove the result by induction on the length of w. The result is true for all fundamental roots by condition KMG5.

Suppose now that $\alpha \in \Phi_+^{re} \setminus \Pi$ and $\alpha = w(\alpha_j) = r_k w'(\alpha_j)$ for some $k \in \underline{n}$ and some $w' \in W$ with $\ell(w') < \ell(w)$. Let $\alpha' = w'(\alpha_j)$. Suppose $\alpha' \in \Phi_-^{re}$. Then $\alpha' \in \Phi_-^{re}$ but $r_k(\alpha') \in \Phi_+^{re}$ and we use Lemma 1.1.25 to conclude that $\alpha' = -\alpha_k$ and therefore $\alpha = \alpha_k$ contradicting our choice of α. Thus $\alpha' \in \Phi_+^{re}$ and

$$\mathrm{Ad}\,(x_{\alpha'}(z)) = \exp \mathrm{ad}\, z e_{\alpha'} \qquad \text{and} \qquad \mathrm{Ad}\,(x_{-\alpha'}(z)) = \exp \mathrm{ad}\, z f_{\alpha'}$$

by induction. Now, the defining relations imply that

$$x_\alpha(z) = n_k x_{\alpha'}(\eta_{k,\alpha'} z) n_k^{-1} \qquad \text{and} \qquad x_{-\alpha}(z) = n_k x_{-\alpha'}(\eta_{k,-\alpha'} z) n_k^{-1}$$

and so we deduce from Lemma 2.4.1 and the fact $\mathrm{Ad}(n_k) = s_k$ that

$$\begin{aligned} \mathrm{Ad}\,(x_\alpha(z)) &= \exp \mathrm{ad}\, s_k(\eta_{k,\alpha'} z e_{\alpha'}) \\ &= \exp \mathrm{ad}\, \eta_{k,\alpha'} z s_k(e_{\alpha'}) \\ &= \exp \mathrm{ad}\, z e_\alpha \end{aligned}$$

and similarly $\mathrm{Ad}\,(x_{-\alpha}(z)) = \exp \mathrm{ad}\, z f_\alpha$. □

Examples

We begin by considering some refinements of the description of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ in terms of generators and relations for particular root data $\mathcal{D}$. We note that since

$$n_i x_\alpha(\mu) n_i^{-1} = x_{r_i(\alpha)}(\eta_{i,\alpha}\mu)$$

where $\eta_{i,\alpha} \in \{1, -1\}$, we may omit the elements $x_\alpha(\mu)$ where $\alpha \notin \pm\Pi$ from the list of generators.

Example 2.5.6

Consider the group associated to the simply-connected root datum $\mathcal{D}_{sc}$ introduced in Example 2.2.1. Since

$$\Lambda^* = \bigoplus_{i \in \underline{n}} \mathbb{Z}\alpha_i^\vee,$$

H is in fact generated by elements of the form

$$h_i(\xi) = \alpha_i^\vee \otimes \xi$$

for $i \in \underline{n}$ and $\xi \in \mathbb{K}^\times$. However, one of the relations in $\mathfrak{G}_{\mathcal{D}_{sc}}(\mathbb{K})$ is that

$$h_i(\xi) = n_i(\xi) n_i^{-1}$$

and

$$n_i(\xi) = x_{\alpha_i}(\xi) x_{-\alpha_i}(\xi^{-1}) x_{\alpha_i}(\xi).$$

Thus $\mathfrak{G}_{\mathcal{D}_{sc}}(\mathbb{K})$ is generated by the elements

$$x_{\alpha_i}(\mu) \quad \text{and} \quad x_{-\alpha_i}(\mu)$$

for $i \in \underline{n}$ and $\mu \in \mathbb{K}$. ◇

Example 2.5.7

Suppose $A = (A_{ij})_{i,j \in \underline{n}}$ is a Cartan matrix and let $\tilde{A} = (A_{ij})_{i,j \in \underline{n}_0}$ be the extended Cartan matrix obtained from it. Consider the group associated to the adjoint root datum, $\mathcal{D}_{ad}(\tilde{A})$ introduced in Example 2.2.2. Recall that

$$\mathcal{D}_{ad}(\tilde{A}) = \left(\sum_{i \in \underline{n}_0} \mathbb{Z}\tilde{\alpha}_i, \{\tilde{\alpha}_i\}_{i \in \underline{n}_0}, \{\alpha_i^\vee\}_{i \in \underline{n}_0} \right)$$

where

$$\tilde{\alpha}_i = (A_{ij})_{j \in \underline{n}_0} \in \mathbb{Z}^{n+1}.$$

Now, since $\tilde{A}$ is singular, its columns are linearly dependent. However, since $\tilde{A}$ has column rank n and no two of its first n columns are linearly dependent we see that

$$\Lambda = \bigoplus_{i \in \underline{n}} \mathbb{Z}\tilde{\alpha}_i.$$

Thus Λ^* has a basis $\{\varpi_i^\vee\}_{i\in\underline{n}}$ defined by

$$\varpi_i^\vee(\tilde{\alpha}_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and so

$$\{h_{\varpi_i^\vee}(\xi) := \varpi_i^\vee \otimes \xi : i \in \underline{n}, \xi \in \mathbb{K}^\times\}$$

is a basis for H. We shall refer to $\{\varpi_i^\vee\}_{i\in\underline{n}}$ as the *fundamental coweights* of $\mathcal{D}_{ad}(\tilde{A})$. Note that we can express $\alpha_i^\vee$ in terms of this basis, namely

$$\alpha_i^\vee = \sum_{j\in\underline{n}} A_{ij}\varpi_j^\vee$$

for each $i \in \underline{n}_0$. Consider the n equations we obtain if we consider the above expression for $i \in \underline{n}$. The matrix for these equations is A. Since A is invertible, we may solve the system of equations and thus express the elements $\{\varpi_i^\vee\}_{i\in\underline{n}}$ as $\mathbb{Q}$-linear combinations of the elements $\{\alpha_i^\vee\}_{i\in\underline{n}}$. Suppose that, for each $i \in \underline{n}$,

$$\varpi_i^\vee = \sum_{j\in\underline{n}} B_{ij}\alpha_j^\vee$$

where $B_{ij} \in \mathbb{Q}$. Then

$$h_{\varpi_i^\vee}(\xi) = \sum_{j\in\underline{n}} \left(B_{ij}\alpha_j^\vee \otimes \xi\right)$$

for all $\xi \in \mathbb{K}^\times$. If $B_{ij} \in \mathbb{Z}$ for all $i, j \in \underline{n}$ then

$$h_{\varpi_i^\vee}(\xi) = \sum_{j\in\underline{n}} \alpha_j^\vee \otimes \xi^{B_{ij}}$$

is well-defined and in such cases we may omit $h \in H$ from our list of generators.

However, in general, the elements

$$x_{\alpha_i}(\mu), x_{-\alpha_i}(\mu), h_{\varpi_j^\vee}(\xi)$$

for $i \in \underline{n}_0$, $j \in \underline{n}$, $\mu \in \mathbb{K}$ and $\xi \in \mathbb{K}^\times$ generate $\mathfrak{G}_{\mathcal{D}_{ad}}(\mathbb{C})$. ◇

Example 2.5.8

Suppose A and $\tilde{A}$ are as in the previous example. Consider the group associated to the minimal adjoint root datum, $\mathcal{D}_m(\tilde{A})$, introduced in Example 2.2.3. Recall that

$$\mathcal{D}_m(\tilde{A}) = \left(\bigoplus_{i\in \underline{n}_0} \tilde{\alpha}_i, \{\tilde{\alpha}_i\}_{i\in\underline{n}_0}, \{\alpha_i^\vee\}_{i\in\underline{n}_0}\right).$$

Thus $\Lambda = \bigoplus_{i\in\underline{n}_0} \tilde{\alpha}_i$ and Λ^* has a basis $\{\varpi_i^\vee\}_{i\in\underline{n}_0}$, given by

$$\varpi_i^\vee(\tilde{\alpha}_j) = \delta_{ij},$$

which we shall refer to as the *fundamental coweights* of $\mathcal{D}_m(\tilde{A})$. Hence H is generated by the elements

$$h_{\varpi_i^\vee}(\xi) = \varpi_i^\vee \otimes \xi$$

for $i \in \underline{n}_0$ and $\xi \in \mathbb{K}^\times$ and the set

$$\{h_{\varpi_i^\vee}(\xi), x_{\alpha_i}(\mu), x_{-\alpha_i}(\mu) : i \in \underline{n}_0, \xi \in \mathbb{K}^\times, \mu \in \mathbb{K}\}$$

is a set of generators for $\mathfrak{G}_{\mathcal{D}_m}(\mathbb{C})$.

An important feature of the minimal adjoint root datum is that the action of $\mathfrak{G}_{\mathcal{D}_m}(\mathbb{C})$ on the Kac-Moody algebra $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ induced by condition KMG5 is faithful. For, suppose there is a non-trivial element $h \in H$ which acts trivially on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$. Such an element $h \in H$ has a unique expression of the form

$$h = h_{\varpi_0^\vee}(\xi_0)h_{\varpi_1^\vee}(\xi_1)\cdots h_{\varpi_n^\vee}(\xi_n)$$

for some $\xi_i \in \mathbb{K}^\times$, $i \in \underline{n}_0$. Since the action of h on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ is trivial we must have

$$\xi_0^{\langle\alpha,\varpi_0^\vee\rangle}\xi_1^{\langle\alpha,\varpi_1^\vee\rangle}\cdots\xi_n^{\langle\alpha,\varpi_n^\vee\rangle} = 1$$

for all $\xi_i \in \mathbb{K}^\times$, $i \in \underline{n}_0$, and $\alpha \in \Phi^{re}(\tilde{A})$. Since h is non-trivial by assumption, there must be at least one $k \in \underline{n}_0$ for which $\xi_k \neq 1$. Choose k so that it is minimal with respect to this property. Thus our original condition on the ξ_i, $i \in \underline{n}_0$ yields the equation

$$\xi_k^{\langle\alpha,\varpi_k^\vee\rangle} = \xi_n^{-\langle\alpha,\varpi_n^\vee\rangle}\cdots\xi_{k+1}^{-\langle\alpha,\varpi_{k+1}^\vee\rangle}$$

for all $\alpha \in \Phi^{re}(\tilde{A})$. However, if we now consider the particular case where $\alpha = \alpha_k$, this equation implies the triviality of ξ_k, which is a contradiction.

Thus we have shown that H, and hence $\mathfrak{G}_{\mathcal{D}_m}(\mathbb{C})$, acts faithfully on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$. ◇

In the following examples we use the notation $\mathfrak{G}^A_{ad}$ and $\mathfrak{G}^A_{sc}$ to denote the group schemes corresponding to $\mathcal{D}_{ad}(A)$ and $\mathcal{D}_{sc}(A)$ respectively.

EXAMPLE 2.5.9

When A is a Cartan matrix, the definition given coincides with that of Chevalley-Demazure group schemes. Thus if $A = A_n$ we see that

$$\mathfrak{G}_{sc}^{A_n}(\mathbb{K}) \cong \mathfrak{SL}_{n+1}(\mathbb{K}) \quad \text{and} \quad \mathfrak{G}_{ad}^{A_n}(\mathbb{K}) \cong \mathfrak{PGL}_{n+1}(\mathbb{K}),$$

where $\mathfrak{PGL}_{n+1}$ denotes the projective general linear group functor. Similarly, if $A = D_n$, then

$$\mathfrak{G}_{sc}^{D_n}(\mathbb{K}) \cong \mathfrak{Spin}_{2n}(\mathbb{K}) \quad \text{and} \quad \mathfrak{G}_{ad}^{D_n}(\mathbb{K}) \cong P(\mathfrak{CO}_{2n}(\mathbb{K})^o),$$

where $\mathfrak{Spin}_{2n}(\mathbb{K})$ denotes the spin group and $P(\mathfrak{CO}_{2n}(\mathbb{K})^o)$ denotes the projective group of the connected component of the conformal orthogonal group of degree $2n$. We note that, in the case of D_n, the special orthogonal group of degree $2n$ appears as the Kac-Moody group corresponding to a root datum which is neither adjoint nor simply-connected. ◇

EXAMPLE 2.5.10

Consider the extended Cartan matrix of type $\tilde{A}_n$. It can be shown that there is a normal subgroup $\mathcal{K}$ of $\mathfrak{G}_{sc}^{\tilde{A}_n}(\mathbb{K})$ isomorphic to $\mathbb{K}^\times$ such that

$$\mathfrak{G}_{sc}^{\tilde{A}_n}(\mathbb{K})/\mathcal{K} \cong \mathfrak{SL}_{n+1}(\mathbb{K}[t, t^{-1}]).$$

This is in fact a special case of the following example. ◇

EXAMPLE 2.5.11

Let A be a Cartan matrix and denote by $\tilde{A}$ the corresponding extended Cartan matrix. Then there is a normal subgroup $\mathcal{K}$ of $\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$ isomorphic to $\mathbb{K}^\times$ such that

$$\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})/\mathcal{K} \cong \mathfrak{G}_{sc}^{A}(\mathbb{K}[t, t^{-1}]).$$

◇

Some Generalized Isogenies

Suppose now that $A = (A_{ij})_{i,j\in n}$ is an arbitrary generalized Cartan matrix. Let $\Phi = \Phi^{re}(A)$ and suppose Π is a set of fundamental roots for Φ. Denote by $\mathfrak{G}_m$, $\mathfrak{G}_{ad}$ and $\mathfrak{G}_{sc}$ the Kac-Moody group functors $\mathfrak{G}_{\mathcal{D}_m(A)}$, $\mathfrak{G}_{\mathcal{D}_{ad}(A)}$ and $\mathfrak{G}_{\mathcal{D}_{sc}(A)}$ respectively. Suppose that

$$\mathfrak{G}_m(\mathbb{K}) = \langle x_\alpha(\mu), h_{\varpi_i^\vee}(\xi) : \alpha \in \Phi, \mu \in \mathbb{K}, i \in n, \xi \in \mathbb{K}^\times \rangle$$

and

$$\mathfrak{G}_{sc}(\mathbb{K}) = \langle x'_\alpha(\mu) : \alpha \in \Phi, \mu \in \mathbb{K} \rangle.$$

We note from §2.4 that the subgroups $\mathfrak{U}_{\pm}(\mathbb{K}) \subseteq \mathfrak{G}_{sc}(\mathbb{K})$ are isomorphic to the subgroups $\mathfrak{U}_{\pm}(\mathbb{K}) \subseteq \mathfrak{G}_{m}(\mathbb{K})$.

We define a map $\imath(\mathbb{K})$ on the generators of $\mathfrak{G}_{sc}(\mathbb{K})$ by letting

$$\imath(\mathbb{K}) : x'_{\alpha}(\mu) \to x_{\alpha}(\mu)$$

for each $\alpha \in \Phi$ and $\mu \in \mathbb{K}$.

Lemma 2.5.12

The map $\imath(\mathbb{K})$ extends to a group homomorphism

$$\imath(\mathbb{K}) : \mathfrak{G}_{sc}(\mathbb{K}) \to \mathfrak{G}_{m}(\mathbb{K}).$$

Proof

In order for the map $\imath(\mathbb{K})$ to extend to a group homomorphism it must preserve the defining relations of $\mathfrak{G}_{sc}(\mathbb{K})$. Thus we must show that the relations

$$\begin{array}{rcll} h'_i(\xi)h'_i(\zeta) & = & h'_i(\xi\zeta) & \\ h'_i(\xi)h'_j(\zeta) & = & h'_j(\zeta)h'_i(\xi) & \\ x'_{\alpha}(\mu)x'_{\alpha}(\nu) & = & x'_{\alpha}(\mu+\nu) & \\ h'x'_i(\mu)h'^{-1} & = & x'_i(\alpha_i(h')\mu) & \\ [x'_{\alpha}(\mu), x'_{\beta}(\nu)] & = & \prod_{\substack{\gamma = i\alpha + j\beta \\ i,j\in\mathbb{N}}} x'_{\gamma}(C_{\alpha\beta\gamma}\mu^i\nu^j), & \gamma\in\Phi^{re},\ \alpha\neq\pm\beta, \\ & & & \{\alpha,\beta\} \text{ prenilpotent} \\ n'_ih'n'^{-1}_i & = & r_i(h') & \text{and} \\ n'_ix'_{\alpha}(\mu)n'^{-1}_i & = & x'_{r_i(\alpha)}(\eta_{i,\alpha}\mu) & \end{array}$$

are preserved, where

$$h'_i(\xi) := n'_i(\xi)n'^{-1}_i$$

and

$$n'_i(\xi) := x'_{\alpha_i}(\xi)x'_{-\alpha_i}(\xi^{-1})x'_{\alpha_i}(\xi).$$

We note that $\imath(\mathbb{K})(n'_i(\xi)) = n_i(\xi)$ and $\imath(\mathbb{K})(h'_i(\xi)) = h_i(\xi)$, so that the first two relations are clearly preserved.

Since the restriction of $\imath(\mathbb{K})$ to $\mathfrak{U}'_{\pm}(\mathbb{K})$ is an isomorphism onto $\mathfrak{U}_{\pm}(\mathbb{K})$, it follows that any relation involving only positive or negative roots is preserved by $\imath(\mathbb{K})$. Furthermore, the construction involving $\mathfrak{St}(\mathbb{K})$ of all Kac-Moody groups associated to A over fields implies that the structure constants $C_{\alpha\beta\gamma}$ are the same in the cases in question. Hence the commutator relations are preserved.

Similarly, the relations of the form

$$n'_ix'_{\alpha}(\mu)n'^{-1}_i = x'_{r_i(\alpha)}(\eta_{i,\alpha}\mu)$$

are preserved since the constants $\eta_{i,\alpha}$ are also independent of the particular root datum in question. Since the action of the Weyl group is also common to all Kac-Moody groups associated to A, the relation

$$n'_ih'n'^{-1}_i = r_i(h')$$

is also preserved.

Finally, it remains to show that the relation

$$h'x'_i(\mu)h'^{-1} = x'_i(\alpha_i(h')\mu)$$

is preserved, for which it is sufficient to show that the relation

$$h'_j(\xi)x'_i(\mu)h'_j(\xi)^{-1} = x'_i(\xi^{\langle \tilde{\alpha}_i, \alpha_j^\vee \rangle}\mu)$$

is preserved. This is the case since all the root data in question are associated to A.

□

Analogous arguments show that the map

$$\imath_{sa}(\mathbb{K}) : \mathfrak{G}_{sc}(\mathbb{K}) \to \mathfrak{G}_{ad}(\mathbb{K})$$

defined in a similar way is also a group homomorphism.

Similarly, we can also construct a group homomorphism

$$\imath_{am}(\mathbb{K}) : \mathfrak{G}_{ad}(\mathbb{K}) \to \mathfrak{G}_{m}(\mathbb{K})$$

though its definition varies with the type of the GCM A.

Chapter 3

Automorphisms of Kac-Moody Groups

The aims of this chapter are twofold. Firstly, we describe the work of Hée on the fixed point subgroups of groups with root data under particular automorphisms, of which Kac-Moody groups over fields are an example. This allows us to give a generating set for the fixed point subgroup of a Kac-Moody group in terms of the fixed point subgroups of the root groups and the stable elements in the maximal torus. This will prove to be of great importance in our later work. Throughout this exposition, we shall denote by $\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{R}^\times$ the set of elements of $\mathbb{R}$ which are ≥ 0, ≤ 0, and $\neq 0$ respectively.

Finally, we describe the work done by Carter and Chen regarding the automorphisms of affine Kac-Moody groups in [CC91]. In particular, we give a classification of the automorphisms of affine Kac-Moody groups.

3.1 Root Bases

Let $\mathcal{B} = (I, V, \boldsymbol{\alpha}, \boldsymbol{\varrho})$ be a quadruple consisting of a set I, a real vector space V, a basis $\boldsymbol{\alpha} = \{\alpha_i\}_{i\in I}$ of V, and a family $\boldsymbol{\varrho} = \{\varrho_i\}_{i\in I}$ of automorphisms of V. We call $\mathcal{B}$ a *root prebasis* if, for each $i \in I$, there exists a linear form ϕ_i on V such that $\phi_i(\alpha_i) = 2$ and $\varrho_i(v) = v - \phi_i(v)\alpha_i$ for all $v \in V$, so that ϱ_i is a reflection in the hyperplane perpendicular to α_i for each $i \in I$.

An *automorphism* of $\mathcal{B}$ is a permutation $\bar{\gamma}$ of I such that if we denote by γ the element of $GL(V)$ defined by

$$\alpha_i \mapsto \alpha_{\bar{\gamma}(i)}$$

for each $i \in I$, we have

$$\gamma \varrho_i \gamma^{-1} = \varrho_{\bar{\gamma}(i)}$$

for each $i \in I$.

The *Cartan matrix* of $\mathcal{B}$ is defined to be

$$A(\mathcal{B}) = A = (A_{ij})_{i,j\in I}$$

where $A_{ij} = \phi_i(\alpha_j)$. For each $i \in I$, $A_{ii} = 2$. Two prebases sharing the same Cartan matrix are isomorphic. Conversely, suppose that

$$A = (A_{ij})_{i,j\in I}$$

is a matrix with coefficients in $\mathbb{R}$, satisfying $A_{ii} = 2$ for all $i \in I$. We shall call such a matrix a *weak Cartan matrix*. Then any weak Cartan matrix is the Cartan matrix of a root prebasis of the form

$$\mathcal{B}(A) = (I, \mathbb{R}^{|I|}, \boldsymbol{\alpha}', \boldsymbol{\varrho}')$$

where $|I| = \operatorname{card} I$, $\boldsymbol{\alpha}'$ is the canonical basis of the restricted product $\mathbb{R}^{|I|}$ and the family $\boldsymbol{\varrho}'$ is entirely determined by A.

Suppose that, for each $i, j \in I$, m_{ij} is the order of $\varrho_i\varrho_j$ in $GL(V)$. Then we define the *Weyl group* of $\mathcal{B}$ to be the group given by the following presentation;

$$W(\mathcal{B}) = W = \langle (r_i)_{i\in I} : (r_ir_j)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty \rangle.$$

The pair $(W, \{r_i : i \in I\})$ is a Coxeter system. The group W acts on V via the representation

$$\rho_{\mathcal{B}} = \rho : W \to GL(V)$$

defined by

$$\rho(r_i) = \varrho_i$$

for each $i \in I$. We then define the action of W on V by taking

$$w(v) = (\rho(w))(v)$$

for $w \in W$ and $v \in V$.

The *roots* of $\mathcal{B}$ are then defined to be the elements of the set

$$\Phi^{re}(\mathcal{B}) = \Phi^{re} = \{w(\alpha_i) : w \in W, i \in I\}.$$

A prebasis $\mathcal{B}$ is said to be *reduced* if for every $i \in I$

$$\Phi^{re} \cap \mathbb{R}\alpha_i = \{\alpha_i, -\alpha_i\}.$$

Suppose we have a subset $J \subseteq I$. Associated to J is the subgroup

$$W_J = \langle r_i : i \in J \rangle$$

of W. Note that $(W_J, \{r_i\}_{i\in J})$ is a Coxeter system. Let V_J be the subspace of V generated by the set $\{\alpha_i\}_{i\in J}$, and denote by $\varrho_i|_{V_J}$ the automorphism of V_J induced by ϱ_i for each $i \in I$. Then

$$\mathcal{B}_J = (J, V_J, \{\alpha_i\}_{i\in J}, \{\varrho_i|_{V_J}\}_{i\in J})$$

is a root prebasis with Weyl group W_J. Let

$$\Phi^{re}_J = \Phi^{re}(\mathcal{B}_J).$$

We say that $J \subseteq I$ is *spherical* if W_J is finite, in which case we denote by w_J the unique longest element of W_J.

For every subset Ψ of V, we denote by Ψ_+ and Ψ_- the set of elements of Ψ whose coordinates with respect to the basis $\boldsymbol{\alpha}$ are ≥ 0 and ≤ 0 respectively. Note that

$$\Phi^{re}_- = -\Phi^{re}_+ \quad \text{and} \quad \Phi^{re}_+ \cap \Phi^{re}_- = \emptyset.$$

We say that a prebasis $\mathcal{B}$ is a *root basis* if

$$\Phi^{re} = \Phi^{re}_+ \cup \Phi^{re}_-.$$

LEMMA 3.1.1

Suppose $\mathcal{B}$ is a root basis. Then the following statements are true.

1. *If $w \in W$ and $w(\alpha_i) \in \Phi^{re}_+$ for all $i \in I$, then $w = 1$.*
2. *The representation $\rho : W \to GL(V)$ is faithful.*
3. *For $i, j \in I$ and $w \in W$ we have*
$$w(\alpha_i) \in \mathbb{R}\alpha_j \Leftrightarrow w r_i w^{-1} = r_j.$$
4. *For each $\alpha \in \Phi^{re}$ there exists at least one pair $(w, i) \in W \times I$ such that*
$$\alpha = w(\alpha_i),$$
and the element
$$r_\alpha = w r_i w^{-1}$$
is independent of the particular choice of such a pair (w, i).

Proof

See [Hé91a, 2.13] for proof. □

LEMMA 3.1.2

Suppose $\mathcal{B}$ is a reduced spherical root basis. Thus $W(\mathcal{B})$ has a unique longest element, which we shall denote by w_I.

1. *The element w_I interchanges Φ_+ and Φ_- and is fixed by* $\mathrm{Aut}(\mathcal{B})$.

2. *If* $\mathrm{Aut}(\mathcal{B})$ *acts transitively on I, then there exists an element $\bar{\gamma} \in$* $\mathrm{Aut}(\mathcal{B})$ *such that for all $i \in I$*
$$w_I(\alpha_i) = -\alpha_{\bar{\gamma}(i)}.$$

Proof

See [H91a, 2.27]. □

Suppose that $\mathcal{B}$ is a reduced root basis. A subset Ψ of V will be referred to as a $\mathcal{B}$*-root system* if it satisfies the following conditions;

- $\Psi \subseteq \{\lambda\alpha : \lambda \in \mathbb{R}^\times, \alpha \in \Phi\}$,
- for each $i \in I$, the set $\Psi \cap \mathbb{R}\alpha_i$ is finite and non-empty, and
- Ψ is stable under the action of $W(\mathcal{B})$.

Thus Φ^{re} is an example of a $\mathcal{B}$-root system.

If Ψ is a $\mathcal{B}$-root system, a subset $\Psi' \subseteq \Psi$ is said to be $\mathcal{B}$*-prenilpotent* if there exist elements w and w' of W such that

$$w(\Psi') \subseteq V_+ \quad \text{and} \quad w'(\Psi') \subseteq V_-.$$

Action of a Finite Group on a Root Basis

Suppose $\mathcal{B}$ is a root basis. The group $\mathrm{Aut}(\mathcal{B})$ of automorphisms of $\mathcal{B}$ can be made to act on W by defining
$$\bar{\gamma}(r_i) = r_{\bar{\gamma}(i)}$$
for $\bar{\gamma} \in \mathrm{Aut}(\mathcal{B})$ and $i \in I$.

Let Γ be a subgroup of $\mathrm{Aut}(\mathcal{B})$. Denote by W^Γ the subgroup of W consisting of elements of W fixed by Γ. Let I^1 be the set of spherical orbits of I under the action of Γ.

Proposition 3.1.3

The pair $(W^\Gamma, \{w_J\}_{J \in I^1})$ is a Coxeter system.

Proof

See [H91a, Proposition 3.4] for proof. □

Suppose Γ is finite. For each $v \in V$, let

$$v^1 = \sum_{\gamma \in \Gamma} \gamma(v).$$

Given any $J \in I^1$, the element α_i^1 where $i \in J$ is independent of the particular choice of $i \in J$. Define $\alpha_J = \alpha_i^1$, $i \in J$. Let V^1 be the subspace of V generated by the vectors $\{\alpha_J\}_{J \in I^1}$.

PROPOSITION 3.1.4

1. *For each $J \in I^1$, w_J induces on V^1 a reflection ϱ_J corresponding to the vector α_J.*
2. *The quadruple*
$$\mathcal{B}^1 = (I^1, V^1, \{\alpha_J\}_{J \in I^1}, \{\varrho_J\}_{J \in I^1})$$
is a root basis.
3. *The groups W^Γ and $W(\mathcal{B}^1)$ are isomorphic. More precisely, the homomorphism of W^Γ into $GL(V^1)$ which associates to every element $w \in W^\Gamma$ the restriction of $\rho_{\mathcal{B}}(w)$ to V^1 is an isomorphism from W^Γ to $\rho_{\mathcal{B}^1}(W(\mathcal{B}^1))$.*

Proof

See [H91a, 3.11] for proof. □

Suppose now that $\mathcal{B}$ is a reduced root basis, and let Ψ be a $\mathcal{B}$-root system which is stable under the action of Γ. Denote by $\Psi_{\Gamma-pren}$ the set of elements of Ψ whose orbit under the action of Γ is prenilpotent. Define
$$\Psi^1 = \{\alpha^1 : \alpha \in \Psi_{\Gamma-pren}\}.$$

LEMMA 3.1.5

1. $\Psi^1 = \bigcup_{J \in I^1} \left\{ w(\alpha^1) : \alpha \in \Psi^+ \cap V_J, w \in W^\Gamma \right\}$.
2. *The set Ψ^1 is a $\mathcal{B}^1$-root system.*

Proof

See [H91a, 3.12] for proof. □

3.2 Root Data Associated to Groups

Throughout this section we assume $\mathcal{B}$ to be a reduced root basis, with root system Φ^{re} and Weyl group W. We suppose furthermore that Ψ is a $\mathcal{B}$-root system.

Suppose that the subgroup Γ of $\mathrm{Aut}(\mathcal{B})$ introduced in §3.1 stabilizes Ψ. We say that Ψ' is an $\mathbb{N}$-*closed* subset of Ψ if Ψ' contains the set
$$\Psi \cap \{\lambda\alpha + \mu\beta : \lambda, \mu \in \mathbb{N}, \alpha, \beta \in \Psi'\}.$$
For each $\alpha^1 \in \Psi^1$ denote by Ψ_{α^1} the smallest $\mathbb{N}$-closed subset of Ψ containing the set
$$\{\beta \in \Psi_{\Gamma-pren} : \beta^1 = \alpha^1\}.$$

Positive Root Systems Associated to Groups

Consider a group G and a family $(X_\alpha)_{\alpha\in\Psi_+}$ of subgroups of G. For each subset $\Omega \subseteq \Psi_+$ let

$$X_\Omega = \langle X_\alpha : \alpha \in \Omega \rangle$$

and define

$$\Omega^{red} = \{\alpha \in \Omega : \mu \in \mathbb{R}, \mu\alpha \in \Omega \Rightarrow \mu \geq 1\}.$$

For each $\alpha \in \Omega^{red}$ let

$$Y_{\alpha,\Omega} = \langle X_{\mu\alpha} : \mu \in \mathbb{R}, \mu\alpha \in \Omega \rangle$$

and for each $w \in W$ define

$$X_w^+ = \langle X_\alpha : \alpha \in \Psi_+, w(\alpha) \in \Psi_+ \rangle, \quad \text{and} \quad X_w^- = \langle X_\alpha : \alpha \in \Psi_+, w(\alpha) \in \Psi_- \rangle.$$

Finally, let

$$Y_\alpha = \langle X_{\mu\alpha} : \mu \in \mathbb{R}, \mu\alpha \in \Psi \rangle$$

for each $\alpha \in \Phi_+^{re}$.

We say that $(X_\alpha)_{\alpha\in\Psi_+}$ is a *positive root system* of type $(\mathcal{B}, \Psi_+, \mathbb{N})$ in G if

PRS1 for all prenilpotent pairs $\{\alpha, \beta\}$ of distinct elements of Ψ_+

$$[X_\alpha, X_\beta] \subseteq \langle X_{i\alpha+j\beta} : i, j \in \mathbb{N}, i\alpha + j\beta \in \Psi \rangle, \text{ and}$$

PRS2 for each $w \in W$, $X_w^+ \cap X_w^- = \{1\}$.

REMARK 3.2.1

Suppose that condition PRS2 is satisfied. Then, since Ψ is a $\mathcal{B}$-root system, this means that for any pair $\{\alpha_0, \beta_0\}$ of distinct elements of Φ_+^{re} there exists a partition $(\mathcal{P}, \mathcal{Q})$ of Φ_+^{re} such that $\alpha_0 \in \mathcal{P}$, $\beta_0 \in \mathcal{Q}$ and

$$\langle Y_\alpha : \alpha \in \mathcal{P} \rangle \cap \langle Y_\beta : \beta \in \mathcal{Q} \rangle = \{1\}.$$

Since $\mathcal{B}$ is reduced, this means that there exists an element $w \in W$ which distinguishes between α_0 and β_0, i.e. such that one of the elements $w(\alpha_0)$, $w(\beta_0)$ is in Φ_+^{re} and the other is in Φ_-^{re}. ◇

Root Data Associated to Groups

Let G be a group and suppose $(X_\alpha)_{\alpha\in\Psi}$ is a family of subgroups of G, N is a subgroup of G and H is a subgroup of N. Define

$$U = \langle X_\alpha : \alpha \in \Psi_+ \rangle, \quad U_- = \langle X_\alpha : \alpha \in \Psi_- \rangle,$$

and

$$Y_\alpha = \langle X_{\lambda\alpha} : \lambda \in \mathbb{R}, \lambda > 0, \lambda\alpha \in \Psi \rangle$$

for each $\alpha \in \Phi^{re}$. For each $w \in W$, let

$$Y_w^- = X_w^- = \langle X_\alpha : \alpha \in \Psi_+, w(\alpha) \in \Psi_- \rangle$$

We say that

$$((X_\alpha)_{\alpha\in\Psi}, N, H)$$

is a *root datum of type* $(\mathcal{B}, \Psi, \mathbb{N})$ *associated to the group* G if the following conditions are satisfied;

RDG1 $G = \langle H, U, U_- \rangle$ and $UH \cap U_- = \{1\} = U \cap U_-H$.

RDG2 For each prenilpotent pair $\{\alpha, \beta\}$ of elements of Ψ,

$$[X_\alpha, X_\beta] \subseteq \langle X_{\lambda\alpha+\mu\beta} : \lambda, \mu \in \mathbb{N}, \lambda\alpha + \mu\beta \in \Psi \rangle.$$

RDG3 There is a surjective homomorphism

$$\begin{array}{cccc} \pi : & N & \to & W \\ & n & \mapsto & w_n \end{array}$$

with kernel H, such that for each $n \in N$ and $\alpha \in \Psi$

$$nX_\alpha n^{-1} = X_{w_n(\alpha)}.$$

RDG4 For each $i \in I$, $Y_{-\alpha_i} \neq \{1\}$ and

$$Y_{-\alpha_i} \setminus \{1\} \subseteq Y_{\alpha_i} N_i Y_{\alpha_i}$$

where

$$N_i = \{n \in N : w_n = r_i\}.$$

The following result establishes a connection between the theory of positive root systems and that of root data.

LEMMA 3.2.2

Suppose $((X_\alpha)_{\alpha\in\Psi}, N, H)$ *is a root datum of type* $(\mathcal{B}, \Psi, \mathbb{N})$ *associated to the group* G. *Then the family* $(X_\alpha)_{\alpha\in\Psi_+}$ *is a positive root system of type* $(\mathcal{B}, \Psi_+, \mathbb{N})$ *in* G.

Proof

See [H91b, Lemme 4.3] □

The following result illuminates some of the structure of a group equipped with such root data.

PROPOSITION 3.2.3

Suppose $((X_\alpha)_{\alpha\in\Psi}, N, H)$ *is a root datum of type* $(\mathcal{B}, \Psi, \mathbb{N})$ *associated to a group* G. *Let* $B = UH$ *and* $B_- = U_-H$.

1. *(B, N) and (B_-, N) are two (B, N)-pairs in G.*
2. *For each $x \in G$ there exists a unique triplet*

$$(u_1, n, u)$$

such that $u_1 \in U$, $n \in N$, $u \in Y^-_{w_n}$ and $x = u_1 n u$.

3. *The map $w \mapsto B_- w B$ is a bijection from W to the set $B_- \backslash G / B$, where W is identified with N/H via π. In particular*

$$G = U_- N U$$

and so G has a Birkhoff decomposition.

Proof
See [H91b, Propositions 2.8–2.10]. □

We refer to a decomposition $x = u_1 n u$ as in the previous result as a *Bruhat decomposition of x*.

The following result is a generalization of [H91b, 2.12], though the proof is analogous.

LEMMA 3.2.4
Let $\mathcal{B}$ be a reduced root basis, $\Phi^{re} = \Phi^{re}(\mathcal{B})$, and

$$((X_\alpha)_{\alpha \in \Phi^{re}}, N, H)$$

be a root datum of type $(\mathcal{B}, \Phi^{re}, \mathbb{N})$ corresponding to a group G. Suppose K is a spherical subset of Φ^{re}. Let V_K be the subspace of V spanned by the elements of K and index K by $\mid K \mid$. Let

$$\mathcal{B}_K = (\mid K \mid, V_K, K, \{\varrho_\alpha\}_{\alpha \in K})$$

where ϱ_α denotes the reflection in the hyperplane perpendicular to α. Let

$$\Phi_K = \Phi^{re}(\mathcal{B}_K) \text{ and } W_K = W(\mathcal{B}_K).$$

We identify Φ_K and W_K with the corresponding subgroups of Φ^{re} and $W(\mathcal{B})$ respectively. Define

$$L_K = \langle H, X_\alpha : \alpha \in \Phi_K \rangle$$

$$X_K = \langle X_\alpha : \alpha \in (\Phi_K)_+ \rangle, \quad X_{-K} = \langle X_\alpha : \alpha \in (\Phi_K)_- \rangle$$

$$N_K = \pi^{-1}(W_K) \quad \text{and} \quad P_K = B N_K B.$$

Then

$$((X_\alpha)_{\alpha \in \Phi_K}, N_K, H)$$

is a root datum of type $(\mathcal{B}_K, \Phi_K, \mathbb{N})$ corresponding to L_K, and

$$L_K = X_K N_K X_K = X_{-K} N_K X_{-K}.$$

Proof

For each $\alpha \in K$ denote by r_α the element of W corresponding to ϱ_α. Let

$$N_\alpha = \pi^{-1}(r_\alpha).$$

Then

$$N_K = \langle H, N_\alpha : \alpha \in K \rangle.$$

However, $H \subseteq L_K$ and

$$N_\alpha \subseteq \langle H, X_\alpha, X_{-\alpha} \rangle \subseteq L_K$$

and so $N_K \subseteq L_K$. Furthermore

$$\Phi^{re} \cap V_K = \Phi(\mathcal{B}_K)$$

since $\mathcal{B}$ is reduced. Thus the restriction of π to N_K satisfies the first of the properties for condition RDG3.

The remaining root data properties follow from the fact that

$$((X_\alpha)_{\alpha \in \Phi^{re}}, N, H)$$

is a root datum of type $(\mathcal{B}, \Phi^{re}, \mathbb{N})$ corresponding to a group G.

Thus $(X_K H, N_K)$ is a (B, N)-pair in L_K and we deduce that

$$L_K = (X_K H) N_K (X_K H) = X_K N_K X_K$$

from Proposition 2.1.7. Similarly we deduce that

$$L_K = X_{-K} N_K X_{-K}.$$

□

We shall refer to L_K as the *Levi subgroup* corresponding to K.

EXAMPLE 3.2.5

Consider the case of a Kac-Moody group as defined in Chapter 2. Recall that we can define a group functor $\mathfrak{G}_D$ on the category of commutative rings such that $\mathfrak{G}_D(\mathbb{K})$ is the Kac-Moody group corresponding to the root datum $D = D(A)$ associated to the GCM A whenever $\mathbb{K}$ is a field.

We construct a root basis $\mathcal{B} = \mathcal{B}(A)$ associated to A. We now define

$$X_\alpha = \mathfrak{U}_\alpha(\mathbb{K}), \qquad H = \mathfrak{T}_D(\mathbb{K}) \qquad \text{and} \qquad N = \langle H, n_i : i \in I \rangle$$

where $n_i = x_i(1)x_{-i}(1)x_i(1)$ in the notation introduced in Chapter 2. Let $\Phi^{re} = \Phi^{re}(A)$. Then, from the results in §2.5, we see that

$$((X_\alpha)_{\alpha \in \Phi^{re}}, N, H)$$

is a root datum of type $(\mathcal{B}, \Phi^{re}, \mathbb{N})$ corresponding to $\mathfrak{G}_D(\mathbb{K})$. ◇

3.3 Twisted Groups

Throughout this section we suppose Γ to be a finite group of automorphisms of $\mathcal{B}$ as introduced in §3.1 which stabilizes Ψ.

Twisted Positive Root Systems

Suppose that $(X_\alpha)_{\alpha\in\Psi^+}$ is a positive root system of type $(\mathcal{B}, \Psi^+, \mathbb{N})$ in a group G.

Let Σ be a set and

$$(\gamma_\sigma)_{\sigma\in\Sigma} \quad \text{and} \quad (\tau_\sigma)_{\sigma\in\Sigma}$$

be two families of endomorphisms of the group G such that for all $\sigma \in \Sigma$

- for all $\alpha \in \Psi^+$, $\tau_\sigma(X_\alpha) \subseteq X_\alpha$ and
- there exists a $\bar{\gamma}_\sigma \in \Gamma$ such that, for each $\alpha \in \Psi^+$, $\gamma_\sigma(X_\alpha) \subseteq X_{\bar{\gamma}_\sigma(\alpha)}$.

Denote by G^1 the set of elements $x \in G$ such that

$$\gamma_\sigma(x) = \tau_\sigma(x)$$

for each $\sigma \in \Sigma$. Let

$$X^1_{\alpha^1} = G^1 \cap X_{\Psi_{\alpha^1}}$$

for each $\alpha^1 \in (\Psi^1)_+$.

Suppose that, for each prenilpotent pair of elements $\{\alpha^1, \beta^1\}$ of $(\Psi^1)_+$, at least one of the following holds: $\Psi_{\alpha^1} \subseteq \Psi_{\beta^1}$, $\Psi_{\beta^1} \subseteq \Psi_{\alpha^1}$, or every pair consisting of one element from Ψ_{α^1} and one from Ψ_{β^1} generates a free abelian subgroup of V of rank 2.

Proposition 3.3.1
Under the above conditions we have:

1. *For each $\mathcal{B}^1$-prenilpotent, $\mathbb{N}$-closed subset Ω^1 of $(\Psi^1)_+$, the set*

$$\Omega = \{\alpha \in \Psi_{\Gamma-pren} : \alpha^1 \in \Omega^1\}$$

is a $\mathcal{B}$-prenilpotent $\mathbb{N}$-closed subset of Ψ_+ stable under the action of Γ and

$$X_\Omega \cap G^1 = \langle X^1_{\alpha^1} : \alpha^1 \in \Omega^1 \rangle.$$

2. *$(X^1_{\alpha^1})_{\alpha^1\in(\Psi^1)_+}$ is a positive root system of type $(\mathcal{B}^1, (\Psi^1)_+, \mathbb{N})$ in G^1.*

Proof
See [H91b, Proposition 3.6]. □

Twisted Root Data

Suppose that $((X_\alpha)_{\alpha\in\Psi}, N, H)$ is a root datum of type $(\mathcal{B}, \Psi, \mathbb{N})$ associated to a group G. We begin by describing some properties of certain endomorphisms of G.

LEMMA 3.3.2

Let ρ be an endomorphism of G satisfying

- $\rho(N) \subseteq N$, *and* $\rho(H) \subseteq H$
- *there exists an automorphism ϕ_ρ of $\mathcal{B}$ such that, for all $\alpha \in \Psi$*

$$\{1\} \neq \rho(Y_\alpha) \subseteq Y_{\phi_\rho(\alpha)}.$$

Then

1. $\rho(U) \subseteq U$, $\rho(U_-) \subseteq U_-$, $\rho(B) \subseteq B$, *and* $\rho(B_-) \subseteq B_-$.
2. *If we identify ϕ_ρ and each element of W with the elements of $GL(V)$ they define then, for all $n \in N$,*

$$w_{\rho(n)} = \phi_\rho w_n \phi_\rho^{-1}, \qquad \text{and} \qquad \rho(Y_{w_n}^-) \subseteq Y_{w_{\rho(n)}}^-.$$

3. *If $x \in G$ has Bruhat decomposition (u_1, n, u) then the Bruhat decomposition of $\rho(x)$ is*

$$(\rho(u_1), \rho(n), \rho(u)).$$

Proof

See [H91b, Lemme 4.3, 2.11]. □

Suppose now that Σ is a set and that we have two families

$$(\gamma_\sigma)_{\sigma\in\Sigma} \quad \text{and} \quad (\tau_\sigma)_{\sigma\in\Sigma}$$

of endomorphisms of the group G satisfying the following conditions for all $\sigma \in \Sigma$:

- $\gamma_\sigma(N) \cup \tau_\sigma(N) \subseteq N$, and $\gamma_\sigma(H) \cup \tau_\sigma(H) \subseteq H$.
- $\gamma_\sigma(Y_\alpha) \neq \{1\}$ and $\tau_\sigma(Y_\alpha) \neq \{1\}$ for each $\alpha \in \Psi$.
- For all $\alpha \in \Psi$, $\tau_\sigma(X_\alpha) \subseteq X_\alpha$.
- There exists a $\bar{\gamma}_\sigma \in \Gamma$ such that $\gamma_\sigma(X_\alpha) \subseteq X_{\bar{\gamma}_\sigma(\alpha)}$ for all $\alpha \in \Psi$.
- The elements $\bar{\gamma}_\sigma$ so defined satisfy the property

$$\Gamma = \langle \bar{\gamma}_\sigma : \sigma \in \Sigma \rangle.$$

Note that the first two conditions mean that $\bar{\gamma}_\sigma$ is unambiguously defined for each $\sigma \in \Sigma$ (see [H91b, 2.11] for details).

We now define

$$G^{\gamma,\tau} = \langle x \in G : \gamma_\sigma(x) = \tau_\sigma(x) \text{ for all } \sigma \in \Sigma \rangle.$$

For each $\alpha^1 \in \Psi^1$, let

$$X^1_{\alpha^1} = G^{\gamma,\tau} \cap \langle X_\beta : \beta \in \Psi_{\alpha^1} \rangle$$

and for each $J \in I^1$, let

$$Y^1_J = G^{\gamma,\tau} \cap \langle Y_\alpha : \alpha \in \Phi^{re}_+ \cap V_J \rangle.$$

Let G^1 be a group such that

$$\langle X^1_{\alpha^1} : \alpha^1 \in \Psi^1 \rangle \leq G^1 \leq G^{\gamma,\tau}$$

and define $N^1 = G^1 \cap N$ and $H^1 = G^1 \cap H$. The following theorem may be referred to as *Hée's Theorem.*

THEOREM 3.3.3

Suppose the following conditions hold;

- *For each $J \in I^1$, $Y^1_J \neq \{1\}$.*
- *For each $\mathcal{B}^1$-prenilpotent pair $\{\alpha^1, \beta^1\}$ of elements of $(\Psi^1)_+$, we have either*

$$\Psi_{\alpha^1} \subseteq \Psi_{\beta^1} \quad \textit{or} \quad \Psi_{\beta^1} \subseteq \Psi_{\alpha^1}$$

  or every pair consisting of an element of Ψ_{α^1} and an element of Ψ_{β^1} generates a free abelian subgroup of V of rank 2.

Then the following are true.

1. *For each $\sigma \in \Sigma$, γ_σ and τ_σ restrict to endomorphisms of U, U_-, B, and B_-.*
2. *For each $n \in N$ and each $\sigma \in \Sigma$, we have*

$$w_{\gamma_\sigma(n)} = \bar{\gamma}_\sigma w_n \bar{\gamma}_\sigma^{-1}, \qquad w_{\tau_\sigma(n)} = w_n,$$

$$\gamma_\sigma(X^-_{w_n}) \subseteq X^-_{w_{\gamma_\sigma(n)}}, \quad \textit{and} \quad \tau_\sigma(X^-_{w_n}) \subseteq X^-_{w_n}$$

   where the elements $\bar{\gamma}_\sigma$ and w_n are identified with the elements of $GL(V)$ they induce. Furthermore, we have

$$w_n \in W^{\Gamma}$$

   for each $n \in N \cap G^{\gamma,\tau}$.

3. *Let x be an element of G and (u_1, n, u) be its Bruhat decomposition. For each $\sigma \in \Sigma$*

$$(\gamma_\sigma(u_1), \gamma_\sigma(n), \gamma_\sigma(u))$$

is the Bruhat decomposition of $\gamma_\sigma(x)$, and

$$(\tau_\sigma(u_1), \tau_\sigma(n), \tau_\sigma(u))$$

is the Bruhat decomposition of $\tau_\sigma(x)$. In particular, if $x \in G^1$,

$$u_1, u \in G^{\gamma,\tau}, \quad n \in N^1 \quad \text{and} \quad w_n \in W^\Gamma.$$

4. *For all $J \in I^1$, let $N^1_J = N^1 \cap N_J$. Then*

$$Y^1_J = \langle X^1_{\alpha^1} : \alpha^1 \in \Psi^1 \cap \mathbb{R}_+\alpha_J \rangle \subseteq G^1,$$

$$Y^1_{-J} = \langle X^1_{\alpha^1} : \alpha^1 \in \Psi^1 \cap \mathbb{R}_-\alpha_J \rangle \subseteq G^1,$$

$$Y^1_{-J} \setminus \{1\} \subseteq Y^1_J N^1_J Y^1_J,$$

and $w_J \in \pi(N^1)$.

5. *The homomorphism*

$$\begin{array}{cccc} \pi^1 & : & N^1 & \rightarrow & W(\mathcal{B}^1) \\ & & n & \mapsto & w^1_n \end{array}$$

induced by the composition of $\pi : N \rightarrow W$ and the canonical isomorphism

$$\begin{array}{ccc} W^\Gamma & \rightarrow & W(\mathcal{B}^1) \\ w & \mapsto & w^1 \end{array}$$

is surjective and $\ker(\pi^1) = H^1$.

6. *For all $n \in N \cap G^{\gamma,\tau}$ and all $\alpha^1 \in \Psi^1$ we have*

$$n(X^1_{\alpha^1})n^{-1} = X^1_{w^1_n(\alpha^1)}.$$

7. *For each $w \in W^\Gamma$,*

$$X^-_w \cap G^{\gamma,\tau} = \langle X^1_{\alpha^1} : \alpha^1 \in (\Psi^1)_+, w(\alpha^1) \in (\Psi^1)_- \rangle \subseteq G^1.$$

8. *We have*

$$G^{\gamma,\tau} \cap U = \langle X^1_{\alpha^1} : \alpha^1 \in (\Psi^1)_+ \rangle, \quad G^{\gamma,\tau} \cap U_- = \langle X^1_{\alpha^1} : \alpha^1 \in (\Psi^1)_- \rangle,$$

and

$$G^{\gamma,\tau} = \langle H \cap G^{\gamma,\tau}, X^1_{\alpha^1} : \alpha^1 \in \Psi^1 \rangle.$$

Thus

$$\left((X^1_{\alpha^1})_{\alpha^1\in\Psi^1}, N^1, H^1\right)$$

is a root datum of type $(\mathcal{B}^1, \Psi^1, \mathbb{N})$ *associated to* G^1. *So*

$$G^1 = \langle H^1, Y^1_J, Y^1_{-J} : J \in I^1\rangle$$

and, in particular

$$G^{\gamma,\tau} = \langle G^{\gamma,\tau} \cap H, Y^1_J, Y^1_{-J} : J \in I^1\rangle.$$

Proof

See [H90] and [H91b, Théorème 4.5]. □

3.4 Automorphisms of Affine Kac-Moody Groups

Finally, we give an exposition of the work of Carter and Chen on the automorphisms of complex simply-connected affine Kac-Moody Groups arising from extended Cartan matrices. All of the results mentioned in this section, together with their proofs, can be found in [CC91].

Let

$$A = (A_{ij})_{i,j\in\underline{n}}$$

be an ordinary Cartan matrix. Let $\Pi = \{\alpha_i\}_{i\in\underline{n}}$ be a set of simple roots corresponding to A and denote by $\Phi(A)$ the finite root system generated by Π. Let

$$\mathcal{R} = a_1\alpha_1 + \cdots + a_n\alpha_n$$

be the expression for the unique highest root of $\Phi(A)$ as a linear combination of the simple roots. Denote by $\mathfrak{G}^A_{sc}(\mathbb{C})$ and $\mathfrak{G}^A_{ad}(\mathbb{C})$ the simply-connected and adjoint Kac-Moody groups over $\mathbb{C}$ associated to A. Let $\mathfrak{G}^A_{sc}(\mathcal{L})$ and $\mathfrak{G}^A_{ad}(\mathcal{L})$ be the corresponding groups of rational points over the ring $\mathcal{L}$ of Laurent polynomials.

Let

$$\tilde{A} = (\tilde{A}_{ij})_{i,j\in\underline{n}_0}$$

be the extended Cartan matrix obtained from A and let $\{\alpha_i\}_{i\in\underline{n}_0} = \Pi \cup \{\alpha_0\}$ be a set of simple roots for the real root system $\Phi^{re}(\tilde{A})$ associated to $\tilde{A}$. Recall that by Propositions 1.3.1 and 1.3.3, $\Phi^{re}(\tilde{A})$ can be expressed as

$$\Phi^{re}(\tilde{A}) = \{\alpha + m\delta : \alpha \in \Phi(A), m \in \mathbb{Z}\}$$

where $\delta = \alpha_0 + \mathcal{R}$. Let $\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C})$ be the simply-connected Kac-Moody group associated to $\tilde{A}$.

Let $\tilde{\mathfrak{g}}$ be the Kac-Moody algebra with GCM $\tilde{A}$ and let $\tilde{\mathfrak{z}}$ be the centre of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}' = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. By the results of Chapter 1

$$\tilde{\mathfrak{z}} \subseteq \tilde{\mathfrak{g}}', \quad \dim \tilde{\mathfrak{z}} = 1, \quad \text{and} \quad \dim \tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}' = 1.$$

By the results of Chapter 2 we have a homomorphism

$$\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C}) \to \mathrm{Aut}(\tilde{\mathfrak{g}}')$$

such that

$$\begin{array}{rcl} x_i(\mu) & \mapsto & \exp \mathrm{ad}\, \mu e_i \\ x_{-i}(\mu) & \mapsto & \exp \mathrm{ad}\, \mu f_i \end{array}$$

for each $i \in \underline{n}_0$ and $\mu \in \mathbb{C}$.

From Chapter 1 we also know that there is a homomorphism

$$\tilde{\mathfrak{g}}' \to \mathfrak{g}(\mathcal{L})$$

with kernel $\tilde{\mathfrak{z}}$ from which we deduce that

$$\tilde{\mathfrak{g}}'/\tilde{\mathfrak{z}} \cong \mathfrak{g}(\mathcal{L}).$$

Thus we have a series of homomorphisms

$$\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C}) \to \mathrm{Aut}(\tilde{\mathfrak{g}}') \to \mathrm{Aut}(\tilde{\mathfrak{g}}'/\tilde{\mathfrak{z}}) \to \mathrm{Aut}(\mathfrak{g}(\mathcal{L})).$$

Combining these homomorphisms, we obtain a homomorphism

$$\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C}) \to \mathfrak{G}^{A}_{ad}(\mathcal{L})$$

under which

$$x_{\alpha+m\delta}(\mu) \mapsto x_\alpha(t^m \otimes \mu)$$

for each $\alpha \in \Phi(A)$, $m \in \mathbb{Z}$, and $\mu \in \mathbb{C}$. Using this homomorphism we can obtain relations between structure constants of $\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C})$ and $\mathfrak{G}^{A}_{ad}(\mathbb{C})$.

Consider first two relations which hold in $\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C})$. For each $\alpha \in \Phi^{re}(\tilde{A})$ and $\mu \in \mathbb{C}$ we have

$$n_i x_\alpha(\mu) n_i^{-1} = x_{r_i(\alpha)}(\eta_{i,\alpha}\mu)$$

for some $\eta_{i,\alpha} \in \{1, -1\}$. Also, for each prenilpotent pair $\{\alpha, \beta\}$ of distinct roots and any $\mu, \nu \in \mathbb{C}$,

$$[x_\alpha(\mu), x_\beta(\nu)] = \prod_{\substack{i\alpha+j\beta \in \Phi^{re}(\tilde{A}) \\ i,j \in \mathbb{N}}} x_{i\alpha+j\beta}(C_{\alpha\beta ij}\mu^i\nu^j),$$

for some $C_{\alpha\beta ij} \in \mathbb{Z}$.

Now consider the corresponding relations which hold in $\mathfrak{G}^A_{ad}(\mathcal{L})$. We define numbers $\eta'_{i,\alpha} \in \{1,-1\}$ for $i \in \underline{n}$ and $\alpha \in \Phi(A)$ and $C'_{\alpha\beta ij} \in \mathbb{Z}$ for $\alpha, \beta \in \Phi(A)$, $i, j \in \mathbb{N}$ with $i\alpha + j\beta \in \Phi(A)$ as follows. Let

$$n_i x_\alpha(\mu) n_i^{-1} = x_{r_i(\alpha)}(\eta'_{i,\alpha}\mu)$$

for $\mu \in \mathbb{C}(t)$ and

$$[x_\alpha(\mu), x_\beta(\nu)] = \prod_{\substack{i\alpha+j\beta\in\Phi(A)\\ i,j\in\mathbb{N}}} x_{i\alpha+j\beta}(C'_{\alpha,\beta,i,j}\mu^i\nu^j), \quad \alpha\neq\pm\beta,$$

for $\mu, \nu \in \mathbb{C}(t)$.

Proposition 3.4.1

1. *For all $\alpha \in \Phi(A)$, $m \in \mathbb{Z}$, and $i \in \underline{n}$,*

$$\eta_{i,\alpha+m\delta} = \eta'_{i,\alpha}.$$

2. *For all $\alpha \in \Phi(A)$ and $m \in \mathbb{Z}$*

$$\eta_{0,\alpha+m\delta} = \eta'_{-\mathcal{R},\alpha},$$

where $\eta'_{-\mathcal{R},\alpha}$ is given by the formula

$$n_{-\mathcal{R}} x_\alpha(\mu) n_{-\mathcal{R}}^{-1} = x_{r_{-\mathcal{R}}(\alpha)}(\eta'_{-\mathcal{R},\alpha}\mu)$$

where $n_{-\mathcal{R}} = x_{-\mathcal{R}}(1)x_{\mathcal{R}}(1)x_{-\mathcal{R}}(1)$.

3. *For all $\alpha + m\delta, \beta + n\delta \in \Phi^{re}(\tilde{A})$ with $\beta \neq \pm\alpha$*

$$C_{\alpha+m\delta,\beta+n\delta,i,j} = C'_{\alpha,\beta,i,j},$$

where the order of the terms on the right hand side in the commutator formulas for $\tilde{\mathfrak{g}}(\mathbb{C})$ and $\mathfrak{G}^A_{ad}(\mathbb{C})$ are chosen to be compatible in an obvious sense.

Proof

See the proof of [CC91, Proposition 3.1]. □

The following result shows that the homomorphism $\mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C}) \to \mathfrak{G}^A_{ad}(\mathcal{L})$ factors through $\mathfrak{G}^A_{sc}(\mathcal{L})$.

Proposition 3.4.2

There is a surjective homomorphism

$$\theta : \mathfrak{G}^{\tilde{A}}_{sc}(\mathbb{C}) \to \mathfrak{G}^A_{sc}(\mathcal{L})$$

under which

$$x_{\alpha+m\delta}(\mu) \mapsto x_\alpha(t^m \otimes \mu)$$

for all $\alpha \in \Phi(A)$, $m \in \mathbb{Z}$, *and* $\mu \in \mathbb{C}$. *Furthermore, the kernel of this homomorphism,* $\ker\theta$, *is a characteristic subgroup of* $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$.

Proof

See Propositions 3.2–3.6 of [CC91]. □

This allows us to define a map

$$\nu : \mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})) \to \mathrm{Aut}(\mathfrak{G}^{A}_{sc}(\mathcal{L}))$$

since each automorphism of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$ leaves $\ker\theta$ invariant and therefore induces an automorphism of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})/\ker\theta \cong \mathfrak{G}^{A}_{sc}(\mathcal{L})$. One of the aims in [CC91] is to prove the following theorem.

THEOREM 3.4.3

The map $\nu : \mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})) \to \mathrm{Aut}(\mathfrak{G}^{A}_{sc}(\mathcal{L}))$ *is an isomorphism.*

In order to do this, the authors classify the automorphisms of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$, so producing a result analogous to the well-known result of Steinberg for Chevalley groups (see [Ste67, Theorem 30]).

Classification of the Automorphisms of Extended Kac-Moody Groups

In order to describe an automorphism of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$, it is sufficient to give its effect on the generators

$$\{x_i(\mu), x_{-i}(\mu) : i \in \underline{n}_0, \mu \in \mathbb{K}\}$$

of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$.

Inner Automorphisms

For each $g \in \mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$ we have a corresponding inner automorphism

$$\tau_g : x \mapsto gxg^{-1}.$$

Let $\tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}) = \{\tau_g : g \in \mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})\}$ be the group of inner automorphisms.

Diagonal Automorphisms

Consider the group $\mathrm{Hom}(\mathbb{Z}\Pi, \mathbb{C}^\times)$ of homomorphisms from the free abelian group $\mathbb{Z}\Pi$ to the multiplicative group of $\mathbb{C}$, where $\Pi = \{\alpha_i\}_{i\in\underline{n}_0}$. We have

$$\mathrm{Hom}(\mathbb{Z}\Pi, \mathbb{C}^\times) \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$$

where there are $n+1$ factors on the right hand side.

For each $\theta \in \mathrm{Hom}(\mathbb{Z}\Pi, \mathbb{C}^\times)$ there is a corresponding automorphism $d(\theta)$ of $\mathfrak{G}_{sc}^{\hat{A}}(\mathbb{C})$ given by

$$d(\theta) \;:\; \begin{array}{lcl} x_i(\mu) & \to & x_i(\xi_i\mu) \\ x_{-i}(\mu) & \to & x_{-i}(\xi_i^{-1}\mu) \end{array}$$

where $\xi_i = \theta(\alpha_i)$. Note that $d(\theta)$ transforms n_i to $h_i(\xi_i)n_i$ and fixes $h_i(\zeta)$ for $\zeta \in \mathbb{C}^\times$.

Some of the automorphisms $d(\theta)$ so defined are in fact inner automorphisms. It can be shown (see [CC91, pp. 23–25] for details) that $d(\theta)$ is an inner automorphism induced by an element of H if and only if

$$\xi_0\xi_1^{a_1}\cdots\xi_n^{a_n} = 1$$

where $\mathcal{R} = \sum_{i\in \underline{n}} a_i\alpha_i$. It is therefore natural to define a diagonal automorphism to be one in which

$$\xi_1 = 1, \ldots, \xi_n = 1 \quad \text{and} \quad \xi_0 \text{ is arbitrary.}$$

Thus for each $\xi \in \mathbb{C}^\times$ we define the diagonal automorphism $d(\xi)$ of $\mathfrak{G}_{sc}^{\hat{A}}(\mathbb{C})$ by

$$d(\xi) \;:\; \begin{array}{lcl} \left.\begin{array}{lcl} x_i(\mu) & \to & x_i(\mu) \\ x_{-i}(\mu) & \to & x_{-i}(\mu) \end{array}\right\} & & i \in \underline{n} \\ x_0(\mu) \;\to\; x_0(\xi\mu) & & \\ x_{-0}(\mu) \;\to\; x_{-0}(\xi^{-1}\mu). & & \end{array}$$

The diagonal automorphisms

$$D = \{d(\xi) : \xi \in \mathbb{C}^\times\}$$

form a subgroup of $\mathrm{Aut}(\mathfrak{G}_{sc}^{\hat{A}}(\mathbb{C}))$ isomorphic to $\mathbb{C}^\times$.

Field Automorphisms

Let f be an automorphism of the field $\mathbb{C}$. Then f induces an element

$$a(f) \in \mathrm{Aut}\,(\mathfrak{G}_{sc}^{\hat{A}}(\mathbb{C}))$$

given by

$$a(f) : x_\alpha(\mu) \to x_\alpha(f(\mu))$$

for $\alpha \in \Phi^{re}$ and $\mu \in \mathbb{C}$. The automorphism $a(f)$ transforms $n_i(\mu)$ into $n_i(f(\mu))$ and $h_i(\zeta)$ into $h_i(f(\zeta))$ for each $i \in \underline{n}_0$, $\mu \in \mathbb{C}$, and $\zeta \in \mathbb{C}^\times$. Let

$$A(\mathbb{C}) = \{a(f) : f \in \mathrm{Aut}(\mathbb{C})\}.$$

Then $A(\mathbb{C})$ is a subgroup of $\mathrm{Aut}(\mathfrak{G}_{sc}^{\hat{A}}(\mathbb{C}))$ isomorphic to $\mathrm{Aut}(\mathbb{C})$.

Graph Automorphisms

Let $\bar{\gamma}$ be a permutation of the set $\underline{n}_0$ such that

$$\hat{A}_{ij} = \hat{A}_{\bar{\gamma}(i)\bar{\gamma}(j)}$$

for each $i, j \in \underline{n}_0$. We call $\bar{\gamma}$ an automorphism of the extended Cartan matrix $\hat{A}$. Then, by [CC91, Proposition 4.2], there is an injective homomorphism

$$\mathrm{Aut}(\hat{A}) \to \mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))$$

mapping $\bar{\gamma}$ to $a(\bar{\gamma})$ where

$$a(\bar{\gamma}): \left.\begin{array}{lcl} x_i(\mu) & \to & x_{\bar{\gamma}(i)}(\varepsilon_i\mu) \\ x_{-i}(\mu) & \to & x_{-\bar{\gamma}(i)}(\varepsilon_i\mu) \end{array}\right\} i \in \underline{n}_0$$

with $\varepsilon_i \in \{1, -1\}$. Note that $a(\bar{\gamma})$ transforms $n_i(\mu)$ to $n_{\bar{\gamma}(i)}(\varepsilon_i\mu)$ and $h_i(\zeta)$ to $h_{\bar{\gamma}(i)}(\zeta)$ for each $i \in \underline{n}_0$, $\mu \in \mathbb{C}$ and $\zeta \in \mathbb{C}^\times$.

Furthermore, we have $\varepsilon_i = 1$ for each $i \in \underline{n}_0$ except when A is of type A_{2l}. In that case let $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathrm{Aut}(\hat{A}_{2l})$ be defined by

$$\bar{\gamma}_1(0) = 1, \bar{\gamma}_1(1) = 2, \ldots \bar{\gamma}_1(2l-1) = 2l, \bar{\gamma}_1(2l) = 0$$

and

$$\bar{\gamma}_2(0) = 0, \bar{\gamma}_2(1) = 2l, \bar{\gamma}_2(2) = 2l-1, \ldots \bar{\gamma}_2(2l-1) = 2, \bar{\gamma}_2(2l) = 1$$

respectively. If $\bar{\gamma} = \bar{\gamma}_1$ then $\varepsilon_i = 1$ for all $i \in \underline{n}_0$. If $\bar{\gamma} = \bar{\gamma}_2$ then $\varepsilon_i = 1$ for all $i \in \underline{n}$ but $\varepsilon_0 = -1$.

Alternatively, we may define $a(\bar{\gamma}_1), a(\bar{\gamma}_2) \in \mathrm{Aut}\,(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))$ by

$$a(\bar{\gamma}_1): \left.\begin{array}{lcl} x_i(\mu) & \to & x_{\bar{\gamma}_1(i)}(\mu) \\ x_{-i}(\mu) & \to & x_{-\bar{\gamma}_1(i)}(\mu) \end{array}\right\} i \in \underline{n}_0$$

and

$$a(\bar{\gamma}_2): \left.\begin{array}{lcl} x_i(\mu) & \to & x_{\bar{\gamma}_2(i)}(-\mu) \\ x_{-i}(\mu) & \to & x_{-\bar{\gamma}_2(i)}(-\mu) \end{array}\right\} i \in \underline{n}_0$$

respectively. Then

$$a(\bar{\gamma}_2)a(\bar{\gamma}_1)a(\bar{\gamma}_2) = a(\bar{\gamma}_1^{-1})$$

and the group $\langle a(\bar{\gamma}_1), a(\bar{\gamma}_2)\rangle$ generated by $a(\bar{\gamma}_1)$ and $a(\bar{\gamma}_2)$ is isomorphic to

$$\langle \bar{\gamma}_1, \bar{\gamma}_2 \rangle = \mathrm{Aut}(\hat{A}_{2l}).$$

Let $\hat{\Delta}$ be the Dynkin diagram of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$. For each symmetry $\bar{\gamma}$ of $\hat{\Delta}$ we have an automorphism of the extended Cartan matrix $\hat{A}$ and hence a corresponding graph automorphism $a(\bar{\gamma})$ of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$. Let

$$\Gamma(\hat{\Delta}) = \{a(\bar{\gamma}) : \bar{\gamma} \in \mathrm{Aut}(\hat{\Delta})\}.$$

Then $\Gamma(\hat{\Delta})$ is a subgroup of $\mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))$ isomorphic to $\mathrm{Aut}(\hat{\Delta})$.

Sign Automorphisms

The group $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$ has an automorphism ω, which we shall refer to as the *Chevalley involution*, such that

$$\omega : x_\alpha(\mu) \to x_{-\alpha}(\mu)$$

for all $\alpha \in \Phi^{re}$. The automorphism ω transforms $n_i(\mu)$ to $n_i(\mu^{-1})$ and $h_i(\zeta)$ to $h_i(\zeta^{-1})$ for $\mu \in \mathbb{C}$ and $\zeta \in \mathbb{C}^\times$.

Note that the corresponding type of automorphism for a Chevalley group is an automorphism involving the inner automorphism induced by the unique longest element of the Weyl group. However, in the affine case the Weyl group is infinite and has no longest element, so the Chevalley involution is not expressible in terms of the automorphisms described hereto.

The Chevalley involution ω satisfies $\omega^2 = 1$ and the group $\Omega = \{1, \omega\}$ it generates will be referred to as the *group of sign automorphisms* of $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$.

Classification Theorem

THEOREM 3.4.4

1. *We have a factorisation*

$$\mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})) = \tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))DA(\mathbb{C})\Gamma(\hat{\Delta})\Omega.$$

Moreover, we have a series of normal subgroups

$$\tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})) \lhd \tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))D \lhd \tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))DA(\mathbb{C})$$
$$\lhd \tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))DA(\mathbb{C})\Gamma(\hat{\Delta}) \lhd \mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})).$$

2. *Every element of* $\mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))$ *is uniquely expressible as a product of automorphisms corresponding to the factorisation given above.*

3. *Define* $\mathrm{Out}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})) = \mathrm{Aut}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))/\tau(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C}))$, *the group of outer automorphisms of* $\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})$. *Then*

$$\mathrm{Out}(\mathfrak{G}^{\hat{A}}_{sc}(\mathbb{C})) \cong \mathrm{Aut}(\mathcal{L}) \times \mathrm{Aut}(\hat{\Delta}).$$

Proof

See Proposition 8.3, Theorem 8.4, and Corollary 8.5 of [CC91]. □

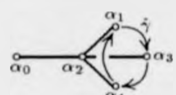
3.5 Examples

We shall give two examples to illustrate some of the applications and limitations of Theorem 3.3.3. Both examples will involve the complex simply-connected group associated to the generalized Cartan matrix of type $\tilde{D}_4$. Thus there will be some details common to both examples and we shall introduce these first.

Let D denote the root datum $D_{sc}(\tilde{D}_4)$ corresponding to $\tilde{D}_4$ introduced in §2.2, $\Phi^{re} = \Phi^{re}(\tilde{D}_4)$, and $\Pi = \{\alpha_i\}_{i\in\underline{4}_0}$ denote a fundamental system of type $\tilde{D}_4$ as labeled on $\Delta(\tilde{D}_4)$ in figure 1.2.12. Let $\mathcal{B} = \mathcal{B}(\tilde{D}_4)$ be a root bases associated to $\tilde{D}_4$ as described in §3.1 and suppose $((X_\alpha)_{\alpha\in\Phi^{re}}, N, H)$ is the root datum of type $(\mathcal{B}, \Phi^{re}, \mathbf{N})$ corresponding to $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$ introduced in §3.2.

Suppose that we are considering a single pair of automorphisms of $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$. Hence, in the notation of §3.3, $\mid \Sigma \mid = 1$ and we may omit the indexing subscripts on γ and τ since no confusion need arise.

Suppose that the automorphism denoted by γ is the graph automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$ induced by the automorphism



of $\Delta(\tilde{D}_4)$. Thus

$$\gamma : \left.\begin{array}{ccc} x_i(\mu) & \to & x_{\bar{\gamma}(i)}(\mu) \\ x_{-i}(\mu) & \to & x_{-\bar{\gamma}(i)}(\mu) \end{array}\right\} i \in \underline{4}_0$$

and we see that $\gamma(H) \subseteq H$ and $\gamma(N) \subseteq N$. Consider $\Gamma = \{1, \bar{\gamma}, \bar{\gamma}^2\}$. This is a finite subgroup of $\mathrm{Aut}(\mathcal{B})$ and stabilizes Φ^{re}. Also,

$$\gamma(X_\alpha) = X_{\bar{\gamma}(\alpha)}$$

for all $\alpha \in \Phi^{re}$.

Since Φ^{re} is reduced, $Y_\alpha = X_\alpha$ for all $\alpha \in \Phi^{re}$ and therefore $Y_\alpha \neq \{1\}$ for all $\alpha \in \Phi^{re}$.

Now, as a permutation of the indexing set of Π, $\bar{\gamma}$ has three orbits, namely

$$\{0\}, \quad \{2\}, \quad \text{and} \quad \{1,3,4\}$$

where we shall denote the last by K. All of these orbits are spherical, since the first two give rise to root systems of type A_1 and K gives rise to a root system of type $A_1 \times A_1 \times A_1$. Similarly, they are all prenilpotent (to see this for K simply take $w = r_{\alpha_0}$ and $w' = r_{\alpha_1} r_{\alpha_3} r_{\alpha_4}$ in the definition of prenilpotency).

Then $\alpha_{\{0\}} = \alpha_0$, $\alpha_{\{2\}} = \alpha_2$ and $\alpha_K = \alpha_1 + \alpha_3 + \alpha_4$. Thus, to use the notation of Theorem 3.3.3,

$$\Psi^1 = \{\alpha_0, \alpha_2, \alpha_K\}$$

and hence,

$$\Psi_{\alpha_0^1} = \{\alpha_0\}, \quad \Psi_{\alpha_2^1} = \{\alpha_2\}, \quad \text{and} \quad \Psi_{\alpha_1^1} = \Psi_{\alpha_3^1} = \Psi_{\alpha_4^1} = \Psi_{\alpha_K} = K.$$

Thus

$$\langle X_\beta : \beta \in \Psi_{\alpha_0^1}\rangle = X_{\alpha_0}, \quad \langle X_\beta : \beta \in \Psi_{\alpha_2^1}\rangle = X_{\alpha_2},$$

and

$$\langle X_\beta : \beta \in \Psi_{\alpha_K}\rangle\langle X_{\alpha_1}X_{\alpha_3}X_{\alpha_4}\rangle = X_{\alpha_1}X_{\alpha_3}X_{\alpha_4}.$$

We now proceed to consider two particular cases.

EXAMPLE 3.5.1

Suppose τ is a field automorphism of $\mathfrak{G}_D(\mathbb{C})$. Let τ be induced by the automorphism f of $\mathbb{C}$. Thus

$$\tau(x_\alpha(\mu)) = x_\alpha(f(\mu))$$

for all $\alpha \in \Phi^{re}$, from which we may deduce

$$\tau(N) \subseteq N, \quad \tau(H) \subseteq H, \quad \text{and} \quad \tau(Y_\alpha) = \tau(X_\alpha) = X_\alpha.$$

Recall that $G^{\gamma,\tau} = \langle x \in G : \gamma(x) = \tau(x)\rangle$. Thus

$$\begin{aligned} X_{\alpha_0}^1 &= G^{\gamma,\tau} \cap X_{\alpha_0} \\ &= \langle x_{\alpha_0}(\mu) : \mu \in \mathbb{C}, \mu = f(\mu)\rangle, \end{aligned}$$

$$\begin{aligned} X_{\alpha_2}^1 &= G^{\gamma,\tau} \cap X_{\alpha_2} \\ &= \langle x_{\alpha_2}(\mu) : \mu \in \mathbb{C}, \mu = f(\mu)\rangle, \quad \text{and} \end{aligned}$$

$$\begin{aligned} X_{\alpha_K}^1 &= G^{\gamma,\tau} \cap X_{\alpha_1}X_{\alpha_3}X_{\alpha_4} \\ &= \langle x_{\alpha_1}(\mu_1)x_{\alpha_3}(\mu_3)x_{\alpha_4}(\mu_4) : \mu_1, \mu_3, \mu_4 \in \mathbb{C}, \\ &\qquad x_{\alpha_1}(\mu_4)x_{\alpha_3}(\mu_1)x_{\alpha_4}(\mu_3) = x_{\alpha_1}(f(\mu_1))x_{\alpha_3}(f(\mu_3))x_{\alpha_4}(f(\mu_4))\rangle \\ &= \langle x_{\alpha_1}(\mu_1)x_{\alpha_3}(\mu_3)x_{\alpha_4}(\mu_4) : \mu_1, \mu_3, \mu_4 \in \mathbb{C}, f(\mu_1) = \mu_4, f(\mu_3) = \mu_1, f(\mu_4) = \mu_3\rangle. \end{aligned}$$

Recall that any automorphism of $\mathbb{C}$ must fix $\mathbb{Q}$. Hence $Y_{\alpha_0}^1$ and $Y_{\alpha_2}^1$ are always nontrivial. Similarly

$$X_{\alpha_K}^1 \supseteq \langle x_{\alpha_1}(\mu)x_{\alpha_3}(\mu)x_{\alpha_4}(\mu) : \mu \in \mathbb{Q}\rangle \neq \{1\}$$

and so $Y_K^1 \neq 1$. We can thus use Theorem 3.3.3 to conclude that

$$G^{\gamma,\tau} = \langle X_{\alpha_0}^1, X_{\alpha_2}^1, X_{\alpha_K}^1\rangle.$$

◇

EXAMPLE 3.5.2

Suppose now that τ is a diagonal automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$. Thus τ is of the form

$$d(\xi) \;:\; \left.\begin{array}{lcl} x_i(\mu) & \to & x_i(\mu) \\ x_{-i}(\mu) & \to & x_{-i}(\mu) \end{array}\right\} \quad i \in \underline{4}$$
$$\begin{array}{lcl} x_0(\mu) & \to & x_0(\xi\mu) \\ x_{-0}(\mu) & \to & x_{-0}(\xi^{-1}\mu) \end{array}$$

for some $\xi \in \mathbb{C}^{\times}$. We know that τ fixes n_i for $i \in \underline{4}$ and transforms n_0 into $h_0(\xi)n_0 \in N$. Thus $\tau(N) \subseteq N$. Since τ fixes each $h_i(\zeta)$ for $\zeta \in \mathbb{C}^{\times}$ we also have $\tau(H) \subseteq H$. Furthermore

$$\tau(Y_\alpha) = \tau(X_\alpha) = X_\alpha.$$

Consider now the groups $X^1_{\alpha^1}$.

$$\begin{aligned} X^1_{\alpha_0} &= G^{\gamma,\tau} \cap X_{\alpha_0} \\ &= \langle x_{\alpha_0}(\mu) : \mu \in \mathbb{C}, \mu = \xi\mu\rangle, \end{aligned}$$

$$\begin{aligned} X^1_{\alpha_2} &= G^{\gamma,\tau} \cap X_{\alpha_2} \\ &= \langle x_{\alpha_2}(\mu) : \mu \in \mathbb{C}\rangle = X_{\alpha_2}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} X^1_{\alpha_K} &= G^{\gamma,\tau} \cap X_{\alpha_1}X_{\alpha_3}X_{\alpha_4} \\ &= \langle x_{\alpha_1}(\mu_1)x_{\alpha_3}(\mu_3)x_{\alpha_4}(\mu_4) : \mu_1,\mu_3,\mu_4 \in \mathbb{C}, \\ &\qquad x_{\alpha_1}(\mu_4)x_{\alpha_3}(\mu_1)x_{\alpha_4}(\mu_3) = x_{\alpha_1}(\mu_1)x_{\alpha_3}(\mu_3)x_{\alpha_4}(\mu_4)\rangle \\ &= \langle x_{\alpha_1}(\mu)x_{\alpha_3}(\mu)x_{\alpha_4}(\mu) : \mu \in \mathbb{C}\rangle. \end{aligned}$$

Thus $X^1_{\alpha_K}$ and $X^1_{\alpha_2}$ are never trivial, but $X^1_{\alpha_0}$ is trivial whenever τ is non-trivial. So Theorem 3.3.3 is only applicable in the degenerate case when τ is the identity. ◇

Chapter 4

The Fixed Point Subgroup of a Graph×Diagonal Automorphism

This chapter is, to the best of my knowledge, entirely original.

In sections 4.1 and 4.2, we extend the results of Hée so that they apply in the case when

- $\tilde{A}$ is a simply-laced extended Cartan matrix,
- γ is a graph automorphism of $\mathfrak{G}_{\mathcal{D}}^{\tilde{A}}(\mathbb{K})$ induced by an automorphism of $\Delta(\tilde{A})$ inherited from an automorphism of $\Delta(A)$, and
- τ is a diagonal automorphism of $\mathfrak{G}_{\mathcal{D}}^{\tilde{A}}(\mathbb{K})$ of the same order as γ.

We do this by first considering the twisted root system constructed following Hée's methods and by omitting all those twisted roots whose associated root subgroups prove to be trivial. We then show that if we restrict our attention to roots with non-trivial root subgroups we can obtain results similar to those of Hée.

Section 4.3 is then concerned with a detailed study of the fixed point subgroup corresponding to $G^{\gamma\cdot\tau}$ obtained in this manner.

In §4.4 we construct isomorphisms between fixed point subgroups of minimal adjoint groups of the types envisaged and certain other Kac-Moody groups of minimal adjoint type. We conclude this section with two brief examples. Section 4.5 then generalizes the results of §4.4 to the simply-connected case.

For the sequel, we suppose $\tilde{A} = (A_{ij})_{i,j\in\underline{n}_0}$ to be a simply-laced extended Cartan matrix of type $\tilde{X}_n$ with fundamental system

$$\Pi = \{\alpha_i\}_{i\in\underline{n}_0},$$

and

$$\mathcal{B} = \mathcal{B}(\tilde{A}) = (\underline{n}_0, V, \boldsymbol{\alpha}, \boldsymbol{\varrho})$$

to be a root base constructed from $\tilde{A}$. Without loss of generality, we may assume that $A = (A_{ij})_{i,j\in\underline{n}}$ is the matrix of type X_n of which $\tilde{A}$ is an extension. Denote by $\mathcal{R}$ the unique highest root in $\Phi(A)$. Note that

$$\mathcal{R} = a_1\alpha_1 + \cdots + a_n\alpha_n$$

where the a_i are the labels on the respective nodes of $\Delta(\tilde{A})$ in Figure 1.2.12. Since A is simply-laced, this also means that

$$\mathcal{R}^\vee = a_1\alpha_1^\vee + \cdots + a_n\alpha_n^\vee.$$

4.1 A Twisted Root System with Non-Trivial Root Subgroups

The aim of the next two sections is to describe an approach which, in some particular cases, allows us to give a set of generators for G^1 even when the condition $Y_J^1 \neq \{1\}$ fails to be satisfied for some $J \in I^1$.

A Comment on Root Bases

Let $\mathcal{B} = (I, V, \boldsymbol{\alpha}, \boldsymbol{\varrho})$ be a root basis. Suppose

$$\boldsymbol{\beta} = \{\beta_i\}_{i\in I}$$

is such that, for each $i \in I$,

$$\alpha_i = c_i\beta_i$$

for some $c_i \in \mathbb{R}^\times$. Then $\boldsymbol{\beta}$ is also a basis of V. Since ϱ_i is a reflection in the hyperplane perpendicular to α_i, it is also a reflection in the hyperplane perpendicular to β_i. Hence

$$\mathcal{B}' = (I, V, \boldsymbol{\beta}, \boldsymbol{\varrho})$$

is a root prebasis satisfying $W(\mathcal{B}') = W(\mathcal{B})$, and there is a natural one-to-one correspondence between $\Phi(\mathcal{B}')$ and $\Phi(\mathcal{B})$, namely that associating $w(\beta_i)$ with $w(\alpha_i)$. Furthermore, if $c_i \in \mathbb{N}$ for all $i \in I$, then $\mathcal{B}'$ is a root basis.

Let $\tilde{\gamma}$ be an automorphism of order k of $\Delta(\tilde{A})$ inherited from an automorphism of $\Delta(A)$, i.e. such that the zeroeth vertex is fixed by $\tilde{\gamma}$. Thus the pair (A, k) will be one of $(A_l, 2)$, $(D_l, 2)$, $(D_4, 3)$ or $(E_6, 2)$ and the automorphisms $\tilde{\gamma}$ will be the automorphisms induced by those considered for Proposition 1.5.3.

We consider first the case when $\tilde{A}$ is not of type $\tilde{A}_{2l}$.

The Twisted Root System For $\tilde{A} \neq \tilde{A}_{2l}$

By Proposition 1.3.1

$$\Phi^{re}(\mathcal{B}) = \{\alpha + m\delta : \alpha \in \Phi(A), m \in \mathbb{Z}\}$$

where $\delta = \alpha_0 + \mathcal{R}$. We note that $\mathcal{R}$ is fixed by $\tilde{\gamma}$ since the height of a root is preserved, whence we deduce that δ is fixed by $\tilde{\gamma}$.

Thus the root system of the root basis

$$\mathcal{B}^1 = \left(\underline{n}_0^1, V^1, \{\alpha_J\}_{J\in\underline{n}_0^1}, \{\varrho_J\}_{J\in\underline{n}_0^1}\right)$$

described in Proposition 3.1.4 is of the form

$$\Phi^{re}(\mathcal{B}^1) = \left\{\alpha^1 + \mid K(\alpha) \mid m\delta : \alpha^1 \in \Phi(A)^1, m \in \mathbb{Z}\right\}$$

where $K(\alpha)$ denotes the $\tilde{\gamma}$-orbit of α.

For each orbit $J \in \underline{n}^1$ take a minimal representative $j \in J$, and let

$$\beta_j = \frac{\alpha_J}{\mid J \mid} = \frac{\alpha_j^1}{\mid J \mid} := \alpha_j^{\gamma}$$

except for when $A = E_6$ and $J = \{6\}$, in which case we let $\beta_4 = \alpha_6$. Denote the set of such representatives j by M. From Proposition 1.5.3, Table 1.5.4 and Tables 1.2.11–1.2.14 we see that $\{\beta_j\}_{j\in M}$ is a fundamental root system of type $\breve{X}_n^{(k)}$, where $X_n^{(k)}$ is given by Table 1.5.5. Denote the set of roots of this system by $\breve{\Phi}^{\gamma}$. Thus

$$\breve{\Phi}^{\gamma} \text{ is a root system of type } \breve{X}_n^{(k)}.$$

Denote by $\mathcal{S}$ the unique highest root in the root system of type $\breve{X}_n^{(k)}$ generated by $\{\beta_j\}_{j\in M}$ and let P be the extended matrix obtained from $\breve{X}_n^{(k)}$. Note that

$$\mathcal{S} = \sum_{j\in M} b_j'\beta_j$$

where the b_j' are the corresponding labels on the relevant nodes of $\Delta(P)$ in Figure 1.2.12.

Let $\beta_0 = \alpha_0$ and define $M_0 = M \cup \{0\}$. Let

$$\boldsymbol{\beta} = \{\beta_j\}_{j\in M_0}$$

and

$$\boldsymbol{\varrho}^{\gamma} = \{\varrho_J\}_{J\in\underline{n}_0^1}.$$

Since

$$\mathcal{B}^1 = \left(\underline{n}_0^1, V^1, \{\alpha_J\}_{J\in\underline{n}_0^1}, \boldsymbol{\varrho}^{\gamma}\right)$$

is a root basis by Proposition 3.1.4, we see that

$$\mathcal{B}^{\gamma} = \left(M_0, V^1, \boldsymbol{\beta}, \boldsymbol{\varrho}^{\gamma}\right)$$

is also a root basis.

In order to show that $\boldsymbol{\beta}$ is a fundamental system of type P, it is sufficient, by Theorem 1.2.20, Proposition 1.3.1, and Proposition 1.3.3, to show that $\mathcal{R}^1 = \mathcal{S}$. Since $\mathcal{R}$ is fixed by $\bar{\gamma}$, it is sufficient to show that

$$b'_j = \sum_{i \in J} a_i$$

which a case by case analysis verifies. Thus $\mathcal{B}^{\gamma}$ is a root basis, entirely determined by M_0 and $\boldsymbol{\beta}$, which we have shown to be isomorphic to $\mathcal{B}(P)$. Furthermore,

$$\Phi^{re}(\mathcal{B}^{\gamma}) = \left\{\alpha^{\gamma} + m\delta : \alpha^{\gamma} \in \check{\Phi}^{\gamma}, m \in \mathbb{Z}\right\}.$$

We state the types of P explicitly in Table 4.1.1.

$\tilde{A}$	$\tilde{A}_{2l-1}$, $(l > 2)$	$\tilde{D}_{l+1}$, $(l > 1)$	$\tilde{D}_4$	$\tilde{E}_6$
P	$\tilde{C}_l$	$\tilde{B}_l$	$\tilde{G}_2$	$\tilde{F}_4$

Table 4.1.1: Types P of root systems $\Phi(\tilde{A})^1$ when $\tilde{A} \neq \tilde{A}_{2l}$.

We note that we have a natural correspondence

$$\alpha^{\gamma} + m\delta \leftrightarrow \alpha^1 + \mid K(\alpha) \mid m\delta$$

between $\Phi^{re}(\mathcal{B}^{\gamma})$ and $\Phi^{re}(\mathcal{B}^1)$, where $\alpha \in \Phi(\tilde{A})$ and $K(\alpha)$ is the $\bar{\gamma}$-orbit of α.

The Nature of the Graph Automorphism When $\tilde{A} \neq \tilde{A}_{2l}$

For each root datum $\mathcal{D} = \mathcal{D}(\tilde{A})$ associated to an extended Cartan matrix $\tilde{A}$ of the type under consideration, let $\Phi^{re} = \Phi^{re}(\tilde{A})$ and

$$((X_{\alpha})_{\alpha \in \Phi^{re}}, N, H)$$

be the root datum of type $(\mathcal{B}, \Phi^{re}, \mathrm{N})$ associated to $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ constructed in Example 3.2.5.

Recall that a presentation for $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ was given in §2.5. Hence in order to describe an automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ it is sufficient to give its effect on the generators. We shall be concerned only in the cases where $\mathcal{D}$ is a root datum of minimal adjoint, adjoint or simply-connected type. Recall that we determined specific generators for groups $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ of these types in Examples 2.5.6–2.5.8. In all of these cases $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is generated by a specified subset of elements $h \in H$ and the elements $x_{\alpha_i}(\mu)$ and $x_{-\alpha_i}(\mu)$ for $i \in \underline{n}_0$ and $\mu \in \mathbb{K}$.

For each root datum $\mathcal{D}$ in question, we let

$$\begin{aligned} \mathfrak{U}_{\mathcal{D}}(\mathbb{K}) &= \langle \mathfrak{U}_+(\mathbb{K}), \mathfrak{U}_-(\mathbb{K}) \rangle \\ &= \langle x_{\alpha_i}(\mu), x_{-\alpha_i}(\mu) \; : \; i \in \underline{n}_0 \, , \, \mu \in \mathbb{K} \rangle \subseteq \mathfrak{G}_{\mathcal{D}}(\mathbb{K}). \end{aligned}$$

Note that $h_{\alpha_i}(\xi) \in \mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ for all $\alpha_i \in \Pi$ and $\xi \in \mathbb{K}^\times$.

We define a map γ on the generators of $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ by letting

$$\begin{aligned} \gamma : \; x_i(\mu) &\mapsto x_{\bar{\gamma}(i)}(\mu) \\ x_{-i}(\mu) &\mapsto x_{-\bar{\gamma}(i)}(\mu) \end{aligned}$$

for $i \in \underline{n}_0$ and $\mu \in \mathbb{K}$. Note that if

$$\begin{aligned} \alpha &= m_0\alpha_0 + m_1\alpha_1 + \cdots + m_n\alpha_n \\ &= m_0\delta + (m_1 - a_1 m_0)\alpha_1 + \cdots + (m_n - a_n m_0)\alpha_n \in \Phi^{re}(\tilde{A}) \end{aligned}$$

then

$$\begin{aligned} \bar{\gamma}(\alpha) &= m_0\alpha_0 + m_1\bar{\gamma}(\alpha_1) + \cdots + m_n\bar{\gamma}(\alpha_n) \\ &= m_0\delta + (m_1 - a_1 m_0)\bar{\gamma}(\alpha_1) + \cdots + (m_n - a_n m_0)\bar{\gamma}(\alpha_n) \in \Phi^{re}(\tilde{A}) \end{aligned}$$

since $\bar{\gamma}$ fixes α_0. Thus the coefficient of δ in $\bar{\gamma}(\alpha)$ is the same as that in α for all $\alpha \in \Phi^{re}$.

Now, for each $\mathsf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $m_i = m_{\bar{\gamma}(i)}$, the set

$$\Pi_{\mathsf{m}} = \{\alpha_i + m_i\delta\}_{i \in \underline{n}}$$

is a fundamental root system of type X_n. Since $\bar{\gamma}$ is an automorphism of $\tilde{A}$ fixing α_0

$$\bar{\gamma}(\Pi_{\mathsf{m}}) = \{\bar{\gamma}(\alpha_i) + m_i\delta\}_{i \in \underline{n}}$$

is also a fundamental root system of type X_n. Hence the complex Lie algebras associated to these systems, which we shall denote by $\mathfrak{g}_{\mathsf{m}}$ and $\gamma(\mathfrak{g}_{\mathsf{m}})$ respectively, are isomorphic. Under this isomorphism

$$e_{\alpha + m\delta} \leftrightarrow \kappa_\alpha e_{\bar{\gamma}(\alpha) + m\delta}$$

where $m \in \mathbb{Z}$ and $\kappa_\alpha \in \{-1, 1\}$ is uniquely determined by α as a result of Lemma 1.4.3. We also note that $\kappa_{-\alpha} = \kappa_\alpha$.

Consider the subgroup schemes of $\mathfrak{G}_{\mathcal{D}}$ given by

$$\mathfrak{G}_{\mathrm{m}}(\mathbb{K}) = \langle x_\alpha(\mu), x_{-\alpha}(\mu) : \alpha \in \Pi_{\mathrm{m}}, \mu \in \mathbb{K}\rangle$$

and

$$\gamma(\mathfrak{G}_{\mathrm{m}})(\mathbb{K}) = \langle x_\alpha(\mu), x_{-\alpha}(\mu) : \alpha \in \bar{\gamma}(\Pi_{\mathrm{m}}), \mu \in \mathbb{K}\rangle.$$

We have natural actions of $\mathfrak{G}_{\mathrm{m}}(\mathbb{C})$ on $\mathfrak{g}_{\mathrm{m}}$ and of $\gamma(\mathfrak{G}_{\mathrm{m}})(\mathbb{C})$ on $\gamma(\mathfrak{g}_{\mathrm{m}})$ respectively. By [Car72, Proposition 12.2.3] the map

$$x_{\alpha+m\delta}(\mu) \mapsto x_{\bar{\gamma}(\alpha)+m\delta}(\kappa_\alpha \mu)$$

for $\alpha \in \Phi(A)$ extends to a group scheme isomorphism

$$\mathfrak{G}_{\mathrm{m}} \to \gamma(\mathfrak{G}_{\mathrm{m}}).$$

Furthermore, the κ_α can be chosen so that $\kappa_\alpha = 1$ if $\alpha \in \pm\Pi$. Note that if we have an orbit of size two, the symmetry of this construction implies that

$$\kappa_\alpha = \kappa_{\bar{\gamma}(\alpha)}.$$

We then extend the map γ to the remaining elements of $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ by letting

$$\gamma : x_{\alpha+m\delta}(\mu) \mapsto x_{\bar{\gamma}(\alpha)+m\delta}(\kappa_\alpha \mu)$$

for all $\alpha \in \Phi(A)$, $m \in \mathbb{Z}$ and $\mu \in \mathbb{K}$. In order for the map

$$\gamma : \mathfrak{U}_{\mathcal{D}}(\mathbb{K}) \to \mathfrak{U}_{\mathcal{D}}(\mathbb{K})$$

so defined to be an automorphism, we must check that γ preserves the defining relations of $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$. When $\mathbb{K} = \mathbb{C}$ this follows directly from the calculations for an arbitrary graph automorphism in [CC91]. However, all the conditions that need to be checked involve equalities of integers and hence the proofs given in [CC91] extend directly to an arbitrary field. Note that $\gamma\left(h_{\alpha_i}(\xi)\right) = h_{\bar{\gamma}(\alpha_i)}(\xi)$ for all $\alpha_i \in \Pi$ and $\xi \in \mathbb{K}^\times$.

If $\mathcal{D}$ is the simply-connected root datum then $\mathfrak{U}_{\mathcal{D}}(\mathbb{K}) = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ and γ is an automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. If $\mathcal{D}$ is of adjoint or minimal adjoint type we extend γ to a map on $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ by defining

$$\gamma : h_{\varpi_i^\vee}(\xi) \mapsto h_{\bar{\gamma}(\varpi_i^\vee)}(\xi)$$

where $\bar{\gamma}(\varpi_i^\vee) = \varpi_{\bar{\gamma}(i)}^\vee$. In order to show that the map

$$\gamma : \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) \longrightarrow \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$$

so defined is an automorphism, we must show that it is well-defined on $H \cap \mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ and that the relations in $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ which are not defining relations for $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ are preserved.

Since $\bar{\gamma}$ is a graph automorphism and $\gamma\,(h_{\alpha_i}(\xi)) = h_{\bar{\gamma}(\alpha_i)}(\xi)$ for all $\alpha_i \in \Pi$ and $\xi \in \mathbb{K}^\times$ it follows that γ is well-defined. The remaining relations all hold because $\bar{\gamma}$ is a graph automorphism.

Defined in this way, straightforward calculations show

$$\gamma : \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) \longrightarrow \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$$

satisfies $\gamma(H) = H$ and $\gamma(N) = N$. We also see that

$$\gamma(X_\alpha) = X_{\bar{\gamma}(\alpha)}$$

for all $\alpha \in \Phi^{re}$.

The Nature of the Diagonal Automorphism

Recall that we are considering the cases when $\mathcal{D} = \mathcal{D}_{sc}(\hat{A})$, $\mathcal{D} = \mathcal{D}_{ad}(\hat{A})$ or $\mathcal{D} = \mathcal{D}_m(\hat{A})$. For each $\xi \in \mathbb{K}^\times$ we define a map $d(\xi)$ on the generators of $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ by letting

$$d(\xi) : x_\alpha(\mu) \mapsto x_\alpha(\xi^{m_0}\mu)$$

where

$$\begin{aligned} \alpha &= m_0\alpha_0 + m_1\alpha_1 + \cdots + m_n\alpha_n \\ &= m_0\delta + (m_1 - a_1)\alpha_1 + \cdots + (m_n - a_n)\alpha_n \in \Phi^{re}(\hat{A}) \end{aligned}$$

and $\mu \in \mathbb{K}$. Note that $d(\xi)$ acts on the elements $x_\alpha(\mu)$ as multiplication of μ by ξ taken to the power of the coefficient of δ in α. In particular, since δ is fixed by $\bar{\gamma}$, $d(\xi)$ affects elements of $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ in the same γ-orbit in the same manner. When $\mathbb{K} = \mathbb{C}$ this is precisely the diagonal automorphism of $\mathfrak{G}_{\mathcal{D}_{sc}}(\mathbb{C})$ introduced in §3.4. Hence $d(\xi)$ extends to an automorphism of $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ since all the structure constants are integers.

If $\mathcal{D}$ is the simply-connected root datum, then $\mathfrak{U}_{\mathcal{D}}(\mathbb{K}) = \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ and $d(\xi)$ is an automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. If $\mathcal{D}$ is of adjoint or minimal adjoint type, we extend the map $d(\xi)$ to the rest of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ by defining

$$d(\xi) : h_{\varpi_i^\vee}(\zeta) \mapsto h_{\varpi_i^\vee}(\zeta)$$

for all $\zeta \in \mathbb{K}^\times$.

LEMMA 4.1.2

The map $d(\xi)$ is an automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ satisfying

$$d(\xi)(H) = H, \quad d(\xi)(X_\alpha) = X_\alpha, \quad \textit{and} \quad d(\xi)(N) = N.$$

Proof

Note that the restriction of $d(\xi)$ to $\mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ is an automorphism. We may use this fact to prove the last two equalities, since $N, X_\alpha \subseteq \mathfrak{U}_{\mathcal{D}}(\mathbb{K})$ for all $\alpha \in \Phi^{re}(\dot{A})$. The fact that $d(\xi)(X_\alpha) = X_\alpha$ follows from the condition $\xi \in \mathbb{K}^\times$. We note that for all

$$\alpha = m_0\delta + (m_1 - a_1)\alpha_1 + \cdots + (m_n - a_n)\alpha_n \in \Phi^{re}$$

we have

$$d(\xi)(x_{-\alpha}(\mu)) = x_{-\alpha}(\xi^{-m_0}\mu),$$

whence we deduce that $d(\xi)$ transforms $n_0(\zeta)$ into $n_0(\xi\zeta)$ for each $\zeta \in \mathbb{K}$ and fixes $n_i(\zeta)$ for all $i \in \underline{n}$ and $\zeta \in \mathbb{K}^\times$. Thus $d(\xi)(N) = N$ since $\xi \in \mathbb{K}^\times$. Furthermore,

$$\begin{aligned} d(\xi)(h_{\alpha_0}(\zeta)) &= d(\xi)\left(n_0(\zeta)n_0^{-1}\right) \\ &= n_0(\xi\zeta)n_0(-\xi) \\ &= h_{\alpha_0}(\zeta) \end{aligned}$$

for all $\zeta \in \mathbb{K}^\times$. Thus $d(\xi)$ is well-defined on $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ since it is well-defined on $H \cap \mathfrak{U}_{\mathcal{D}}(\mathbb{K})$. The defining relations of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ are now easily seen to be preserved and so $d(\xi)$ extends to an automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. Furthermore, since $d(\xi)$ fixes every element of H, the condition $d(\xi)(H) = H$ is trivially satisfied. □

Note that $d(\xi)^{-1} = d(\xi^{-1})$.

Suppose that char $\mathbb{K} \neq 2$ and that

$\mathbb{K}$ has three cube roots of unity

if $(\dot{A}, k) = (\dot{D}_4, 3)$.

Let $\epsilon \in \mathbb{K}$ be a primitive kth root of unity

and consider the automorphism $\tau = d(\epsilon)$ of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$. By the conditions imposed on $\mathbb{K}$ such an automorphism exists and is non-trivial.

Constructing Another Twisted Root System When $\dot{A} \neq \dot{A}_{2l}$

We begin by considering the root subgroups associated to roots in the original twisted root system $\Phi^{re}(\mathcal{B}^1)$. We then consider the set obtained by omitting those roots whose root subgroups fail the non-triviality condition. We show that those roots retained form a root system of type $X_n^{(k)}$.

Since all the original root systems under consideration are reduced, we have $Y_\alpha = X_\alpha$ for all $\alpha \in \Phi^{re}$. Furthermore, all the $\bar{\gamma}$-orbits of roots form root systems of types A_1, $A_1 \times A_1$, or $A_1 \times A_1 \times A_1$ and so are prenilpotent. Hence, for each $\alpha^1 \in \Phi^{re}(\dot{A})^1$

$$\Phi^{re}_{\alpha^1} = \left\{\beta \in \Phi^{re} : \beta^1 = \alpha^1\right\}$$

is just the $\bar{\gamma}$-orbit of α. Also, given any $\bar{\gamma}$-orbit K and any two (not necessarily distinct) elements $\alpha, \beta \in K$ we have

$$x_\alpha(\mu)x_\beta(\nu) = x_\beta(\nu)x_\alpha(\mu)$$

for all $\mu, \nu \in \mathbb{K}$. Hence

$$\langle X_\beta : \beta \in \Phi^{re}_{\alpha^1} \rangle = \prod_{\beta \in \Phi^{re}_{\alpha^1}} X_\beta$$

is abelian for all $\alpha^1 \in \Phi^{re}(\tilde{A})^1$.

We now consider the elements

$$x \in \langle X_\beta : \beta \in \Phi^{re}_{\alpha^1} \rangle$$

which satisfy $\gamma(x) = \tau(x)$. Let

$$\mathfrak{G}^{\gamma,\tau}(\mathbb{K}) = \{x \in \mathfrak{G}_D(\mathbb{K}) : \gamma(x) = \tau(x)\}.$$

Denote $\bar{\gamma}(\alpha)$ by $\bar{\alpha}$ for all $\alpha \in \Phi^{re}$ and let $K(\alpha) = \Phi^{re}_{\alpha^1}$ be the $\bar{\gamma}$-orbit of $\alpha \in \Phi^{re}$.

Suppose $\alpha + m\delta \in \Phi^{re}$ with $\alpha \in \Phi(A)$. Recall that by Theorem 1.5.6 the subset of $\Phi^{re}(B^\gamma)$ consisting of roots $\alpha^\gamma + m\delta$ such that

$$\begin{array}{ll} \text{either} & |K(\alpha)| > 1 \\ \text{or} & |K(\alpha)| = 1 \quad \text{and } m \text{ is divisible by } k \end{array}$$

is a root system of type $X_n^{(k)}$. We denote this subsystem of $\Phi^{re}(B^\gamma)$ by Φ^σ or by $\Phi^\sigma(\tilde{A})$ if there is any risk of confusion.

We also exploit the one-to-one correspondence between $\Phi^{re}(B^\gamma)$ and $\Phi^{re}(B^1)$ to define

$$X_{\alpha^\gamma + m\delta} = X^1_{\alpha^1 + |K(\alpha)|m\delta}.$$

Proposition 4.1.3

Suppose $\tilde{A} \neq \tilde{A}_{2l}$, $\alpha + m\delta \in \Phi^{re}$ with $\alpha \in \Phi(A)$, $\mathfrak{g} = \mathfrak{g}_A(\mathbb{C})$ and $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{g}_A(\mathbb{C})$ described in §1.5. Then

$$X_{\alpha^\gamma + m\delta} \neq \{1\} \quad \Leftrightarrow \quad \alpha^\gamma + m\delta \in \Phi^\sigma.$$

Furthermore, we can describe the elements of $X_{\alpha^\gamma + m\delta}$.

1. *If $K(\alpha + m\delta) = \{\alpha + m\delta\}$ then*

$$X_{\alpha^\gamma + m\delta} = \begin{cases} X_{\alpha + m\delta} & \text{if } m \equiv 0 \bmod k \\ 1 & \text{otherwise.} \end{cases}$$

2. If $K(\alpha + m\delta) = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ then

$$X_{\alpha^\gamma + m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\epsilon^{-m}\mu) : \mu \in \mathbb{K}\}$$

where $\kappa \in \{-1, 1\}$ is uniquely determined by the condition

$$e_\alpha + \kappa e_{\bar{\alpha}} \in \mathfrak{g}_0.$$

3. If $K(\alpha + m\delta) = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \bar{\bar{\alpha}} + m\delta\}$, then

$$X_{\alpha^\gamma + m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\epsilon^{-m}\mu)x_{\bar{\bar{\alpha}}+m\delta}(\kappa'\epsilon^{-2m}\mu) : \mu \in \mathbb{K}\}$$

where $\kappa, \kappa' \in \{-1, 1\}$ are uniquely determined by the condition

$$e_\alpha + \kappa e_{\bar{\alpha}} + \kappa' e_{\bar{\bar{\alpha}}} \in \mathfrak{g}_0.$$

Proof

We begin by making some general comments which we will find of use in our calculations. We first note that the actions of $\mathfrak{U}_\pm(\mathbb{C})$ on $\mathfrak{g}_A(\mathbb{C})$ are faithful.

We also note that in all cases under consideration $N_{\alpha,\beta} \in \{1, -1\}$. Hence, by Theorem 1.4.4

$$N_{\alpha,\beta} = N_{-\alpha,-\beta}.$$

Furthermore, the structure constants involved in the calculations for $K(-\alpha + m\delta)$ whenever $\alpha \in \Phi_+^{re}$ and $m \in \mathbb{Z}$ depend only on those for $K(-\alpha)$ by Lemma 1.4.3, which by the above observation depend only on those for $K(\alpha)$. Thus it is sufficient to consider $\bar{\gamma}$-orbits of the form $K(\alpha + m\delta)$ for $\alpha \in \Phi_+^{re}$ and $m \in \mathbb{Z}$. We are now in a position to proceed with the proof.

Suppose $K(\alpha + m\delta) = \{\alpha + m\delta\}$. Then in all cases

$$\gamma\left(x_{\alpha+m\delta}(\mu)\right) = x_{\alpha+m\delta}(\kappa_\alpha\mu) \quad \text{and} \quad \tau\left(x_{\alpha+m\delta}(\mu)\right) = x_{\alpha+m\delta}(\epsilon^m\mu)$$

for each $\mu \in \mathbb{K}$. Thus

$$x_{\alpha+m\delta}(\mu) \in \mathfrak{G}^{\gamma\cdot\tau}(\mathbb{K}) \iff \kappa_\alpha\epsilon^{-m} = 1.$$

Hence either

$$\kappa_\alpha = 1 \text{ and } m \equiv 0 \bmod k$$

or

$$\kappa_\alpha = -1,\ \epsilon = -1 \text{ and } m \equiv 1 \bmod 2.$$

Now, property KMG5 endows $\mathfrak{G}_D(\mathbb{C})$ with an action on $\mathfrak{g}_{\hat{A}}(\mathbb{C})$ such that, for all $z \in \mathbb{C}$,

$$x_{\alpha+m\delta}(z) \text{ corresponds to the application of } \exp \operatorname{ad} z(t^m \otimes e_\alpha)$$

under the realization of $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ given in §1.4. Thus

$$\gamma\left(x_{\alpha+m\delta}(z)\right) \text{ acts as } \exp \operatorname{ad} z\kappa_\alpha(t^m \otimes e_\alpha)$$

and

$$\tau\left(x_{\alpha+m\delta}(z)\right) \text{ acts as } \exp \operatorname{ad} z\epsilon^m(t^m \otimes e_\alpha).$$

Hence we obtain the implication

$$(x_{\alpha+m\delta}(z)) \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K}) \Rightarrow \kappa_\alpha\epsilon^{-m}(t^m \otimes e_\alpha) = t^m \otimes e_\alpha.$$

We note that, in the notation of §1.5, this is precisely the necessary condition that ensures

$$t^m \otimes e_\alpha \in \hat{\mathcal{L}}(\mathfrak{g}, \gamma, k)_{\alpha^\gamma+m\delta}.$$

Thus in particular either

$$e_\alpha \in \mathfrak{g}_0 \text{ and } \alpha^\gamma + km\delta \in \Phi^{re}\left(X_n^{(k)}\right) \text{ for all } m \in \mathbb{Z}$$

or

$$e_\alpha \in \mathfrak{g}_1 \text{ and } \alpha^\gamma + (2m-1)\delta \in \Phi^{re}\left(X_n^{(k)}\right) \text{ for all } m \in \mathbb{Z} \text{ but } \alpha \notin \Phi^{re}\left(X_n^{(k)}\right).$$

By Proposition 1.3.1 only the first of these cases occurs whenever $\tilde{A} \neq \tilde{A}_{2l}$. Hence the result follows.

Suppose next that

$$K(\alpha + m\delta) = \{\alpha + m\delta, \bar{\alpha} + m\delta\}.$$

Then

$$\gamma\left(x_{\alpha+m\delta}(\mu)x_{\alpha+m\delta}(\nu)\right) = x_{\alpha+m\delta}(\kappa_\alpha\nu)x_{\bar{\alpha}+m\delta}(\kappa_\alpha\mu)$$

and

$$\tau\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)\right) = x_{\alpha+m\delta}(\epsilon^m\nu)x_{\bar{\alpha}+m\delta}(\epsilon^m\mu)$$

for all $\mu, \nu \in \mathbb{K}$. However, orbits of size two only occur if $k = 2$ and thus $\epsilon = -1$ and $\epsilon^m = \epsilon^{-m}$. Also, since we must have

$$\kappa_\alpha = \kappa_{\bar{\alpha}},$$

we obtain the single condition

$$x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu) \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad \nu = \kappa_\alpha\epsilon^{-m}\mu.$$

When $\mu, \nu \in \mathbb{K}^\times$, this is once again precisely the condition for

$$t^m \otimes (e_\alpha + \kappa_\alpha\epsilon^{-m}e_{\bar{\alpha}}) \in \hat{\mathcal{L}}(\mathfrak{g}, \gamma, k)_{\alpha^\gamma+m\delta}$$

and so

$$\alpha^{\gamma} + m\delta \in \Phi^{re}\left(X_n^{(k)}\right).$$

Considering the case of $m = 0$ we also obtain the condition

$$e_\alpha + \kappa_\alpha e_{\bar{\alpha}} \in \mathfrak{g}_0.$$

Finally, consider the case when

$$K(\alpha + m\delta) = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \bar{\bar{\alpha}} + m\delta\}.$$

Then

$$\gamma\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\bar{\bar{\alpha}}+m\delta}(\eta)\right) = x_{\alpha+m\delta}(\kappa_{\bar{\bar{\alpha}}}\eta)x_{\bar{\alpha}+m\delta}(\kappa_\alpha\mu)x_{\bar{\bar{\alpha}}+m\delta}(\kappa_{\bar{\alpha}}\nu)$$

and

$$\tau\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\bar{\bar{\alpha}}+m\delta}(\eta)\right) = x_{\alpha+m\delta}(\epsilon^m\nu)x_{\bar{\alpha}+m\delta}(\epsilon^m\mu)x_{\bar{\bar{\alpha}}+m\delta}(\epsilon^m\eta)$$

for all $\mu, \nu \in \mathbb{K}$. Hence

$$x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\bar{\bar{\alpha}}+m\delta}(\eta) \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad \nu = \kappa_\alpha\epsilon^{-m}\mu \text{ and } \eta = \kappa_{\bar{\alpha}}\epsilon^{-m}\nu$$

where we note that once again this is precisely the requirement that

$$t^m \otimes (e_\alpha + \kappa_\alpha\epsilon^{-m}e_{\bar{\alpha}} + \kappa_\alpha\kappa_{\bar{\alpha}}\epsilon^{-2m}e_{\bar{\bar{\alpha}}}) \in \tilde{\mathcal{L}}(\mathfrak{g}, \gamma, k)_{\alpha^{\gamma}+m\delta}.$$

Once again this implies that

$$\alpha^{\gamma} + m\delta \in \Phi^{re}\left(X_n^{(k)}\right)$$

and consideration of the case $m = 0$ yields

$$e_\alpha + \kappa_\alpha e_{\bar{\alpha}} + \kappa_\alpha\kappa_{\bar{\alpha}}e_{\bar{\bar{\alpha}}} \in \mathfrak{g}_0.$$

Thus the result is proved. □

We now proceed to consider the remaining case, namely $\tilde{A} = \tilde{A}_{2l}$.

The Case $\tilde{A} = \tilde{A}_{2l}$

We first make an observation about the root system of type ${}^*\hat{C}_l$. We note that there are several ways in which this root system could be expressed in terms of root systems of finite type. We concentrate on two, namely the expression in terms of δ and the roots of the finite system of type C_l obtained by omitting the zeroeth node (as in Proposition 1.3.1), and that in terms of δ and the roots of the finite system obtained by omitting the lth node (as would seem natural from the construction in Proposition 1.5.3). Suppose

$$\{\beta_i\}_{i\in\underline{l}_0}$$

is a fundamental root system of type $^*\tilde{C}_l$ conforming with the labeling of $\Delta(^*\tilde{C}_l)$ in Figure 1.2.13. Then, by Proposition 1.3.1, the roots of $\Phi^{re}(^*\tilde{C}_l)$ are given by

$$\begin{aligned}\Phi^{re}(^*\tilde{C}_l) &= \left\{\frac{1}{2}(\beta+(2m-1)\delta) \,:\, \beta\in\bar{\Phi}_l,\ m\in\mathbf{Z}\right\}\cup\left\{\beta+m\delta \,:\, \beta\in\bar{\Phi}_s,\ m\in\mathbf{Z}\right\}\\ &\qquad\cup\left\{\beta+2m\delta \,:\, \beta\in\bar{\Phi}_l,\ m\in\mathbf{Z}\right\}.\end{aligned}$$

where $\bar{\Phi}_l$ and $\bar{\Phi}_s$ are the long and short roots, respectively, of the root system of type A_1 if $l=1$ or C_l if $l>1$ generated by the elements

$$\{\beta_i\}_{i\in\underline{l}}$$

and

$$\delta = 2\beta_0+\cdots+2\beta_{l-1}+\beta_l$$

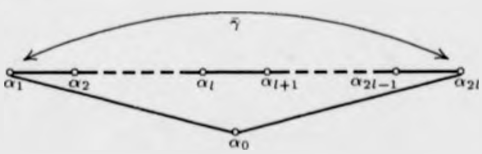
by Proposition 1.3.3. By using these equalities we can substitute for β_l and thus yield the following alternative expression of the root system $\Phi^{re}(^*\tilde{C}_l)$;

$$\Phi^{re}(^*\tilde{C}_l) = \{\beta+m\delta : \beta\in\Phi(B_l), m\in\mathbf{Z}\}\cup\{2\beta+(2m-1)\delta : \beta\in\Phi(B_l)_s, m\in\mathbf{Z}\}$$

where $\Phi(B_l)$ is the root system of type B_l generated by the elements

$$\{\beta_i\}_{i\in\underline{l-1}_0}.$$

We shall be concerned with the automorphism



of $\Delta(\tilde{A}_{2l})$. We recall that our initial root basis is

$$\mathcal{B} = \left(\underline{2l}_0, V, \{\alpha_i\}_{i\in\underline{2l}_0}, \{\varrho_i\}_{i\in\underline{2l}_0}\right).$$

By Proposition 1.3.1

$$\Phi^{re}(\mathcal{B}) = \{\alpha+m\delta : \alpha\in\Phi(A_{2l}), m\in\mathbf{Z}\}$$

where

$$\delta = \sum_{i\in\underline{2l}_0}\alpha_i = \alpha_0+\sum_{i\in\underline{l}}(\alpha_i+\alpha_{2l-i+1}),$$

obtained by omitting the lth node (as would seem natural from the construction in Proposition 1.5.3). Suppose

$$\{\beta_i\}_{i\in \underline{l}_0}$$

is a fundamental root system of type ${}^*\tilde{C}_l$ conforming with the labeling of $\Delta({}^*\tilde{C}_l)$ in Figure 1.2.13. Then, by Proposition 1.3.1, the roots of $\Phi^{re}({}^*\tilde{C}_l)$ are given by

$$\Phi^{re}({}^*\tilde{C}_l) = \left\{\frac{1}{2}(\beta+(2m-1)\delta) \,:\, \beta\in\check{\Phi}_l,\ m\in\mathbb{Z}\right\}\cup\left\{\beta+m\delta \,:\, \beta\in\check{\Phi}_s,\ m\in\mathbb{Z}\right\}$$
$$\cup\left\{\beta+2m\delta \,:\, \beta\in\check{\Phi}_l,\ m\in\mathbb{Z}\right\}.$$

where $\check{\Phi}_l$ and $\check{\Phi}_s$ are the long and short roots, respectively, of the root system of type A_1 if $l=1$ or C_l if $l>1$ generated by the elements

$$\{\beta_i\}_{i\in\underline{l}}$$

and

$$\delta = 2\beta_0+\cdots+2\beta_{l-1}+\beta_l$$

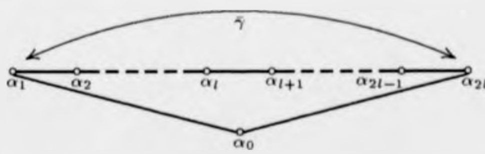
by Proposition 1.3.3. By using these equalities we can substitute for β_l and thus yield the following alternative expression of the root system $\Phi^{re}({}^*\tilde{C}_l)$;

$$\Phi^{re}({}^*\tilde{C}_l) = \{\beta+m\delta : \beta\in\Phi(B_l), m\in\mathbb{Z}\}\cup\{2\beta+(2m-1)\delta : \beta\in\Phi(B_l)_s, m\in\mathbb{Z}\}$$

where $\Phi(B_l)$ is the root system of type B_l generated by the elements

$$\{\beta_i\}_{i\in\underline{l-1}_0}.$$

We shall be concerned with the automorphism



of $\Delta(\tilde{A}_{2l})$. We recall that our initial root basis is

$$\mathcal{B} = \left(\underline{2l}_0, V, \{\alpha_i\}_{i\in\underline{2l}_0}, \{\varrho_i\}_{i\in\underline{2l}_0}\right).$$

By Proposition 1.3.1

$$\Phi^{re}(\mathcal{B}) = \{\alpha+m\delta : \alpha\in\Phi(A_{2l}), m\in\mathbb{Z}\}$$

where

$$\delta = \sum_{i\in\underline{2l}_0}\alpha_i = \alpha_0+\sum_{i\in\underline{l}}(\alpha_i+\alpha_{2l-i+1}),$$

and so is fixed by $\bar{\gamma}$. Thus

$$\Phi^{re}(\mathcal{B}^1) = \left\{\alpha^1 + \mid K(\alpha) \mid m\delta : \alpha^1 \in \Phi(A_{2l})^1, m \in \mathbb{Z}, \right\}$$

where $\mathcal{B}^1$ is the root basis identified in Proposition 3.1.4 and $K(\alpha)$ denotes the $\bar{\gamma}$-orbit of α. Let

$$\beta_i = \frac{1}{2}(\alpha_{l-i} + \alpha_{l+i+1}) = \alpha_i^{\gamma}$$

for $i \in \underline{l-1}$ and $\beta_0 = \alpha_l + \alpha_{l+1}$. We denote by $\breve{\mathcal{B}}$ the root basis corresponding to $\{\alpha_i\}_{i \in \underline{2l}}$ which embeds into $\mathcal{B}$ in the natural manner and by $\breve{\mathcal{B}}^1$ the twisted root basis induced by $\bar{\gamma}$. Then we can use the set

$$\boldsymbol{\beta} = \{\beta_i\}_{i \in \underline{l-1}_0}$$

to construct a new root basis $\breve{\mathcal{B}}^{\gamma}$ such that $W(\breve{\mathcal{B}}^{\gamma}) = W(\breve{\mathcal{B}}^1)$. By [Ste67, Theorem 32] we know that $\boldsymbol{\beta}$ is a fundamental system of type Q and

$$\breve{\Phi}^{\gamma} = \Phi(\breve{\mathcal{B}}^{\gamma})$$

is a non-reduced root system of type T given by Table 4.1.4.

	Q	T
$l = 1$	A_1	A_1
$l > 1$	B_l	BC_l

Table 4.1.4: Types of root systems $\breve{\Phi}^{\gamma}$.

Furthermore

$$\delta = \alpha_0 + \mathcal{S}$$

where $\mathcal{S}$ is the unique highest root in the system $\breve{\Phi}^{\gamma}$. Hence, by letting $\beta_l = \alpha_0$ we can construct a root basis $\mathcal{B}^{\gamma}$ from $\mathcal{B}^1$ such that $W(\mathcal{B}^{\gamma}) = W(\mathcal{B}^1)$,

$$\Phi^{re}(\mathcal{B}^{\gamma}) = \left\{\alpha^{\gamma} + m\delta : \alpha^{\gamma} \in \breve{\Phi}^{\gamma}, m \in \mathbb{Z}\right\}$$

and there is a one-to-one correspondence between $\Phi^{re}(\mathcal{B}^{\gamma})$ and $\Phi^{re}(\mathcal{B}^1)$ such that

$$\alpha^{\gamma} + m\delta \leftrightarrow \alpha^1 + \mid K(\alpha) \mid m\delta.$$

For each root datum $\mathcal{D} = \mathcal{D}(\tilde{A}_{2l})$, let $\Phi^{re} = \Phi^{re}(\tilde{A}_{2l})$ and

$$((X_\alpha)_{\alpha \in \Phi^{re}}, N, H)$$

be the root datum of type $(\mathcal{B}, \Phi^{re}, \mathbb{N})$ associated to $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ constructed in Example 3.2.5. Once again, we limit ourselves to the cases when $\mathcal{D}$ is of simply-connected, adjoint, or minimal adjoint type. In these cases we define a graph automorphism γ such that

$$\gamma : \begin{array}{lcl} \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) & \to & \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) \\ x_i(\mu) & \mapsto & x_{\bar{\gamma}(i)}(\mu) \\ x_{-i}(\mu) & \mapsto & x_{-\bar{\gamma}(i)}(\mu) \\ x_0(\mu) & \mapsto & x_0(-\mu) \\ x_{-0}(\mu) & \mapsto & x_{-0}(-\mu) \\ h_{\varpi_j^\vee}(\xi) & \mapsto & h_{\varpi^\vee_{\bar{\gamma}(j)}}(\xi) \end{array} \quad \left.\right\} i \in \underline{2l}$$

for $\mu \in \mathbb{K}$, $\xi \in \mathbb{K}^\times$ and fundamental coweights $\varpi_j^\vee$. Thus the coefficient of δ in $\bar{\gamma}(\alpha) = \bar{\alpha}$ is the same as that in α for all $\alpha \in \Phi^{re}$. Also, given $\alpha + m\delta \in \Phi^{re}$ with $\alpha \in \Phi(A_{2l})$, we define κ_α in a manner analogous to that used for the previous cases. We note that

$$\kappa_{\alpha_i+\cdots+\alpha_{2l-i+1}} = -1$$

for all $i \in \underline{l}$ and $\kappa_\alpha = \kappa_{\bar{\alpha}}$ for all $\alpha \in \Phi^{re}$. We once again deduce that γ is an automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ such that

$$\gamma\left(x_{\alpha+m\delta}(\mu)\right) = x_{\bar{\alpha}+m\delta}(\kappa_\alpha \mu).$$

We assume char $\mathbb{K} \neq 2$ and define a diagonal automorphism $d(-1)$ of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ as for the other cases where γ has order 2.

Since $\Phi^{re}(\bar{A}_{2l})$ is a reduced root system we once again have $Y_\alpha = X_\alpha$ in the notation of §3.2. All the $\bar{\gamma}$-orbits of Φ^{re} form root systems of types A_1, $A_1 \times A_1$ or A_2, and hence are prenilpotent. Define

$$K(\alpha) = \{\bar{\gamma}^j(\alpha)\}_{j \in \underline{2}}.$$

Suppose $\alpha \in \Phi^{re}$. Whenever $K(\alpha)$ forms a root system of type A_1 or $A_1 \times A_1$,

$$\Phi^{re}_{\alpha^1} = K(\alpha)$$

and the root subgroups corresponding to the roots in the orbit generate an abelian group. However, if $K(\alpha)$ forms a root system of type A_2, then

$$\Phi^{re}_{\alpha^1} = \{\alpha,\ \bar{\alpha},\ \alpha + \bar{\alpha}\}$$

and the group

$$X = \langle X_\beta : \beta \in \Phi^{re}_{\alpha^1} \rangle$$

is not abelian. Nevertheless, since $x_{\alpha+\bar{\alpha}}(\eta)$ commutes with both $x_\alpha(\mu)$ and $x_{\bar{\alpha}}(\nu)$ for all $\eta, \mu, \nu \in \mathbb{K}$, every element of X may be expressed as a product of the form

$$x_\alpha(\mu)x_{\bar{\alpha}}(\nu)x_{\alpha+\bar{\alpha}}(\eta)$$

for some $\mu, \nu, \eta \in \mathbb{K}$. Furthermore, by Proposition 2.3.1, such an expression is unique.

Suppose $\alpha + m\delta \in \Phi^{re}(\tilde{A}_{2l})$ with $\alpha \in \Phi(A_{2l})$. Recall that by Theorem 1.5.6 the subset of $\Phi^{re}(\mathcal{B}^\gamma)$ consisting of roots $\alpha^\gamma + m\delta$ such that

$$\begin{array}{lll} \text{either} & \mid K(\alpha) \mid > 1 & \\ \text{or} & \mid K(\alpha) \mid = 1 & \text{and } m \text{ is odd} \end{array}$$

is a root system of type ${}^*\tilde{A}_1$ if $l = 1$ or of type ${}^*\tilde{C}_l$ if $l > 1$. Denote this subsystem of $\Phi^{re}(\mathcal{B}^\gamma)$ by Φ^σ or $\Phi^\sigma(\tilde{A})$ if there is any danger of confusion. We again exploit the one-to-one correspondence between $\Phi(\mathcal{B}^1)$ and $\Phi(\mathcal{B}^\gamma)$ and define

$$X_{\alpha^\gamma + m\delta} = X^1_{\alpha^1 + |K(\alpha)| m\delta}.$$

We are now in a position to prove the following result.

PROPOSITION 4.1.5

Suppose $\alpha + m\delta \in \Phi^{re}(\tilde{A}_{2l})$ with $\alpha \in \Phi(A_{2l})$. Then

$$X_{\alpha^\gamma + m\delta} \neq \{1\} \quad \Leftrightarrow \quad \alpha^\gamma + m\delta \in \Phi^\sigma.$$

Furthermore, we can describe the elements of $X_{\alpha^\gamma + m\delta}$.

1. *If $\Phi^{re}_{\alpha + m\delta^1} = \{\alpha + m\delta\}$ then*

$$X_{\alpha^\gamma + m\delta} = \begin{cases} X_{\alpha + m\delta} & \text{if } m \equiv 1 \bmod 2 \\ 1 & \text{otherwise.} \end{cases}$$

2. *If $\Phi^{re}_{\alpha + m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ then*

$$X_{\alpha^\gamma + m\delta} = \{x_{\alpha + m\delta}(\mu) x_{\bar{\alpha} + m\delta}((-1)^m \kappa\mu) : \mu \in \mathbb{K}\}$$

where $\kappa \in \{-1, 1\}$ is uniquely determined by the condition

$$e_\alpha + \kappa e_{\bar{\alpha}} \in \mathfrak{g}_0.$$

3. *If $\Phi^{re}_{\alpha + m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \alpha + \bar{\alpha} + 2m\delta\}$ then*

$$X_{\alpha^\gamma + m\delta} = \{x_{\alpha + m\delta}(\mu) x_{\bar{\alpha} + m\delta}((-1)^m \kappa\mu) x_{\alpha + \bar{\alpha} + 2m\delta}(-(-1)^m \kappa\mu^2/2) : \mu \in \mathbb{K}\}$$

where $\kappa \in \{-1, 1\}$ is uniquely determined by the condition

$$e_\alpha + \kappa e_{\bar{\alpha}} \in \mathfrak{g}_0.$$

Furthermore, if we define

$$x_{\alpha^\gamma + m\delta}(\mu) = x_{\alpha + m\delta}(\mu) x_{\bar{\alpha} + m\delta}((-1)^m \kappa\mu) x_{\alpha + \bar{\alpha} + 2m\delta}(-(-1)^m \kappa\mu^2/2)$$

for each $\mu \in \mathbb{K}$, then

$$x_{\alpha^\gamma + m\delta}(\mu) x_{\alpha^\gamma + m\delta}(\nu) = x_{\alpha^\gamma + m\delta}(\mu + \nu)$$

for all $\mu, \nu \in \mathbb{K}$.

Proof

All the comments we made at the beginning of our proof of Proposition 4.1.3 are still relevant and so we once again restrict our attention to $\bar{\gamma}$-orbits of the type $K(\alpha + m\delta)$ where $\alpha \in \Phi_+^{re}$ and $m \in \mathbb{Z}$.

We prove the result about the root systems by constructing the elements of the groups $X_{\alpha^\gamma + m\delta}$.

We begin by noting that any single element orbit is of the form

$$\{\alpha_i + \cdots + \alpha_{2l-i+1} + m\delta\}$$

for some $i \in \underline{l}$. Let $\alpha = \alpha_i + \cdots + \alpha_{2l-i+1}$. Then

$$\gamma(x_{\alpha+m\delta}(\mu)) = x_{\alpha+m\delta}(-\mu)$$

and

$$\tau(x_{\alpha+m\delta}(\mu)) = x_{\alpha+m\delta}((-1)^m\mu)$$

for any $\mu \in \mathbb{K}$. Thus

$$x_{\alpha+m\delta}(\mu) \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad m \text{ is odd.}$$

We note that this is precisely the condition that

$$t^m \otimes e_\alpha \in \hat{\mathcal{L}}(\mathfrak{g}, \gamma, 2)_{2\beta_0+\cdots+2\beta_{l-i}+(m-1)\delta}.$$

Any orbit of type $A_1 \times A_1$ is of the form

$$\{\alpha_i + \cdots + \alpha_j + m\delta,\ \alpha_{2l-j+1} + \cdots + \alpha_{2l-i+1} + m\delta\}$$

for some $i \in \underline{l-1}$, $j \in \underline{2l} \setminus \{l\}$. Let $\alpha = \alpha_i + \cdots + \alpha_j$ and $\alpha_{2l-j+1} + \cdots + \alpha_{2l-i+1}$. Then

$$\gamma(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)) = x_{\alpha+m\delta}((-1)^{j-i}\mu)x_{\bar{\alpha}+m\delta}((-1)^{j-i}\nu)$$

and

$$\tau(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)) = x_{\alpha+m\delta}((-1)^m\mu)x_{\bar{\alpha}+m\delta}((-1)^m\nu)$$

for any $\mu, \nu \in \mathbb{K}$. Thus

$$x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu) \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad \nu = (-1)^{m+j-i}\mu.$$

We note that this is precisely the condition that $\alpha^\gamma + m\delta \in \Phi^{re}({}^*\tilde{C}_l)$ and consideration of the case $m = 0$ yields

$$e_\alpha + (-1)^{j-i}e_{\bar{\alpha}} \in \mathfrak{g}_0.$$

For the last of the cases, we note that any orbit of type A_2 is of the form

$$\{\alpha_i + \cdots + \alpha_l + m\delta,\ \alpha_{l+1} + \cdots + \alpha_{2l-i+1} + m\delta\}$$

for some $i \in \underline{l}$. Let $\alpha = \alpha_i + \cdots + \alpha_l$ and $\bar{\alpha} = \alpha_{l+1} + \cdots + \alpha_{2l-i+1}$. Then

$$\begin{aligned}
&\gamma(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)) \\
&\quad = x_{\bar{\alpha}+m\delta}((-1)^{l-i}\mu)x_{\alpha+m\delta}((-1)^{l-i}\nu)x_{\alpha+\bar{\alpha}+2m\delta}(-\eta) \\
&\quad = x_{\alpha+m\delta}((-1)^{l-i}\nu)x_{\alpha+m\delta}((-1)^{l-i}\mu)x_{\alpha+\bar{\alpha}+2m\delta}(-\mu\nu)x_{\alpha+\alpha+2m\delta}(-\eta) \\
&\quad = x_{\alpha+m\delta}((-1)^{l-i}\nu)x_{\bar{\alpha}+m\delta}((-1)^{l-i}\mu)x_{\alpha+\bar{\alpha}+2m\delta}(-\eta-\mu\nu)
\end{aligned}$$

and

$$\tau(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)) = x_{\alpha+m\delta}((-1)^m\mu)x_{\bar{\alpha}+m\delta}((-1)^m\nu)x_{\alpha+\alpha+2m\delta}(\eta)$$

for any $\mu, \nu, \eta \in \mathbb{K}$. Thus

$$\begin{aligned}
&x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta) \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K}) \\
&\qquad \Leftrightarrow \quad \nu = (-1)^{m+l-i}\mu \text{ and } \eta = \frac{-(-1)^{m+l-i}\mu^2}{2}.
\end{aligned}$$

We note that η is uniquely determined by μ and ν. Furthermore, the condition imposed on μ and ν corresponds precisely to the condition

$$e_\alpha + (-1)^{l-i}e_{\bar{\alpha}} \in \mathfrak{g}_0.$$

Thus it remains only to show that the necessary condition is satisfied by any two generators of $X_{\alpha^\gamma+m\delta}$. Let $\kappa' = (-1)^{m+l-i}$. Then

$$\begin{aligned}
&x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa'\mu)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\mu^2/2)x_{\alpha+m\delta}(\nu)x_{\bar{\alpha}+m\delta}(\kappa'\nu)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\nu^2/2) \\
&\quad = x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa'\mu)x_{\alpha+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\mu^2/2) \\
&\qquad\qquad x_{\bar{\alpha}+m\delta}(\kappa'\nu)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\nu^2/2) \\
&\quad = x_{\alpha+m\delta}(\mu)x_{\alpha+m\delta}(\nu)x_{\bar{\alpha}+m\delta}(\kappa'\mu)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\mu\nu/2)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\mu^2/2) \\
&\qquad\qquad x_{\bar{\alpha}+m\delta}(\kappa'\nu)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'\nu^2/2) \\
&\quad = x_{\alpha+m\delta}(\mu+\nu)x_{\bar{\alpha}+m\delta}(\kappa'(\mu+\nu))x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'(\mu^2/2+\mu\nu+\nu^2/2)) \\
&\quad = x_{\alpha+m\delta}(\mu+\nu)x_{\bar{\alpha}+m\delta}(\kappa'(\mu+\nu))x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa'(\mu+\nu)^2/2)
\end{aligned}$$

since $\operatorname{char}\mathbb{K} \neq 2$, and the result is proved. □

4.2 Groups Fixed by the Graph×Diagonal Automorphism

We now return to the general situation, where $\tilde{A}$ is any simply-laced extended Cartan matrix.

The Root Base and Weyl Group of the New Twisted Root System

Let Γ be the subgroup of $\mathrm{Aut}(\mathcal{B})$ consisting of the powers of $\bar{\gamma}$ and W^{Γ} be the subgroup of $W(\tilde{A})$ consisting of elements fixed by Γ. By 3.1.4,

$$W^{\Gamma} \cong W(\mathcal{B}^1) = W(\mathcal{B}^{\gamma}).$$

The following result is essentially a more general form of the first statement in Proposition 3.1.4. The proof is nonetheless entirely analogous.

LEMMA 4.2.1

For any $\bar{\gamma}$-orbit $K \subseteq \Phi^{re}$, let $\mathcal{B}_K$ denote the spherical root basis constructed from it. Let w_K denote the unique longest element of $W(\mathcal{B}_K)$. Then w_K induces on V^1 a reflection in the hyperplane perpendicular to the vector α^1.

Proof

By Lemma 3.1.2, $w_K(\alpha^1) = -\alpha^1$. So it is sufficient to show that for any $J \in \underline{n}_0$

$$v = w_K(\alpha_J) - \alpha_J \in \mathbb{K}\alpha^1.$$

Now $v \in V_K$ and is fixed by Γ since Γ fixes w_K and α_J. Hence $v \in \mathbb{K}\alpha^1$. □

Consider the subsystem Φ^{σ} of $\Phi^{re}(\mathcal{B}^{\gamma})$ we constructed in §4.1. Recall that $\Phi^{\sigma}(\tilde{X}_n)$ is a root system of type $X_n^{(k)}$. The reflections corresponding to these roots form a Coxeter group of type $X_n^{(k)}$, which we shall denote by W^{σ}. Thus, in particular,

$$w^{\sigma}(\alpha^{\gamma}) \in \Phi^{\sigma} \text{ whenever } w^{\sigma} \in W^{\sigma}, \alpha^{\gamma} \in \Phi^{\sigma}.$$

We note that in all cases Φ^{σ} is a reduced root system. Let

$$\Pi^{\sigma} = \left\{ \beta_l = \delta - \sum_{i \in \underline{l-1}_0} 2\beta_i \,,\ \beta_i \ \ i \in \underline{l-1}_0 \right\}$$

if $\tilde{A} = \tilde{A}_{2l}$ and

$$\Pi^{\sigma} = \left\{ \beta_0 = \delta - \sum_{i \in M} b_i \beta_i \,,\ \beta_i \ \ i \in M \right\}$$

if $\tilde{A} \neq \tilde{A}_{2l}$, where the elements b_i are the labels on the corresponding nodes of $\Delta\left(X_n^{(k)}\right)$ in Figures 1.2.13–1.2.14.

LEMMA 4.2.2

Suppose $j \in M_0$ represents the $\bar{\gamma}$-orbit J in Π^{σ} and let $r_{\beta_j} = w_J$ for each $j \in M_0$. Then the pair

$$\left(W^{\sigma}, \{r_{\beta_j}\}_{j \in M_0}\right)$$

is a Coxeter system. In particular

$$W_J \cap W^\sigma = \{1, w_J\}$$

for all $j \in M_0$.

Proof

By the results of §1.5, the set Π^σ is a fundamental root system for Φ^σ. Thus the elements of W^σ corresponding to these roots form a set of Coxeter generators of W^σ.

□

A Generalization of Hée's Theorem

Let V^σ be the subspace of V^1 generated by Π^σ and, for each $j \in M_0$, denote by ϱ_j^σ the automorphism of V^σ induced by ϱ_J where j is the representative of J in M_0. Let

$$\boldsymbol{\varrho}^\sigma = \{\varrho_j^\sigma\}_{j \in M_0}.$$

Then

$$\mathcal{B}^\sigma = (M_0, V^\sigma, \Pi^\sigma, \boldsymbol{\varrho}^\sigma)$$

is a root base of type $X_n^{(k)}$. Furthermore,

$$W^\sigma \cong W(\mathcal{B}^\sigma).$$

Recall that

$$((X_\alpha)_{\alpha \in \Phi^{re}}, N, H)$$

is the root datum of type $(\mathcal{B}, \Phi^{re}, \mathbb{N})$ associated to $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ constructed in Example 3.2.5. Let $\mathfrak{G}^\sigma(\mathbb{K})$ be a subgroup of $\mathfrak{G}^{\gamma,\tau}(\mathbb{K})$ such that

$$\langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle \subseteq \mathfrak{G}^\sigma(\mathbb{K})$$

and let

$$N^\sigma = N \cap \mathfrak{G}^\sigma(\mathbb{K}) \qquad \text{and} \qquad H^\sigma = H \cap \mathfrak{G}^\sigma(\mathbb{K}).$$

We note that $H^\sigma \subseteq N^\sigma$.

Proposition 4.2.3

The triplet

$$((X_{\alpha^\gamma})_{\alpha^\gamma \in \Phi^\sigma}, N^\sigma, H^\sigma)$$

is a root datum of type $(\mathcal{B}^\sigma, \Phi^\sigma, \mathbb{N})$ associated to $\mathfrak{G}^\sigma(\mathbb{K})$.

Proof

We follow an analogous line of reasoning to that used by Hée in his proof of Theorem 3.3.3.

We recall first that $\gamma(H) = H$, $\gamma(N) = N$, $\tau(H) = H$, and $\tau(N) = N$. Also, since

$$\gamma(X_\alpha) = X_{\gamma(\alpha)} \quad \text{and} \quad \tau(X_\alpha) = X_\alpha,$$

both γ and τ fix U, U_-, B, and B_-. Both γ and τ thus satisfy the conditions of Lemma 3.3.2, with $\phi_\gamma = \bar{\gamma}$ and ϕ_τ being the identity automorphism of $\mathcal{B}$. Hence

$$w_{\gamma(n)} = \bar{\gamma} w_n \bar{\gamma}^{-1}, \quad w_{\tau_\sigma(n)} = w_n,$$

$$\gamma(X^-_{w_n}) \subseteq X^-_{w_{\gamma(n)}}, \quad \text{and} \quad \tau(X^-_{w_n}) \subseteq X^-_{w_n}$$

where the elements $\bar{\gamma}$ and w_n are identified with the elements of $GL(V)$ they induce. Now, if $n \in N \cap \mathfrak{G}^{\gamma,\tau}(\mathbb{K})$ then $\gamma(n) = \tau(n)$ and so

$$\bar{\gamma} w_n \bar{\gamma}^{-1} = w_n,$$

leading us to the conclusion that $w_n \in W^\Gamma$.

Suppose that $x \in \mathfrak{G}_\mathcal{D}(\mathbb{K})$ has Bruhat decomposition (u_1, n, u). By Lemma 3.3.2 the Bruhat decompositions of $\gamma(x)$ and $\tau(x)$ are then

$$(\gamma(u_1), \gamma(n), \gamma(u))$$

and

$$(\tau(u_1), \tau(n), \tau(u))$$

respectively. Thus, if $x \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K})$ then we must have $u_1, n, u \in \mathfrak{G}^{\gamma,\tau}(\mathbb{K})$ and $w_n \in W^\Gamma$.

Recall that

$$U = \langle X_\alpha : \alpha \in \Phi^{re}_+ \rangle$$

and let

$$U^\sigma = U \cap \mathfrak{G}^{\gamma,\tau}(\mathbb{K}).$$

Since $(X_\alpha)_{\alpha \in \Phi_+}$ is a positive root system in U and the conditions of Proposition 3.3.1 are satisfied we conclude that $(X^1_{\alpha^1})_{\alpha^1 \in \Phi^1_+}$ is a positive root system of type $\left(\mathcal{B}^1, \Phi^1_+, \mathbb{N}\right)$ in U^1. Thus we deduce that

$$(X_{\alpha^\gamma})_{\alpha^\gamma \in \Phi^\sigma_+}$$

is a positive root system of type $\left(\mathcal{B}^\sigma, \Phi^\sigma_+, \mathbb{N}\right)$ in U^σ. Thus

$$U^\sigma = \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma_+ \rangle \subseteq \mathfrak{G}^\sigma(\mathbb{K}).$$

Similarly,

$$U^\sigma_- = \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma_- \rangle \subseteq \mathfrak{G}^\sigma(\mathbb{K})$$

where $U_-^\sigma = U_- \cap \mathfrak{G}^{\gamma,\tau}(\mathbb{K})$.

We are now in a position to demonstrate that the conditions RDG1–RDG4 hold. We begin by showing that condition RDG4 holds.

Since $\mathcal{B}^\sigma$ is reduced in all cases,

$$Y_{-\beta_j} = X_{-\beta_j} \subseteq \mathfrak{G}^\sigma(\mathbb{K})$$

for all $j \in M_0$. Recall that β_j corresponds to a spherical $\bar{\gamma}$-orbit, J say, of elements of Φ^{re}. For each element $\alpha \in J$, denote by ϱ_α the reflection of V in the hyperplane perpendicular to α, and by r_α the element of W corresponding to ϱ_α. Let

$$W_J = \langle r_\alpha : \alpha \in J \rangle$$

and $N_J = \{n \in N : w_n \in W_J\}$. Denote by w_J the unique longest element in W_J.

We thus require to show that

$$\{1\} \neq X_{-\beta_j} \subseteq X_{\beta_j} N_j X_{\beta_j}$$

where $N_j = \{n \in N^\sigma : w_n = w_J\}$. Now $X_{-\beta_j} \neq \{1\}$ by construction. Hence let

$$x \in X_{-\beta_j} \setminus \{1\}.$$

We consider the Bruhat decomposition of x in L_J. Suppose

$$x = u_1 n u$$

with $u_1 \in X_J$, $n \in N_J$ and $u \in X_J \cap X_{w_n}^-$. Hence $u_1, u \in U^\sigma \subseteq \mathfrak{G}^\sigma(\mathbb{K})$, and we deduce that

$$n = u_1^{-1} x u^{-1} \in \mathfrak{G}^\sigma(\mathbb{K}) \text{ and } w_n \in W^\Gamma.$$

Thus $n \in N_J^\sigma = N_J \cap \mathfrak{G}^\sigma(\mathbb{K})$ and $w_n \in W_J \cap W^\Gamma$. Now, by Lemma 4.2.2,

$$W_J \cap W^\Gamma = \{1, w_J\}.$$

Suppose, if possible, that $w_n = 1$. Then $X_{w_n}^- = \{1\}$, and so the Bruhat decomposition of x would reduce to

$$x = u_1 n \in UH$$

since $\ker \pi = H$. However,

$$x \in X_{-\beta_j} \setminus 1 \subseteq U_- \setminus \{1\}$$

by assumption. Hence

$$x \in UH \cap U_- = \{1\},$$

which is a contradiction. Thus we must have $w_n \neq \{1\}$, whence $w_n = w_J = r_{\beta_j}$, and condition RDG4 is satisfied.

We proceed to establish the existence of an epimorphism

$$\pi^\sigma : N^\sigma \to W(\mathcal{B}^\sigma).$$

Let π^σ be the composition of

$$\pi : N \to W$$

and the canonical isomorphism

$$\begin{array}{ccc} W^\sigma & \to & W(\mathcal{B}^\sigma) \\ w & \mapsto & w^\sigma. \end{array}$$

Now

$$W^\sigma = \langle r_{\beta_j} : j \in M_0 \rangle$$

and, for each $j \in M_0$, $r_{\beta_j} \in \pi(N^\sigma)$. Thus π^σ is surjective. Furthermore

$$\ker \pi^\sigma = (\ker \pi) \cap N^\sigma = H \cap N^\sigma = H^\sigma.$$

Thus π^σ is an epimorphism with kernel H^σ. Suppose $n \in N^\sigma$, $\alpha^\gamma \in \Phi^\sigma$ and

$$\beta^\gamma = w_n^\sigma(\alpha^\gamma).$$

Since $w_n \in W^\Gamma$, we have

$$w_n(\Phi^{re}_{\alpha^1}) = \Phi^{re}_{\beta^1}.$$

Thus

$$n(X_{\alpha^\gamma})n^{-1} = n\left(X_{\Phi^{re}_{\alpha^1}} \cap \mathfrak{G}^\sigma(\mathbb{K})\right)n^{-1} = X_{\Phi^{re}_{\beta^1}} \cap \mathfrak{G}^\sigma(\mathbb{K}) = X_{\beta^\gamma}.$$

Hence π^σ satisfies all of the conditions of RDG3.

We now proceed to demonstrate that

$$\mathfrak{G}^\sigma(\mathbb{K}) = \langle H^\sigma, X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle.$$

Suppose that $x \in \mathfrak{G}^\sigma(\mathbb{K})$ has Bruhat decomposition (u_1, n, u) in $\mathfrak{G}^{\gamma,\tau}(\mathbb{K})$. Then we have shown that in fact

$$u_1, u \in \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle \qquad \text{and} \qquad n \in N^\sigma.$$

Thus it is sufficient to show that

$$N^\sigma \subseteq \langle H^\sigma, X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle.$$

Since $W^\sigma = \langle r_{\beta_j} : j \in M_0 \rangle$ and $N_j = N^\sigma \cap (\pi^\sigma)^{-1}(r_{\beta_j})$ is non-empty for each $j \in M_0$, we deduce that

$$N^\sigma \subseteq \langle H^\sigma, N_j : j \in M_0 \rangle$$

from the properties of π^σ.

We have also shown that

$$\emptyset \neq X_{-\beta_j} \setminus \{1\} \subseteq X_{\beta_j} N_j X_{\beta_j}$$

and we know that

$$\langle X_{\beta_j}, X_{-\beta_j} \rangle \subseteq \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle \subseteq \mathfrak{G}^\sigma(\mathbb{K}).$$

Hence at least one element $n \in N_j$ satisfies

$$n = u_1^{-1} x u^{-1} \in \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle$$

and, since $\ker \pi^\sigma = H^\sigma$, this means

$$N_j \subseteq \langle H^\sigma, X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle$$

and consequently

$$N^\sigma \subseteq \langle H^\sigma, X_{\alpha^\gamma} : \alpha^\gamma \in \Phi^\sigma \rangle$$

as required. Thus we have shown that

$$\mathfrak{G}^\sigma(\mathbb{K}) = \langle H^\sigma, X_{\alpha^\gamma} : \alpha \in \Phi^\sigma \rangle$$

and, in particular,

$$\mathfrak{G}^\sigma(\mathbb{K}) = \langle H^\sigma, U^\sigma, U_-^\sigma \rangle.$$

We note that the conditions

$$U^\sigma H^\sigma \cap U_-^\sigma = \{1\} = U^\sigma \cap U_-^\sigma H^\sigma$$

follow directly from the corresponding conditions on U, U_- and H. Thus condition RDG1 is also satisfied.

Finally, condition RDG2 follows from Proposition 3.3.1, the fact that $(X_{\alpha^\gamma})_{\alpha^\gamma \in \Phi_+^\sigma}$ is a positive root system of type $\left(B^\sigma, \Phi_+^\sigma, \mathbb{N}\right)$ in U^σ, and the conditions on positive root systems. □

The following result is a direct corollary of Propositions 3.2.3 and 4.2.3.

Corollary 4.2.4

Let $\mathfrak{G}^\sigma(\mathbb{K})$, N^σ, and H^σ be as in Proposition 4.2.3. Define

$$B^\sigma = \langle H^\sigma, X_{\alpha^\gamma} : \alpha^\gamma \in \Phi_+^\sigma \rangle$$

and

$$B_-^\sigma = \langle H^\sigma, X_{\alpha^\gamma} : \alpha^\gamma \in \Phi_-^\sigma \rangle.$$

Then (B^σ, N^σ) and (B_-^σ, N^σ) are two (B, N)-pairs in $\mathfrak{G}^\sigma(\mathbb{K})$.

The property

$$\mathfrak{G}^\sigma(\mathbb{K}) = \langle H^\sigma, X_{\alpha^\gamma} : \alpha \in \Phi^\sigma \rangle$$

which we established whilst proving Proposition 4.2.3 will be of great use in the sequel. When required we shall refer to this as the *generation property* of the groups $\mathfrak{G}^\sigma(\mathbb{K})$.

The Groups H^σ

In order to study the structure of $\mathfrak{G}^\sigma(\mathbb{K})$, and in particular its (B, N)-structure, it is necessary to study the subgroup H^σ in more detail. Since $H^\sigma = H \cap \mathfrak{G}^\sigma(\mathbb{K})$, we begin by considering

$$H^{\gamma,\tau} = H \cap \mathfrak{G}^{\gamma,\tau}(\mathbb{K}).$$

Recall that, if $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is a Kac-Moody group corresponding to a root datum

$$\mathcal{D} = \left(\Lambda, \{\tilde{\alpha}_i\}_{i\in\underline{n}_0}, \{\alpha_i^\vee\}_{i\in\underline{n}_0}\right)$$

associated to $\tilde{A}$, then the subgroup H is the image in $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ of $\mathfrak{T}_{\mathcal{D}}(\mathbb{K})$. In other words

$$H \cong \Lambda^* \otimes \mathbb{K}^\times \cong \mathbb{K}^\times \times \cdots \times \mathbb{K}^\times$$

where there are rank Λ^* terms in the above product. Thus, given a generating set $\{\lambda_i\}$ of Λ^*, the set corresponding to

$$\{\lambda_i \otimes \xi : \xi \in \mathbb{K}^\times\}$$

is a generating set for H. We note that H is abelian in all cases.

We consider the minimal adjoint, adjoint and simply-connected cases as separate examples.

Example 4.2.5

If $\mathcal{D} = \mathcal{D}_{sc}(\tilde{A})$ is the simply-connected root datum

$$\Lambda^* = \bigoplus_{i\in\underline{n}_0} \mathbb{Z}\alpha_i^\vee$$

and hence $H_{sc} \cong \mathfrak{T}_{D_{sc}}(K)$ is generated by the elements

$$h_{\alpha_i}(\xi) = \alpha_i^\vee \otimes \xi$$

for $\xi \in \mathbb{K}^\times$. Recall that,

$$\gamma : h_{\alpha_i}(\xi) \mapsto h_{\bar{\gamma}(\alpha_i)}(\xi)$$

and

$$\tau : h_{\alpha_i}(\xi) \mapsto h_{\alpha_i}(\xi)$$

for all $i \in \underline{n}_0$ and $\xi \in \mathbb{K}^\times$. We deduce that

$$h \in H_{sc}^{\gamma,\tau} \quad \Leftrightarrow \quad h = h_{\alpha_0}(\xi_0) \prod_{J\in\underline{n}^\prime} \left(\prod_{i\in J} h_{\alpha_i^\vee}(\xi_J)\right)$$

for some $\xi_0, \xi_J \in \mathbb{K}^\times$. ◇

EXAMPLE 4.2.6
If $\mathcal{D} = \mathcal{D}_{ad}(\hat{A})$ is the adjoint root datum, then by Example 2.5.8

$$\Lambda = \bigoplus_{i \in \underline{n}} \mathbb{Z}\tilde{\alpha}_i$$

and so Λ^* has a generating set, $\{\varpi_i^\vee\}_{i \in \underline{n}}$, defined by

$$\varpi_i^\vee(\alpha_j) = \delta_{ij}$$

for all $j \in \underline{n}_0$, where δ_{ij} is the Kronecker delta. Thus $H_{ad} \cong \mathfrak{T}_{\mathcal{D}_{ad}}(\mathbb{K})$ is generated by elements of the form

$$h_{\varpi_i^\vee}(\xi) = \varpi_i^\vee \otimes \xi$$

for $i \in \underline{n}$ and $\xi \in \mathbb{K}^\times$. Recall that

$$\gamma : h_{\varpi_i^\vee}(\xi) \mapsto h_{\bar{\gamma}(\varpi_i^\vee)}(\xi)$$

where $\bar{\gamma}(\varpi_i^\vee) = \varpi_{\bar{\gamma}(i)}^\vee$, and

$$\tau : h_{\varpi_i^\vee}(\xi) \mapsto h_{\varpi_i^\vee}(\xi)$$

for all $i \in \underline{n}$ and $\xi \in \mathbb{K}^\times$. We thus obtain the condition

$$h \in H_{ad}^{\gamma,\tau} \quad \Leftrightarrow \quad h = \prod_{J \in \underline{n}^1} \left(\prod_{i \in J} h_{\varpi_i^\vee}(\xi_J) \right)$$

for some $\xi_J \in \mathbb{K}^\times$. ◇

EXAMPLE 4.2.7
If $\mathcal{D} = \mathcal{D}_m(\hat{A})$ is the minimal adjoint root datum

$$\Lambda = \bigoplus_{i \in \underline{n}_0} \mathbb{Z}\hat{\alpha}_i = \bigoplus_{i \in \underline{n}_0} \mathbb{Z}\alpha_i$$

and so Λ^* has basis $\{\varpi_i^\vee\}_{i \in \underline{n}_0}$ where

$$\varpi_i^\vee(\alpha_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. Thus $H_m \cong \mathfrak{T}_{\mathcal{D}_m}(\mathbb{K})$ is generated by elements of the form

$$h_{\varpi_i^\vee}(\xi) = \varpi_i^\vee \otimes \xi$$

for $i \in \underline{n}_0$ and $\xi \in \mathbb{K}^\times$. Recall that

$$\gamma : h_{\varpi_i^\vee}(\xi) \mapsto h_{\bar{\gamma}(\varpi_i^\vee)}(\xi)$$

where $\bar{\gamma}(\varpi_i^\vee) = \varpi_{\bar{\gamma}(i)}^\vee$, and

$$\tau : h_{\varpi_i^\vee}(\xi) \mapsto h_{\varpi_i^\vee}(\xi)$$

for all $i \in \underline{n}_0$ and $\xi \in \mathbb{K}^\times$. We thus obtain the condition

$$h \in H_m^{\gamma,\tau} \quad \Leftrightarrow \quad h = h_{\varpi_0^\vee}(\xi_0) \prod_{J \in \underline{n}^1} \left(\prod_{i \in J} h_{\varpi_i^\vee}(\xi_J) \right)$$

for some $\xi_0, \xi_J \in \mathbb{K}^\times$. ◇

4.3 A Comparison of the Action of Two Groups on a Kac-Moody Algebra

Suppose $\mathcal{D}$ is a root datum associated to a GCM and $\mathfrak{G}_{\mathcal{D}}$ is the Kac-Moody group functor corresponding to $\mathcal{D}$. Recall that condition KMG5 endows $\mathfrak{G}_{\mathcal{D}}(\mathbb{C})$ with an action on the corresponding complex Kac-Moody algebra. Furthermore, if $\mathcal{D}$ is of minimal adjoint type this action is faithful.

Suppose that $\tilde{A}$ is a simply-laced extended Cartan matrix of type $\tilde{X}_n$ and maintain the notation introduced in the first two sections of this chapter.

The Algebra in Question

Recall that in §1.5 we constructed a subalgebra $\tilde{\mathcal{L}}(\mathfrak{g}, \gamma, k)$ of $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ which was the fixed point subalgebra of $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ under the action of an automorphism $\tilde{\sigma}$. We shall hereafter denote $\tilde{\mathcal{L}}(\mathfrak{g}, \gamma, k)$ by $\mathfrak{g}^{\tilde{\sigma}}$.

By Theorem 1.5.6 we have

$$\mathfrak{g}^{\tilde{\sigma}} \cong \mathfrak{g}_{X_n^{(k)}}(\mathbb{C}),$$

where $X_n^{(k)}$ is determined by Table 1.5.5.

Let $B = (B_{ij})_{i,j\in \underline{m}_0}$ be a GCM of type $X_n^{(k)}$ and let $\mathfrak{g}_B(\mathbb{C})$ be the Kac-Moody algebra associated to B. Denote by $\Pi^B = \{\beta_i\}_{i\in\underline{m}_0}$ and Φ^B the set of fundamental roots and the real root system of $\mathfrak{g}_B(\mathbb{C})$ respectively. Let

$$\{e_{\beta_i}, f_{\beta_i}, \beta_j^\vee : i \in \underline{m}_0, j \in \underline{m+1}_0\}$$

be a Chevalley basis of $\mathfrak{g}_B(\mathbb{C})$.

Suppose

$$\phi' : \mathfrak{g}^{\tilde{\sigma}} \to \mathfrak{g}_B(\mathbb{C}),$$

is a Lie algebra isomorphism of the type constructed in §1.5. Given a fixed set of structure constants for $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$, this entails a particular choice of structure constants and root vectors in $\mathfrak{g}_B(\mathbb{C})$. We shall refer to such root vectors in $\mathfrak{g}_B(\mathbb{C})$ as *distinguished* root vectors.

The Lie algebra isomorphism ϕ' induces a group isomorphism

$$\phi : \mathrm{Aut}\ \mathfrak{g}^{\tilde{\sigma}} \to \mathrm{Aut}\ \mathfrak{g}_B(\mathbb{C})$$

such that $\phi(\exp \mathrm{ad}\, zx) = \exp \mathrm{ad}\, z\phi'(x)$ for all $x \in \mathfrak{g}^{\tilde{\sigma}}$ and $z \in \mathbb{C}$.

A Faithful Action

Denote by $\mathfrak{G}_m^B$ the Kac-Moody group functor $\mathfrak{G}_{\mathcal{D}_m(B)}$. Thus $\mathfrak{G}_m^B(\mathbb{C})$ acts faithfully on $\mathfrak{g}_B(\mathbb{C})$ and so the map

$$\mathrm{Ad}_B : \mathfrak{G}_m^B(\mathbb{C}) \to \mathrm{Aut}\ \mathfrak{g}_B(\mathbb{C})$$

determined by condition KMG5 is injective.

Denote the generators of $\mathfrak{G}_m^B(\mathbb{C})$ determined in Example 2.5.8 by

$$y_{\beta_i}(z),\ y_{-\beta_i}(z), \text{ and } h_{\varpi_{\beta_i}^\vee}(\xi)$$

for $i \in \underline{m}_0$, $z \in \mathbb{C}$, and $\xi \in \mathbb{C}^\times$. Then

$$\mathrm{Ad}_B\,(y_{\beta_i}(z)) = \exp \mathrm{ad}\, ze_{\beta_i} \qquad \text{and} \qquad \mathrm{Ad}_B\,(y_{-\beta_i}(z)) = \exp \mathrm{ad}\, zf_{\beta_i}$$

for all $z \in \mathbb{C}$. We also wish to determine the image under Ad_B of the generators $h_{\varpi_{\beta_i}^\vee}(\xi)$. Recall that $h_{\varpi_{\beta_i}^\vee}(\xi)$ acts on each root vector e_β as multiplication by $\xi^{\langle \beta, \varpi_{\beta_i}^\vee \rangle}$. Thus

$$\mathrm{Ad}_B\left(h_{\varpi_{\beta_i}^\vee}(\xi)\right) = \mathcal{M}_{\beta_i}^\xi$$

where $\mathcal{M}_{\beta_i}^\xi$ is the automorphism of $\mathfrak{g}_B(\mathbb{C})$ defined by

$$\begin{array}{rrcll} \mathcal{M}_{\beta_i}^\xi : & \mathfrak{g}_B(\mathbb{C}) & \to & \mathfrak{g}_B(\mathbb{C}) & \\ & e_{\beta_i} & \mapsto & \xi e_{\beta_i} & \\ & e_{\beta_{i'}} & \mapsto & e_{\beta_{i'}} & \text{if } i' \in \underline{m}_0 \setminus \{i\} \\ & f_{\beta_i} & \mapsto & \xi^{-1} f_{\beta_i} & \\ & f_{\beta_{i'}} & \mapsto & f_{\beta_{i'}} & \text{if } i' \in \underline{m}_0 \setminus \{i\} \\ & \beta_j^\vee & \mapsto & \beta_j^\vee & \text{for all } j \in \underline{m+1}_0. \end{array}$$

An Action of the Twisted Groups

Denote by $\mathfrak{G}_m^{\hat{A}}$ the Kac-Moody group functor $\mathfrak{G}_{\mathcal{D}_m(\hat{A})}$. We begin by showing that the subgroups of $\mathfrak{G}_m^{\hat{A}}$ which consist of elements $g \in \mathfrak{G}_m^{\hat{A}}$ satisfying $\gamma(g) = \tau(g)$ act on the algebra $\mathfrak{g}^\sigma$.

LEMMA 4.3.1

The groups $X_{\alpha^\gamma}(\mathbb{C})$ act on $\mathfrak{g}^\sigma$ for all $\alpha^\gamma \in \Phi^\sigma$.

Proof

It is sufficient to show that, for all $\alpha^\gamma \in \Phi^\sigma$, the generators of $X_{\alpha^\gamma}(\mathbb{C})$ act on $\mathfrak{g}^\sigma$. In particular, if we can show that every generator of $X_{\alpha^\gamma}(\mathbb{C})$ acts on $\mathfrak{g}_{\hat{A}}(\mathbb{C})$ as $\exp \mathrm{ad}\, zx$ for some $z \in \mathbb{C}$ and some $x \in \mathfrak{g}^\sigma$, then the result will follow.

Suppose $\alpha + m\delta \in \Phi^{re}(\hat{A})$ with $\alpha \in \Phi^{re}(A)$ and $\alpha^\gamma + m\delta \in \Phi^\sigma$.

We note first that if two elements $x, y \in \mathfrak{g}_{\tilde{A}}(\mathbb{C})$ satisfy $[x, y] = 0$, then ad zx and ad $z'y$ are commuting locally nilpotent operators on the complex vector space $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ and so

$$\exp \operatorname{ad} zx . \exp \operatorname{ad} z'y = \exp \operatorname{ad} (zx + z'y)$$

for all $z, z' \in \mathbb{C}$. Thus, whenever $\Phi^{re}_{\alpha^1} = K(\alpha)$, the calculations involved are straightforward. We consider these cases first.

Case 1: $K(\alpha + m\delta) = \{\alpha + m\delta\}$.

In this case $x_{\alpha+m\delta}(z)$ acts on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ as $\exp \operatorname{ad} ze_{\alpha+m\delta}$ for all $z \in \mathbb{C}$, where it is understood that, if $\alpha + m\delta \in \Phi^{re}_{-}(\tilde{A})$, then $e_{\alpha+m\delta}$ denotes the generator $f_{-\alpha-m\delta}$ of $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$. Furthermore, from the proofs of Propositions 4.1.3 and 4.1.5 it is clear that $e_{\alpha+m\delta}, f_{-\alpha-m\delta} \in \mathfrak{g}^{\tilde{\sigma}}$.

Case 2 : $\Phi^{re}_{(\alpha+m\delta)^1} = K(\alpha + m\delta) = \{\alpha + m\delta\, , \bar{\alpha} + m\delta\}$.

We consider the action of the element

$$x_{\alpha+m\delta}(z) x_{\bar{\alpha}+m\delta}(\kappa z) \in X_{\alpha^1+m\delta}(\mathbb{C})$$

on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$. Recall that such an element acts on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ as

$$\exp \operatorname{ad} ze_{\alpha+m\delta} . \exp \operatorname{ad} z\kappa e_{\bar{\alpha}+m\delta} = \exp \operatorname{ad} z(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta}).$$

However,

$$e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta} \in \mathfrak{g}^{\tilde{\sigma}}$$

by Propositions 4.1.3 and 4.1.5.

Case 3 : $K(\alpha + m\delta) = \{\alpha + m\delta\, , \bar{\alpha} + m\delta\, , \bar{\bar{\alpha}} + m\delta\}$.

This case is proved in an entirely analogous manner to Case 2.

Finally, we consider the only case where $\Phi^{re}_{(\alpha+m\delta)^1} \neq K(\alpha + m\delta)$.

Case 4 : $\Phi^{re}_{(\alpha+m\delta)^1} = \{\alpha + m\delta\, , \bar{\alpha} + m\delta\, , \alpha + \bar{\alpha} + 2m\delta\}$.

We consider the action of an element

$$x_{\alpha+m\delta}(z) x_{\bar{\alpha}+m\delta}(\kappa z) x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa z^2/2) \in X_{\alpha^1+m\delta}(\mathbb{C})$$

on $\mathfrak{g}_{A}(\mathbb{C})$. Such an element acts on $\mathfrak{g}_{A}(\mathbb{C})$ as

$$\exp \operatorname{ad} (ze_{\alpha+m\delta}) \cdot \exp \operatorname{ad} (z\kappa e_{\bar{\alpha}+m\delta}) \cdot \exp \operatorname{ad} \left(\frac{-\kappa z^2}{2} e_{\alpha+\bar{\alpha}+2m\delta}\right)$$

which simplifies to

$$\exp \operatorname{ad} z\,(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta})$$

by Lemma 1.1.19. However,

$$e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta} \in \mathfrak{g}^{\tilde{\tilde{\sigma}}}$$

by Proposition 4.1.5.

Thus $X_{\alpha^\gamma}(\mathbb{C})$ acts on $\mathfrak{g}^{\tilde{\tilde{\sigma}}}$ for all $\alpha^\gamma \in \Phi^\sigma$. □

LEMMA 4.3.2

The group $H_m^{\gamma,\tau}(\mathbb{C})$ acts on $\mathfrak{g}^{\tilde{\sigma}}$.

Proof

Recall that every element of $H_m(\mathbb{C})$ acts diagonally on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$. Thus in order to show that $H_m^{\gamma,\tau}(\mathbb{C})$ acts on $\mathfrak{g}^{\tilde{\sigma}}$ we must show that the action of $(\tau^{-1}\circ\gamma)$-stable elements of $H_m(\mathbb{C})$ is constant on $\tilde{\sigma}$-orbits of $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$. Furthermore, it is sufficient to show that this is the case for the generators of $H_m^{\gamma,\tau}(\mathbb{C})$.

From Example 4.2.7, we see that

$$\left\{ h_{\varpi_0^\vee}(\xi_0)\ ,\ \prod_{i\in J} h_{\varpi_i^\vee}(\xi_J)\ :\ J \in \underline{n}^1\ ,\ \xi_0, \xi_J \in \mathbb{C}^\times \right\}$$

is a set of generators for $H_m^{\gamma,\tau}(\mathbb{C})$. We note that, for any $e_\alpha \in \mathfrak{g}_{\tilde{A}}(\mathbb{C})$ with $\alpha \in \Phi^{re}(\tilde{A})$,

$$h_{\varpi_0^\vee}(\xi_0) \text{ acts on } e_\alpha \text{ as multiplication by } \xi_0^{\langle\alpha,\varpi_0^\vee\rangle}$$

and

$$\prod_{i\in J} h_{\varpi_i^\vee}(\xi_J) \text{ acts on } e_\alpha \text{ as multiplication by } \xi_J^{\sum_{i\in J}\langle\alpha,\varpi_i^\vee\rangle}.$$

Thus by the results of §1.4, we need consider only $\tilde{\sigma}$-orbits of $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ consisting of elements of the form e_α where $\alpha \in \Phi^{re}(A)$. Suppose this is the case. We consider several possibilities.

If $\Phi^{re}_{\alpha^1} = \{\alpha\}$ the claim is clearly true.

Suppose now that $\Phi^{re}_{\alpha^1} = \{\alpha, \bar{\alpha}\}$ or $\{\alpha, \bar{\alpha}, \alpha+\bar{\alpha}\}$. In each of these cases we have $k = 2$ and

$$e_\alpha + \kappa e_{\bar{\alpha}} \in \mathfrak{g}_{\bar{0}}$$

for a unique $\kappa \in \{1, -1\}$. We need to show that

$$H_m^{\gamma,\tau}(\mathbb{C}).(e_\alpha + \kappa e_{\bar{\alpha}}) \subseteq \mathbb{C}(e_\alpha + \kappa e_{\bar{\alpha}}).$$

Hence, we require to show that

$$\langle\alpha, \varpi_0^\vee\rangle = \langle\bar{\alpha}, \varpi_0^\vee\rangle = \langle\bar{\gamma}(\alpha), \bar{\gamma}(\varpi_0^\vee)\rangle$$

and that for each $J \in \underline{n}^1$

$$\sum_{i\in J}\langle\alpha, \varpi_i^\vee\rangle = \sum_{i\in J}\langle\bar{\alpha}, \varpi_i^\vee\rangle = \sum_{i\in J}\langle\bar{\gamma}(\alpha), \varpi_i^\vee\rangle.$$

However, by the definition of $\bar{\gamma}$,

$$\langle \alpha, \varpi_j^\vee \rangle = \langle \bar{\gamma}(\alpha), \bar{\gamma}(\varpi_j^\vee) \rangle$$

for all $\alpha \in \Phi^{re}(\tilde{A})$ and $j \in \underline{n}_0$. Thus the required equalities do in fact hold.

The case of $\Phi^{re}_{\alpha^\gamma} = \{\alpha, \bar{\alpha}, \bar{\bar{\alpha}}\}$ is proved in an entirely analogous manner to the above. □

Let

$$\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) = \left(\mathfrak{G}_m^{\tilde{A}}(\mathbb{C})\right)^{\gamma,\tau}.$$

LEMMA 4.3.3

The group $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ acts on $\mathfrak{g}^{\sigma}$, i.e. there is a homomorphism

$$\mathrm{Ad}_{\gamma,\tau} : \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) \to \mathrm{Aut}\ \mathfrak{g}^{\sigma}.$$

Thus, given any subgroup $\mathfrak{G}_m^{\sigma}(\mathbb{C}) \le \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$, the map

$$\mathrm{Ad}_{\sigma} : \mathfrak{G}_m^{\sigma}(\mathbb{C}) \to \mathrm{Aut}\ (\mathfrak{g}^{\sigma})$$

which is the restriction of $\mathrm{Ad}_{\gamma,\tau}$ to $\mathfrak{G}_m^{\sigma}(\mathbb{C})$ is a homomorphism. In particular, every subgroup of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ acts on $\mathfrak{g}^{\sigma}$.

Proof

It is sufficient to check that the generators of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ act on $\mathfrak{g}^{\sigma}$. Since, by Proposition 4.2.3,

$$\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) = \langle H_m^{\gamma,\tau}\ ,\ X_{\alpha^\gamma}\ :\ \alpha^\gamma \in \Phi^{\sigma} \rangle$$

this means we need only check that the generators of $H_m^{\gamma,\tau}$ and of X_{α^γ} act on $\mathfrak{g}^{\sigma}$. Thus the result now follows from Lemmas 4.3.1 and 4.3.2. □

An Induced Action of the Twisted Groups

Since $\mathfrak{g}^{\sigma} \cong \mathfrak{g}_B(\mathbb{C})$, Lemma 4.3.3 leads us to the conclusion that every subgroup of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ acts on $\mathfrak{g}_B(\mathbb{C})$. Thus, given any $\mathfrak{G}_m^{\sigma}(\mathbb{C}) \le \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$, there is a homomorphism

$$\varsigma_{\sigma} : \mathfrak{G}_m^{\sigma}(\mathbb{C}) \to \mathrm{Aut}\ \mathfrak{g}_B(\mathbb{C})$$

such that the diagram

$$\begin{array}{ccc} \mathfrak{G}_m^{\sigma}(\mathbb{C}) & \xrightarrow{\ \mathrm{Ad}_{\sigma}\ } & \mathrm{Aut}\ \mathfrak{g}^{\sigma} \\ & \searrow^{\varsigma_{\sigma}} & \downarrow \phi \\ & & \mathrm{Aut}\ \mathfrak{g}_B(\mathbb{C}) \end{array}$$

commutes. We wish to determine the images of the generators of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ under the homomorphism $\varsigma = \varsigma_{\gamma,\tau}$. We first consider the images of the generators of the subgroups $X_{\alpha^\gamma}(\mathbb{C}) \leq \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$.

The Action of the Groups X_{α^γ}

At this point it is convenient to fix our notation for the generators of the groups X_{α^γ}. Suppose that $\alpha^\gamma + m\delta \in \Phi^\sigma$ with $\alpha \in \Phi(A)$. There may be several roots $\alpha' + m\delta \in \Phi^{re}(\hat{A})$ such that $(\alpha' + m\delta)^\gamma = \alpha^\gamma + m\delta$. Henceforth we suppose the representative $\alpha + m\delta \in \Phi^{re}(\hat{A})$ is chosen so that it is the first in the natural lexicographic ordering we can define on the set of roots satisfying such a condition.

We consider the images of the generators of the subgroups $X_{\alpha^\gamma}(\mathbb{C}) \leq \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ in a case by case analysis.

Case 1: $K(\alpha + m\delta) = \{\alpha + m\delta\}$.

In this case the diagram

$$\begin{array}{ccc} x_{\alpha+m\delta}(z) & \xrightarrow{\mathrm{Ad}_{\gamma,\tau}} & \exp \mathrm{ad}\, z e_{\alpha+m\delta} \\ & \searrow^{\varsigma} & \downarrow \phi \\ & & \exp \mathrm{ad}\, z\phi'(e_{\alpha+m\delta}) \end{array}$$

commutes, and so $\varsigma\,(x_{\alpha+m\delta}(z)) = \exp \mathrm{ad}\, z\phi'(e_{\alpha+m\delta})$. However, by the results of §1.5, whenever $\alpha + m\delta \in \Phi^\sigma$, $\phi'(e_{\alpha+m\delta})$ is simply a distinguished root vector of $\mathfrak{g}_H(\mathbb{C})$. Thus $x_{\alpha+m\delta}(z)$ acts on $\mathfrak{g}_H(\mathbb{C})$ as $\exp \mathrm{ad}\, z e_\beta$ for some $\beta \in \Phi^H$.

Case 2 : $\Phi^{re}_{(\alpha+m\delta)^1} = K(\alpha + m\delta) = \{\alpha + m\delta\, ,\, \bar{\alpha} + m\delta\}$.

This time the diagram

$$\begin{array}{ccc} x_{\alpha+m\delta}(z)x_{\bar{\alpha}+m\delta}(\kappa z) & \xrightarrow{\mathrm{Ad}_{\gamma,\tau}} & \exp \mathrm{ad}\, z(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta}) \\ & \searrow^{\varsigma} & \downarrow \phi \\ & & \exp \mathrm{ad}\, z\phi'(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta}) \end{array}$$

commutes. Once again the results of §1.5 show that $\phi'(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta})$ is a distinguished root vector of $\mathfrak{g}_H(\mathbb{C})$.

Case 3 : $K(\alpha + m\delta) = \{\alpha + m\delta\, ,\, \bar{\alpha} + m\delta\, ,\, \bar{\bar{\alpha}} + m\delta\}$.

In this case the diagram

$$\begin{array}{ccc} & & \exp \operatorname{ad} z(e_{\alpha+m\delta} + \kappa\epsilon^{-m} e_{\bar{\alpha}+m\delta} + \kappa'\epsilon^{-2m} e_{\bar{\bar{\alpha}}+m\delta}) \\ & \nearrow^{\operatorname{Ad}_{\gamma,\tau}} & \\ x_{\alpha+m\delta}(z)x_{\bar{\alpha}+m\delta}(\kappa\epsilon^{-m}z)x_{\bar{\bar{\alpha}}+m\delta}(\kappa'\epsilon^{-2m}z) & & \downarrow \phi \\ & \searrow_{\varsigma} & \\ & & \exp \operatorname{ad} z\phi'(e_{\alpha+m\delta} + \kappa\epsilon^{-m} e_{\bar{\alpha}+m\delta} + \kappa'\epsilon^{-2m} e_{\bar{\bar{\alpha}}+m\delta}) \end{array}$$

commutes. Example 1.5.9 explicitly shows that $\phi'(e_{\alpha+m\delta}+\kappa\epsilon^{-m}e_{\bar{\alpha}+m\delta}+\kappa'\epsilon^{-2m}e_{\bar{\bar{\alpha}}+m\delta})$ is a distinguished root vector in $\mathfrak{g}_B(\mathbb{C})$.

Case 4 : $\Phi^{re}_{(\alpha+m\delta)^1} = \{\alpha + m\delta\,,\, \bar{\alpha} + m\delta\,,\, \alpha + \bar{\alpha} + 2m\delta\}$.
Finally, the diagram

$$\begin{array}{ccc} & & \exp \operatorname{ad} z(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta}) \\ & \nearrow^{\operatorname{Ad}_{\gamma,\tau}} & \\ x_{\alpha+m\delta}(z)x_{\bar{\alpha}+m\delta}(\kappa z)x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa z^2/2) & & \downarrow \phi \\ & \searrow_{\varsigma} & \\ & & \exp \operatorname{ad} z\phi'(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta}) \end{array}$$

commutes. However, as was demonstrated in Example 1.5.11, in this case

$$\phi'(e_{\alpha+m\delta} + \kappa e_{\bar{\alpha}+m\delta}) = \frac{1}{\sqrt{2}} e_\beta$$

where e_β is a distinguished root vector in $\mathfrak{g}_B(\mathbb{C})$.

Moreover, a case by case analysis shows that in all cases $\exp \operatorname{ad} e_\beta$ is in the image of ς whenever e_β is a distinguished root vector.

The Action of $H_m^{\gamma,\tau}$

We note that the Lie algebra isomorphism ϕ' induces a bijective correspondence between Φ^σ and Φ^B. Suppose $\beta \in \Phi^B$ is identified with $\alpha^\gamma \in \Phi^\sigma$ under this correspondence. Thus we may write the root vector $e_\beta \in \mathfrak{g}_B(\mathbb{C})$ as e_{α^γ}. Recall that

$$\left\{ h_{\varpi_0^\vee}(\xi_0)\,,\ \prod_{i\in J} h_{\varpi_i^\vee}(\xi_J)\ :\ J \in \underline{n}^1\,,\ \xi_0, \xi_J \in \mathbb{C}^\times \right\}$$

is a set of generators for $H_m^{\gamma,\tau}(\mathbb{C})$. We note that

$$\varsigma\left(h_{\varpi_0^\vee}(\xi_0)\right) \text{ acts on } e_{\alpha^\gamma} \text{ as multiplication by } \xi_0^{\langle\alpha,\varpi_0^\vee\rangle}$$

and

$$\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi_J)\right) \text{ acts on } e_{\alpha^\gamma} \text{ as multiplication by } \xi_J^{\sum_{i\in J}\langle\alpha,\varpi_i^\vee\rangle}$$

where $\alpha \in \Phi^{re}(\hat{A})$ is a representative of the orbit corresponding to $\alpha^\gamma \in \Phi^\sigma$.

LEMMA 4.3.4

Suppose first that $(\hat{A}, k) \neq (\hat{A}_n, 2)$. *Then, for each* $\xi \in \mathbb{C}^\times$,

$$\varsigma\left(h_{\varpi_0^\vee}(\xi)\right) = \mathcal{M}_{\beta_0}^\xi \qquad \text{and} \qquad \varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right) = \mathcal{M}_{\beta_0}^\xi \mathcal{M}_{\beta_j}^\xi$$

where $J \in \underline{n}^1$, *and* $\beta_j \in \Pi^H$ *corresponds to* $\alpha_j^\gamma \in \Phi^\sigma$ *with two exceptions;*

$$\varsigma\left(h_{\varpi_1^\vee}(\xi)\right) = \mathcal{M}_{\beta_1}^\xi \qquad \text{when } (\hat{A}, k) = (\hat{D}_{l+1}, 2)$$

and

$$\varsigma\left(h_{\varpi_1^\vee}(\xi) h_{\varpi_5^\vee}(\xi)\right) = \mathcal{M}_{\beta_1}^\xi \qquad \text{when } (\hat{A}, k) = (\hat{E}_6, 2).$$

Suppose next that $(\hat{A}, k) = (\hat{A}_{2l-1}, 2)$. *Then, for each* $\xi \in \mathbb{C}^\times$,

$$\varsigma\left(h_{\varpi_0^\vee}(\xi)\right) = \mathcal{M}_{\beta_0}^\xi, \quad \varsigma\left(h_{\varpi_1^\vee}(\xi) h_{\varpi_{2l-1}^\vee}(\xi)\right) = \mathcal{M}_{\beta_0}^\xi \mathcal{M}_{\beta_1}^\xi,$$

and

$$\varsigma\left(h_{\varpi_i^\vee}(\xi) h_{\varpi_{2l-i}^\vee}(\xi)\right) = \mathcal{M}_{\beta_i}^\xi$$

for $i \in \underline{l} \setminus \{1\}$.

Finally, if $(\hat{A}, k) = (\hat{A}_{2l}, 2)$ *then*

$$\varsigma\left(h_{\varpi_0^\vee}(\xi)\right) = \mathcal{M}_{\beta_l}^\xi \qquad \text{and} \qquad \varsigma\left(h_{\varpi_i^\vee}(\xi) h_{\varpi_{2l-i+1}^\vee}(\xi)\right) = \mathcal{M}_{\beta_{l-i}}^\xi$$

for $\xi \in \mathbb{C}^\times$ *and* $i \in \underline{l}$.

Proof

We begin by considering the element of Φ^σ associated to $\beta_0 \in \Phi^H$. This element corresponds to an orbit of roots in $\Phi(\hat{A})$. We give an explicit description of this orbit in terms of α_i for $i \in \underline{n}_0$ in Table 4.3.5.

$(\tilde{A}, k)$	Orbit corresponding to β_0
$(\tilde{A}_{2l}, 2)$	$\{\alpha_l + \alpha_{l+1}\}$
$(\tilde{A}_{2l-1}, 2)$	$\{\alpha_0 + \alpha_1\ ,\ \alpha_0 + \alpha_{2l-1}\}$
$(\tilde{D}_{l+1}, 2)$	$\{\alpha_0 + \alpha_2 + \cdots + \alpha_{l-1} + \alpha_l\ ,\ \alpha_0 + \alpha_2 + \cdots + \alpha_{l-1} + \alpha_{l+1}\}$
$(\tilde{D}_4, 3)$	$\{\alpha_0 + \alpha_1 + \alpha_2\ ,\ \alpha_0 + \alpha_2 + \alpha_3\ ,\ \alpha_0 + \alpha_2 + \alpha_4\}$
$(\tilde{E}_6, 2)$	$\{\alpha_0 + \alpha_2 + \alpha_3 + \alpha_6\ ,\ \alpha_0 + \alpha_3 + \alpha_4 + \alpha_6\}$

Table 4.3.5: Orbit of roots in $\Phi(\tilde{A})$ corresponding to $\beta_0 \in \Phi^B$.

Suppose that $(\tilde{A}, k) \neq (\tilde{A}_{2l}, 2)$.

Consider the element of Φ^σ associated to $\beta_0 \in \Phi^B$ in these cases. This corresponds to an orbit of roots, detailed in Table 4.3.5, where α_0 occurs with multiplicity one in each root. For example in the case $(\tilde{A}, k) = (\tilde{A}_{2l-1}, 2)$ the orbit in question is

$$\{-\alpha_1 - \cdots - \alpha_{2l-2} + \delta, -\alpha_2 - \cdots - \alpha_{2l-1} + \delta\} = \{\alpha_0 + \alpha_1, \alpha_0 + \alpha_{2l-1}\}.$$

Hence, whenever $(\tilde{A}, k) \neq (\tilde{A}_{2l}, 2)$, $\varsigma\left(h_{\varpi_0^\vee}(\xi)\right)$ will act on e_{β_0} as multiplication by ξ and on f_{β_0} as multiplication by ξ^{-1}. Furthermore, since the remaining elements of Π^B correspond to orbits independent of α_0, $\varsigma\left(h_{\varpi_0^\vee}(\xi)\right)$ will act trivially on the remaining Chevalley generators of $\mathfrak{g}_B(\mathbb{C})$. Thus we conclude that

$$\varsigma\left(h_{\varpi_0^\vee}(\xi)\right) = \mathcal{M}^{\xi}_{\beta_0}$$

whenever $(\tilde{A}, k) \neq (\tilde{A}_{2l}, 2)$.

When considering elements of the form $\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right)$, we separate root orbits into two types: those whose roots appear with non-zero multiplicities in the orbit corresponding to β_0 and those whose roots don't appear in the orbit corresponding to β_0.

Consider the element $\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right)$ where $J \in \underline{n}^1$ and $K(\alpha_i)$ is such that its roots appear with multiplicity one in the orbit corresponding to β_0 when the elements of that orbit are expressed in terms of α_i for $i \in \underline{n}_0$. Then $\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right)$

will act as multiplication by ξ on both e_{β_0} and $e_{\alpha_i^\gamma}$, as multiplication by ξ^{-1} on f_{β_0} and $f_{\alpha_i^\gamma}$, and trivially on the remaining Chevalley generators of $\mathfrak{g}_H(\mathbb{C})$. Thus in these cases

$$\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right) = \mathcal{M}^\xi_{\beta_0}\mathcal{M}^\xi_{\beta_j}$$

where β_j corresponds to α_i^γ.

Whenever $K(\alpha_i)$ is such that its roots do not appear in the orbit corresponding to β_0, the element $\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right)$ will act as multiplication by ξ on e_{β_j}, as multiplication by ξ^{-1} on f_{β_j} and trivially on the remaining Chevalley generators of $\mathfrak{g}_H(\mathbb{C})$. Thus in these cases

$$\varsigma\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi)\right) = \mathcal{M}^\xi_{\beta_j}$$

where β_j corresponds to α_i^γ.

Finally, the case $(\tilde{A}, k) = (\tilde{A}_{2l}, 2)$ is argued similarly. □

We note that in particular, $\mathcal{M}^\xi_{\beta_i}$ lies in the image of ς for each $i \in \underline{m}_0$.

Lemma 4.3.6

The action of $H^{\gamma,\tau}_m$ on $\mathfrak{g}_H(\mathbb{C})$ is faithful.

Proof

In order to show that $H^{\gamma,\tau}_m$ acts faithfully on $\mathfrak{g}_H(\mathbb{C})$ it is sufficient to show that only the identity element of $H^{\gamma,\tau}_m$ acts trivially on $\mathfrak{g}_H(\mathbb{C})$.

Recall that

$$h \in H^{\gamma,\tau}_m \quad\Leftrightarrow\quad h = h_{\varpi_0^\vee}(\xi_0)\prod_{J\in\underline{n}^1}\left(\prod_{i\in J} h_{\varpi_i^\vee}(\xi_J)\right)$$

for some $\xi_0, \xi_J \in \mathbb{K}^\times$ and that such an expression is unique. Thus such an element h acts on $\mathfrak{g}_H(\mathbb{C})$ as

$$\mathcal{M}^{\xi_K}_{\beta_0}\prod_{J\in\underline{n}^1}\mathcal{M}^{\xi_J}_{\beta_j}$$

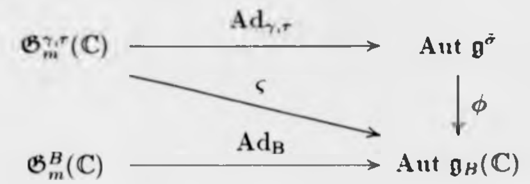
where $\beta_j \in \Pi^H$ corresponds to $\alpha_J^\gamma \in \Phi^\sigma$, ξ_K is a product of ξ_0 and some of the elements ξ_J for $J \in \underline{n}^1$ if $(\tilde{A}, k) \neq (\tilde{A}_{2l}, 2)$, and $\xi_K = \xi_{\{l,l+1\}}$ if $(\tilde{A}, k) = (\tilde{A}_{2l}, 2)$. Hence h acts trivially on $\mathfrak{g}_H(\mathbb{C})$ if and only if

$$\xi_K = 1, \qquad \text{and} \qquad \xi_J = 1$$

for all $J \in \underline{n}^1$. Now if $\xi_J = 1$ for all $J \in \underline{n}^1$and h acts trivially on $\mathfrak{g}_H(\mathbb{C})$, then $\xi_0 = \xi_K = 1$ and h must be the identity. □

4.4 An Isomorphism Between Minimal Adjoint Kac-Moody Groups

We now have details of the maps schematically described in the following diagram.



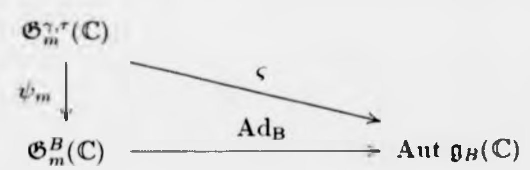
From the generation property of the groups $\mathfrak{G}^{\sigma}(\mathbb{K})$, Example 4.2.7, and the results of §4.3 we can deduce that

$$\varsigma\left(\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})\right) \subseteq \mathrm{Ad}_{\mathrm{B}}\left(\mathfrak{G}_m^{B}(\mathbb{C})\right).$$

Using the fact that Ad_{B} is a monomorphism, we may therefore define a map

$$\psi_m : \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) \to \mathfrak{G}_m^{B}(\mathbb{C})$$

such that the diagram



commutes.

Lemma 4.4.1

The map ψ_m is a group homomorphism.

Proof

We need to show that

$$\psi_m(xy) = \psi_m(x)\psi_m(y)$$

for any two elements $x, y \in \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$. We know that both ς and Ad_{B} are group homomorphisms and that the above diagram commutes. Hence

$$\begin{aligned} \mathrm{Ad}_{\mathrm{B}}\left(\psi_m(xy)\right) &= \varsigma(xy) = \varsigma(x)\varsigma(y) \\ &= \mathrm{Ad}_{\mathrm{B}}\left(\psi_m(x)\right)\mathrm{Ad}_{\mathrm{B}}\left(\psi_m(y)\right) \\ &= \mathrm{Ad}_{\mathrm{B}}\left(\psi_m(x)\psi_m(y)\right). \end{aligned}$$

However, since Ad_B is injective, we deduce that

$$\psi_m(xy) = \psi_m(x)\psi_m(y)$$

as required. □

Furthermore, the homomorphism ψ_m is surjective since every generator of $\mathfrak{G}_m^B(\mathbb{C})$ lies in the image of ψ_m. We aim to show that ψ_m is in fact an isomorphism. To do this we use the properties of a (B, N)-pair.

LEMMA 4.4.2

The homomorphism ψ_m is injective.

Proof

We show first that

$$\ker \psi_m \subseteq H_m \cap \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) = H_m^{\gamma,\tau}.$$

Since Ad_B is injective, $\ker \psi_m$ consists of those elements of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ which act trivially on $\mathfrak{g}_B(\mathbb{C})$. We aim to identify these elements.

Recall that

$$U^\sigma = \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi_+^\sigma \rangle \quad \text{and} \quad U_-^\sigma = \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi_-^\sigma \rangle.$$

Let

$$B^{\gamma,\tau} = U^\sigma H_m^{\gamma,\tau} \quad \text{and} \quad B_-^{\gamma,\tau} = U_-^\sigma H_m^{\gamma,\tau}.$$

Corollary 4.2.4 implies that $(B^{\gamma,\tau}, N^{\gamma,\tau})$ and $(B_-^{\gamma,\tau}, N^{\gamma,\tau})$ are two (B, N)-pairs in $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$. Thus

$$\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) = B^{\gamma,\tau} N^{\gamma,\tau} B^{\gamma,\tau} \quad \text{and} \quad \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) = B_-^{\gamma,\tau} N^{\gamma,\tau} B_-^{\gamma,\tau}$$

are two Bruhat decompositions of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$.

We consider the first of these. As a result of the third part of Proposition 2.1.7 we know that, for any $n, n' \in N^{\gamma,\tau}$,

$$B^{\gamma,\tau} n B^{\gamma,\tau} = B^{\gamma,\tau} n' B^{\gamma,\tau} \Leftrightarrow \pi^\sigma(n) = \pi^\sigma(n') \in W^\sigma$$

where π^σ denotes the projection of $N^{\gamma,\tau}$ onto W^σ. Thus all the elements of $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C})$ which act trivially on $\mathfrak{g}_B(\mathbb{C})$ must lie in the double coset corresponding to the identity element of W^σ, and hence we deduce that

$$\ker \psi_m \subseteq B^{\gamma,\tau}.$$

However, an entirely analogous argument holds when we consider the Bruhat decomposition $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) = B_-^{\gamma,\tau} N^{\gamma,\tau} B_-^{\gamma,\tau}$ instead. Thus

$$\ker \psi_m \subseteq B^{\gamma,\tau} \cap B_-^{\gamma,\tau} = H_m^{\gamma,\tau}.$$

We know $H_m^{\gamma,\tau}$ acts faithfully on $\mathfrak{g}_B(\mathbb{C})$ as a result of Lemma 4.3.6. Hence $\ker \psi_m$ is trivial and we deduce that ψ_m is injective. □

We thus have the following result.

PROPOSITION 4.4.3

There is a group functor isomorphism

$$\Psi_m : \mathfrak{G}_m^{\gamma,\tau} \to \mathfrak{G}_m^B,$$

where the group functors in question are defined on the category of fields $\mathbb{K}$ *with* $\operatorname{char}\mathbb{K} \neq 2$ *and such that*

$$\begin{array}{cccc} \sqrt{2} \in \mathbb{K} & \textit{if} & (\tilde{A}, k) = (\tilde{A}_{2l}, 2), & \textit{and} \\ \operatorname{char}\mathbb{K} \neq 3 & \textit{and } \mathbb{K} \text{ contains a primitive cube root of unity} & \textit{if} & (\tilde{A}, k) = (\dot{D}_4, 3). \end{array}$$

Proof

We have shown that there is an isomorphism

$$\Psi_m(\mathbb{C}) = \psi_m : \mathfrak{G}_m^{\gamma,\tau}(\mathbb{C}) \to \mathfrak{G}_m^B(\mathbb{C}).$$

Thus it remains to show that there exists an isomorphism

$$\Psi_m(\mathbb{K}) : \mathfrak{G}_m^{\gamma,\tau}(\mathbb{K}) \to \mathfrak{G}_m^B(\mathbb{K})$$

where $\mathbb{K}$ is an arbitrary field. We do this by considering the map

$$\Psi_m'(\mathbb{K}) : \mathfrak{G}_m^B(\mathbb{K}) \to \mathfrak{G}_m^{\gamma,\tau}(\mathbb{K})$$

defined on the generators of $\mathfrak{G}_m^B(\mathbb{K})$ by

$$\Psi_m'(\mathbb{K})(y_\beta(\mu)) = \prod_{\alpha \in \Phi_\beta} x_\alpha(\kappa_\alpha \mu)$$

for $\mu \in \mathbb{K}$ whenever

$$\Psi_m(\mathbb{C}) \left(\prod_{\alpha \in \Phi_\beta} x_\alpha(\kappa_\alpha \nu) \right) = y_\beta(\nu)$$

for all $\nu \in \mathbb{C}$ and

$$\Psi_m'(\mathbb{K}) \left(h_{\varpi_{\beta_i}^\vee}(\xi) \right) = \prod_{\varpi_j^\vee \in \Omega(\varpi_{\beta_i}^\vee)} h_{\varpi_j^\vee}(\xi)$$

for $\xi \in \mathbb{K}^\times$ whenever

$$\Psi_m(\mathbb{C}) \left(\prod_{\varpi_j^\vee \in \Omega(\varpi_{\beta_i}^\vee)} h_{\varpi_j^\vee}(\zeta) \right) = h_{\varpi_{\beta_i}^\vee}(\zeta)$$

for all $\zeta \in \mathbb{C}^\times$. We note that the set $\Omega\left(\varpi^\vee_{\beta_i}\right)$ is determined by Lemma 4.3.4. Furthermore, if either $(\tilde{A}, k) = (\tilde{A}_{2l}, 2)$ and $\sqrt{2} \notin \mathbb{K}$ or $(\tilde{A}, k) = (\tilde{D}_4, 3)$ and then $\Psi'_m(\mathbb{K})$ is not defined. *K contains no primitive cube root of unity*

To show that $\Psi'_m(\mathbb{K})$ is a group homomorphism we show that the defining relations are preserved. That is to say, we need to show that the relations

$$\begin{array}{lcll}
h_{\varpi^\vee_{\beta_i}}(\xi)h_{\varpi^\vee_{\beta_i}}(\zeta) & = & h_{\varpi^\vee_{\beta_i}}(\xi\zeta) & \\
h_{\varpi^\vee_{\beta_i}}(\xi)h_{\varpi^\vee_{\beta_j}}(\zeta) & = & h_{\varpi^\vee_{\beta_j}}(\zeta)h_{\varpi^\vee_{\beta_i}}(\xi) & \\
y_\beta(\mu)y_\beta(\nu) & = & y_\beta(\mu+\nu) & \\
h_{\varpi^\vee_{\beta_i}}(\xi)y_{\beta_j}(\mu)h_{\varpi^\vee_{\beta_i}}(\xi)^{-1} & = & y_{\beta_j}(\xi^{\delta_{ij}}\mu) & \\
n_{\beta_j}h_{\varpi^\vee_{\beta_i}}(\xi)n_{\beta_j}^{-1} & = & r_{\beta_j}\left(h_{\varpi^\vee_{\beta_i}}(\xi)\right) & \\
n_{\beta_i}y_\beta(\mu)n_{\beta_i}^{-1} & = & y_{r_{\beta_i}(\beta)}(\eta_{\beta_i,\beta}\mu) & \text{and} \\
[y_\beta(\mu), y_{\beta'}(\nu)] & = & \prod_{\substack{\gamma = i\beta + j\beta' \\ i,j \in \mathbb{N}}} y_\gamma(C_{\beta\beta'\gamma}\mu^i\nu^j), & \gamma \in \Phi^B,\ \beta \neq \pm\beta', \\
 & & & \{\beta,\beta'\} \text{ prenilpotent}
\end{array}$$

are preserved for all $i, j \in \underline{m}_0$, $\mu, \nu \in \mathbb{K}$, and $\xi, \zeta \in \mathbb{K}^\times$, where δ_{ij} denotes the Kronecker delta.

The first two relations are preserved since

$$\Psi'_m(\mathbb{K})\left(h_{\varpi^\vee_{\beta_i}}(\xi)\right) \in H_m^{\gamma,\tau}(\mathbb{K})$$

for all $i \in \underline{m}_0$ and $\xi \in \mathbb{K}^\times$. We note that whenever $\beta \in \pm\Pi^B \setminus \{\pm\beta_0\}$, the elements

$$\{x_\alpha(\kappa_\alpha\mu) : \alpha \in \Phi_\beta\}$$

commute pairwise and so the relation

$$y_\beta(\mu)y_\beta(\nu) = y_\beta(\mu+\nu)$$

is preserved. If $\beta \in \{\pm\beta_0\}$, the preservation of the above relation is a consequence of the third part of Proposition 4.1.5. We split the calculations for the remaining relations up into Lemmas for convenience.

LEMMA 4.4.4

The relation

$$h_{\varpi^\vee_{\beta_i}}(\xi)y_{\beta_j}(\mu)h_{\varpi^\vee_{\beta_i}}(\xi)^{-1} = y_{\beta_j}(\xi^{\delta_{ij}}\mu)$$

is preserved by $\Psi'_m(\mathbb{K})$ *for all* $i, j \in \underline{m}_0$, $\xi \in \mathbb{K}^\times$, *and* $\mu \in \mathbb{K}$.

Proof

We note that

$$\Psi'_m(\mathbb{K})\left(h_{\varpi^\vee_{\beta_i}}(\xi)y_{\beta_j}(\mu)h_{\varpi^\vee_{\beta_i}}(\xi)^{-1}\right)$$

$$= \left(\prod_{\varpi_{i'}^\vee \in \Omega(\varpi_{\beta_i}^\vee)} h_{\varpi_{i'}^\vee}(\xi)\right)\left(\prod_{\alpha\in\Phi_\beta} x_\alpha(\kappa_\alpha\mu)\right)\left(\prod_{\varpi_{i'}^\vee \in \Omega(\varpi_{\beta_i}^\vee)} h_{\varpi_{i'}^\vee}(\xi)\right)^{-1}$$

$$= \prod_{\alpha\in\Phi_\beta} x_\alpha\left(\left(\prod_{\varpi_{i'}^\vee \in \Omega(\varpi_{\beta_i}^\vee)} \xi^{\langle\alpha,\varpi_{i'}^\vee\rangle}\right)\kappa_\alpha\mu\right)$$

and

$$\Psi'_m(\mathbb{K})\left(y_{\beta_j}\left(\xi^{\delta_{ij}}\mu\right)\right) = \prod_{\alpha\in\Phi_\beta} x_\alpha\left(\xi^{\delta_{ij}}\kappa_\alpha\mu\right).$$

We thus require to prove that

$$\sum_{\varpi_{i'}^\vee \in \Omega\left(\varpi_{\beta_i}^\vee\right)} \langle\alpha,\varpi_{i'}^\vee\rangle = \delta_{ij}$$

for all $\alpha\in\Phi_{\beta_j}$. However, this is a condition independent of the field and therefore must be satisfied since $\Psi'_m(\mathbb{C})$ preserves the relation. □

Lemma 4.4.5

The relation

$$n_{\beta_j} h_{\varpi_{\beta_i}^\vee}(\xi) n_{\beta_j}^{-1} = r_{\beta_j}\left(h_{\varpi_{\beta_i}^\vee}(\xi)\right)$$

is preserved for all $i,j\in \underline{m}_0$, $\xi\in\mathbb{K}^\times$.

Proof

We note that

$$\Psi'_m(\mathbb{K})(n_{\beta_j}) = \prod_{\alpha\in\Phi_{\beta_j}} n_\alpha(\kappa_\alpha)$$

where

$$n_\alpha(\kappa_\alpha) := x_\alpha(\kappa_\alpha)x_{-\alpha}(\kappa_\alpha^{-1})x_\alpha(\kappa_\alpha).$$

We note that $\alpha = w(\alpha')$ for some $w\in W^\sigma$ and $\alpha'\in\Pi(\tilde{A})$. Thus

$$n_\alpha(\kappa_\alpha) = n_w n_{\alpha'}(\kappa_{\alpha'}\kappa_\alpha)n_w^{-1}$$

where $\kappa_{\alpha'}\in\mathbb{K}^\times$ and $n_w\in N^\sigma$ is an element such that $\pi(n_w) = w$. We also note that

$$\begin{aligned} n_{\alpha_j}(\zeta)h_{\varpi_i^\vee}(\xi)n_{\alpha_j}(\zeta)^{-1} &= h_{\alpha_j}(\zeta)n_\alpha h_{\varpi_i^\vee}(\xi)n_{\alpha_j}^{-1}h_{\alpha_j}(\zeta)^{-1} \\ &= h_{\alpha_j}(\zeta)r_{\alpha_j}\left(h_{\varpi_i^\vee}(\xi)\right)h_{\alpha_j}(\zeta)^{-1} \\ &= r_{\alpha_j}\left(h_{\varpi_i^\vee}(\xi)\right) \end{aligned}$$

for all $i, j \in \underline{n}_0$ and $\xi, \zeta \in \mathbb{K}^\times$. Thus

$$\begin{aligned}
&\Psi'_m(\mathbb{K})\left(n_{\beta_j} h_{\varpi^\vee_{\beta_i}}(\xi) n_{\beta_j}^{-1}\right) \\
&\quad = \left(\prod_{\alpha\in\Phi_{\beta_j}} n_\alpha(\kappa_\alpha)\right)\left(\prod_{\varpi^\vee_{i'}\in\Omega(\varpi^\vee_{\beta_i})} h_{\varpi^\vee_{i'}}(\xi)\right)\left(\prod_{\alpha\in\Phi_{\beta_j}} n_\alpha(\kappa_\alpha)\right)^{-1} \\
&\quad = \left(\prod_{\alpha\in\Phi_{\beta_j}} r_\alpha\right)\left(\prod_{\varpi^\vee_{i'}\in\Omega(\varpi^\vee_{\beta_i})} h_{\varpi^\vee_{i'}}(\xi)\right) \\
&\quad = \left(\prod_{\varpi^\vee_{i'}\in\Omega(\varpi^\vee_{\beta_i})} h_{\varpi^\vee_{i'}}(\xi)\right)\left(\prod_{\alpha\in\Phi_{\beta_j}} h_\alpha\left(\prod_{\varpi^\vee_{i'}\in\Omega(\varpi^\vee_{\beta_i})} \xi^{-\langle\alpha,\varpi^\vee_{i'}\rangle}\right)\right)
\end{aligned}$$

where $\prod_{\alpha\in\Phi_{\beta_j}} r_\alpha$ is well-defined since the roots in the orbit corresponding to β_j, and hence the reflections corresponding to them, all commute. However,

$$\begin{aligned}
\Psi'_m(\mathbb{K})\left(r_{\beta_j}\left(h_{\varpi^\vee_{\beta_i}}(\xi)\right)\right) &= \Psi'_m(\mathbb{K})\left(h_{\varpi^\vee_{\beta_i}}(\xi) h_{\beta_j}\left(\xi^{-\delta_{ij}}\right)\right) \\
&= \left(\prod_{\varpi^\vee_{i'}\in\Omega(\varpi^\vee_{\beta_i})} h_{\varpi^\vee_{i'}}(\xi)\right)\left(\prod_{\alpha\in\Phi_{\beta_j}} h_\alpha\left(\xi^{\delta_{ij}}\right)\right).
\end{aligned}$$

Thus it is sufficient to show that

$$\sum_{\varpi^\vee_{i'}\in\Omega\left(\varpi^\vee_{\beta_i}\right)} \langle\alpha, \varpi^\vee_{i'}\rangle = \delta_{ij}$$

for all $\alpha \in \Phi_{\beta_j}$, which is a fact we have already established. □

Since the structure constants are all integers, the preservation of the last two relations involves integral equalities which, since they are satisfied in $\mathbb{C}$, must certainly be satisfied modulo a prime.

Thus $\Psi'_m(\mathbb{K})$ is a group homomorphism. Furthermore, since every generator of $\mathfrak{G}^{\tau,r}_m(\mathbb{K})$ described in §4.2 appears in its image, $\Psi'_m(\mathbb{K})$ is surjective.

We thus require to show that $\Psi'_m(\mathbb{K})$ is injective. Let

$$U^B = \langle X_\beta : \beta \in \Phi^B_+\rangle, \quad U^B_- = \langle X_\beta : \beta \in \Phi^B_-\rangle,$$
$$B' = U^B H^B_m \quad \text{and} \quad B'_- = U^B_- H^B_m.$$

Recall that (B', N^B) and (B'_-, N^B) are two (B, N)-pairs in $\mathfrak{G}^B_m(\mathbb{K})$.

We note that we may define Kac-Moody algebras over arbitrary fields in a manner analogous to that used in the classical case. Furthermore, provided the field $\mathbb{K}$ is of a suitable type, we may again obtain isomorphisms between $\mathfrak{g}_B(\mathbb{K})$

and $\mathfrak{g}^{\sigma}(\mathbb{K})$. We may also define actions of the groups $\mathfrak{G}_m^B(\mathbb{K})$ and $\mathfrak{G}_m^{\gamma,\tau}(\mathbb{K})$ on $\mathfrak{g}_B(\mathbb{K}) \cong \mathfrak{g}^{\sigma}(\mathbb{K})$ and the diagram

$$\begin{array}{ccc} \mathfrak{G}_m^B(\mathbb{K}) & & \\ {\scriptstyle \Psi'_m(\mathbb{K})}\downarrow & \searrow & \\ \mathfrak{G}_m^{\gamma,\tau}(\mathbb{K}) & \longrightarrow & \mathrm{Aut}\ \mathfrak{g}_B(\mathbb{K}) \end{array}$$

commutes. Thus a non-trivial element

$$x \in \ker \Psi'_m(\mathbb{K})$$

must certainly act trivially on $\mathfrak{g}_B(\mathbb{K})$. Recall that by Proposition 3.2.3 there is a unique decomposition

$$x = u_1 n u$$

such that $u_1 \in U^B$, $n \in N^B$ and

$$u \in X_{w_n}^- = \langle X_\alpha : \alpha \in \Phi_+^B, w_n(\alpha) \in \Phi_-^B \rangle$$

where $w_n = \pi(n) \in W^B$. Since x acts trivially on $\mathfrak{g}_B(\mathbb{K})$ we conclude that the action of $u_1 n$ is the same as that of u^{-1}. Thus $u_1 n$ must leave $\mathfrak{n}_+(\mathbb{K})$ invariant. Hence we deduce that $w_n = 1$ and so $n \in H$. Thus $x \in B'$ and we conclude that

$$\ker \Psi'_m(\mathbb{K}) \leq B'.$$

Similarly, consideration of the (B, N)-pair (B'_-, N^B) leads us to the conclusion that

$$\ker \Psi'_m(\mathbb{K}) \leq B'_-.$$

Thus

$$\ker \Psi'_m(\mathbb{K}) \leq B' \cap B'_- = H_m^B.$$

Recall that every element $h \in H_m^B$ has a unique expression of the form

$$h = \prod_{i \in \underline{m}_0} h_{\varpi_{\beta_i}^\vee}(\xi_i)$$

for some $\xi_i \in \mathbb{K}^\times$. Suppose that such an element h is a non-trivial element of $\ker \Psi'_m(\mathbb{K})$, so that $\xi_i \neq 1$ for at least one $i \in \underline{m}_0$, but the element

$$\Psi'_m(\mathbb{K})(h) = \prod_{i \in \underline{m}_0} \Psi'_m(\mathbb{K})\left(h_{\varpi_{\beta_i}^\vee}(\xi_i)\right) = 1.$$

Since $\xi_i \neq 1$ for at least one $i \in \underline{m}_0$

$$\Psi'_m(\mathbb{K})\left(h_{\varpi^\vee_{\beta_i}}(\xi_i)\right) = \prod_{\varpi^\vee_j \in \Omega(\varpi^\vee_{\beta_i})} h_{\varpi^\vee_j}(\xi_i) \neq 1$$

for at least one $i \in \underline{m}_0$. However, we note that the orbits $\Omega(\varpi^\vee_{\beta_i})$ for $i \in \underline{m}_0$ are linearly independent. We thus conclude that $\Psi'_m(\mathbb{K})(h) \neq 1$ contradicting our assumption on h. Hence $\Psi'_m(\mathbb{K})$ is injective.

Thus $\Psi'_m(\mathbb{K})$ is a group isomorphism for all suitable $\mathbb{K}$. □

Examples

We build on the examples of Kac-Moody algebra isomorphisms we gave in §1.5. We first note the general facts that

$$\beta_i = \sum_{j \in \underline{m}_0} B_{ij} \varpi^\vee_{\beta_j}$$

for each $i \in \underline{m}_0$ and

$$h_{a\varpi^\vee_{\beta_i}}(\xi) h_{b\varpi^\vee_{\beta_j}}(\xi) = h_{a\varpi^\vee_{\beta_i} + b\varpi^\vee_{\beta_j}}(\xi)$$

for all $i, j \in \underline{m}_0$, $a, b \in \mathbb{N}_0$ and $\xi \in \mathbb{K}^\times$.

EXAMPLE 4.4.6

We shall first consider the case when $(\tilde{A}, k) = (\tilde{D}_4, 3)$, and $B = {}^t\tilde{G}_2$. Thus we suppose that $\mathbb{K}$ is a field such that

$$\operatorname{char} \mathbb{K} \neq 2, 3 \qquad \text{and} \qquad \epsilon = e^{\frac{2\pi i}{3}} \in \mathbb{K}.$$

The results of the preceding sections and Example 1.5.9 lead us to the conclusion that the map $\Psi'_m(\mathbb{K})$ defined by

$$\begin{array}{lcl}
y_{\beta_0}(\mu) & \mapsto & x_{\alpha_0+\alpha_2+\alpha_4}(\mu) x_{\alpha_0+\alpha_1+\alpha_2}(\epsilon\mu) x_{\alpha_0+\alpha_2+\alpha_3}(\epsilon^2\mu) \\
y_{\beta_1}(\mu) & \mapsto & x_{\alpha_1}(\mu) x_{\alpha_3}(\mu) x_{\alpha_4}(\mu) \\
y_{\beta_2}(\mu) & \mapsto & x_{\alpha_2}(\mu) \\
y_{-\beta_0}(\mu) & \mapsto & x_{-\alpha_0-\alpha_2-\alpha_4}(\mu) x_{-\alpha_0-\alpha_1-\alpha_2}(\epsilon\mu) x_{-\alpha_0-\alpha_2-\alpha_3}(\epsilon^2\mu) \\
y_{-\beta_1}(\mu) & \mapsto & x_{-\alpha_1}(\mu) x_{-\alpha_3}(\mu) x_{-\alpha_4}(\mu) \\
y_{-\beta_2}(\mu) & \mapsto & x_{-\alpha_2}(\mu) \\
h_{\varpi^\vee_{\beta_0}}(\xi) & \mapsto & h_{\varpi^\vee_0}(\xi) \\
h_{\varpi^\vee_{\beta_1}}(\xi) & \mapsto & h_{\varpi^\vee_0}(\xi^{-1}) h_{\varpi^\vee_1}(\xi) h_{\varpi^\vee_3}(\xi) h_{\varpi^\vee_4}(\xi) \\
h_{\varpi^\vee_{\beta_2}}(\xi) & \mapsto & h_{\varpi^\vee_0}(\xi^{-1}) h_{\varpi^\vee_2}(\xi)
\end{array}$$

for $\mu \in \mathbb{K}$ ad $\xi \in \mathbb{K}^\times$, extends to an isomorphism

$$\Psi'_m(\mathbb{K}) : \mathfrak{G}^B_m(\mathbb{K}) \to \mathfrak{G}^{A,\tau}_m(\mathbb{K}).$$

Note that since

$$\begin{aligned} \beta_0 &= 2\varpi^\vee_{\beta_0} - \varpi^\vee_{\beta_1} \\ \beta_1 &= -\varpi^\vee_{\beta_0} + 2\varpi^\vee_{\beta_1} - 3\varpi^\vee_{\beta_2} \qquad \text{and} \\ \beta_2 &= -\varpi^\vee_{\beta_1} + 2\varpi^\vee_{\beta_2} \end{aligned}$$

we deduce that

$$\begin{aligned} \mathfrak{h}_{\beta_0}(\xi) &\mapsto h_{3\varpi^\vee_{\beta_0} - \varpi^\vee_{\beta_1} - \varpi^\vee_{\beta_3} - \varpi^\vee_{\beta_4}}(\xi) &&= h_{\alpha_0+\alpha_1+\alpha_2}(\xi)h_{\alpha_0+\alpha_2+\alpha_3}(\xi)h_{\alpha_0+\alpha_2+\alpha_4}(\xi) \\ h_{\beta_1}(\xi) &\mapsto h_{2\varpi^\vee_{\beta_1} - 3\varpi^\vee_{\beta_2} + 2\varpi^\vee_{\beta_3} + 2\varpi^\vee_{\beta_4}}(\xi) &&= h_{\alpha_1}(\xi)h_{\alpha_3}(\xi)h_{\alpha_4}(\xi) \qquad \text{and} \\ \mathfrak{h}_{\beta_2}(\xi) &\mapsto h_{-\varpi^\vee_{\beta_0} - \varpi^\vee_{\beta_1} + 2\varpi^\vee_{\beta_2} - \varpi^\vee_{\beta_3} - \varpi^\vee_{\beta_4}}(\xi) &&= h_{\alpha_2}(\xi). \end{aligned}$$

In particular, since $\tilde{D}_4$ is simply-laced

$$h_{a\alpha_i+b\alpha_j}(\xi) = h_{\alpha_i}(\xi^a)h_{\alpha_j}(\xi^b)$$

for all $i, j \in \underline{4}_0$, $a, b \in \mathbb{N}_0$ and $\xi \in \mathbb{K}^\times$ and hence

$$\mathfrak{h}_{\beta_0}(\xi) \mapsto h_{\alpha_0}(\xi^3)h_{\alpha_2}(\xi^3)h_{\alpha_1}(\xi)h_{\alpha_3}(\xi)h_{\alpha_4}(\xi).$$

◇

EXAMPLE 4.4.7

We next consider the case when $(\tilde{A}, k) = (\tilde{A}_4, 2)$, and $B = {}^{*}\tilde{C}_2$. Thus we suppose that $\mathbb{K}$ is a field such that

$$\operatorname{char}\mathbb{K} \neq 2 \qquad \text{and} \qquad \sqrt{2} \in \mathbb{K}.$$

The results of the preceding sections and Example 1.5.11 lead us to the conclusion that the map $\Psi'_m(\mathbb{K})$ defined by

$$\begin{aligned} y_{\beta_0}(\mu) &\mapsto x_{\alpha_2}(\sqrt{2}\mu)x_{\alpha_3}(\sqrt{2}\mu)x_{\alpha_2+\alpha_3}(-\mu^2) \\ y_{\beta_1}(\mu) &\mapsto x_{\alpha_1}(\mu)x_{\alpha_4}(\mu) \\ y_{\beta_2}(\mu) &\mapsto x_{\alpha_0}(\mu) \\ y_{-\beta_0}(\mu) &\mapsto x_{-\alpha_2}(\sqrt{2}\mu)x_{-\alpha_3}(\sqrt{2}\mu)x_{-\alpha_2-\alpha_3}(-\mu^2) \\ y_{-\beta_1}(\mu) &\mapsto x_{-\alpha_1}(\mu)x_{-\alpha_4}(\mu) \\ y_{-\beta_2}(\mu) &\mapsto x_{-\alpha_0}(\mu) \\ h_{\varpi^\vee_{\beta_0}}(\xi) &\mapsto h_{\varpi^\vee_2}(\xi)h_{\varpi^\vee_3}(\xi) \\ h_{\varpi^\vee_{\beta_1}}(\xi) &\mapsto h_{\varpi^\vee_1}(\xi)h_{\varpi^\vee_4}(\xi) \\ h_{\varpi^\vee_{\beta_2}}(\xi) &\mapsto h_{\varpi^\vee_0}(\xi) \end{aligned}$$

for $\mu \in \mathbb{K}$ and $\xi \in \mathbb{K}^\times$, extends to an isomorphism

$$\Psi'_m(\mathbb{K}) : \mathfrak{G}^B_m(\mathbb{K}) \to \mathfrak{G}^{\gamma,\tau}_m(\mathbb{K}).$$

Once again we note that

$$\begin{aligned} \beta_0 &= 2\varpi^\vee_{\beta_0} - 2\varpi^\vee_{\beta_1}(\xi) \\ \beta_1 &= -\varpi^\vee_{\beta_0} + 2\varpi^\vee_{\beta_1}(\xi) - 2\varpi^\vee_{\beta_2}(\xi) \qquad \text{and} \\ \beta_2 &= -\varpi^\vee_{\beta_1} + 2\varpi^\vee_{\beta_2}(\xi). \end{aligned}$$

Hence we may deduce that

$$\begin{aligned} \mathfrak{h}_{\beta_0}(\xi) &\mapsto h_{2\varpi^\vee_{\beta_2}+2\varpi^\vee_{\beta_3}-2\varpi^\vee_{\beta_1}-2\varpi^\vee_{\beta_4}}(\xi) &= h_{\alpha_2}(\xi^2)h_{\alpha_3}(\xi^2) \\ h_{\beta_1}(\xi) &\mapsto h_{-\varpi^\vee_{\beta_2}-\varpi^\vee_{\beta_3}+2\varpi^\vee_{\beta_1}+2\varpi^\vee_{\beta_4}-2\varpi^\vee_{\beta_0}}(\xi) &= h_{\alpha_1}(\xi)h_{\alpha_4}(\xi) \qquad \text{and} \\ \mathfrak{h}_{\beta_2}(\xi) &\mapsto h_{-\varpi^\vee_{\beta_1}-\varpi^\vee_{\beta_4}+2\varpi^\vee_{\beta_0}}(\xi) &= h_{\alpha_0}(\xi). \end{aligned}$$

◇

The Images of the Elements $h_{\beta_i}(\xi)$ Under Ψ'_m

We note a general observation.

LEMMA 4.4.8
Let $\beta_i \in \Pi^B$ and suppose $K(\alpha)$ is the orbit corresponding to β_i. Let

$$\alpha^1 = \sum_{i \in k} \gamma^i(\alpha) = \sum_{j \in \underline{n}_0} k_j \alpha_j.$$

Then

$$\Psi'_m(\mathbb{K})(h_{\beta_i}(\xi)) = \prod_{j \in \underline{n}_0} h_{\alpha_j}\left(\xi^{k_j}\right)$$

for all $\xi \in \mathbb{K}^\times$.

Proof
This can be shown by straightforward case-by-case analysis such as in the preceding examples. □

4.5 An Isomorphism Between Simply-Connected Kac-Moody Groups

We shall here exploit the existence of generalized isogenies, described in §2.5, between simply-connected Kac-Moody groups and adjoint Kac-Moody groups corresponding to the same generalized Cartan matrices.

A Restricted Generalized Isogeny

Denote by $\mathfrak{G}_{sc}^{\tilde{A}}$ the Kac-Moody group functor $\mathfrak{G}_{\mathcal{D}_{sc}(\tilde{A})}$ and suppose that

$$\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K}) = \langle x'_\alpha(\mu) : \alpha \in \Phi^{re}(\tilde{A}), \mu \in \mathbb{K} \rangle.$$

By the results of §2.5, the map defined on the generators of $\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$ by

$$\imath_{\tilde{A}}(\mathbb{K}) : x'_\alpha(\mu) \rightarrow x_\alpha(\mu)$$

for all $\alpha \in \Phi^{re}(\tilde{A})$ and $\mu \in \mathbb{K}$, extends to a group homomorphism

$$\imath_{\tilde{A}}(\mathbb{K}) : \mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K}) \rightarrow \mathfrak{G}_{m}^{\tilde{A}}(\mathbb{K}).$$

We note also that we may define automorphisms of $\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$, γ' and τ', analogous to the automorphisms γ and τ defined for $\mathfrak{G}_{m}^{\tilde{A}}(\mathbb{K})$, but with $x_\alpha(\mu)$ replaced by $x'_\alpha(\mu)$. We shall denote by $\mathfrak{G}_{sc}^{\gamma',\tau'}(\mathbb{K})$ the subgroup of $\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$ consisting of elements $x' \in \mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$ satisfying $\gamma'(x') = \tau'(x')$.

Since the action of $\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$ on $\mathfrak{g}_{\tilde{A}}(\mathbb{C})$ factors through $\mathfrak{G}_{m}^{\tilde{A}}(\mathbb{K})$, the results of Propositions 4.1.3 and 4.1.5 also hold for the simply-connected case. Hence in particular, the homomorphism $\imath_{\tilde{A}}(\mathbb{K})$ maps elements of $\mathfrak{G}_{sc}^{\tilde{A}}(\mathbb{K})$ which are stable under the automorphism $\sigma' = (\tau'^{-1} \circ \gamma')$ to elements of $\mathfrak{G}_{m}^{\tilde{A}}(\mathbb{K})$ which are stable under the automorphism $\sigma = (\tau^{-1} \circ \gamma)$. Thus the generalized isogeny $\imath_{\tilde{A}}(\mathbb{K})$ restricts to a homomorphism

$$\imath_\sigma(\mathbb{K}) : \mathfrak{G}_{sc}^{\gamma',\tau'}(\mathbb{K}) \rightarrow \mathfrak{G}_{m}^{\gamma,\tau}(\mathbb{K})$$

and enables us to define a group functor homomorphism

$$\imath_\sigma : \mathfrak{G}_{sc}^{\gamma',\tau'} \rightarrow \mathfrak{G}_{m}^{\gamma,\tau}$$

defined over the category of all fields in which γ' and τ' are defined.

An Induced Generalized Isogeny

Denote by $\mathfrak{G}_{sc}^{B}$ the Kac-Moody group functor $\mathfrak{G}_{\mathcal{D}_{sc}(B)}$ and suppose that

$$\mathfrak{G}_{m}^{B}(\mathbb{K}) = \langle y'_\beta(\mu) : \beta \in \Phi^{B}, \mu \in \mathbb{K} \rangle.$$

From the results of §2.5 we have a group functor homomorphism

$$\imath_B : \mathfrak{G}_{sc}^{B} \rightarrow \mathfrak{G}_{m}^{B}$$

defined over all fields, such that the diagram

$$\begin{array}{ccc} \mathfrak{G}^B_{sc}(\mathbb{C}) & \xrightarrow{\imath_B(\mathbb{C})} & \mathfrak{G}^B_m(\mathbb{C}) \\ & \searrow^{\mathrm{Ad}_{sc}} & \downarrow \mathrm{Ad}_B \\ & & \mathrm{Aut}\ \mathfrak{g}_B(\mathbb{C}) \end{array}$$

commutes, where the maps Ad_B and Ad_{sc} are those determined by condition KMG5. Hence

$$\ker \imath_B(\mathbb{C}) = \ker \mathrm{Ad}_{sc}$$

since $\mathfrak{G}^B_m(\mathbb{C})$ acts faithfully on $\mathfrak{g}_B(\mathbb{C})$. Thus $\ker \imath_B(\mathbb{C})$ consists of all those elements which act trivially on $\mathfrak{g}_B(\mathbb{C})$, whence

$$\ker \imath_B(\mathbb{C}) \subseteq H^B_{sc}$$

by condition KMG5. Now every element $h \in H^B_{sc}$ has a unique expression

$$h = h_{\beta_0}(\xi_0)h_{\beta_1}(\xi_1)\cdots h_{\beta_m}(\xi_m)$$

where $\xi_i \in \mathbb{C}^\times$ for all $i \in \underline{m}$. Such an element will act trivially on e_{β_i} if and only if

$$\xi_0^{B_{0i}}\xi_1^{B_{1i}}\cdots\xi_m^{B_{mi}} = 1.$$

Thus an element h which acts trivially on $\mathfrak{g}_B(\mathbb{C})$ satisfies a set of $m+1$ equations in $m+1$ variables. Consider the equations

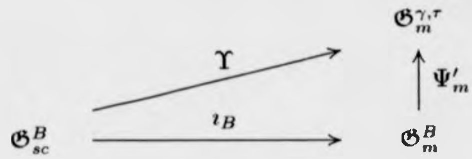
$$\begin{array}{ccccc} \xi_1^{B_{11}} & \cdots & \xi_m^{B_{m1}} & = & \xi_0^{-B_{01}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_1^{B_{1m}} & \cdots & \xi_m^{B_{mm}} & = & \xi_0^{-B_{0m}}. \end{array}$$

Since $\check{B}$ is non-singular and $\mathbb{C}$ is algebraically closed these equations have a simultaneous solution. The remaining equation is the then automatically satisfied by this solution as a result of the linear dependence in the rows of B. Thus $\ker \imath_B(\mathbb{C})$ is certainly non-trivial.

Now, by §4.4 we also have a group functor isomorphism

$$\Psi'_m : \mathfrak{G}^B_m \to \mathfrak{G}^{\gamma,\tau}_m$$

defined over the category of all fields in which γ and τ are defined. We may thus define a group functor homomorphism Υ such that the diagram



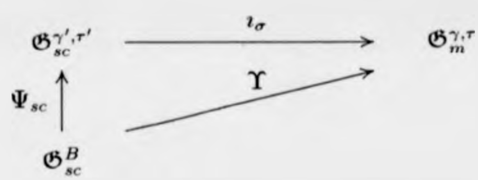
commutes. We note that the action of $\Upsilon(\mathbb{K})$ on the generator $y'_\beta(\mu)$ of $\mathfrak{G}_{sc}^B(\mathbb{K})$ is

$$\Upsilon\left(y'_\beta(\mu)\right) = \Psi'_m\left(y_\beta(\mu)\right)$$

for each $\beta \in \Phi^B$ and $\mu \in \mathbb{K}$.

A Homomorphism Between Simply-Connected Groups

We would now like to define a group functor homomorphism Ψ_{sc} such that the diagram



commutes. We do this by defining $\Psi_{sc}(\mathbb{K})$ on the generators of $\mathfrak{G}_{sc}^B(\mathbb{K})$ and proving that this definition extends to a group homomorphism by showing that the defining relations of $\mathfrak{G}_{sc}^B(\mathbb{K})$ are preserved.

We begin by defining $\Psi_{sc}(\mathbb{K})$ on the generators of $\mathfrak{G}_{sc}^B(\mathbb{K})$. Suppose that for $\beta \in \pm\Pi^B$

$$\Upsilon(\mathbb{K})\left(y'_\beta(\mu)\right) = \prod_{\alpha\in\Phi^{re}(\dot{A})} x_\alpha(\mu_\alpha)$$

for $\mu, \mu_\alpha \in \mathbb{K}$. We define

$$\Psi_{sc}(\mathbb{K})\left(y'_\beta(\mu)\right) = \prod_{\alpha\in\Phi^{re}(\dot{A})} x'_\alpha(\mu_\alpha).$$

We begin by considering the images of the elements $n'_{\beta_i}(\xi)$ and $h'_{\beta_i}(\xi)$ for $i \in \underline{m}_0$ and $\xi \in \mathbb{K}^\times$.

The Images of $n'_{\beta_i}(\xi)$ and $h'_{\beta_i}(\xi)$ under Ψ_{sc}

Suppose

$$\Psi_{sc}(\mathbb{K})\left(y'_{\beta_i}(\xi)\right) = \prod_{\alpha\in\Phi^{re}(\tilde{A})} x'_\alpha(\kappa_\alpha\xi)$$

and

$$\Psi_{sc}(\mathbb{K})\left(y'_{-\beta_i}(\xi^{-1})\right) = \prod_{\alpha\in\Phi^{re}(\tilde{A})} x'_{-\alpha}\left(\kappa'_{-\alpha}\xi^{-1}\right)$$

for some $\kappa_\alpha, \kappa'_{-\alpha} \in \mathbb{K}$. Then

$$\Psi_{sc}(\mathbb{K})\left(n'_{\beta_i}(\xi)\right) = \left(\prod_{\alpha\in\Phi^{re}(\tilde{A})} x'_\alpha(\kappa_\alpha\xi)\right)\left(\prod_{\alpha\in\Phi^{re}(\tilde{A})} x'_{-\alpha}\left(\kappa'_{-\alpha}\xi^{-1}\right)\right)\left(\prod_{\alpha\in\Phi^{re}(\tilde{A})} x'_\alpha(\kappa_\alpha\xi)\right).$$

However, whenever $\beta_i \neq \beta_0$ the the orbit of roots, $K(\alpha)$ say, corresponding to β_i consists of commuting fundamental roots and hence

$$\Psi_{sc}(\mathbb{K})\left(n'_{\beta_i}(\xi)\right) = \prod_{\alpha_j\in\Phi_{\alpha^1}}\left(x'_{\alpha_j}(\kappa_{\alpha_j}\xi)x'_{-\alpha_j}\left(\kappa'_{-\alpha_j}\xi^{-1}\right)x'_{\alpha_j}(\kappa_{\alpha_j}\xi)\right).$$

Furthermore, in these cases $\kappa'_{-\alpha_i} = \kappa_{\alpha_i}^{-1} \in \mathbb{K}^\times$ for all $\alpha_i \in \Phi_{\alpha^1}$ and hence

$$\Psi_{sc}(\mathbb{K})\left(n'_{\beta_i}(\xi)\right) = \prod_{\alpha_j\in\Phi_{\alpha^1}} n'_{\alpha_j}(\kappa_{\alpha_j}\xi).$$

We now use the fact that

$$n'_{\alpha_j}(\zeta)n'_{\alpha_j}(\xi) = h'_{\alpha_j}\left(-\xi^{-1}\zeta\right)$$

for all $j \in \underline{n}_0$ and $\zeta, \xi \in \mathbb{K}^\times$ to deduce that

$$\Psi_{sc}(\mathbb{K})\left(h'_{\beta_i}(\xi)\right) = \prod_{\alpha_j\in\Phi_{\alpha^1}} h'_{\alpha_j}(\xi)$$

whenever $\beta_i \neq \beta_0$.

The Case $\beta_i = \beta_0$

Suppose now that $\beta_i = \beta_0$. Recall that the orbits of roots, which we shall denote by Φ_{β_0}, corresponding to β_0 were described in Table 4.3.5. Whenever $(\tilde{A}, k) \neq (\tilde{A}_{2l}, 2)$, Φ_{β_0} consists of pairwise commuting non-fundamental roots, each of which also commutes with the negatives of the remaining roots in the orbit. Thus

$$\Psi_{sc}(\mathbb{K})\left(n'_{\beta_0}(\xi)\right) = \prod_{\alpha\in\Phi_{\beta_0}}\left(x'_\alpha(\kappa_\alpha\xi)x'_{-\alpha}\left(\kappa'_{-\alpha}\xi^{-1}\right)x'_\alpha(\kappa_\alpha\xi)\right)$$

and once again we note that $\kappa'_{-\alpha} = \kappa_\alpha^{-1}$. Suppose that $\alpha = w(\alpha_j)$ for some $j \in \underline{n}_0$ and $w \in W(\hat{A})$. Let $n'_w \in N_{sc}^{\hat{A}}$ be a product of elements of the form n'_{α_i} for $i \in \underline{n}_0$ such that $\pi(n'_w) = w$. Then

$$\begin{aligned} x'_\alpha(\kappa_\alpha\xi) &= n'_w x'_{\alpha_j}(\eta_{w,\alpha_j}\kappa_\alpha\xi){n'_w}^{-1} \quad \text{and} \\ x'_{-\alpha}\left(\kappa_\alpha^{-1}\xi^{-1}\right) &= n'_w x'_{-\alpha_j}\left(\eta_{w,\alpha_j}\kappa_\alpha^{-1}\xi^{-1}\right){n'_w}^{-1} \end{aligned}$$

for some $\eta_{w,\alpha_j} \in \{1,-1\}$. Hence

$$\Psi_{sc}(\mathbb{K})\left(n'_{\beta_0}(\xi)\right) = \prod_{\alpha\in\Phi_{\beta_0}} \left(n'_{w_\alpha} n'_{\alpha_{j_\alpha}}(\eta_{w_\alpha,\alpha_{j_\alpha}}\kappa_\alpha\xi){n'_{w_\alpha}}^{-1}\right)$$

where $\alpha = w_\alpha(\alpha_{j_\alpha})$.

We now use the fact that

$$n'_{\alpha_j}(\zeta)n'_{\alpha_j}(\xi) = h'_{\alpha_j}\left(-\xi^{-1}\zeta\right)$$

for all $j \in \underline{n}_0$ and $\zeta, \xi \in \mathbb{K}^\times$ to deduce that

$$\Psi_{sc}(\mathbb{K})\left(h'_{\beta_0}(\xi)\right) = \prod_{\alpha\in\Phi_{\beta_0}} \left(n'_{w_\alpha} h'_{\alpha_{j_\alpha}}(\xi){n'_{w_\alpha}}^{-1}\right) = \prod_{\alpha\in\Phi_{\beta_0}} h'_\alpha(\xi) \in H_{sc}^{\gamma',\tau'}.$$

Since $\hat{A}$ is simply-laced in all cases under consideration, we deduce that

$$\Psi_{sc}(\mathbb{K})\left(h'_{\beta_0}(\xi)\right) = \prod_{j\in\underline{n}_0} h'_{\alpha_j}\left(\xi^{k_j}\right)$$

where $\alpha^1 = \sum_{j\in\underline{n}_0} k_j\alpha_j$ for every $\alpha \in \Phi_{\beta_0}$.

The Case $\beta_i = \beta_0$ When $\hat{A} = \hat{A}_{2l}$

Finally, we consider the case $\beta_i = \beta_0$ when $(\hat{A}, k) = (\hat{A}_{2l}, 2)$. We recall that the orbit corresponding to β_0 in this case is $\{\alpha_l, \alpha_{l+1}\}$ and that

$$\begin{aligned} \Psi_{sc}(\mathbb{K})\left(n'_{\beta_0}(\xi)\right) &= x'_{\alpha_l}(\sqrt{2}\xi)x'_{\alpha_{l+1}}(\sqrt{2}\xi)x'_{\alpha_l+\alpha_{l+1}}(-\xi^2) \\ &\quad x'_{-\alpha_l}(\sqrt{2}\xi^{-1})x'_{-\alpha_{l+1}}(\sqrt{2}\xi^{-1})x'_{-\alpha_l-\alpha_{l+1}}(-\xi^{-2}) \\ &\quad x'_{\alpha_l}(\sqrt{2}\xi)x'_{\alpha_{l+1}}(\sqrt{2}\xi)x'_{\alpha_l+\alpha_{l+1}}(-\xi^2). \end{aligned}$$

We also note that there is a group homomorphism

$$\iota_{sl} : \mathfrak{SL}_3(\mathbb{K}) \to \langle X_{\alpha_l}, X_{\alpha_{l+1}}, X_{-\alpha_l}, X_{-\alpha_{l+1}}\rangle \subseteq \mathfrak{G}_{sc}^{\hat{A}_{2l}}(\mathbb{K})$$

such that

$$\imath_{sl}: \begin{pmatrix} 1 & \mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto x'_{\alpha_l}(\mu)$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} \mapsto x'_{\alpha_{l+1}}(\mu)$$
$$\begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto x'_{\alpha_l+\alpha_{l+1}}(\mu)$$
$$\begin{pmatrix} 1 & 0 & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto x'_{-\alpha_l}(\mu)$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\mu & 1 \end{pmatrix} \mapsto x'_{-\alpha_{l+1}}(\mu)$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mu & 0 & 1 \end{pmatrix} \mapsto x'_{-\alpha_l-\alpha_{l+1}}(\mu)$$
$$\begin{pmatrix} 0 & \xi & 0 \\ -\xi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto n'_{\alpha_l}(\xi)$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\xi^{-1} & 0 \end{pmatrix} \mapsto n'_{\alpha_{l+1}}(\xi)$$

for all $\mu \in \mathbb{K}$ and $\xi \in \mathbb{K}^\times$.

Consider the matrix

$$M = \begin{pmatrix} 0 & 0 & \xi^2 \\ 0 & -1 & 0 \\ \xi^{-2} & 0 & 0 \end{pmatrix}$$

for $\xi \in \mathbb{K}^\times$. Since

$$\begin{aligned} M \;=\; & \begin{pmatrix} 1 & \sqrt{2}\xi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2}\xi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\xi^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}\xi^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\sqrt{2}\xi^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi^{-2} & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} 1 & \sqrt{2}\xi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2}\xi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\xi^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

we deduce that $\imath_{sl}(M) = \Psi_{sc}(\mathbb{K})\left(n'_{\beta_0}(\xi)\right)$. Furthermore, since

$$\begin{aligned} M &= \begin{pmatrix} 0 & \xi & 0 \\ -\xi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\xi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi & 0 \\ -\xi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\xi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi & 0 \\ -\xi^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\xi^{-1} & 0 \end{pmatrix}, \end{aligned}$$

we may deduce that

$$\begin{aligned} \Psi_{sc}(\mathbb{K})\left(n'_{\beta_0}(\xi)\right) &= n'_{\alpha_l}(\xi)n'_{\alpha_{l+1}}(\xi)n'_{\alpha_l}(\xi) \qquad \text{and} \\ &= n'_{\alpha_{l+1}}(\xi)n'_{\alpha_l}(\xi)n'_{\alpha_{l+1}}(\xi). \end{aligned}$$

We note that

$$\begin{aligned} \Psi_{sc}(\mathbb{K})\left(h'_{\beta_0}(\xi)\right) &= n'_{\alpha_l}(\xi)n'_{\alpha_{l+1}}(\xi)n'_{\alpha_l}(\xi)\left(n'_{\alpha_l}n'_{\alpha_{l+1}}n'_{\alpha_l}\right)^{-1} \\ &= n'_{\alpha_l}(\xi)n'_{\alpha_{l+1}}(\xi)h'_{\alpha_l}(\xi){n'_{\alpha_{l+1}}}^{-1}{n'_{\alpha_l}}^{-1} \\ &= n'_{\alpha_l}(\xi)h'_{\alpha_{l+1}}(\xi)n'_{\alpha_{l+1}}h'_{\alpha_l}(\xi){n'_{\alpha_{l+1}}}^{-1}{n'_{\alpha_l}}^{-1} \\ &= n'_{\alpha_l}(\xi)h'_{\alpha_{l+1}}(\xi)h'_{\alpha_l}(\xi)h'_{\alpha_{l+1}}(\xi){n'_{\alpha_l}}^{-1} \\ &= h'_{\alpha_l}(\xi)n'_{\alpha_l}h'_{\alpha_l}(\xi)h'_{\alpha_{l+1}}(\xi^2){n'_{\alpha_l}}^{-1} \\ &= h'_{\alpha_l}(\xi)h'_{\alpha_l}h'_{\alpha_l}(\xi^{-2})h'_{\alpha_{l+1}}(\xi^2)h'_{\alpha_l}(\xi^2) \\ &= h'_{\alpha_l}(\xi^2)h'_{\alpha_{l+1}}(\xi^2) \end{aligned}$$

as expected.

In particular, we note that in all cases the natural analogue of Lemma 4.4.8 holds in the simply-connected case.

The Preservation of the Relations

We must show that the relations

$$\begin{aligned} h'_{\beta_i}(\xi)h'_{\beta_i}(\zeta) &= h'_{\beta_i}(\xi\zeta) \\ h'_{\beta_i}(\xi)h'_{\beta_j}(\zeta) &= h'_{\beta_j}(\zeta)h'_{\beta_i}(\xi) \\ y'_\beta(\mu)y'_\beta(\nu) &= y'_\beta(\mu+\nu) \\ h'_{\beta_i}(\xi)y'_{\beta_j}(\mu)h'_{\beta_i}(\xi)^{-1} &= y'_{\beta_j}\left(\xi^{B_{ij}}\mu\right) \\ n'_{\beta_j}h'_{\beta_i}(\xi){n'_{\beta_j}}^{-1} &= r_{\beta_j}\left(h'_{\beta_i}(\xi)\right) = h'_{\beta_i}(\xi)h'_{\beta_j}\left(\xi^{-B_{ij}}\right) \\ n'_{\beta_i}y'_\beta(\mu){n'_{\beta_i}}^{-1} &= y'_{r_{\beta_i}(\beta)}(\eta_{\beta_i,\beta}\mu) \qquad \text{and} \\ [y'_\beta(\mu), y'_{\beta'}(\nu)] &= \prod_{\substack{\gamma=i\beta+j\beta' \\ i,j\in\mathbb{N}}} y'_\gamma\left(C_{\beta\beta'\gamma}\mu^i\nu^j\right), \qquad \gamma\in\Phi^B,\ \beta\neq\pm\beta',\ \{\beta,\beta'\}\text{ prenilpotent} \end{aligned}$$

are preserved for all $i \in \underline{m}_0$, $\mu, \nu \in \mathbb{K}$, and $\xi, \zeta \in \mathbb{K}^\times$, where

$$h'_{\beta_i}(\xi) := n'_{\beta_i}(\xi) {n'_{\beta_i}}^{-1}$$

and

$$n'_{\beta_i}(\xi) := x'_{\beta_i}(\xi) x'_{-\beta_i}\left(\xi^{-1}\right) x'_{\beta_i}(\xi).$$

Since $\Psi_{sc}(\mathbb{K})\left(h'_{\beta_i}(\xi)\right) \in H_{sc}^{\gamma',\tau'}$ for all $i \in \underline{m}_0$ and $\xi \in \mathbb{K}^\times$ it is now straightforward to show that the relations

$$\begin{aligned} h'_{\beta_i}(\zeta) h'_{\beta_i}(\xi) &= h'_{\beta_i}(\zeta\xi) \quad \text{and} \\ h'_{\beta_i}(\zeta) h'_{\beta_j}(\xi) &= h'_{\beta_i}(\xi) h'_{\beta_i}(\zeta) \end{aligned}$$

are preserved by $\Psi_{sc}(\mathbb{K})$. The relation

$$y'_\beta(\mu) y'_\beta(\nu) = y'_\beta(\mu + \nu)$$

for $\beta \in \Phi^B$ and $\mu, \nu \in \mathbb{K}$ is preserved since the relation

$$y_\beta(\mu) y_\beta(\nu) = y_\beta(\mu + \mu)$$

in $\mathfrak{G}_m^B(\mathbb{K})$ is preserved by $\Psi'_m(\mathbb{K})$ and $\mathfrak{U}_\pm^{\gamma',\tau'}(\mathbb{K}) \cong \mathfrak{U}_\pm^{\gamma,\tau}(\mathbb{K})$.

In order to verify the relations

$$h'_{\beta_i}(\xi) y'_{\beta_j}(\mu) h'_{\beta_i}(\xi)^{-1} = y'_{\beta_j}\left(\xi^{B_{ij}}\mu\right)$$

and

$$n'_{\beta_j} h'_{\beta_i}(\xi) {n'_{\beta_j}}^{-1} = r_{\beta_j}\left(h'_{\beta_i}(\xi)\right) = h'_{r_{\beta_j}(\beta_i)}(\xi) = h'_{\beta_i}(\xi) h'_{\beta_j}\left(\xi^{-B_{ij}}\right)$$

we consider the nature of the correspondence between the matrices B and $\hat{A}$.

Lemma 4.5.1

The relation

$$h'_{\beta_i}(\xi) y'_{\beta_j}(\mu) h'_{\beta_i}(\xi)^{-1} = y'_{\beta_j}\left(\xi^{B_{ij}}\mu\right)$$

is preserved for all $i, j \in \underline{m}_0$, $\xi \in \mathbb{K}^\times$ and $\mu \in \mathbb{K}$.

Proof

Suppose

$$\Psi_{sc}(\mathbb{K})\left(h'_{\beta_i}(\xi)\right) = \prod_{\alpha_{i'} \in \Pi(\hat{A})} h'_{\alpha_{i'}}\left(\xi^{k_{i'}}\right)$$

and

$$\Psi_{sc}(\mathbb{K})\left(y'_{\beta_j}(\mu)\right) = \prod_{\alpha \in \Phi_{\beta_j}} x'_\alpha(\kappa_\alpha \mu)$$

where Φ_{β_j} denotes the orbit of roots in $\Phi^{re}(\tilde{A})$ corresponding to β_i. Then the required relation is preserved if and only if

$$\left(\prod_{\alpha_{i'}\in\Pi(\tilde{A})} h'_{\alpha_{i'}}\left(\xi^{k_{i'}}\right)\right) x'_\alpha(\kappa_\alpha\mu)\left(\prod_{\alpha_{i'}\in\Pi(\tilde{A})} h'_{\alpha_{i'}}\left(\xi^{k_{i'}}\right)\right)^{-1} = x'_\alpha\left(\xi^{B_{ij}}\kappa_\alpha\mu\right)$$

for all $\alpha \in \Phi_{\beta_j}$. Using Lemma 2.5.1, we may reformulate this condition as

$$x'_\alpha\left(\xi^{\sum_{\alpha_{i'}\in\Pi(\tilde{A})} k_{i'}\langle\alpha,\alpha_{i'}^\vee\rangle}\kappa_\alpha\mu\right) = x'_\alpha\left(\xi^{B_{ij}}\kappa_\alpha\mu\right)$$

for all $\alpha \in \Phi_{\beta_j}$. Thus it is sufficient to show that

$$\sum_{\alpha_{i'}\in\Pi(\tilde{A})} k_{i'}\langle\alpha,\alpha_{i'}^\vee\rangle = B_{ij}$$

for all $\alpha \in \Phi_{\beta_j}$. However, this is a condition on the Cartan matrices. Furthermore, this condition must hold since the relation

$$h_{\beta_i}(\xi)y_{\beta_j}(\mu)h_{\beta_i}(\xi)^{-1} = y_{\beta_j}\left(\xi^{B_{ij}}\mu\right)$$

is preserved by $\Psi'_m(\mathbb{C})$. □

LEMMA 4.5.2

The relation

$$n'_{\beta_j}h'_{\beta_i}(\xi)n'^{\,-1}_{\beta_j} = h'_{\beta_i}(\xi)h'_{\beta_j}\left(\xi^{-B_{ij}}\right)$$

is preserved for all $i, j \in \underline{m}_0$ *and* $\xi \in \mathbb{K}^\times$.

Proof

Suppose first that $\beta_j \neq \beta_0$. Thus

$$\begin{aligned}\Psi_{sc}(\mathbb{K})&\left(n'_{\beta_j}h'_{\beta_i}(\xi)n'^{\,-1}_{\beta_j}\right)\\ &= \left(\prod_{\alpha_{j'}\in\Phi_{\beta_j}} n'_{\alpha_{j'}}\left(\kappa_{\alpha_{j'}}\right)\right)\left(\prod_{\alpha\in\Phi_{\beta_i}} h'_\alpha(\xi_\alpha)\right)\left(\prod_{\alpha_{j'}\in\Phi_{\beta_j}} n'_{\alpha_{j'}}\left(\kappa_{\alpha_{j'}}\right)\right)^{-1}\\ &= \left(\prod_{\alpha\in\Phi_{\beta_i}} h'_{w(\alpha)}(\xi_\alpha)\right)\end{aligned}$$

where $w = \prod_{\alpha_{j'}\in\Phi_{\beta_j}} r_{\alpha_{j'}} \in W(\tilde{A})$. We note that w is well-defined since the roots $\alpha_{j'} \in \Phi_{\beta_j}$, and hence the fundamental reflections corresponding to them, all commute. However

$$\Psi_{sc}(\mathbb{K})\left(h'_{r_{\beta_j}(\beta_i)}(\xi)\right) = \prod_{\alpha'\in\Phi_{r_{\beta_j}(\beta_i)}} h'_{\alpha'}(\xi_{\alpha'})$$

where $\Phi_{r_{\beta_j}(\beta_i)}$ denotes the orbit of roots in $\Phi(\hat{A})$ corresponding to $r_{\beta_j}(\beta_i)$.

Hence in order to show that the relation is preserved, it is sufficient to show that

$$\Phi_{r_{\beta_j}(\beta_i)} = w(\Phi_{\beta_i})$$

where $w = \prod_{\alpha_{j'} \in \Phi_{\beta_j}} r_{\alpha_{j'}}$ and that $\xi_{\alpha'} = \xi_\alpha$ whenever $\alpha' = w(\alpha)$.

The former is a condition on the root system and the Weyl group of $\hat{A}$. Since these are independent of the type of the group in question,

$$\Psi'_m(\mathbb{K})(h_{\beta_i}(\xi)) = \left(\prod_{\alpha \in \Phi_{\beta_i}} h_\alpha(\xi_\alpha)\right),$$

and

$$\Psi_{sc}(\mathbb{K})\left(h_{r_{\beta_j}(\beta_i)}(\xi)\right) = \prod_{\alpha' \in \Phi_{r_{\beta_j}(\beta_i)}} h_{\alpha'}(\xi_{\alpha'}),$$

we use the fact that the relation

$$n_{\beta_j} h_{\beta_i}(\xi) n_{\beta_j}{}^{-1} = h_{\beta_i}(\xi) h_{\beta_j}\left(\xi^{-B_{ij}}\right)$$

is preserved by $\Psi'_m(\mathbb{K})$ to conclude that its analogue in $\mathfrak{G}^B_{sc}(\mathbb{K})$ is preserved by $\Psi_{sc}(\mathbb{K})$.

A similar but lengthier argument shows this also to be the case for $\beta_j = \beta_0$. □

Lemma 4.5.3
The relation

$$n'_{\beta_i} y'_\beta(\mu) n'_{\beta_i}{}^{-1} = y'_{r_{\beta_i}(\beta)}(\eta_{\beta_i,\beta}\mu)$$

is preserved for all $i \in \underline{m}_0$, $\beta \in \Phi^B$ *and* $\mu \in \mathbb{K}$.

Proof

We note that using the relation

$$n'_{\alpha_j}(\xi) = h'_{\beta_j}(\xi) n'_{\alpha_j}{}^{-1}$$

we may deduce that

$$n'_{\alpha_j}(\xi) x'_\alpha(\mu) n'_{\alpha_j}(\xi)^{-1} = x'_{r_j(\alpha)}\left(\xi^{-\langle\alpha,\alpha_j^\vee\rangle}\eta_{i,\alpha}\mu\right)$$

for all $j \in \underline{n}_0$, $\xi \in \mathbb{K}^\times$ and $\mu \in \mathbb{K}$.

Suppose first that $\beta_i \neq \beta_0$. Then

$$\Psi_{sc}(\mathbb{K})\left(n'_{\beta_i} y'_\beta(\mu) n'_{\beta_i}{}^{-1}\right) = \left(\prod_{\alpha_j \in \Phi_{\beta_i}} n'_{\alpha_j}(\kappa_{\alpha_j})\right)\left(\prod_{\alpha \in \Phi_\beta} x'_\alpha(\kappa_\alpha \mu)\right)\left(\prod_{\alpha_j \in \Phi_{\beta_i}} n'_{\alpha_j}(\kappa_{\alpha_j})\right)^{-1}$$

and

$$\Psi_{sc}(\mathbb{K})\left(y'_{r_{\beta_i}(\beta)}(\eta_{\beta_i,\beta}\mu)\right) = \prod_{\alpha' \in \Phi_{r_{\beta_i}(\beta)}} x'_{\alpha'}(\kappa_{\alpha'}\eta_{\beta_i,\beta}\mu).$$

Let $w = \prod_{\alpha_j \in \Phi_{\beta_i}} r_j$. Then it is sufficient to show that

$$w(\Phi_\beta) = \Phi_{r_{\beta_i}(\beta)}$$

and that, for each $\alpha \in \Phi_\beta$,

$$\left(\prod_{\alpha_j \in \Phi_{\beta_i}} n'_{\alpha_j}(\kappa_{\alpha_j})\right) x'_\alpha(\kappa_\alpha \mu) \left(\prod_{\alpha_j \in \Phi_{\beta_i}} n'_{\alpha_j}(\kappa_{\alpha_j})\right)^{-1} = x'_{\alpha'}(\kappa_{\alpha'}\eta_{\beta_i,\beta}\mu)$$

where $\alpha' = w(\alpha)$. We may fix a natural lexicographic order on Φ_{β_i} since it consists of fundamental roots. Then the equality we require reduces to

$$x'_{\alpha'}\left(\kappa_{\alpha_{j_k}}^{-\langle r_{\alpha_{j_k}}(\alpha'),\alpha_{j_k}^\vee\rangle} \eta_{j_k, r_{\alpha_{j_k}}(\alpha')} \cdots \kappa_{\alpha_{j_1}}^{-\langle \alpha,\alpha_{j_1}^\vee\rangle} \eta_{j_1,\alpha}\kappa_\alpha \mu\right) = x'_{\alpha'}(\kappa_{\alpha'}\eta_{\beta_i,\beta}\mu),$$

for which it is sufficient to show that

$$\kappa_{\alpha_{j_k}}^{-\langle r_{\alpha_{j_k}}(\alpha'),\alpha_{j_k}^\vee\rangle} \eta_{j_k, r_{\alpha_{j_k}}(\alpha')} \cdots \kappa_{\alpha_{j_1}}^{-\langle \alpha,\alpha_{j_1}^\vee\rangle} \eta_{j_1,\alpha}\kappa_\alpha = \kappa_{\alpha'}\eta_{\beta_i,\beta}.$$

Thus we have reduced the problem to properties of the root system, the field, and constants which are independent of the type of group. Furthermore, we note that

$$\begin{aligned}
\Psi'_m(\mathbb{K})(n_{\beta_i}) &= \prod_{\alpha_j \in \Phi_{\beta_i}} n_{\alpha_j}(\kappa_{\alpha_j}) \\
\Psi'_m(\mathbb{K})(y_\beta(\mu)) &= \prod_{\alpha \in \Phi_\beta} x_\alpha(\kappa_\alpha \mu)
\end{aligned}$$

and

$$\Psi'_m(\mathbb{K})\left(y_{r_{\beta_i}(\beta)}(\eta_{\beta_i,\beta}\mu)\right) = \prod_{\alpha' \in \Phi_{r_{\beta_i}(\beta)}} x_{\alpha'}(\kappa_{\alpha'}\eta_{\beta_i,\beta}\mu).$$

Since $\Psi'_m(\mathbb{K})$ preserves the relation

$$n_{\beta_i} y_\beta(\mu) n_{\beta_i}^{-1} = y_{r_{\beta_i}(\beta)}(\eta_{\beta_i,\beta}\mu)$$

the conditions we require on the root system and the structure constants must be satisfied.

A similar argument holds for the case $\beta_i = \beta_0$. □

In order to show that the commutator formula is preserved we make use of the (B, N)-pair structure of $\mathfrak{G}^B_{sc}(\mathbb{K})$.

LEMMA 4.5.4

For all prenilpotent pairs $\{\beta, \beta'\}$ such that $\beta \neq \pm\beta'$, the relation

$$[y'_\beta(\mu), y'_{\beta'}(\nu)] = \prod_\gamma y'_\gamma \left(C_{\beta\beta'\gamma}\mu^i\nu^j\right)$$

with $\mu, \nu \in \mathbb{K}$ and where the product is taken over all roots $\gamma = i\beta + j\beta' \in \Phi^B$ with $i, j \in \mathbb{N}$, is preserved by $\Psi_{sc}(\mathbb{K})$.

Proof

Since $\{\beta, \beta'\}$ is a prenilpotent pair there exists an element $w \in W^B$ such that

$$w^{-1}(\beta), w^{-1}(\beta') \in \Phi^B_+.$$

Let $n'_w \in N^B_{sc} \subseteq \mathfrak{G}^B_{sc}(\mathbb{K})$ be such that $\pi(n'_w) = w$. Thus

$$[y'_\beta(\mu), y'_{\beta'}(\nu)] \left(\prod_\gamma y'_\gamma \left(C_{\beta\beta'\gamma}\mu^i\nu^j\right)\right)^{-1} \in n'_w \mathfrak{U}^B_{sc}(\mathbb{K}) {n'_w}^{-1}.$$

Recall that

$$\mathfrak{U}^B_{sc}(\mathbb{K}) \cap H^B_{sc} = 1 \qquad \text{and} \qquad nH^B_{sc}n^{-1} = H^B_{sc}$$

for all $n \in N^B_{sc}$. Thus

$$n\left(\mathfrak{U}^B_{sc}(\mathbb{K})\right)n^{-1} \cap H^B_{sc} = 1$$

for all $n \in N^B_{sc}$. Hence

$$n\left(\mathfrak{U}^B_{sc}(\mathbb{K})\right)n^{-1} \cap \ker \imath_B = 1$$

and we may deduce that

$$n\left(\mathfrak{U}^B_{sc}(\mathbb{K})\right)n^{-1} \cong \imath_B(n)\left(\mathfrak{U}^B_{ad}(\mathbb{K})\right)\imath_B(n)^{-1}$$

for all $n \in N^B_{sc}$. Thus $\Psi_{sc}(\mathbb{K})$ must preserve the relation since $\Psi'_m(\mathbb{K})$ does. □

We have thus shown that $\Psi_{sc}(\mathbb{K})$ is a group homomorphism for all $\mathbb{K}$.

An Isomorphism of Simply-Connected Groups

We will show that the group homomorphism $\Psi_{sc}(\mathbb{K})$ described above is in fact bijective.

We note first that every generator of $\mathfrak{G}^{\gamma',\tau'}_{sc}(\mathbb{K})$ appears as the image of some element of $\mathfrak{G}^B_{sc}(\mathbb{K})$ so that $\Psi_{sc}(\mathbb{K})$ is surjective.

LEMMA 4.5.5

The group homomorphism

$$\Psi_{sc}(\mathbb{K}) : \mathfrak{G}^B_{sc}(\mathbb{K}) \to \mathfrak{G}^{\gamma',\tau'}_{sc}(\mathbb{K})$$

is injective.

Proof

Since the diagram

$$\begin{array}{ccc} \mathfrak{G}_{sc}^{\gamma',\tau'} & \xrightarrow{\imath_\sigma} & \mathfrak{G}_m^{\gamma,\tau} \\ \Psi_{sc}\uparrow & \nearrow_{\Upsilon} & \\ \mathfrak{G}_{sc}^{B} & & \end{array}$$

commutes, it follows that

$$\ker \Psi_{sc}(\mathbb{K}) \subseteq H_{sc}^B(\mathbb{K}).$$

Suppose there were a non-trivial element $h \in H_{sc}^B(\mathbb{K})$ such that $\Psi_{sc}(\mathbb{K})(h) = 1$. Recall that h has a unique expression

$$h = h_{\beta_0}(\xi_0)h_{\beta_1}(\xi_1)\cdots h_{\beta_m}(\xi_m)$$

with $\xi_i \in \mathbb{K}^\times$ for all $i \in \underline{m}_0$ and such that $\xi_j \neq 1$ for at least one $j \in \underline{m}_0$. Since $h \in \ker \Psi_{sc}(\mathbb{K})$ we must have

$$\Psi_{sc}(\mathbb{K})(h_{\beta_0}(\xi_0))\,\Psi_{sc}(\mathbb{K})(h_{\beta_1}(\xi_1))\cdots\Psi_{sc}(\mathbb{K})(h_{\beta_m}(\xi_m)) = 1.$$

However, we note that

$$\Psi_{sc}(\mathbb{K})(h_{\beta_i}(\xi_i)) = \prod_{\alpha\in\Phi_{\beta_i}} h_\alpha(\xi_i)$$

in all cases except $(\hat{A}, k, \beta_j) = (\hat{A}_{2l}, 2, \beta_0)$, in which case

$$\Psi_{sc}(\mathbb{K})(h_{\beta_0}(\xi_0)) = \prod_{\alpha\in\Phi_{\beta_0}} h_\alpha(\xi_0^2).$$

In particular, $\Psi_{sc}(\mathbb{K})\left(h_{\beta_j}(\xi_j)\right) \neq 1$ for at least one $j \in \underline{m}_0$ in the above decomposition of $\Psi_{sc}(\mathbb{K})(h)$. Furthermore, since the orbits Φ_{β_i} for $i \in \underline{m}_0$ are linearly independent, the elements $\Psi_{sc}(\mathbb{K})(h_{\beta_i}(\xi_i))$ are linearly independent. Thus we can not have

$$\Psi_{sc}(\mathbb{K})(h_{\beta_0}(\xi_0))\,\Psi_{sc}(\mathbb{K})(h_{\beta_1}(\xi_1))\cdots\Psi_{sc}(\mathbb{K})(h_{\beta_m}(\xi_m)) = 1$$

where $\Psi_{sc}(\mathbb{K})\left(h_{\beta_j}(\xi_j)\right) \neq 1$ for at least one $j \in \underline{m}_0$ and we deduce that

$$\ker \Psi_{sc}(\mathbb{K}) = 1$$

as required. □

We thus have the following result.

PROPOSITION 4.5.6
There is a group functor isomorphism

$$\Psi_{sc} : \mathfrak{G}_{sc}^{B} \to \mathfrak{G}_{sc}^{\gamma',\tau'},$$

where the group functors in question are defined on the category of fields $\mathbb{K}$ *with* $\operatorname{char}\mathbb{K} \neq 2$ *and such that*

$$\sqrt{2} \in \mathbb{K} \quad \textit{if} \quad (\tilde{A}, k) = (\tilde{A}_{2l}, 2), \qquad \textit{and}$$
$$\operatorname{char}\mathbb{K} \neq 3 \quad \textit{and } \mathbb{K} \text{ contains a primitive cube root of unity} \quad \textit{if} \quad (\tilde{A}, k) = (\tilde{D}_4, 3).$$

Proof
We have shown that whenever $\mathbb{K}$ is a field admitting the definition of γ' and τ', there is an isomorphism

$$\Psi_{sc}(\mathbb{K}) : \mathfrak{G}_{sc}^{B}(\mathbb{K}) \to \mathfrak{G}_{sc}^{\gamma',\tau'}(\mathbb{K}).$$

We recall that if either $(\tilde{A}, k) = (\tilde{A}_{2l}, 2)$ and $\sqrt{2} \notin \mathbb{K}$ or $(\tilde{A}, k) = (\tilde{D}_4, 3)$ and $\mathbb{K}$ contains no primitive cube root of unity, then $\Psi_{sc}(\mathbb{K})$ is not defined. □

4.6 An Isomorphism Between Adjoint Kac-Moody Groups

We shall here extend our results to adjoint Kac-Moody groups.

Generalized Isogenies Associated to B

Denote by $\mathfrak{G}_{ad}^{B}$ the Kac-Moody group functor $\mathfrak{G}_{\mathcal{D}_{ad}(B)}$ and suppose

$$\mathfrak{G}_{ad}^{B}(\mathbb{K}) = \langle \hat{y}_{\beta}(\mu), \hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi) : \beta \in \Phi^{B}, \mu \in \mathbb{K}, i \in \underline{m}, \xi \in \mathbb{K}^{\times} \rangle.$$

By the results of §2.5, the map defined on the generators of $\mathfrak{G}_{sc}^{B}(\mathbb{K})$ by

$$\iota_{sa} : y'_{\beta}(\mu) \to \hat{y}_{\beta}(\mu)$$

for all $\beta \in \Phi^{B}$ and $\mu \in \mathbb{K}$ extends to a homomorphism

$$\iota_{sa} : \mathfrak{G}_{sc}^{B}(\mathbb{K}) \to \mathfrak{G}_{ad}^{B}(\mathbb{K}).$$

Furthermore, the generalized isogeny

$$\imath_B : \mathfrak{G}^B_{sc}(\mathbb{K}) \to \mathfrak{G}^B_m(\mathbb{K})$$

factors through $\imath_{sa}$ so that there is a group homomorphism $\imath_{am}$ making the diagram

$$\begin{array}{ccc} \mathfrak{G}^B_{sc}(\mathbb{K}) & \xrightarrow{\imath_{sa}} & \mathfrak{G}^B_{ad}(\mathbb{K}) \\ & \searrow^{\imath_B} & \downarrow \imath_{am} \\ & & \mathfrak{G}^B_m(\mathbb{K}) \end{array}$$

commute and we note that $\ker \imath_{sa} \subseteq H^B_{sc}$.

Generalized Isogenies Associated to $\bar{A}$

Denote by $\mathfrak{G}^{\bar{A}}_{ad}$ the Kac-Moody group functor $\mathfrak{G}_{\mathcal{D}_{ad}(\bar{A})}$ and suppose

$$\mathfrak{G}^{\bar{A}}_{ad}(\mathbb{K}) = \langle \hat{x}_\alpha(\mu), \hat{h}_{\varpi_i^\vee}(\xi) : \alpha \in \Phi^{re}(\bar{A}), \mu \in \mathbb{K}, i \in \underline{n}, \xi \in \mathbb{K}^\times \rangle.$$

Once again we have group homomorphisms $\imath'_{sa}$ and $\imath'_{am}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{G}^{\bar{A}}_{sc}(\mathbb{K}) & \xrightarrow{\imath'_{sa}} & \mathfrak{G}^{\bar{A}}_{ad}(\mathbb{K}) \\ & \searrow^{\imath_{\bar{A}}} & \downarrow \imath'_{am} \\ & & \mathfrak{G}^{\bar{A}}_m(\mathbb{K}) \end{array}$$

commutes. We may also define automorphisms $\hat{\gamma}$ and $\hat{\tau}$ of $\mathfrak{G}^{\bar{A}}_{ad}(\mathbb{K})$ analogous to the automorphisms γ and τ defined for $\mathfrak{G}^{\bar{A}}_m(\mathbb{K})$, with $\hat{x}_\alpha(\mu)$ replacing $x_\alpha(\mu)$ and $\hat{h}_{\varpi_i^\vee}(\xi)$ replacing $h_{\varpi_i^\vee}(\xi)$ for $i \in \underline{n}$. Denote by $\mathfrak{G}^{\hat{\gamma},\hat{\tau}}_{ad}(\mathbb{K})$ the subgroup of $\mathfrak{G}^{\bar{A}}_{ad}(\mathbb{K})$ consisting of elements $\hat{x} \in \mathfrak{G}^{\bar{A}}_{ad}(\mathbb{K})$ satisfying $\hat{\gamma}(\hat{x}) = \hat{\tau}(\hat{x})$. We note that the homomorphism $\imath'_{sa}$ maps elements of $\mathfrak{G}^{\bar{A}}_{sc}(\mathbb{K})$ which are stable under the automorphism $\sigma' = (\tau'^{-1} \circ \gamma')$ to elements of $\mathfrak{G}^{\bar{A}}_{ad}(\mathbb{K})$ which are stable under the automorphism $\hat{\sigma} = (\hat{\tau}^{-1} \circ \hat{\gamma})$. Thus the generalized isogeny $\imath'_{sa}$ restricts to a group homomorphism

$$\imath_{\sigma'} : \mathfrak{G}^{\gamma',\tau'}_{sc}(\mathbb{K}) \to \mathfrak{G}^{\hat{\gamma},\hat{\tau}}_{ad}(\mathbb{K}).$$

Similarly, the generalized isogeny $\imath'_{am}$ restricts to a group homomorphism

$$\imath_{\hat{\sigma}} : \mathfrak{G}^{\hat{\gamma},\hat{\tau}}_{ad}(\mathbb{K}) \to \mathfrak{G}^{\gamma,\tau}_m(\mathbb{K}).$$

A Homomorphism of Adjoint Groups

We would like to define a group homomorphism $\Psi_{ad}(\mathbb{K})$ making the diagram

$$\begin{array}{ccc} \mathfrak{G}_{sc}^{\gamma',\tau'}(\mathbb{K}) & \xrightarrow{\ \iota_{\sigma'}\ } & \mathfrak{G}_{ad}^{\hat{\gamma},\hat{\tau}}(\mathbb{K}) \\ \Psi_{sc}(\mathbb{K}) \uparrow & & \uparrow \Psi_{ad}(\mathbb{K}) \\ \mathfrak{G}_{sc}^{B}(\mathbb{K}) & \xrightarrow{\ \iota_{sa}\ } & \mathfrak{G}_{ad}^{B}(\mathbb{K}) \end{array}$$

commute. We do this by defining $\Psi_{ad}(\mathbb{K})$ on the generators of $\mathfrak{G}_{ad}^{B}(\mathbb{K})$ and showing that the relations are preserved.

Firstly, we note that there is a group homomorphism

$$\Upsilon_{ad}(\mathbb{K}) : \mathfrak{G}_{ad}^{B}(\mathbb{K}) \to \mathfrak{G}_{m}^{\gamma,\tau}(\mathbb{K})$$

whose kernel lies in $H_{ad}^{B}(\mathbb{K})$. Suppose that for $\beta \in \pm\Pi$ and $\mu \in \mathbb{K}$

$$\Upsilon_{ad}(\mathbb{K})\,(\hat{y}_{\beta}(\mu)) = \prod_{\alpha \in \Phi_{\beta}} x_{\alpha}(\mu_{\alpha})$$

for some $\mu_{\alpha} \in \mathbb{K}$. and that for $i \in \underline{m}$ and $\xi \in \mathbb{K}^{\times}$

$$\Upsilon_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\right) = \prod_{j \in \underline{n}_0} h_{\varpi_j}(\xi_j)$$

for some $\xi_j \in \mathbb{K}^{\times}$. Then we define

$$\Psi_{ad}(\mathbb{K})\,(\hat{y}_{\beta}(\mu)) = \prod_{\alpha \in \Phi_{\beta}} \hat{x}_{\alpha}(\mu_{\alpha})$$

and

$$\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\right) = \prod_{j \in \underline{n}_0} \hat{h}_{\varpi_j}(\xi_j).$$

The Preservation of the Relations

We note that any relations common to both $\mathfrak{G}_{sc}^{B}(\mathbb{K})$ and $\mathfrak{G}_{ad}^{B}(\mathbb{K})$ are preserved by $\Psi_{ad}(\mathbb{K})$ since they are preserved by $\Psi_{sc}(\mathbb{K})$. Thus we need only show that the relations

$$\begin{aligned} \hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\hat{h}_{\varpi^{\vee}_{\beta_i}}(\zeta) &= \hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi\zeta) \\ \hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\hat{h}_{\varpi^{\vee}_{\beta_{i'}}}(\zeta) &= \hat{h}_{\varpi^{\vee}_{\beta_{i'}}}(\zeta)\hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi) \\ \hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\hat{y}_{\beta_j}(\mu)\hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)^{-1} &= \hat{y}_{\beta_j}(\xi^{\delta_{ij}}\mu) \\ \hat{n}_{\beta_j}\hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\hat{n}_{\beta_j}^{-1} &= r_{\beta_j}\left(\hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\right) = \hat{h}_{\varpi^{\vee}_{\beta_i}}(\xi)\hat{h}_{\beta_j}\left(\xi^{-\delta_{ij}}\right) \end{aligned}$$

are preserved, where $i, i' \in \underline{m}$, $j \in \underline{m}_0$, $\xi, \zeta \in \mathbb{K}^\times$, $\mu \in \mathbb{K}$ and δ_{ij} denotes the Kronecker delta.

Since $\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\right) \in H^{\hat{\jmath},+}_{ad}$ for all $i \in \underline{m}$ and $\xi \in \mathbb{K}^\times$ it is straightforward to show that the relations

$$\begin{aligned} \hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{h}_{\varpi^\vee_{\beta_i}}(\zeta) &= \hat{h}_{\varpi^\vee_{\beta_i}}(\xi\zeta) \quad \text{and} \\ \hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{h}_{\varpi^\vee_{\beta_{i'}}}(\zeta) &= \hat{h}_{\varpi^\vee_{\beta_{i'}}}(\zeta)\hat{h}_{\varpi^\vee_{\beta_i}}(\xi) \end{aligned}$$

are preserved.

Note that for each $i \in \underline{m}$,

$$\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_i}}(\xi_i)\right) = \prod_{\varpi^\vee_j \in \Omega(\varpi^\vee_{\beta_i})} \hat{h}_{\varpi^\vee_{\beta_j}}(\xi)$$

where $\Omega(\varpi^\vee_{\beta_i})$ denotes the set of fundamental coweights of $\mathcal{D}_{ad}(\hat{A})$ corresponding to $\varpi^\vee_{\beta_i}$ deduced from Lemma 4.3.4.

Lemma 4.6.1

The relation

$$\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{y}_{\beta_j}(\mu)\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)^{-1} = \hat{y}_{\beta_j}\left(\xi^{\delta_{ij}}\mu\right)$$

is preserved for all $i \in \underline{m}$, $j \in \underline{n}_0$, $\xi \in \mathbb{K}^\times$ *and* $\mu \in \mathbb{K}$.

Proof

Note that

$$\begin{aligned} &\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{y}_{\beta_j}(\mu)\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)^{-1}\right) \\ &= \left(\prod_{\varpi^\vee_{j'} \in \Omega(\varpi^\vee_{\beta_i})} \hat{h}_{\varpi^\vee_{\beta_{j'}}}(\xi)\right)\left(\prod_{\alpha \in \Phi_{\beta_j}} \hat{x}_\alpha(\kappa_\alpha\mu)\right)\left(\prod_{\varpi^\vee_{j'} \in \Omega(\varpi^\vee_{\beta_i})} \hat{h}_{\varpi^\vee_{\beta_{j'}}}(\xi)\right)^{-1} \\ &= \prod_{\alpha \in \Phi_{\beta_j}} \hat{x}_\alpha\left(\left(\prod_{\varpi^\vee_{j'} \in \Omega(\varpi^\vee_{\beta_i})} \xi_{j'}^{\langle\alpha, \varpi^\vee_{j'}\rangle}\right)\kappa_\alpha\mu\right) \end{aligned}$$

and

$$\Psi_{ad}(\mathbb{K})\left(\hat{y}_{\beta_j}\left(\xi^{\delta_{ij}}\mu\right)\right) = \prod_{\alpha \in \Phi_{\beta_j}} \hat{x}_\alpha\left(\xi^{\delta_{ij}}\kappa_\alpha\mu\right).$$

Hence it is sufficient to show that

$$\sum_{\varpi^\vee_{j'} \in \Omega(\varpi^\vee_{\beta_i})} \langle\alpha, \varpi^\vee_{j'}\rangle = \delta_{ij}$$

for all $\alpha \in \Phi_{\beta_j}$.

Thus the problem reduces to a condition on the root system which must be satisfied since $\Psi'_m(\mathbb{K})$ preserves the corresponding relation in $\mathfrak{G}^B_m(\mathbb{K})$. □

LEMMA 4.6.2
The relation

$$\hat{n}_{\beta_j}\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{n}_{\beta_j}^{-1} = r_{\beta_j}\left(\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\right) = \hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{h}_{\beta_j}\left(\xi^{-\delta_{ij}}\right)$$

is preserved for all $i \in \underline{m}$, $j \in \underline{m}_0$ and $\xi \in \mathbb{K}^\times$.

Proof
Suppose first that $\beta_j \neq \beta_0$. Thus

$$\begin{aligned}
&\Psi_{ad}(\mathbb{K})\left(\hat{n}_{\beta_j}\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{n}_{\beta_j}^{-1}\right)\\
&= \left(\prod_{\alpha\in\Phi_{\beta_j}} \hat{n}_\alpha(\kappa_\alpha)\right)\left(\prod_{\varpi^\vee_j\in\Omega(\varpi^\vee_{\beta_i})} \hat{h}_{\varpi^\vee_{\beta_j}}(\xi)\right)\left(\prod_{\alpha\in\Phi_{\beta_j}} \hat{n}_\alpha(\kappa_\alpha)\right)^{-1}\\
&= \left(\prod_{\varpi^\vee_j\in\Omega(\varpi^\vee_{\beta_i})} \hat{h}_{w(\varpi^\vee_{\beta_j})}(\xi)\right)\\
&= \left(\prod_{\varpi^\vee_j\in\Omega(\varpi^\vee_{\beta_i})} \hat{h}_{\varpi^\vee_{\beta_j}}(\xi)\right)\left(\prod_{\alpha\in\Phi_{\beta_j}} \hat{h}_\alpha\left(\prod_{\varpi^\vee_j\in\Omega(\varpi^\vee_{\beta_i})} \xi^{-\langle\alpha,\varpi^\vee_j\rangle}\right)\right)
\end{aligned}$$

where $w = \prod_{\alpha\in\Phi_{\beta_j}} r_\alpha \in W^\sigma$. On the other hand,

$$\begin{aligned}
\Psi_{ad}(\mathbb{K})\left(\hat{h}_{r_{\beta_j}(\varpi^\vee_{\beta_i})}(\xi)\right) &= \Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_i}}(\xi)\hat{h}_{\beta_j}\left(\xi^{-\delta_{ij}}\right)\right)\\
&= \left(\prod_{\varpi^\vee_j\in\Omega(\varpi^\vee_{\beta_i})} \hat{h}_{\varpi^\vee_{\beta_j}}(\xi)\right)\left(\prod_{\alpha\in\Phi_{\beta_j}} \hat{h}_\alpha\left(\xi^{-\delta_{ij}}\right)\right).
\end{aligned}$$

Thus it is sufficient to show that

$$\sum_{\varpi^\vee_j\in\Omega(\varpi^\vee_{\beta_i})} \langle\alpha,\varpi^\vee_j\rangle = \delta_{ij}$$

for all $\alpha \in \Phi_{\beta_j}$. We have therefore once again reduced the preservation of the relation to a condition on the root system and we deduce that the relation is preserved since its analogue is preserved by $\Psi'_m(\mathbb{K})$.

The case $\beta_j = \beta_0$ follows similarly. □

Hence we have established the existence of a homomorphism

$$\Psi_{ad}(\mathbb{K}) : \mathfrak{G}^B_{ad}(\mathbb{K}) \to \mathfrak{G}^{\hat{\cdot},\dagger}_{ad}(\mathbb{K}).$$

We proceed to show that it is in fact an isomorphism in an analogous manner to that used for $\Psi_{sc}(\mathbb{K})$.

An Isomorphism of Adjoint Groups

Once again we use the fact that every generator of $\mathfrak{G}_{ad}^{\hat{\gamma},\hat{\tau}}(\mathbb{K})$ appears in the image of $\Psi_{ad}(\mathbb{K})$ to show that $\Psi_{ad}(\mathbb{K})$ is in fact an epimorphism.

LEMMA 4.6.3

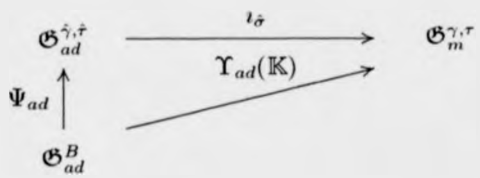
The group homomorphism

$$\Psi_{ad}(\mathbb{K}) : \mathfrak{G}_{ad}^{B}(\mathbb{K}) \to \mathfrak{G}_{ad}^{\hat{\gamma},\hat{\tau}}(\mathbb{K})$$

is injective.

Proof

Since the diagram



commutes, it follows that

$$\ker \Psi_{ad}(\mathbb{K}) \subseteq H_{ad}^{B}(\mathbb{K}).$$

Suppose there were a non-trivial element $h \in H_{ad}^{B}(\mathbb{K})$ such that $\Psi_{ad}(\mathbb{K})(h) = 1$. Recall that h has a unique expression

$$h = \hat{h}_{\varpi_{\beta_1}^{\vee}}(\xi_1) \cdots \hat{h}_{\varpi_{\beta_m}^{\vee}}(\xi_m)$$

with $\xi_i \in \mathbb{K}^{\times}$ for all $i \in \underline{m}$ and such that $\xi_j \neq 1$ for at least one $j \in \underline{m}$. Since $h \in \ker \Psi_{ad}(\mathbb{K})$ we must have

$$\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi_{\beta_1}^{\vee}}(\xi_1)\right) \cdots \Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi_{\beta_m}^{\vee}}(\xi_m)\right) = 1.$$

Recall that for each $i \in \underline{m}$,

$$\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi_{\beta_i}^{\vee}}(\xi_i)\right) = \prod_{\varpi_j^{\vee} \in \Omega(\varpi_{\beta_i}^{\vee})} \hat{h}_{\varpi_{\beta_j}^{\vee}}(\xi)$$

where $\Omega(\varpi_{\beta_i}^{\vee})$ denotes the set of fundamental coweights of $\mathcal{D}_{ad}(\hat{A})$ corresponding to $\varpi_{\beta_i}^{\vee}$ deduced from Lemma 4.3.4. Thus we see that

$$\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi_{\beta_i}^{\vee}}(\xi_i)\right) \neq 1 \quad \Leftrightarrow \quad \xi_i \neq 1,$$

whence we deduce that

$$\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_1}}(\xi_1)\right)\cdots\Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_m}}(\xi_m)\right)=1$$
$$\Leftrightarrow\quad \Psi_{ad}(\mathbb{K})\left(\hat{h}_{\varpi^\vee_{\beta_i}}(\xi_i)\right)=1 \qquad \text{for all} \qquad i\in\underline{m}.$$

Thus $\Psi_{ad}(\mathbb{K})$ is injective. □

We have thus proved the following result.

PROPOSITION 4.6.4

There is a group functor isomorphism

$$\Psi_{ad}:\mathfrak{G}^{B}_{ad}\to\mathfrak{G}^{\hat{\gamma},\hat{\tau}}_{ad},$$

where the group functors in question are defined on the category of fields $\mathbb{K}$ *with* $\operatorname{char}\mathbb{K}\neq 2$ *and such that*

$$\sqrt{2}\in\mathbb{K} \quad \textit{if} \quad (\hat{A},k)=(\hat{A}_{2l},2), \quad \textit{and}$$
$$\operatorname{char}\mathbb{K}\neq 3 \quad \textit{and } \mathbb{K} \text{ contains a primitive cube root of unity} \quad \textit{if} \quad (\hat{A},k)=(\hat{D}_4,3).$$

Proof

We have shown that whenever $\mathbb{K}$ is a field admitting the definition of $\hat{\gamma}$ and $\hat{\tau}$, there is an isomorphism

$$\Psi_{ad}(\mathbb{K}):\mathfrak{G}^{B}_{ad}(\mathbb{K})\to\mathfrak{G}^{\hat{\gamma},\hat{\tau}}_{ad}(\mathbb{K}).$$

We recall that if either $(\hat{A},k)=(\hat{A}_{2l},2)$ and $\sqrt{2}\notin\mathbb{K}$ or $(\hat{A},k)=(\hat{D}_4,3)$ and $\mathbb{K}$ contains no primitive cube root of unity, then $\Psi_{ad}(\mathbb{K})$ is not defined. □

Chapter 5

The Fixed Point Subgroup of a Graph×Field Automorphism

In this chapter we consider the implications of the results of Hée in the case when

- $\hat{A}$ is a simply-laced extended Cartan matrix,
- γ is a graph automorphism of $\mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K})$ induced by an automorphism of $\Delta(\hat{A})$ inherited from an automorphism of $\Delta(A)$, and
- τ is a field automorphism of $\mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K})$ of the same order as γ.

We consider the twisted root system obtained in this manner and describe the fixed point subgroups associated to the twisted roots.

Finally, we complete our description of the generators of the fixed point subgroup $G^{\gamma\cdot\tau}$ by giving an explicit description of $H_{\mathcal{D}}^{\gamma\cdot\tau}(\mathbb{K})$ for $\mathcal{D} = \mathcal{D}_m$, $\mathcal{D}_{ad}$, and $\mathcal{D}_{sc}$.

5.1 The Automorphisms in Question

Suppose γ is a graph automorphism of $\mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K})$ of order k such as described in Chapter 4. We recall that the action of γ gives rise to twisted root systems Φ^{γ} of types described in §4.1. We recall these in Table 5.1.1, where $\hat{N}_1$ and $\hat{BC}_l$ denote the root systems obtained by extending the non-reduced root systems of type A_1 and BC_l respectively by the $\delta - \theta$, where θ is the highest root in that system.

Let φ be a field automorphism of $\mathbb{K}$ which is also of order k. Denote by $\mathbb{K}^{\varphi}$ the set of φ-stable elements of $\mathbb{K}$. We note that the prime subfield of $\mathbb{K}$ is stable under all automorphisms of $\mathbb{K}$, and hence $\mathbb{K}^{\varphi}$ is always non-trivial.

We may use φ to define a map, τ, on the generators of $\mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K})$ by

$$\tau : x_{\alpha}(\mu) \mapsto x_{\alpha}(\varphi(\mu))$$

$\hat{A}$	$\hat{A}_2$	$\hat{A}_{2l}$, $(l>1)$	$\hat{A}_{2l-1}$, $(l>2)$	$\hat{D}_{l+1}$, $(l>1)$	$\hat{D}_4$	$\hat{E}_6$
R	$\hat{N}_1$	$\hat{BC}_l$	$\hat{C}_l$	$\hat{B}_l$	$\hat{G}_2$	$\hat{F}_4$

Table 5.1.1: Types R of root systems Φ^{γ}.

for all $\alpha \in \pm\Pi(\hat{A})$, $\mu \in \mathbb{K}$, and

$$\tau : h_{\varpi_i^\vee}(\xi) \mapsto h_{\varpi_i^\vee}(\varphi(\xi))$$

for $\xi \in \mathbb{K}^\times$ and $\varpi_i^\vee$ a fundamental coweight in the case of $\mathcal{D} = \mathcal{D}_m$ or $\mathcal{D}_{ad}$. It is straightforward to check that $\tau \in \operatorname{Aut} \mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K})$.

We note also that

$$\tau(n_{\alpha_i}(\xi)) = n_{\alpha_i}(\varphi(\xi)) \quad \text{and} \quad \tau(h_{\alpha_i}(\xi)) = h_{\alpha_i}(\varphi(\xi))$$

for all $i \in \underline{n}_0$ and $\xi \in \mathbb{K}^\times$. Thus

$$\tau(N) = N, \quad \tau(H_{\mathcal{D}}) = H_{\mathcal{D}}, \quad \text{and} \quad \tau(X_\alpha) = X_\alpha \neq 1$$

for all $\alpha \in \Phi(\hat{A})$.

5.2 The Twisted Root Subgroups

Suppose that $\alpha + m\delta \in \Phi^{re}(\hat{A})$ with $\alpha \in \Phi(A)$ and let $K(\alpha)$ denote the $\bar{\gamma}$-orbit of α. We define

$$\begin{aligned} X_{\alpha^\gamma+m\delta} &= X^1_{\alpha^1+|K(\alpha)|m\delta} = \mathfrak{G}_{\mathcal{D}}^{\gamma,\tau}(\mathbb{K}) \cap X_{\Phi_{\alpha+m\delta}} \\ &= \langle g \in \mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K}) : \gamma(g) = \tau(g) \rangle \quad \cap \quad \langle X_\beta : \beta \in \Phi_{\alpha+m\delta'} \rangle. \end{aligned}$$

In order to study the groups $X_{\alpha^\gamma+m\delta}$, we consider the cases $\hat{A} \neq \hat{A}_{2l}$ and $\hat{A} = \hat{A}_{2l}$ separately.

The Case $\hat{A} \neq \hat{A}_{2l}$

In this case the automorphism γ satisfies

$$\begin{array}{rccc} \gamma : & \mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K}) & \to & \mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K}) \\ & x_i(\mu) & \mapsto & x_{\bar{\gamma}(i)}(\mu) \\ & x_{-i}(\mu) & \mapsto & x_{-\bar{\gamma}(i)}(\mu) \\ & h_{\varpi_j^\vee}(\xi) & \mapsto & h_{\varpi_{\bar{\gamma}(j)}^\vee}(\xi) \end{array}$$

for all $i \in \underline{n}_0$, $\mu \in \mathbb{K}$, $\xi \in \mathbb{K}^\times$ and fundamental coweights $\varpi_j^\vee$.

LEMMA 5.2.1
The elements of $X_{\alpha^\gamma+m\delta}$ are as follows;

1. *If $K(\alpha + m\delta) = \{\alpha + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu) : \mu \in \mathbb{K}^\varphi\}.$$

2. *If $K(\alpha + m\delta) = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\varphi(\mu)) : \mu \in \mathbb{K}\}.$$

3. *If $K(\alpha + m\delta) = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \bar{\bar{\alpha}} + m\delta\}$, then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\varphi(\mu))x_{\bar{\bar{\alpha}}+m\delta}(\varphi^2(\mu)) : \mu \in \mathbb{K}\}.$$

Proof
This follows from a direct comparison of $\gamma(x)$ and $\tau(x)$ for general $x \in X_{\alpha^\gamma+m\delta}$, bearing in mind that $X_{\alpha^\gamma+m\delta}$ is abelian in the cases under consideration. □

In particular, we note that $X_{\alpha^\gamma+m\delta} \neq 1$ for all $\alpha \in \Phi(A)$ since $\mathbb{K}^\varphi$ is always nontrivial. Thus we may apply Theorem 3.3.3 directly.

The Case $\tilde{A} = \tilde{A}_{2l}$

In this case the automorphism γ satisfies

$$\gamma : \begin{array}{lcl} \mathfrak{G}_{\mathcal{D}}^{\tilde{A}}(\mathbb{K}) & \rightarrow & \mathfrak{G}_{\mathcal{D}}^{\tilde{A}}(\mathbb{K}) \\ x_i(\mu) & \mapsto & x_{\bar{\gamma}(i)}(\mu) \\ x_{-i}(\mu) & \mapsto & x_{-\bar{\gamma}(i)}(\mu) \\ x_0(\mu) & \mapsto & x_0(-\mu) \\ x_{-0}(\mu) & \mapsto & x_{-0}(-\mu) \\ h_{\varpi_j^\vee}(\xi) & \mapsto & h_{\varpi_{\bar{\gamma}(j)}^\vee}(\xi) \end{array} \quad \left.\right\} i \in \underline{2l}$$

for all $\mu \in \mathbb{K}$, $\xi \in \mathbb{K}^\times$ and fundamental coweights $\varpi_j^\vee$.

LEMMA 5.2.2
The elements of $X_{\alpha^\gamma+m\delta}$ are as follows;

1. *If $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu) : \mu \in \mathbb{K}^{-\varphi}\}$$

where $\mathbb{K}^{-\varphi}$ denotes the -1-eigenspace of $\mathbb{K}$ with respect to φ.

2. *If* $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ *then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\varphi(\mu)) : \mu \in \mathbb{K}\}.$$

3. *If* $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \alpha + \bar{\alpha} + 2m\delta\}$ *where*

$$\alpha = \alpha_i + \cdots + \alpha_l$$

for some $i \in \underline{l}$ *then*

$$\begin{aligned} X_{\alpha^\gamma+m\delta} \;=\; & \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(\eta) : \\ & \mu, \eta \in \mathbb{K} \text{ with } \varphi(\eta) + \eta = -\kappa\mu\varphi(\mu)\} \end{aligned}$$

where $\kappa = (-1)^{l-i}$. *Furthermore, if we define*

$$x_{\alpha^\gamma+m\delta}(\mu,\eta) = x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(\eta) \in X_{\alpha^\gamma+m\delta}$$

then

$$x_{\alpha^\gamma+m\delta}(\mu,\eta)x_{\alpha^\gamma+m\delta}(\nu,\lambda) = x_{\alpha^\gamma+m\delta}(\mu+\nu, \eta+\lambda-\kappa\varphi(\mu)\nu)$$

for all suitable $\mu, \eta, \nu, \lambda \in \mathbb{K}$.

Proof

We begin by making some general comments which we will find of use in our calculations. We note that $N_{\alpha,\beta} \in \{1,-1\}$ for all $\alpha, \beta \in \Phi^{re}(\tilde{A})$. Hence, by Theorem 1.4.4

$$N_{\alpha,\beta} = N_{-\alpha,-\beta}.$$

Furthermore, the structure constants involved in the calculations for $K(-\alpha + m\delta)$ whenever $\alpha \in \Phi^{re}_+$ and $m \in \mathbb{Z}$ depend only on those for $K(-\alpha)$ by Lemma 1.4.3, which by the above observation depend only on those for $K(\alpha)$. Thus it is sufficient to consider $\bar{\gamma}$-orbits of the form $K(\alpha + m\delta)$ for $\alpha \in \Phi^{re}_+$ and $m \in \mathbb{Z}$. We are now in a position to proceed with the proof.

Suppose first that $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta\}$. Thus

$$\alpha = \alpha_i + \cdots + \alpha_{2l-i+1}$$

for some $i \in \underline{l}$ and

$$\gamma : x_{\alpha+m\delta}(\mu) \mapsto x_{\alpha+m\delta}(-\mu)$$

for all $\mu \in \mathbb{K}$. However,

$$\tau : x_{\alpha+m\delta}(\mu) \mapsto x_{\alpha+m\delta}(\varphi(\mu))$$

for all $\mu \in \mathbb{K}$, and hence

$$x_{\alpha+m\delta}(\mu) \in \mathfrak{G}^{\gamma,\tau}_{\mathcal{D}}(\mathbb{K}) \Leftrightarrow \varphi(\mu) = -\mu$$

which is precisely the condition that $\mu \in \mathbb{K}^{-\varphi}$.

The result for $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ is straightforward.

Finally, suppose $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \alpha + \bar{\alpha} + 2m\delta\}$ with

$$\alpha = \alpha_i + \cdots + \alpha_l$$

for some $i \in \underline{l}$. Then, whenever $\mu, \nu, \eta \in \mathbb{K}$,

$$\begin{aligned}&\gamma\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)\right)\\ &\quad = x_{\alpha+m\delta}((-1)^{l-i}\nu)x_{\bar{\alpha}+m\delta}((-1)^{l-i}\mu)x_{\alpha+\bar{\alpha}+2m\delta}(-\eta-\mu\nu)\end{aligned}$$

from calculations made in the proof of Proposition 4.1.5. However,

$$\begin{aligned}&\tau\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)\right)\\ &\quad = x_{\alpha+m\delta}(\varphi(\mu))x_{\bar{\alpha}+m\delta}(\varphi(\nu))x_{\alpha+\bar{\alpha}+2m\delta}(\varphi(\eta)),\end{aligned}$$

so that

$$\begin{aligned}&x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta) \in \mathfrak{G}_{\mathcal{D}}^{\gamma,\tau}(\mathbb{K})\\ &\Leftrightarrow\ \varphi(\mu) = (-1)^{l-i}\nu \text{ and } \varphi(\eta) = -\eta - \mu\nu\\ &\Leftrightarrow\ \varphi(\mu) = (-1)^{l-i}\nu \text{ and } \varphi(\eta) + \eta = -(-1)^{l-i}\mu\varphi(\mu).\end{aligned}$$

Thus it remains only to show that the necessary condition is satisfied by any two generators of $X_{\alpha^\gamma+m\delta}$. Let $x_{\alpha^\gamma+m\delta}(\mu,\eta), x_{\alpha^\gamma+m\delta}(\nu,\lambda) \in X_{\alpha^\gamma+m\delta}$ and $\kappa = (-1)^{l-i}$. Then

$$\begin{aligned}&x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(\eta)x_{\alpha+m\delta}(\nu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\nu))x_{\alpha+\bar{\alpha}+2m\delta}(\lambda)\\ &\quad = x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\nu))x_{\alpha+\bar{\alpha}+2m\delta}(\lambda)\\ &\quad = x_{\alpha+m\delta}(\mu)x_{\alpha+m\delta}(\nu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(-\kappa\varphi(\mu)\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)\\ &\qquad\quad x_{\bar{\alpha}+m\delta}(\kappa\varphi(\nu))x_{\alpha+\bar{\alpha}+2m\delta}(\lambda)\\ &\quad = x_{\alpha+m\delta}(\mu+\nu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu+\nu))x_{\alpha+\bar{\alpha}+2m\delta}(\eta+\lambda-\kappa\varphi(\mu)\nu).\end{aligned}$$

Now

$$\begin{aligned}\varphi(\eta+\lambda-\kappa\varphi(\mu)\nu) &= \varphi(\eta)+\varphi(\lambda)-\kappa\mu\varphi(\nu)\\ &= -\eta-\kappa\mu\varphi(\mu)-\lambda-\kappa\nu\varphi(\nu)-\kappa\mu\varphi(\nu)\\ &= -\eta-\lambda+\kappa\varphi(\mu)\nu-\kappa(\mu+\nu)\varphi(\mu+\nu)\end{aligned}$$

so that

$$x_{\alpha^\gamma+m\delta}(\mu,\eta)x_{\alpha^\gamma+m\delta}(\nu,\lambda) = x_{\alpha^\gamma+m\delta}(\mu+\nu,\eta+\lambda-\kappa\varphi(\mu)\nu) \in X_{\alpha^\gamma+m\delta}$$

as required. □

LEMMA 5.2.3

Suppose $\varphi \in \operatorname{Aut}\mathbb{K}$ *has order two. Denote by* $\mathbb{K}^{-\varphi}$ *the* -1*-eigenspace of* φ*. Then* $\mathbb{K}^{-\varphi}$ *is non-trivial.*

Proof

We need to show that there exists a $\xi \in \mathbb{K}^\times$ with $\varphi(\xi) = -\xi$. We note that if $\operatorname{char}\mathbb{K} = 2$ then all non-trivial elements of the prime subfield satisfy this condition. So suppose that $\operatorname{char}\mathbb{K} \neq 2$. Let $\zeta \in \mathbb{K}^\times$ satisfy

$$\varphi(\zeta) \neq \zeta.$$

Such a ζ exists because φ is non-trivial. If

$$\varphi(\zeta) = -\zeta$$

then we are done. Otherwise, consider the element

$$\xi = \varphi(\zeta) - \zeta \in \mathbb{K}^\times.$$

Then $\varphi(\xi) = -\xi$ and $\mathbb{K}^{-\varphi}$ is non-trivial as required. □

Lemmas 5.2.2 and 5.2.3 lead us to the conclusion that $X_{\alpha^\gamma+m\delta}$ is non-trivial for all $\alpha \in \Phi^{re}(\dot{A})$ whenever φ is non-trivial.

Thus we may again use Theorem 3.3.3 directly.

5.3 The Groups $H^{\gamma,\tau}$

We return to the general case and now consider the fixed point subgroup of $H_{\mathcal{D}}(\mathbb{K})$ under the automorphism $\sigma = \tau^{-1}\circ\gamma$. We consider the minimal adjoint, adjoint, and simply-connected cases separately.

The Minimal Adjoint Case

Recall that every element $h \in H_m(\mathbb{K})$ has a unique expression of the form

$$h = h_{\varpi_0^\vee}(\xi_0)\cdots h_{\varpi_n^\vee}(\xi_n)$$

where $\xi_i \in \mathbb{K}^\times$ for all $i \in \underline{n}_0$. Since

$$\gamma : h_{\varpi_i^\vee}(\xi_i) \mapsto h_{\varpi_{\gamma(i)}^\vee}(\xi_i)$$

and

$$\tau : h_{\varpi_i^\vee}(\xi_i) \mapsto h_{\varpi_i^\vee}(\varphi(\xi_i))$$

for every $i \in \underline{n}_0$, we deduce that

$$\gamma(h) = h_{\varpi_{\gamma(0)}^\vee}(\xi_0)\cdots h_{\varpi_{\gamma(n)}^\vee}(\xi_n)$$

and

$$\tau(h) = h_{\varpi_0^\vee}(\varphi(\xi_0)) \cdots h_{\varpi_n^\vee}(\varphi(\xi_n)).$$

Thus

$$h \in H_m^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad h = h_{\varpi_0^\vee}(\xi_0) \prod_{J \in \underline{n}^1} \left(\prod_{i \in \underline{k}} h_{\varpi^\vee_{\gamma^i(j)}}(\varphi^{-i}(\xi_J)) \right)$$

where $\xi_0 \in \mathbb{K}^\varphi \cap \mathbb{K}^\times$, j is a representative of $J \in \underline{n}^1$, and $\xi_J \in \mathbb{K}^\times$ for each $J \in \underline{n}^1$.

The Adjoint Case

Recall that every element $h \in H_{ad}(\mathbb{K})$ has a unique expression of the form

$$h = h_{\varpi_1^\vee}(\xi_1) \cdots h_{\varpi_n^\vee}(\xi_n)$$

where $\xi_i \in \mathbb{K}^\times$ for all $i \in \underline{n}$. Thus

$$h \in H_{ad}^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad h = \prod_{J \in \underline{n}^1} \left(\prod_{i \in \underline{k}} h_{\varpi^\vee_{\gamma^i(j)}}(\varphi^{-i}(\xi_J)) \right)$$

where j is a representative of $J \in \underline{n}^1$ and $\xi_J \in \mathbb{K}^\times$ for each $J \in \underline{n}^1$.

The Simply-Connected Case

Recall that every element $h \in H_{sc}(\mathbb{K})$ has a unique expression of the form

$$h = h_{\alpha_0}(\xi_0) \cdots h_{\alpha_n}(\xi_n)$$

where $\xi_i \in \mathbb{K}^\times$ for all $i \in \underline{n}_0$. Since

$$\gamma : h_{\alpha_i}(\xi_i) \mapsto h_{\alpha_{\gamma(i)}}(\xi_i)$$

and

$$\tau : h_{\alpha_i}(\xi_i) \mapsto h_{\alpha_i}(\varphi(\xi_i))$$

for every $i \in \underline{n}_0$, we deduce that

$$\gamma(h) = h_{\alpha_{\gamma(0)}}(\xi_0) \cdots h_{\alpha_{\gamma(n)}}(\xi_n)$$

and

$$\tau(h) = h_{\alpha_0}(\varphi(\xi_0)) \cdots h_{\alpha_n}(\varphi(\xi_n)).$$

Thus

$$h \in H_m^{\gamma,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad h = h_{\alpha_0}(\xi_0) \prod_{J \in \underline{n}^1} \left(\prod_{i \in \underline{k}} h_{\alpha^\vee_{\gamma^i(j)}}(\varphi^{-i}(\xi_J)) \right)$$

where $\xi_0 \in \mathbb{K}^\varphi \cap \mathbb{K}^\times$, j is a representative of the orbit $J \in \underline{n}^1$, and $\xi_J \in \mathbb{K}^\times$ for each $J \in \underline{n}^1$.

5.4 The Application of Hée's Theorem

Suppose $A = (A_{ij})_{i,j\in\underline{n}}$ is a Cartan matrix of type A_l, D_l, or E_6 and that

$$\tilde{A} = (A_{ij})_{i,j\in\underline{n}_0}$$

is the extended Cartan matrix obtained from it.

Suppose $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ is a Kac-Moody group of type $\tilde{A}$ and let $\gamma \in \mathrm{Aut}\, \mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ be induced by an automorphism $\bar{\gamma}$ of $\Delta(\tilde{A})$ inherited from $\Delta(A)$. Suppose γ has order k and let $\varphi \in \mathrm{Aut}\, \mathbb{K}$ also have order k. Denote by τ the automorphism of $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$ induced by φ. Define

$$\mathfrak{G}_{\mathcal{D}}^{\gamma,\tau}(\mathbb{K}) = \langle x \in \mathfrak{G}_{\mathcal{D}}(\mathbb{K}) : \gamma(x) = \tau(x)\rangle.$$

Recall the root base B^{γ} and the twisted root system Φ^{γ} of type R constructed from $\Phi(\tilde{A})$ in Chapter 4 and described in Table 5.1.1. Define

$$U^{\gamma,\tau} = \langle X_{\alpha^{\gamma}} : \alpha^{\gamma} \in \Phi_{+}^{\gamma}\rangle \qquad \text{and} \qquad U_{-}^{\gamma,\tau} = \langle X_{\alpha^{\gamma}} : \alpha^{\gamma} \in \Phi_{-}^{\gamma}\rangle.$$

Suppose that $\mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$ is a subgroup of $\mathfrak{G}_{\mathcal{D}}^{\gamma,\tau}(\mathbb{K})$ such that

$$\langle U^{\gamma,\tau}, U_{-}^{\gamma,\tau}\rangle \subseteq \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K}).$$

Define

$$H_{\mathcal{D}}^{\sigma} = H_{\mathcal{D}} \cap \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K}) = H_{\mathcal{D}}^{\gamma,\tau} \cap \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$$

and

$$N^{\sigma} = N \cap \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K}).$$

Let

$$B^{\sigma} = U^{\gamma,\tau} H_{\mathcal{D}}^{\sigma} \qquad \text{and} \qquad B_{-}^{\sigma} = U_{-}^{\gamma,\tau} H_{\mathcal{D}}^{\sigma}.$$

PROPOSITION 5.4.1

The triple

$$((X_{\alpha^{\gamma}})_{\alpha^{\gamma}} \in \Phi^{\gamma}, N^{\sigma}, H^{\sigma})$$

is a root datum of type $(B^{\gamma}, \Phi^{\gamma}, \mathbb{N})$ *associated to* $\mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$. *Furthermore*

$$(B^{\sigma}, N^{\sigma}) \qquad \textit{and} \qquad \left(B_{-}^{\sigma}, N^{\sigma}\right)$$

are two (B, N)*-pairs in* $\mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$ *with Weyl group* W^{γ}.

Proof

This is a direct result of Theorem 3.3.3 and calculations made in Chapter 4. □

The Group $N^{\gamma,\tau}$

Let j be any representative of $J \in \underline{n}_0^1$. Whenever the roots in $K(\alpha_j)$ are independent, define

$$n_{\alpha_J^\gamma} = \prod_{i \in J} n_{\alpha_i}.$$

If $(\tilde{A}, k) = (\tilde{A}_{2l}, 2)$ and $J = \{l, l+1\}$, define

$$n_{\alpha_J^\gamma} = n_{\alpha_l} n_{\alpha_{l+1}} n_{\alpha_l} = n_{\alpha_{l+1}} n_{\alpha_l} n_{\alpha_{l+1}}.$$

The proof of Theorem 3.3.3 and the calculations of §4.5 then lead us to the conclusion that

$$N^{\gamma,\tau} = \langle H_{\mathfrak{p}}^{\gamma,\tau}, n_{\alpha_J^\gamma} \rangle.$$

Chapter 6

The Fixed Point Subgroup of a Graph×Diagonal×Field Automorphism

Finally, in this chapter we investigate the fixed point subgroup under an automorphism which is a combination of a graph, a field and a diagonal automorphism of Kac-Moody groups of the types envisaged in the previous two chapters.

6.1 The Automorphisms in Question

Suppose $\tilde{A}$, A, γ and τ are as in Chapter 4. Let ϕ be a field automorphism of $\mathfrak{G}_{\mathcal{D}}^{A}(\mathbb{K})$ induced by an automorphism φ of $\mathbb{K}$ of the same order, k, as γ.

Suppose $\mathbb{K}$ contains a primitive kth root of unity, say ϵ

. Denote by $\mathbb{K}^{\varphi}$ the fixed point subspace of $\mathbb{K}$ with respect to φ and by $\mathbb{K}^{\epsilon^m\varphi}$ the ϵ^m-eigenspace of $\mathbb{K}$ with respect to φ for each integer m.

LEMMA 6.1.1
With the above notation, $\mathbb{K}^{\epsilon^m\varphi}$ is non-trivial.

Proof
The case $k = 2$ was considered in Lemma 5.2.3. Thus it remains to show that

$$\mathbb{K}^{\epsilon\varphi} \neq 1 \quad \text{and} \quad \mathbb{K}^{\epsilon^2\varphi} \neq 1 \quad \text{when} \quad \epsilon^3 = 1 \ .$$

We begin by showing that $\mathbb{K}^{\epsilon\varphi} \neq 1$. Since φ is non-trivial, we may find an element $\xi \in \mathbb{K}^{\times}$ satisfying $\varphi(\xi) \neq \xi$. If $\varphi(\xi) = \epsilon\xi$, then we are done. Suppose that $\varphi(\xi) \neq \epsilon\xi$, and consider the element

$$\zeta = \xi + \epsilon^2\varphi(\xi) + \epsilon\varphi^2(\xi) \in \mathbb{K}^{\times} \ .$$

Now $\varphi(\zeta) = \epsilon\zeta$ so that $\zeta \in \mathbb{K}^{\epsilon\varphi}$ and $\mathbb{K}^{\epsilon\varphi}$ is non-trivial as required.

A similar argument using

$$\zeta = \xi + \epsilon\varphi(\xi) + \epsilon^2\varphi^2(\xi) \in \mathbb{K}^{\times}$$

shows that $\mathbb{K}^{\epsilon^2\varphi}$ is non-trivial. □

The Case $\tilde{A} \neq \tilde{A}_{2l}$

Consider the automorphism

$$\gamma\phi := \gamma\circ\phi = \phi\circ\gamma$$

defined on the generators of $\mathfrak{G}_{\mathcal{D}}^{\tilde{A}}(\mathbb{K})$ by

$$\begin{aligned} \gamma\phi:\ & x_i(\mu) && \mapsto && x_{\bar\gamma(i)}(\varphi(\mu)) \\ & x_{-i}(\mu) && \mapsto && x_{-\bar\gamma(i)}(\varphi(\mu)) \\ & h_{\varpi_j^\vee}(\xi) && \mapsto && h_{\varpi^\vee_{\bar\gamma(j)}}(\xi) \end{aligned}$$

for all $i \in \underline{n}_0$, $\mu \in \mathbb{K}$, $\xi \in \mathbb{K}^{\times}$ and fundamental coweights $\varpi_j^\vee$ if $\mathcal{D} = \mathcal{D}_m$ or $\mathcal{D}_{ad}$. Thus

$$\gamma\phi(n_{\alpha_i}(\xi)) = n_{\alpha_{\bar\gamma(i)}}(\varphi(\xi))$$

and we note that

$$\gamma\phi(N) = N, \quad \gamma\phi(H) = H, \quad \text{and} \quad \gamma\phi(X_\alpha) = X_{\bar\gamma(\alpha)}$$

for each $\alpha \in \Phi^{re}(\tilde{A})$.

The Case $\tilde{A} = \tilde{A}_{2l}$

This time the automorphism

$$\gamma\phi := \gamma\circ\phi = \phi\circ\gamma$$

is defined on the generators of $\mathfrak{G}_{\mathcal{D}}^{\tilde{A}}(\mathbb{K})$ by

$$\begin{aligned} \gamma\phi:\ & x_i(\mu) && \mapsto && x_{\bar\gamma(i)}(\varphi(\mu)) \\ & x_{-i}(\mu) && \mapsto && x_{-\bar\gamma(i)}(\varphi(\mu)) \\ & x_0(\mu) && \mapsto && x_0(-\varphi(\mu)) \\ & x_{-0}(\mu) && \mapsto && x_{-0}(-\varphi(\mu)) \\ & h_{\varpi_j^\vee}(\xi) && \mapsto && h_{\varpi^\vee_{\bar\gamma(j)}}(\xi) \end{aligned}$$

for all $i \in \underline{n}$, $\mu \in \mathbb{K}$, $\xi \in \mathbb{K}^{\times}$ and fundamental coweights $\varpi_j^\vee$ if $\mathcal{D} = \mathcal{D}_m$ or $\mathcal{D}_{ad}$. Once again we note that

$$\gamma\phi(N) = N, \quad \gamma\phi(H) = H, \quad \text{and} \quad \gamma\phi(X_\alpha) = X_{\bar\gamma(\alpha)}$$

for each $\alpha \in \Phi^{re}(\tilde{A})$.

We note that in all cases the automorphism $\gamma\phi$ satisfies the conditions imposed on the family $(\gamma_\sigma)_{\sigma\in\Sigma}$ of automorphisms in §3.3.

6.2 The Twisted Root Subgroups

Define

$$\mathfrak{G}_{\mathcal{D}}^{\gamma\phi,\tau}(\mathbb{K}) = \langle g \in \mathfrak{G}_{\mathcal{D}}^{\hat{A}}(\mathbb{K}) : \gamma\phi(g) = \tau(g)\rangle$$

and let

$$X_{\alpha^\gamma+m\delta} = X^1_{\alpha^1+|K(\alpha)|m\delta} = \mathfrak{G}_{\mathcal{D}}^{\gamma\phi,\tau}(\mathbb{K}) \cap \langle X_\beta : \beta \in \Phi_{\alpha+m\delta}\rangle.$$

We must once again study the cases $\tilde{A} \neq \tilde{A}_{2l}$ and $\tilde{A} = \tilde{A}_{2l}$ separately.

The Case $\tilde{A} \neq \tilde{A}_{2l}$

LEMMA 6.2.1
The elements of $X_{\alpha^\gamma+m\delta}$ are as follows;

1. *If $K(\alpha + m\delta) = \{\alpha + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \left\{x_{\alpha+m\delta}(\mu) : \mu \in \mathbb{K}^{\epsilon^m\varphi}\right\}$$

   where $\mathbb{K}^{\epsilon^m\varphi}$ denotes the ϵ^m-eigenspace of $\mathbb{K}$ with respect to φ.

2. *If $K(\alpha + m\delta) = \{\alpha + m\delta, \bar{\alpha} + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\epsilon^{-m}\varphi(\mu)) : \mu \in \mathbb{K}\}.$$

3. *If $K(\alpha + m\delta) = \{\alpha + m\delta, \bar{\alpha} + m\delta, \bar{\bar{\alpha}} + m\delta\}$, then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\epsilon^{-m}\varphi(\mu))x_{\bar{\bar{\alpha}}+m\delta}(\epsilon^{-2m}\varphi^2(\mu)) : \mu \in \mathbb{K}\}.$$

Proof
Consider the case $K(\alpha + m\delta) = \{\alpha + m\delta\}$. Then

$$\gamma\phi : x_{\alpha+m\delta}(\mu) \longmapsto x_{\alpha+m\delta}(\varphi(\mu))$$

and

$$\tau : x_{\alpha+m\delta}(\mu) \longmapsto x_{\alpha+m\delta}(\epsilon^m\mu).$$

Thus

$$x_{\alpha+m\delta}(\mu) \in \mathfrak{G}_{\mathcal{D}}^{\gamma\phi,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad \varphi(\mu) = \epsilon^m\mu$$

which corresponds precisely to the condition that $\mu \in \mathbb{K}^{\epsilon^m\varphi}$.
If $K(\alpha + m\delta) = \{\alpha + m\delta, \bar{\alpha} + m\delta\}$ then

$$\gamma\phi : x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu) \longmapsto x_{\bar{\alpha}+m\delta}(\varphi(\mu))x_{\alpha+m\delta}(\varphi(\nu))$$

and

$$\tau : x_{\alpha+m\delta}(\mu)x_{\alpha+m\delta}(\nu) \mapsto x_{\alpha+m\delta}(\epsilon^m\mu)x_{\bar{\alpha}+m\delta}(\epsilon^m\mu).$$

Thus

$$x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)) \in \mathfrak{G}_D^{\gamma\phi,\tau}(\mathbb{K}) \quad \Leftrightarrow \quad \nu = \epsilon^{-m}\varphi(\mu)$$

as required.

Finally, suppose $K(\alpha + m\delta) = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \bar{\bar{\alpha}} + m\delta\}$. Then

$$\gamma\phi : x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\bar{\bar{\alpha}}+m\delta}(\eta) \mapsto x_{\bar{\alpha}+m\delta}(\varphi(\mu))x_{\bar{\bar{\alpha}}+m\delta}(\varphi(\nu))x_{\alpha+m\delta}(\varphi(\eta))$$

and

$$\tau : x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\bar{\bar{\alpha}}+m\delta}(\eta) \mapsto x_{\alpha+m\delta}(\epsilon^m\mu)x_{\bar{\alpha}+m\delta}(\epsilon^m\nu).x_{\bar{\bar{\alpha}}+m\delta}(\epsilon^m\eta)$$

leading us to the conclusion that

$$\begin{aligned} & x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu))x_{\bar{\bar{\alpha}}+m\delta}(\eta) \in \mathfrak{G}_D^{\gamma\phi,\tau}(\mathbb{K}) \\ \Leftrightarrow \quad & \nu = \epsilon^{-m}\varphi(\mu), \quad \text{and} \quad \eta = \epsilon^{-2m}\varphi^2(\mu) \end{aligned}$$

as required. □

We note that $X_{\alpha^\gamma+m\delta} \neq 1$ in all cases as a result of Lemma 6.1.1. Hence we may use Hée's Theorem directly.

The Case $\bar{A} = \bar{A}_{2l}$

LEMMA 6.2.2

The elements of $X_{\alpha^\gamma+m\delta}$ are as follows;

1. *If $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu) : \mu \in \mathbb{K}^{\kappa\varphi}\}$$

where $\kappa = (-1)^{m+1}$ and $\mathbb{K}^{\kappa\varphi}$ denotes the κ-eigenspace of $\mathbb{K}$ with respect to φ.

2. *If $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ then*

$$X_{\alpha^\gamma+m\delta} = \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}((-1)^m\varphi(\mu)) : \mu \in \mathbb{K}\}.$$

3. *If $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \alpha + \bar{\alpha} + 2m\delta\}$ where*

$$\alpha = \alpha_i + \cdots + \alpha_l$$

for some $i \in \underline{l}$ then

$$\begin{aligned} X_{\alpha^\gamma+m\delta} = & \{x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(\eta) : \\ & \mu, \eta \in \mathbb{K} \text{ with } \varphi(\eta) + \eta = -\kappa\mu\varphi(\mu)\} \end{aligned}$$

where $\kappa = (-1)^{m+l-i}$. *Furthermore, if we define*

$$x_{\alpha^\gamma+m\delta}(\mu,\eta) = x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(\eta) \in X_{\alpha^\gamma+m\delta}$$

then

$$x_{\alpha^\gamma+m\delta}(\mu,\eta)x_{\alpha^\gamma+m\delta}(\nu,\lambda) = x_{\alpha^\gamma+m\delta}(\mu+\nu,\eta+\lambda-\kappa\varphi(\mu)\nu)$$

for all suitable $\mu,\eta,\nu,\lambda \in \mathbb{K}$.

Proof

We once again assume without loss of generality that $\alpha \in \Phi_+^{re}$ and $m \in \mathbb{Z}$.

Suppose first that $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta\}$. Thus

$$\alpha = \alpha_i + \cdots + \alpha_{2l-i+1}$$

for some $i \in \underline{l}$ and

$$\gamma\phi : x_{\alpha+m\delta}(\mu) \mapsto x_{\alpha+m\delta}(-\varphi(\mu))$$

for all $\mu \in \mathbb{K}$. However,

$$\tau : x_{\alpha+m\delta}(\mu) \mapsto x_{\alpha+m\delta}((-1)^m\mu)$$

for all $\mu \in \mathbb{K}$, and hence

$$x_{\alpha+m\delta}(\mu) \in \mathfrak{G}_{\mathcal{D}}^{\gamma,\tau}(\mathbb{K}) \Leftrightarrow \varphi(\mu) = \kappa\mu$$

for $\kappa = (-1)^{m+1}$, which is precisely the condition that $\mu \in \mathbb{K}^{\kappa\varphi}$.

The result for $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta\}$ is proved in precisely the same manner as the corresponding part of Lemma 6.2.1.

Finally, suppose $\Phi^{re}_{\alpha+m\delta^1} = \{\alpha + m\delta,\ \bar{\alpha} + m\delta,\ \alpha + \bar{\alpha} + 2m\delta\}$ with

$$\alpha = \alpha_i + \cdots + \alpha_l$$

for some $i \in \underline{l}$. Then, whenever $\mu,\nu,\eta \in \mathbb{K}$,

$$\begin{aligned}&\gamma\phi\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)\right)\\ &\quad= x_{\alpha+m\delta}((-1)^{l-i}\varphi(\nu))x_{\bar{\alpha}+m\delta}((-1)^{l-i}\varphi(\mu))x_{\alpha+\bar{\alpha}+2m\delta}(-\varphi(\eta)-\varphi(\mu\nu)).\end{aligned}$$

However,

$$\begin{aligned}&\tau\left(x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta)\right)\\ &\quad= x_{\alpha+m\delta}((-1)^m\mu)x_{\alpha+m\delta}((-1)^m\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta),\end{aligned}$$

so that

$$\begin{aligned}&x_{\alpha+m\delta}(\mu)x_{\bar{\alpha}+m\delta}(\nu)x_{\alpha+\bar{\alpha}+2m\delta}(\eta) \in \mathfrak{G}_{\mathcal{D}}^{\gamma\phi,\tau}(\mathbb{K})\\ &\quad\Leftrightarrow\ \varphi(\mu) = (-1)^{m+l-i}\nu \text{ and } \eta = -\varphi(\eta) - \varphi(\mu\nu)\\ &\quad\Leftrightarrow\ \varphi(\mu) = (-1)^{m+l-i}\nu \text{ and } \varphi(\eta) + \eta = -(-1)^{m+l-i}\mu\varphi(\mu).\end{aligned}$$

Thus it remains only to show that the necessary condition is satisfied by any two generators of $X_{\alpha^\gamma+m\delta}$. Let $x_{\alpha^\gamma+m\delta}(\mu,\eta), x_{\alpha^\gamma+m\delta}(\nu,\lambda) \in X_{\alpha^\gamma+m\delta}$ and $\kappa = (-1)^{m+l-\iota}$. Then calculations similar to those in the proof of Lemma 6.2.2 lead us to the conclusion that

$$\begin{aligned} x_{\alpha+m\delta}(\mu)x_{\bar\alpha+m\delta}(\kappa\varphi(\mu))x_{\alpha+\bar\alpha+2m\delta}(\eta)x_{\alpha+m\delta}(\nu)x_{\bar\alpha+m\delta}(\kappa\varphi(\nu))x_{\alpha+\bar\alpha+2m\delta}(\lambda) \\ = \quad x_{\alpha+m\delta}(\mu+\nu)x_{\bar\alpha+m\delta}(\kappa\varphi(\mu+\nu))x_{\alpha+\bar\alpha+2m\delta}(\eta+\lambda-\kappa\varphi(\mu)\nu). \end{aligned}$$

where

$$\varphi(\eta+\lambda-\kappa\varphi(\mu)\nu) = -\eta-\lambda+\kappa\varphi(\mu)\nu-\kappa(\mu+\nu)\varphi(\mu+\nu).$$

Thus

$$x_{\alpha^\gamma+m\delta}(\mu,\eta)x_{\alpha^\gamma+m\delta}(\nu,\lambda) = x_{\alpha^\gamma+m\delta}(\mu+\nu,\eta+\lambda-\kappa\varphi(\mu)\nu) \in X_{\alpha^\gamma+m\delta}$$

as required. □

Once again we have $X_{\alpha^\gamma+m\delta} \neq 1$ in all cases and we may thus use Hée's Theorem directly.

Returning to the general case we note that the twisted root system involved is thus none other than Φ^γ, described in Table 5.1.1.

6.3 The Groups $H^{\gamma\phi,\tau}$

Let $H_{\mathcal{D}}^{\gamma\phi,\tau}(\mathbb{K})$ denote the fixed point subgroup of $H_{\mathcal{D}}(\mathbb{K})$ under the automorphism $\sigma = \tau^{-1}\circ\gamma\phi$. We note the general fact about diagonal automorphisms that

$$\tau(h) = h \qquad \text{for all} \qquad h \in H_{\mathcal{D}}(\mathbb{K})$$

so that

$$h \in H_{\mathcal{D}}(\mathbb{K}) \text{ is } \sigma\text{-stable} \quad \Leftrightarrow \quad h \text{ is } \gamma\phi\text{-stable} .$$

Thus we may use the results of §5.3 to deduce the following result.

Lemma 6.3.1

1. *Every element $h \in H_{m}^{\gamma\phi,\tau}(\mathbb{K})$ has a unique expression of the form*

$$h = h_{\varpi_0^\vee}(\xi_0)\prod_{J\in\underline{n}^1}\left(\prod_{i\in\underline{k}} h_{\varpi^\vee_{\gamma^i(j)}}(\varphi^i(\xi_J))\right)$$

where $\xi_0 \in \mathbb{K}^\varphi \cap \mathbb{K}^\times$, j is a representative of $J \in \underline{n}^1$, and $\xi_J \in \mathbb{K}^\times$ for each $J \in \underline{n}^1$.

2. *Every element* $h \in H_{ad}^{\gamma\phi,\tau}(\mathbb{K})$ *has a unique expression of the form*

$$h = \prod_{J \in \underline{n}^1} \left(\prod_{i \in \underline{k}} h_{\varpi^\vee_{\gamma^i(j)}}(\varphi^i(\xi_J)) \right)$$

where j *is a representative of* $J \in \underline{n}^1$ *and* $\xi_J \in \mathbb{K}^\times$ *for each* $J \in \underline{n}^1$.

3. *Every element* $h \in H_{sc}^{\gamma\phi,\tau}(\mathbb{K})$ *has a unique expression of the form*

$$h = h_{\alpha_0}(\xi_0) \prod_{J \in \underline{n}^1} \left(\prod_{i \in \underline{k}} h_{\alpha^\vee_{\gamma^i(j)}}(\varphi^i(\xi_J)) \right)$$

where $\xi_0 \in \mathbb{K}^\varphi \cap \mathbb{K}^\times$, j *is a representative of the orbit* $J \in \underline{n}^1$, *and* $\xi_J \in \mathbb{K}^\times$ *for each* $J \in \underline{n}^1$.

6.4 The Application of Hée's Theorem

Suppose $A = (A_{ij})_{i,j \in \underline{n}}$ is a Cartan matrix of type A_l, D_l, or E_6 and that

$$\tilde{A} = (A_{ij})_{i,j \in \underline{n}_0}$$

is the extended Cartan matrix obtained from it.

Suppose $\mathfrak{G}_\mathcal{D}(\mathbb{K})$ is a Kac-Moody group of type $\tilde{A}$ and let $\gamma \in \operatorname{Aut} \mathfrak{G}_\mathcal{D}(\mathbb{K})$ be induced by an automorphism $\bar{\gamma}$ of $\Delta(\tilde{A})$ inherited from $\Delta(A)$. Suppose γ has order k and let $\varphi \in \operatorname{Aut} \mathbb{K}$ also have order k. Denote by ϕ the automorphism of $\mathfrak{G}_\mathcal{D}(\mathbb{K})$ induced by φ. Define

$$\gamma\phi := \gamma \circ \phi = \phi \circ \gamma$$

and denote by τ the diagonal automorphism $d(\epsilon)$ of $\mathfrak{G}_\mathcal{D}(\mathbb{K})$ where

$$\epsilon = e^{\frac{2\pi i}{k}}.$$

Define

$$\mathfrak{G}_\mathcal{D}^{\gamma\phi,\tau}(\mathbb{K}) = \langle x \in \mathfrak{G}_\mathcal{D}(\mathbb{K}) : \gamma\phi(x) = \tau(x) \rangle.$$

Recall the root base B^γ and the twisted root system Φ^γ of type R constructed from $\Phi(\tilde{A})$ in Chapter 4 and described in Table 5.1.1. Define

$$U^{\gamma\phi,\tau} = \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi_+^\gamma \rangle \quad \text{and} \quad U_-^{\gamma\phi,\tau} = \langle X_{\alpha^\gamma} : \alpha^\gamma \in \Phi_-^\gamma \rangle.$$

Suppose that $\mathfrak{G}_\mathcal{D}^\sigma(\mathbb{K})$ is a subgroup of $\mathfrak{G}_\mathcal{D}^{\gamma\phi,\tau}(\mathbb{K})$ such that

$$\langle U^{\gamma\phi,\tau}, U_-^{\gamma\phi,\tau} \rangle \subseteq \mathfrak{G}_\mathcal{D}^\sigma(\mathbb{K}).$$

Define

$$H_{\mathcal{D}}^{\sigma} = H_{\mathcal{D}} \cap \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K}) = H_{\mathcal{D}}^{\gamma\phi,\tau} \cap \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$$

and

$$N^{\sigma} = N \cap \mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K}).$$

Let

$$B^{\sigma} = U^{\gamma\phi,\tau} H_{\mathcal{D}}^{\sigma} \qquad \text{and} \qquad B_{-}^{\sigma} = U_{-}^{\gamma\phi,\tau} H_{\mathcal{D}}^{\sigma}.$$

PROPOSITION 6.4.1

The triple

$$((X_{\alpha^{\gamma}})_{\alpha^{\gamma}} \in \Phi^{\gamma}, N^{\sigma}, H^{\sigma})$$

is a root datum of type $(\mathcal{B}^{\gamma}, \Phi^{\gamma}, \mathbb{N})$ *associated to* $\mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$. *Furthermore*

$$(B^{\sigma}, N^{\sigma}) \qquad \textit{and} \qquad \left(B_{-}^{\sigma}, N^{\sigma}\right)$$

are two (B, N)*-pairs in* $\mathfrak{G}_{\mathcal{D}}^{\sigma}(\mathbb{K})$ *with Weyl group* W^{Γ}.

Proof

This is a direct result of Theorem 3.3.3 and calculations made in Chapter 4.
□

The Group $N^{\gamma\phi,\tau}$

Let j be any representative of $J \in \underline{n}_0^1$. Whenever the roots in $K(\alpha_j)$ are independent, define

$$n_{\alpha_j^{\gamma}} = \prod_{i \in J} n_{\alpha_i}.$$

If $(\tilde{A}, k) = (\tilde{A}_{2l}, 2)$ and $J = \{l, l+1\}$, define

$$n_{\alpha_j^{\gamma}} = n_{\alpha_l} n_{\alpha_{l+1}} n_{\alpha_l} = n_{\alpha_{l+1}} n_{\alpha_l} n_{\alpha_{l+1}}.$$

The proof of Theorem 3.3.3 and the calculations of §4.5 then lead us to the conclusion that

$$N^{\gamma\phi,\tau} = \langle H_{\mathcal{D}}^{\gamma\phi,\tau}, n_{\alpha_j^{\gamma}} \rangle.$$

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