Characterizations of Homotopy 3-Spheres

by

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Thesis submitted for the degree of Doctor of Philosophy at the University of Warwick in 1988

March 1988

Ackowlegment

As a post-graduate student in Warwick University
I had a grant from INVOTAN (Portugal).

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to Jessica, Sancho and Sara:

A line – the ultimate paradigm. From its intangible continuity
spring the roots of the tree of infinity where spirit is form
We see it waving on a plane while the artist's loving hand
continuously move in space
And when the hand gets tired the line becomes the artist's
conscious continuously weaving the time.
And he wonders if it is his hand that moves
or if the line and the space moving and the vision hand
are but a ripple shaping instant
on the continuous stream of time.
And when finally the artist gets tired the hand humbly
returns to the loving task of tendering his thoughts
in her own living time.
And so she writes, in little shaped fragments of the ultimate
paradigm:
It's where my heart lies that any line hurts.

Introduction

"Once and for all there is a great deal I do not want to know - Wisdom sets bounds even to knowledge".
(F Nietzsche - Twilight of the Idols)

This thesis consists of several characterizations of homotopy 3-spheres in general and of S^3 in particular It is divided in sections, grouped in chapters, and the results are numbered according to the section where they are. Those characterizations are in terms of a blend between Heegaard-decompositions (H-decompositions) and surgery-presentations of a special kind associated to a (longitudinal) surgery in a pure braid in S^3 (i.e. a link whose components are once transverse to the pages of the standard fibration of the unknot in S^3) there is a "special" (in the sense of [B-P]) Heegaard-decomposition of the 3-manifold, M^3, obtained through that surgery. This special decomposition corresponds in a simple way to a planar open book decomposition of M^3 (one whose pages are homeomorphic to a planar surface). Reciprocally any planar open book decomposition (pob-decomposition) of a 3-manifold leads in a direct way to presentations of that manifold as surgery in pure braids in S^3. The basic results about this relationship between pure braid surgeries, pob-decompositions and associated special H-decompositions, as well as all the relevant definitions, are presented at the beginning of Chapter 2 together with the so called "Kirby calculus" (a procedure to alter and generate all possible surgery presentations of a given 3-manifold, [K], [F-R]) of which we will make extensive use throughout this work.

The characterizations we obtain could be generally located in the 3-manifold world as variations on some Haken's themes. There was hope sometime in the sixties that W Haken and/or V.Poénaru would then provide proofs of the Poincaré conjecture. That didn't happen but work by Haken from the same period was seminal for the subsequent development of 3-manifold theory - additivity of Heegaard genus, sufficiently-large 3-manifolds, incompressible-surfaces and hierarchies, etc. [H_1].

Concerning the "central question" for 3-manifolds - their classification and decision of the homeomorphism problem - (and after uniqueness of "prime-decomposition". [Kn], [M]), the works of Haken [H_2], Waldhausen [W_1], Jaco and Shalen [J-S], Johanson [J] and Thurston [T] essentially solve the problem for the class of sufficiently-large 3-manifolds. In particular the geometrization theory of Thurston apart from giving a general and unifying treatment of previous classifications for certain classes of 3-manifolds (e.g. Seifert fibred spaces [S-T] (see the survey on geometries of 3-manifolds by P. Scott [Sc]) extends to other classes like irreducible 3-manifolds with symmetry and manifolds which fiber over the circle.

But Thurston's work also unveiled the existence of "many" 3-manifolds which are not sufficiently-large, for these in the absence of an incompressible surface, carrying a significant part of their structure, to start splitting them there is still no standard procedure of decomposing into simpler pieces that will reveal that structure. The non-sufficiently large 3-manifolds (which include the homotopy spheres) still appear as one-piece blocks and some people think (or feel...) that, despite the enormous progress of recent years and the power of the techniques now available, the essence of those mysterious manifolds (if they are at all tangible) is still out of reach.

While the combinatorial technics developed by Haken to work with surfaces in 3-manifolds have been so widely used in the past twenty years leading by more or less direct adaptations to many useful and interesting topological results - for instance the equivalence of H-decompositions for Lens-spaces [B-O], extending the results of Waldhausen for S^3 [W_2], many criteria for reducing Heegaard splittings (e.g. [C-G]) and many recent results using Haken's normal forms in conexion with branched surfaces - the same didn't seem to happen with his results on homotopy 3-spheres [H_3],[H_4].In one of his papers [H_4] Haken shows that the difficulties with homotopy 3-spheres are not essentially of an algebraic nature by proving that for any homotopy 3-sphere, H^3, there exists a cell-decomposition whose "corresponding" presentation of the fundamental group $\pi_1(H^3) = \{ g_1, \dots, g_n | w_1, \dots, w_k \}$ is "almost trivial" in the sense that it can be reduced to the trivial presentation $\{ g_1, \dots, g_n | g_1, \dots, g_n \}$ by successive cancellations in the words w_i, of syllables of the type $g_k g_k^{-1}$ or $g_k^{-1} g_k$, or syllables which are themselves words (as Haken remarks the problem is the lack of correspondence between these transformations of the group presentation and transformations of the cell-decomposition). The topological result that is behind that fact is the following: any homotopy 3-sphere, H^3, has a H-decomposition $H^3 = H \cup_h H'$ where H has a "system of longitudes" $l_1, \dots, l_n$ such that each l_i bounds an embedded disc D_i in H^3 - a <u>system of longitudes or longitudinal system</u> for H is a set of disjoint embedded curves ($l_1, \dots, l_n$) in ∂H such that there exists a complete system of discs $C_1, \dots, C_n$ for H with $l_i \cap \partial C_i = pt.$ and $l_i \cap \partial C_j = \phi$ if $i \neq j$ (definitions in Chapter 2). Furthermore we can assume the D_i's meet, if they meet at all, in <u>clasp</u>-form intersections (figure 2, Chapter 2) and there are no triple points (i.e. the clasps are disjoint) - note that by Waldhausen [W_2] any H-decomposition of S^3 has a system of longitudes bounding disjoint embedded discs. It is this result that we extend in Chapter 2. We prove (Theorem 4.1) that any homotopy 3-sphere, H^3, has a pob-decomposition with the components of the binding link $P=(p_1, \dots, p_n, p_{n+1})$ bounding embedded discs D_i $i=1, \dots, n, n+1$ in H^3 any two of which meet, if at all, in a set of clasps and all the clasps are disjoint. Haken's result is recovered (Corollary 4.1.1) by noting that any n boundary components of the planar surface constitute a set of longitudes for the associated H-decomposition.

Our approach however is very different from Haken's one (we were actually unaware of his result when we proved Theorem 4.1). While Haken used simple combinatorial arguments we work with surgery presentations and use the Kirby calculus to change them.

We start with a presentation of an arbitrary homotopy 3-sphere, H^3, as surgery in a link in S^3, $\mathcal{L}_{S^3}(\ )$, and consider the "dual" presentation of S^3 as surgery in H^3, $\mathcal{L}_{H^3}(\ )$. The proof is done by alternatingly unknotting both pictures, using the Kirby calculus, and observing at each stage how a change in one of the pictures reflects in a change in the dual picture. In the end we get a simple description of H^3 which simultaneously embodies three types of representation: a surgery (in a pure braid in S^3) and associated pob-decomposition and H-decomposition, with the required properties - these will be called primitive H-decompositions and primitive surgery presentations of H^3.

These primitive representations of H^3 as surgery in a pure braid in S^3 have the (apparent) advantage of presenting all the important elements of Haken's result - the system of longitudes and respective embedded discs and the singular set (the set of clasps) - in a single global picture in our three-dimensional field of vision (for instance the discs are seen as planar surfaces whose boundary components are longitudes on the boundaries of the surgery tori). This may help us to have a better idea of what are the difficulties of a global nature in the homotopy 3-spheres and the flexibility of the Kirby calculus might allow us to execute more significant and global changes than we could hope for by using simple combinatorial techniques. So far as we know there has not been a serious treatment oh homotopy 3-spheres using surgery (in spite of Bing's comment [B,p.36] that the existence of surgery presentations "might lead to a solution of the Poincaré hypothesis").

Chapter 2 in conjunction with Chapters 3 and 4, whose contents we describe next, can be seen as part of a program for a proof of the Poincare conjecture.

in the aforementioned paper by Haken and in relation to the result we have been referring to, he states that if the set of clasps forms a "trivial link" in the complement of the handlebody H, then the homotopy 3-sphere is in fact S^3.

In Chapter 4 we present general techniques to unknot and unlink the set of clasps while staying within our class of primitive representations. That is done (§10) using the surgery picture in S^3, by forcing the clasps to be monotone in relation to an height function, through a change of crossings with the introduction of new surgery components applying the Kirby calculus. But that "trivialization" of the set of clasps is done at the expense of introducing a new type of singularities, ribbons (§ 2 - figure 13), which do not form with the set of clasps a trivial link - to put things in perspective we note that the techniques of §10 used in a negative way (to complicate instead of simplifying) allows us to see how in the situation of Haken's result we can start with a set of clasps as far from forming a trivial link as we wish.

So far, all the attempts to get a trivial singular set (consisting of clasps or ribbons) have failed and I personally believe that is not possible (although it is good fun to play around with those moves) - the work of Connor [Co] should be mentioned in this context.

We try to circumvent this difficulty by using a characterization of S^3 that is provided in Chapter 3. We prove (Theorems 6.1, 7.1, 8.1) that if the singular set consists of clasps forming a trivial link and in addition a certain ordering condition is satisfied then the homotopy 3-sphere is indeed S^3. The proof consists of collapsing the discs of the primitive H-decomposition first, in a certain way, and then cancelling 1-handles corresponding to the clasps using the ordering condition and the Loop-Theorem to resolve immersed discs that are created with the initial collapse from the set of discs that trivialize the clasps - for the initial collapse to be possible we also require a technical condition : that in the primitive

H-decomposition the system of discs intersects the handlebody H only in meridional discs at the end- points of clasps.

Now it is a fact (Theorem 9.1) that we can always assume in our primitive representations that the ordering condition on the s * of clasps as well as that technical condition are satisfied. We can then make the clasps monotone as mentioned above (§10) at the expense of introducing ribbons, but still preserving the ordering condition. In this situation we can try to apply the proof of Theorem 6.1 (7.1, 8.1) working only with the trivial set of clasps and ignoring the badly behaved ribbons since these do not prevent the collapse of certain discs and after some cancellations of 1-handles they can be removed from the picture (§11-11.0). The problem that subsists now is that the new surgery components introduced to make the clasps monotone have to be joined to the previous ones (in the standard way of passing from a pure-braid surgery to a H-decomposition) and in doing so we might create new intersections with the discs (the so called base-intersections) thus loosing the above technical condition that allows the necessary collapses for our characterization of S^3.

The remaining of Chapter 4, §11, presents a general technique - tagging - to try to incorporate those new base intersections in the whole process. The process of tagging consists, shortly, in joining each base intersection to the boundary of the disc where it lies by an imaginary line (a tag) which avoids all the clasps and all other tags in that disc. Tags are then treated essentially as clasps (and there is a generalized notion of order for clasps and tags - §§7 and 8) although the process of making them trivial (11.3) is easier than for clasps.

In this process of tagging we can always assume that the order is preserved but then when we make them trivial new base intersections are created which in turn will need to be tagged and there is no guarantee that the process will finish On the other hand we can make make the process of tagging finite by dealing with the relative minima of the planar surfaces in the surgery picture in S^3 (in relation to an height function), but then the ordering may be lost. The vanishing of the order in this situation is closely related to the way those relative minima are distributed in relation to the height function and we can assume that under certain conditions (see §11-11.2) it is preserved.

There is some hope that a proof of Poincaré conjecture can be carried on along these lines, either by forcing the ordering condition in the tagging and finding the right notion of complexity for the whole picture (planar surfaces, their relative minima, etc.) to make the process finite or, more likely, by improving the initial primitive presentation so as to assure that the original surfaces are nice enough for some simple tagging to preserve the order.

The difficulties with this approach plus some directions and details for further developments are presented in §11 under the tittle - **A project.**

Chapter 1 is independent of the other chapters. It contains work done previously to the remaining of the thesis and is reproduced here in the form it has for publication [R-R$_1$] including its own references and numbering of results and figures. The basic definitions concerning H-diagrams and H-decompositions are to be found in this chapter. The results of Chapter 1 were obtained thinking about the possibility of algorithmically searching for counter-examples to the Poincaré conjecture and two more characterizations of homotopy 3-spheres are given.

The first characterization is stated in a paper by Haken [H$_5$] and attributed to Moise and others. It says that a handlebody H of genus n with a system of non-separating disjoint embedded curves $(c_1, \dots, c_n)$ in ∂H represents a H-diagram for a homotopy 3-sphere if and only if there is an embedding of H in S^3 such that the curves c_i bound disjoint properly embedded orientable surfaces in $Cl(S^3-H)$. Such an embedding together with the orientable surfaces represents simple a <u>transversality diagram</u> for a degree one map $S^3 \dashrightarrow H$, and is a well know fact that if the embedding is standard (i.e. $CL(S^3-H)$ is also a handlebody) then $H^3=S^3$. We give a simple direct proof of that characterization using standard transversality techniques. The second characterization is derived from the first one by stabilizing and is given in terms of a relation that associates to an arbitrary H-diagram of genus n of a homotopy 3-sphere H^3 a certain H-diagram of $H^3 \# T$ where T is a handlebody of genus say k. This H-diagram of $H^3 \# T$ consists of a handlebody H of genus n+k standardly embedded in S^3 and of n non-separating disjoint embedded curves in ∂H bounding disjoint orientable surfaces in $Cl(S^3-H)$. As explained in Chapter 1 this characterization gives an effective way, using some computer program, of generating all possible H-diagrams of homotopy 3-spheres.

It was not until recently that we realized that the primitive representations of homotopy 3-spheres also contain useful information on transversality diagrams. In Chapter 5 we derive from primitive representations and using techniques from previous chapters some more results about H-diagrams and transversality diagrams analogous to those in Chapter 1.

The contents of this thesis represent work done jointly with my supervisor Dr. Colin Rourke mainly during my final year at Warwick and most of it exists in pre-print form [R-R_2]. The rest existed somehow in "blackboard-form".

I want to thank Dr. Colin Rourke for all the mathematics he taught me but most important for the spirit and the atmosphere of that period of work that allowed me to experience for the first (and perhaps last) time the joy and excitement of mathematical research and that brought me closer to becoming a mathematician than I would ever have hoped for.

I also want to thank my ex-colleagues at Warwick, John Klipenstein and Luciano Lomonaco for being such nice persons to have shared an office with. Finally I thank Alexis Marin and Eugénia César de Sá who first introduced me to 3-manifolds (although sometimes I blame them for that) and Rúben Azevedo for patiently helping me with PC ("personal computer" not Poincaré conjecture).

A final word : I believe this thesis has an unusual number of pictures.

I quote Haken (from the preface in "Some Results on Surfaces in 3-Manifolds", [H_1]):

" ... The geometric methods have the special feature of appealing to visualization, and it is a matter of taste whether one regards this feature as an advantage or a disadvantage. If the author of a topological paper relies on pictures instead of precise verbal proofs, he may produce an especially short and understandable treatment of his topic, but at the same time this may lead to serious errors. So some special care is required in this respect. On the other hand, visualization enables one to maintain a survey over rather complicated configurations whose verbal description would be lengthy and require elaborate notation. So one may regard visualization as a valuable tool for finding topological results and attacking problems which are extremely difficult and complex."

Chapter 1 - Heegaard Diagrams and Homotopy 3-Spheres

Topology Vol. 00, No. 0, pp. 000–000, 1987
Printed in Great Britain

0040-9383/87 $3.00+.00
Pergamon Journals Ltd.

HEEGAARD DIAGRAMS AND HOMOTOPY 3-SPHERES

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(Received 30 November 1986)

ALL SURFACES and three-manifolds are assumed to be orientable throughout the paper.

A *complete system* (CS) $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ on a closed surface S means a set of simple closed curves on S which are:

(1) mutually disjoint; and
(2) do not separate S

and such that the set is maximal with respect to properties (1) and (2). [It follows that n is the genus of S and that the result of surgering S along $\mathbf{x}$ (i.e. along each x_i) is a 2-sphere.]

Given $(S, \mathbf{x})$, where $\mathbf{x}$ is a CS on S, we can construct a solid handle body $T(\mathbf{x})$ as follows: glue a (thickened) 2-disc to S along each x_i, thereby "realizing" the surgery of S along $\mathbf{x}$, and then glue in a 3-ball to the 2-sphere which results from this surgery:

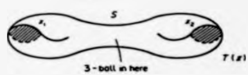

Fig. 1

A *Heegaard diagram* (H-diagram) is a closed surface S with two complete systems $\mathbf{x}$, $\mathbf{y}$. The H-diagram defines a 3-manifold

$$M(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}) \cup_S T(\mathbf{y}).$$

$M(\mathbf{x}, \mathbf{y})$ has a handle presentation in which the thickened discs attached to the x_i (resp. y_i) are the 2-handles (resp. 1-handles), and the 3-ball which completes $T(\mathbf{x})$ (resp. $T(\mathbf{y})$) is the single 3-handle (resp. 0-handle). It follows from standard results (on existence of nice handle presentations, etc.) that any closed (orientable) 3-manifold can be obtained in this way.

This paper is concerned with the possibility of using H-diagrams to find counterexamples to the Poincare conjecture.

There is a very elegant characterization of a homotopy 3-sphere in terms of any corresponding H-diagram:

THEOREM 1. *$S(\mathbf{x}, \mathbf{y})$ is an H-diagram of a homotopy 3-sphere if and only if there is an embedding of $T(\mathbf{y})$ in S^3 such that $x_1, x_2, \ldots, x_n$ bound disjoint orientable surfaces $S_1, S_2, \ldots, S_n$ in $\overline{S^3 - T(\mathbf{y})}$.*

Theorem 1 is stated in Haken's paper [1] and attributed to Moise and others. For completeness, we give a proof of Theorem 1 in §1.

Elegant though Theorem 1 is, it is useless for detecting a possible counterexample to the Poincare conjecture because it does not provide a computable way to recognize an H-diagram which represents a homotopy 3-sphere. Given an H-diagram there is no effective way to search through all possible knotted embeddings of $T(\mathbf{y})$ in S^3 or to check for a given knotted embedding whether $x_1, x_2, \ldots, x_n$ bound disjoint surfaces.

However, as we will show in this paper, Theorem 1 can be used to derive another characterization (Theorem 2) of an H-diagram of a homotopy 3-sphere, which leads at once to a computer program to list all such diagrams.

Before stating Theorem 2, we need to prove some, more-or-less well-known, facts about systems of curves on a surface.

We say complete systems $\mathbf{x}$, $\mathbf{y}$ are *equivalent* (written $\mathbf{x} \sim \mathbf{y}$) if $T(\mathbf{x})$ is homeomorphic to $T(\mathbf{y})$ by a homeomorphism fixed on S. (Clearly $M(\mathbf{x}, \mathbf{y})$ depends only on the $\sim$ classes of $\mathbf{x}$ and $\mathbf{y}$.)

By a super-complete system (SCS) on S we mean any set of disjoint simple closed curves on S which contains a CS. Given an SCS $\mathbf{x}$ on S, then $T(\mathbf{x})$ is constructed exactly as for a CS: the only difference is that there may be several 3-balls to be glued in at the end, and there is therefore a similar notion of equivalence ($\sim$) for SCSs.

Remark. If $\mathbf{x}_+ \supset \mathbf{x}$ are SCSs on S, then $\mathbf{x}_+ \sim \mathbf{x}$.

Proof. The new discs and 3-balls in $T(\mathbf{x}_+)$ can be found inside the 3-ball(s) of $T(\mathbf{x})$. Q.E.D.

The remark implies that insertions and deletions of curves in an SCS does not change the $\sim$ class. We now prove the converse.

Lemma 1. *Suppose $\mathbf{x}$ is an SCS on S and $\mathbf{y}$ is any set of disjoint simple closed curves on S such that each y_i bounds a disc in $T(\mathbf{x})$. Then $\mathbf{x} \sim \mathbf{y}_+$ by insertions and deletions where $\mathbf{y}_+ \supset \mathbf{y}$.*

Proof. Suppose y_1 bounds the disc D in $T(\mathbf{x})$ and let $D_1, \ldots, D_n$ denote the discs bounded by $\mathbf{x} = \{x_1, \ldots, x_n\}$. We simplify the transverse intersection

$$Q = D \cap (D_1 \cup \ldots \cup D_n).$$

If Q contains a closed curve, then the usual "push across a 3-ball" argument isotopes D so as to simplify Q. If not, then Q consists of arcs. Choose an outermost arc α with endpoints a, b in x_i, say. Let β be the arc in y_1 from a to b on the outside. Define two new curves x_i', x_i'' by cutting x_i at a, b, inserting β and then pushing away from D and x_i a little:

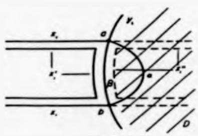

Fig. 2

Then if we replace x_i by x_i' and x_i'' in $\mathbf{x}$ (two insertions and one deletion), we have x_i', x_i'' bounding discs D_i', D_i'' such that $Q' = D \cap (D_1 \cup \dots \cup D_i' \cup D_i'' \cup \dots ; D_q)$ is simpler. By a finite number of such moves we have $Q = \emptyset$ and y_1 can be inserted. We continue in the same way to insert $y_2, y_3, \dots$ and since the only deleted curves are ones which meet a y_i and since the y_i are mutually disjoint, the process ends with the required system $\mathbf{y}$. Q.E.D.

Corollary 1. $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \mathbf{x}$ *is obtained from* $\mathbf{y}$ *by insertions and deletions.*

Proof. $\Leftarrow$ See Remark. $\Rightarrow$ Identify $T(\mathbf{x})$ and $T(\mathbf{y})$ by the homeomorphism, then $\mathbf{y}$ satisfies the hypotheses of the lemma and hence $\mathbf{y}_* \supset \mathbf{y}$ is obtained from $\mathbf{x}$ by insertion and deletion. But $\mathbf{y}_*$ then deletes down to $\mathbf{y}$. Q.E.D.

Suppose now that we have any set of simple closed curves $\mathbf{x} = \{x_1, \dots, x_n\}$ on a surface S and that α is a further curve. Then, by orienting S and all the curves, we can read off a (cyclic) word $\omega(\alpha, \mathbf{x})$ in the symbols $x_1, \dots, x_n$ by traversing α once (in the given direction) and reading x_i or x_i^{-1} for each transverse crossing with x_i by the rules:

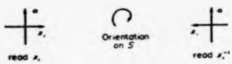

Fig 3.

If $\omega(\alpha, \mathbf{x})$ reduces by cancellation to the empty word, then we write $\omega(\alpha, \mathbf{x}) = e$ and clearly this statement about ω is independent of all chosen orientations.

Lemma 2. *Suppose* $\mathbf{x}$ *is an SCS on* S, *then* α *bounds a disc in* $T(\mathbf{x}) \Leftrightarrow \omega(\alpha, \mathbf{x}) = e$.

Proof. $\Rightarrow$ Suppose α bounds D and $x_1, \dots, x_n$ bound $D_1, \dots, D_n$. The triviality of $\omega(\alpha, \mathbf{x})$ follows from inspecting the arcs in the transverse intersection $Q = D \cap (D_1 \cup \dots \cup D_n)$:

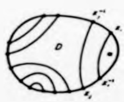

Fig 4.

An outermost arc α corresponds to a subword $x_i x_i^{-1}$ or $x_i^{-1} x_i$ in $\omega = \omega(\alpha, \mathbf{x})$ which cancels to yield ω', say. Then by induction on the number of arcs in Q, $\omega' = e$ and hence $\omega = e$.

$\Leftarrow$ $\omega(\alpha, \mathbf{x}) = e$ implies that α represents 1 in $\pi_1(T(\mathbf{x}))$ and hence α bounds a disc by Dehn's lemma. (Actually, this is a very special case of Dehn's lemma which has an elementary proof using handle slides (see [5]).) Q.E.D.

Combining Lemmas 1 and 2 we have the combinatorial criterion for equivalence of SCSs:

Corollary 2. $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \omega(y_i, \mathbf{x}) = e$ *for each* $y_i \in \mathbf{y}$.

Proof. ⇒ Follows at once from Lemma 2. ⇐ By Lemma 2, each y_i bounds a disc in $T(\mathbf{x})$ and hence $\mathbf{x} \sim \mathbf{y}_* \supset \mathbf{y}$ by Lemma 1. Q.E.D.

Now let T denote the standard solid handlebody of genus n embedded in R^3. The boundary of T is denoted S, and there are two standard CSs on S, $\mathbf{a} = \{a_1, \ldots, a_n\}$ bounding discs outside T and $\mathbf{b} = \{b_1, \ldots, b_n\}$ bounding discs inside T:

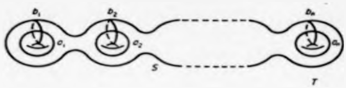

Fig 5

An H-system is a CS $\{x_1, \ldots, x_t, y_{t+1}, \ldots, y_n\}$ on S such that

(1) $\omega(a_i, \mathbf{x}) = e$, $i = 1, 2, \ldots, n$ where $\mathbf{x} = \{x_1, \ldots, x_t\}$, and

(2) $\omega(y_i, \mathbf{b}) = e$, $i = t+1, t+2, \ldots, n$.

By Lemma 2, condition (2) is equivalent to saying that each y_i bounds a disc in T, and hence by Lemma 1 we can extend $y_{t+1}, \ldots, y_n$ to a CS $\mathbf{y}_* = \{y_1, \ldots, y_n\}$ equivalent to $\mathbf{b}$ (and clearly $\mathbf{y}_*$ is determined up to equivalence). Now let S' be the result of surgering S along $y_{t+1}, \ldots, y_n$, then there are two CSs on S', namely $\mathbf{x}$ and $\mathbf{y} = \{y_1, \ldots, y_t\}$ (and $\mathbf{y}$ is again clearly determined up to equivalence). Thus the H-system gives rise to *the associated H-diagram* $S'(\mathbf{x}, \mathbf{y})$.

Theorem 2. *Let $S'(\mathbf{x}, \mathbf{y})$ be an H-diagram associated to an H-system. Then $M(\mathbf{x}, \mathbf{y})$ is a homotopy 3-sphere and every H-diagram for a homotopy 3-sphere arises in this way.*

Since the data for an H-system are clearly effectively computable and since the process of constructing $S'(\mathbf{x}, \mathbf{y})$ from the data is algorithmic, Theorem 2 leads at once to a computer program to list all H-diagrams of homotopy 3-spheres (at least up to equivalence of one of the systems) and hence to list all candidates for a counterexample to Poincare conjecture.

There is also an algorithm to compute the Rohlin invariant of a homotopy 3-sphere from its H-diagram. (Lickorish's proof [3] that $\Omega_3 = 0$ and the new proof [4] are both algorithmic: they provide algorithms to convert an H-diagram into a surgery description. From the surgery description the Rohlin invariant can be computed, see [2].) Therefore there is a computer program which would effectively search for a strong counterexample to the Poincare conjecture, i.e. a homotopy 3-sphere of Rohlin invariant 1.

§II. PROOF OF THEOREM 1

Theorem 1 is proved using transversality and the following well-known lemma:

Lemma 3. *M^3 is a homotopy 3-sphere if and only if there is a degree 1 map $f: S^3 \to M^3$.*

Proof of Lemma 3. Suppose M^3 is a homotopy 3-sphere. Choose standard embeddings of D^3 in S^3 and M^3. $\overline{M^3 - D^3}$ is contractible (by Whitehead's theorem, homology version) and hence there is a homotopy to zero of $S^2 = \partial D^3$ in $\overline{M^3 - D^3}$. Define the degree 1 map

$f: S^3 \to M^3$ by mapping D^3 to D^3 by the identity and $\overline{S^3 - D^3}$ to $\overline{N^3 - D^3}$ by the homotopy of S^2 to zero.

Conversely, suppose $f: S^3 \to M^3$ is a degree 1 map. If M^3 is non-simply-connected, then there is a lift $\tilde{f}: S^3 \to \tilde{M}^3$. Now either $\tilde{M}^3$ is compact (in which case $\tilde{M} \to M$ has finite degree > 1) or $\tilde{M}^3$ is non-compact (in which case $H_3(\tilde{M}^3) = 0$). In either case, $f_*: H_3(S^3) \to H_3(M^3)$ fails to be an isomorphism, which is a contradiction. So, we can assume M^3 is simply-connected and then, since f is a homology equivalence (using duality), f is a homotopy equivalence by Whitehead. Q.E.D.

Proof of Theorem 1. Suppose M^3 is a homotopy 3-sphere and

$$M^3 = D^3 \cup h_1 \cup \ldots \cup h_n \cup j_1 \cup \ldots \cup j_n \cup B^3$$

is a given nice handle decomposition of M, where h_j are 1-handles and j_k are 2-handles. That is, $S(x, y)$ is an H-diagram for M^3 when $S = \partial(D^3 \cup h_1 \cup \ldots \cup h_n)$, y_i is the b-sphere of h_i and x_k is the a-sphere of j_k, for each i, k.

The notation core (h_i) denotes the (1-dimensional) core of h_i and, similarly, core (j_k), the (2-dimensional) core of j_k.

Using the proof of Lemma 3 there is a degree 1 map $f: S^3 \to M^3$ such that $f: f^{-1}(D^3) \to D^3$ is a homeomorphism. Make f transverse to the cores of the 1-handles $h_1, h_2, \ldots, h_n$, then f^{-1} (core (h_i)), for each i, consists of an arc starting and ending on ∂D^3 and a number of "spare" circles. By thinking of h_i as a disc bundle over the core, it may be assumed that $f^{-1}(h_i)$ is a thickened version of f^{-1} (core (h_i)), i.e. it consists of a "tube" (the arc thickened) and a number of "spare tubes" which are the thickened "spare" circles: the "spare tubes" are, in fact, solid tori.

Next make f transverse to the cores of the 2-handles and then f^{-1} (core (j_k)), for each k, is a surface with boundary lying on the "tubes" and "spare tubes" $f^{-1}(h_i)$ and on ∂D^3.

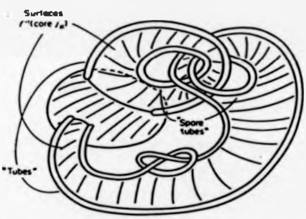

Fig. 6.

Call the collection D^3, $f^{-1}(h_i)$, f^{-1}(core (j_k)) a *transversality diagram* for f. Then the diagram determines f up to homotopy since each element of the diagram is mapped to a contractible subset of M^3. It follows that we can make "abstract" changes to the diagram and then use the changed diagram to redefine the map f. In particular, any free components of the diagram may be deleted and hence it may be assumed that the diagram is connected.

We now explain how to eliminate the "spare tubes". Since the diagram is connected there must be a connected surface S in the diagram (part of f^{-1} (core (j_k)), say) with one boundary component α meeting a tube and another lying on a "spare tube". At this point we need to observe that D^3 together with the tubes (*not* the spare tubes) is in fact a copy of $T(y) = D^3 \cup h_1 \cup \ldots \cup h_n$ and hence α is a copy of x_k (the a-sphere of j_k). So, we can choose a point of S on the "spare tube" and join it by an arc β in S to a *corresponding* point on α (that is, a point which has the same image under f in x_k). Now perform a bridge move on the diagram using the arc β as pictured in Fig. 7. The figure explains how the surface S and any other surfaces (typified by S') incident to the tubes are modified.

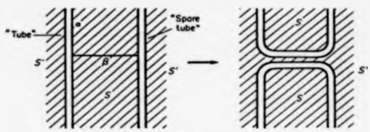

Fig. 7.

(This move can in fact be realized by a homotopy of f). The move reduces the number of "spare tubes" by one and hence, by induction, we can assume that there are no spare tubes. After eliminating the spare tubes, $f^{-1}(T(y)) = D^3 \cup$ tubes, is a copy of $T(y)$ and f^{-1} (core (j_k)), $k = 1, 2, \ldots, n$, are disjoint surfaces spanning the copies of x_k; in other words, we have found the required embedding of $T(y)$ in S^3 so that $x_1, x_2, \ldots, x_n$ bound disjoint surfaces in $\overline{S^3 - T(y)}$.

For the converse, if such an embedding is given, then it may be regarded as a transversality diagram and hence it defines a degree 1 map $S^3 \to M^3$, and therefore M^3 is a homotopy sphere by Lemma 3. Q.E.D.

§2. PROOF OF THEOREM 2

Throughout this section, T, S, **a**, **b** are the standard objects and systems as in the definition of an H-system. We need a geometrical interpretation of condition (1) in that definition.

LEMMA 4. *Let* $\mathbf{x} = \{x_1, \ldots, x_n\}$ *be a set of disjoint curves on* S, *then*

$$\omega(a_i, \mathbf{x}) = e, \qquad i = 1, 2, \ldots, n,$$

$\Leftrightarrow$ $x_1, \ldots, x_n$ *bound disjoint surfaces*

$$S_1, S_2, \ldots, S_n \quad \text{in} \quad \mathbb{R}^3 - T.$$

Proof. $\Rightarrow$ Denote by D_i the disc bounded by a_i outside T. The triviality of $\omega(a_i, \mathbf{x})$ follows by inspecting the transverse intersection $D_i \cap (S_1 \cup \ldots \cup S_n)$ exactly as in the first half of the proof of Lemma 2.

$\Leftarrow$ The triviality of $\omega(a_i, \mathbf{x})$ means that we can find disjoint arcs in D_i for each i which joins pairs (x_j, x_j^{-1}) in $a_i \cap \mathbf{x}$ and such that all points of $a_i \cap \mathbf{x}$ are paired off as in Fig. 4. (That is, we have transverse intersection sets, *as if* the required surfaces existed.) Now thicken each D_i and thicken the arcs as well, then in the complement we have a 3-ball with a number of disjoint closed curves in its boundary which can be capped by disjoint 2-balls to complete the required surfaces. Q.E.D.

As an aside here, we remark that if we combine Lemmas 2 and 4 we see that a *complete* system $\mathbf{x}$ on S which bounds disjoint surfaces outside T is equivalent to the standard system $\mathbf{a}$.

Proceeding now with the proof of Theorem 2, suppose that $x_1, \ldots, x_t, y_{t+1}, \ldots, y_n$ is an H-diagram and that $\mathbf{y}_* = y_1 \ldots y_n$ is a complete system equivalent to $\mathbf{b}$. Therefore the y_i bound disjoint discs $D_1 \ldots D_n$ inside T. Now let T' be the result of cutting T along $D_{t+1}, \ldots, D_n$, then T' is a copy of $T(\mathbf{y})$, where $\mathbf{y} = y_1 \ldots y_t$ and T' is embedded in S^3. But by Lemma 4, $x_1 \ldots x_t$ bound disjoint surfaces in $\overline{S^3 - T}$ and therefore in $\overline{S^3 - T'}$, and it follows from Theorem 1 that $M(\mathbf{x}, \mathbf{y})$ is a homotopy 3-sphere.

Conversely, suppose that $M(\mathbf{x}, \mathbf{y})$ is a homotopy sphere where $\mathbf{x}, \mathbf{y}$ are complete systems on S', say (of genus t), and let T' denote $T(\mathbf{y})$. By Theorem 1 there is an embedding of T' in S^3 such that $x_1 \ldots x_t$ bound disjoint surfaces $S_1, S_2, \ldots, S_t$ in $\overline{S^3 - T'}$. Now T' can be regarded as the regular neighbourhood of a 1-dimensional complex K', say, and by choosing a triangulation of S^3, whose 1-skeleton K'' contains K' as a subset, we can extend T' to an unknotted handlebody T'', say; and we can think of T'' as obtained from T' by attaching a sequence of 1-handles $h_{t+1}, h_{t+2}, \ldots, h_n$, say. We need to choose T'' so that the h_i miss the surfaces S_i. To do this, we start by assuming that the chosen triangulation of S^3 includes each S_i as a subcomplex, then $K' \subset K_1 \subset K''$, where $K_1 - T'$ lies in $\cup S_i$. Thus we can think of T'' as obtained from T' by first attaching handles $h_{t+1}, \ldots, h_s$ whose cores lie in $\cup S_i$ and then attaching handles $h_{s+1}, \ldots, h_n$ disjoint from $\cup S_i$. Now slide the attaching tubes of the $h_{s+1}, \ldots, h_n$ off the $h_{t+1}, \ldots, h_s$ (by the usual "reordering" argument for handles) and then push the $h_{t+1}, \ldots, h_s$ to one side of the S_i. Thus, we now have T'' (unknotted) obtained from T' by attaching 1-handles $h_{t+1}, \ldots, h_n$ disjoint from $\cup S_i$. Since T'' is unknotted, we can ambiently isotope it to standard position, i.e. to T. Write y_i for the belt sphere of h_i, $i = t+1, \ldots, n$, then y_i bounds the co-core D_i of h_i, and if we cut T along each of the D_i we regain T' (up to ambient isotopy). Now $x_1, \ldots, x_t$ bound disjoint surfaces in $\overline{S^3 - T}$ and $y_{t+1}, \ldots, y_n$ bound disjoint discs in T. Therefore by Lemmas 2 and 4, $x_1, \ldots, x_t, y_{t+1}, \ldots, y_n$ is an H-system and the theorem is proved. Q.E.D.

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Chapter 2 – A Characterization of Homotopy 3-Spheres

Let H be the standard handlebody of genus n. We can regard H as obtained by forming the boundary connected sum of a 3-ball (called the base) with n copies of the solid torus $S^1 \times D^2$ (see figure 1). A curve $c = S^1 \times \{p\} \subset \partial H$ $(p \in \partial D^2)$ in one copy of $S^1 \times D^2$ will be called a longitudinal curve for H and a properly embedded disc $\{q\} \times D^2 \subset H$ in the same copy will be called a meridional disc for H, transverse to c.

A choice of n longitudinal curves (one for each copy of $S^1 \times D^2$) is called a longitudinal system (or a system of longitudes) for H and the associated system of transverse meridional discs a complete system of discs for H.

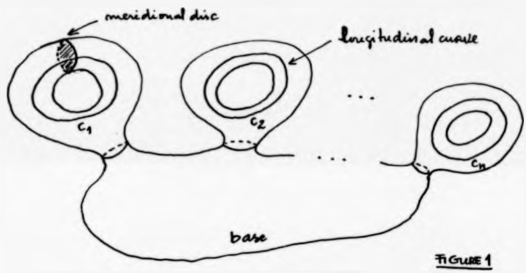


FIGURE 1

It is well-known that any closed orientable 3-manifold has a Heegaard decomposition (H-decomposition) $M^3 = H \cup_h H'$ where H, H' are handlebodies and $h: \partial H \to \partial H'$ is a homeomorphism. A system of longitudes $\{c_1, \ldots, c_n\}$ for H can then be seen as a set of generators for $\pi_1(M^3)$. Therefore M^3 is a homotopy 3-sphere if and only if each c_i bounds a (singular) disc in M^3. The Poincaré conjecture can be formulated in terms of longitudinal systems as follows:

Conjecture. Any homotopy 3-sphere, H^3, has a H-decomposition $H^3 = H \cup_h H'$ with a longitudinal system $\{c_1, \dots, c_n\}$ for H such that there are disjoint embedded discs $D_1, \dots, D_n$ in H^3 with $\partial D_i = c_i$ for each i. The proof that this is equivalent to the Poincaré conjecture will be given in §4 (Theorem 4.2).

The main result of this chapter is an extension of the following result by Haken [H4]

Theorem: Any homotopy 3-sphere has a H-decomposition $H^3 = H \cup_h H'$ with a system of longitudes $\{c_1, \dots, c_n\}$ for H such that each c_i bounds an embedded disc D_i in H^3. Moreover we can assume that any two of the D_i meet, if they meet at all, in clasp-like double arcs (simply called "clasps") (figure 2) and there are no triple points.

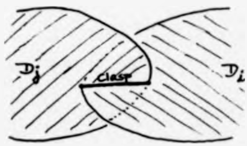


FIGURE 2

What we actually prove (Theorem 4.1) is that any homotopy 3-sphere, H^3, has an "open book decomposition" whose pages are planar surfaces with boundary components $b_0, b_1, \dots, b_n$ bounding embedded discs $D_0, D_1, \dots, D_n$ in H^3 that intersect as in the previous theorem. Haken's result is recovered by noting that any n of those boundary components form a system of longitudes for a certain H-decomposition of H^3 (details in §1).

§1. Definitions and Preliminary Observations

Planar open-book decompositions

A link $\mathcal{L}=(L_1, \ldots, L_n)$ in a 3-manifold M^3 is fibred if there exists a surface F embedded in M^3 with boundary the given link, which is 2-sided and such that the result of cutting M^3 open along F is a manifold homeomorphic to F x I, I the unit interval. We can then write M^3 as the identification space $F \times I/_h$ where $F \times I$ is $F \times I$ with each stalk $a \times I$, $a \in \partial F$ identified to a point and $h: F \to F$ is a homeomorphism which restricts to the identity on ∂F. This defines what is usually called an open book decomposition of M^3 with binding link $\mathcal{L}$ and pages F_t ($t \in [0,1]$). We will refer to such a decomposition with pages homeomorphic to a planar surface as a planar open book decomposition (pob-decomposition).

Surgery presentations and the "Kirby calculus"

Definition: Surgery will always mean "longitudinal surgery" described as follows. Suppose N^3 is a 3-manifold $L \subset N^3$ a simple closed curve and $h: S^1 \times D^2 \to N(L)$ a "framing" of L (i.e. h is a homeomorphism of $S^1 \times D^2$ onto a regular neighbourhood of L such that $h(S^1 \times \{0\})=L$). Then the manifold M^3 obtained by surgery on N^3 along the curve L is defined by $M^3 = (N^3 - N(L)) \cup_{h|S^1 \times S^1} (D^2 \times S^1)$ i.e. the image of h is removed and glued back with factors interchanged. Note that the framing h of L is equivalent to a choice of a parallel curve f to L (i.e. a curve such that L, f cobound an annulus). This is because the framing determines the parallel curve $h(S^1 \times \{p\})$, $p \in D^2$, $p \neq 0$ and conversely the parallel determines h (up to an isotopy and a possible reflection of S^1 - neither of which affect the result of the surgery).

If we have a link $\mathcal{L}=(L_1, \ldots, L_n)$ in N^3 then $\mathcal{L}_{N^3}(\ )$ represents the 3-manifold M^3 obtained by surgery on N^3 along each component of the link with some prescribed framing, for each component. We call $\mathcal{L}_N{}^3(\ )$ a surgery presentation of M^3 (in terms of N^3).

Convention: when dealing with surgery presentations we will often confuse components of the surgery link with their regular neighbourhoods (the surgery tori) to simplify notation - the context always makes clear what is being referred.

It is a fact [L1],[R] that any closed orientable 3-manifold can be obtained by performing (longitudinal) surgeries on the components of some link in S^3, $\mathcal{L}=(L_1, \dots ,L_n)$ (and therefore can be given a surgery presentation in terms of any other closed orientable 3-manifold). Such a surgery can be represented by drawing disjoint annuli A_i $(i=1, \dots ,n)$ with $\partial A_i=L_i \cup f_i$ where f_i (seen as a longitude in the boundary of a solid torus with centre curve L_i) represents the "framing" for the surgery on the component L_i. Alternatively we can represent each framing f_i simply by labelling L_i with a framing number (or surgery coefficient) $s_i \in Z$. This integer represents the number of times f_i twists to the right or to the left of the preferred longitude(i.e. the only longitude nulhomologous in the complement of L_i), with the convention that we count the twists to the right as the positive ones [F-R]. Note that framing numbers are still well defined if we perform surgeries in any homotopy-sphere (or homology-sphere) or more generally in an arbitrary 3-manifold for all curves that are unknotted (i.e. that bound a disc).

It is sometimes convenient to indicate the trivial surgery by the label ∞ and denote the manifold obtained by surgery on the link $\mathcal{L}=(L_1, \dots ,L_n)$ with framing numbers s_i $(i=1, \dots ,n)$ by $\mathcal{L}(s_1, \dots ,s_n)$. When no reference to the framings of a surgery is needed we simply write $\mathcal{L}(\ )$, as in the above definition.

We will make,throughout this thesis, extensive use of the moves of the Kirby calculus [K],[F-R] to alter the surgery presentations of a 3-manifold M^3.

Given a surgery presentation $M^3= \mathcal{L}(s_1, \dots ,s_n)$ $\mathcal{L}=(L_1, \dots ,L_n)$ we have the following moves:

(a) - Introduction or deletion of an unknotted component labelled ±1 and separated from the rest of the link by a 2-sphere (This does not change the homeomorphism type since surgery on the unknot with ±1 coefficients yields S^3)

(b) - The Band Sum of one component say L_1 to another say L_2. Let A_1,A_2 be annuli representing the surgeries along L_1 and L_2 respectively, and A_2' another annulus extending A_2 (as shown in figure 3-a). The move consists of substituting A_1 by its band sum with A_2' as indicated in figure 3-b using a band B that does not intersect any component of the link. The new annulus thus obtained defines the new surgery component (that substitutes A_1 in the picture) and respective surgery framing.

This move also does not change the homeomorphism type as it corresponds, as indicated in figure 3-c ,to an ambient isotopy of L_1 in the manifold $\mathcal{L}(\infty, s_2, \ldots, s_n)$ across the solid torus that corresponds to the surgery along L_2

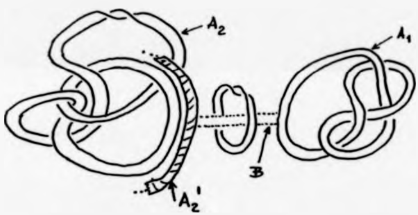


FIGURE 3-a

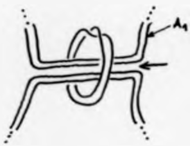


FIGURE 3-b

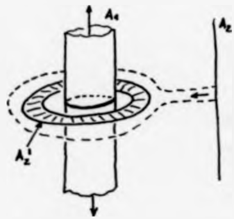


FIGURE 3-c

There is a useful simple consequence of these two moves which is convenient to be thought as a "third move".

(c) - We can introduce an unknotted component c, labelled ±1 not necessarily separated from the rest of the link, as long as we twist all the curves (link components and respective framing curves) that pierce the disc bounded by c (twist to the right if the label is 1, to the left if it is −1). See figure 3-d.

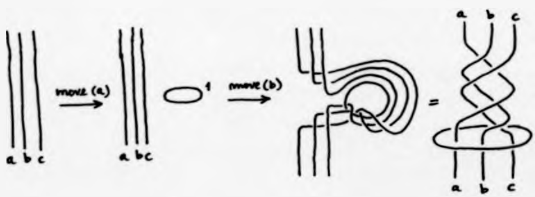


FIGURE 3-d

There is also an alternative move to (a) that will be used very frequently

(a') - introduction or deletion of an even number of 0-labelled components separated from the rest of the link by a 2-sphere and linked between them as shown in figure 3-e. We will call such a link of 0-labelled components (in even number) a daisy-chain.

Figure 3-e

It is easy to see that the result of surgery in S^3 by a daisy-chain yields S^3 and therefore this alternative move also does not change the homeomorphism type.

The fundamental fact [K],[F-R] about this calculus is that two surgery presentations $\mathcal{L}(s_1, \ldots, s_n)$ and $\mathcal{L}(s_1', \ldots, s_n')$ represent the same manifold iff we can pass from one to the other by successive applications of moves (a) and (b).

Remark:

Another fundamental fact about surgery is that it is reversible, i.e. if $N^3 = \mathcal{L}_M{}^3(\ )$, $\mathcal{L} = (L_1, \ldots, L_n)$, then we have the obvious dual presentation $M^3 = \mathcal{L}'_N{}^3(\ )$ where $\mathcal{L}' = (L_1', \ldots, L_n')$ and for each $i = 1, \ldots, n$, L_i' corresponds to L_i. Since the moves of the Kirby calculus make sense in any surgery presentation (not only surgeries in S^3) we must see how in this more general situation one of those moves reflects in the dual presentation.

move (a) - this move is in a sense self-dual: it is easy to see that the introduction of an unknotted curve labelled +1 (-1) in $\mathcal{L}$ corresponds to the introduction of an unknotted curve labeled -1 (+1) in $\mathcal{L}'$.

move (b) - if in $\mathcal{L}_M{}^3(\ )$, L_1 is band-sum to L_2 (as in figure 3-a), using a band B, then that corresponds in $\mathcal{L}'_N{}^3(\ )$ to band-sum L_2' to L_1' along a band B' obtained from B by two 90 degrees twists at the extremities and in opposite directions (we are of course identifying B with its image by the canonical homeomorphism between $M^3 - \mathcal{L}$ and $N^3 - \mathcal{L}'$ provided by the above definition of surgery). This is immediate considering the intermediate surgery $\mathcal{L}(\infty, \ldots)$ as in figure 3-c: the result of isotoping L_1 across the meridional disc of L_2', along the band B, is the same (up to isotopy) as isotoping L_2' across the meridional disc of L_1 along a band B' obtained from B as explained.

We will make extensive use of these dual moves later on in the case where $M^3 = S^3$ and N^3 is a homotopy 3-sphere and the dual move of (a') (introduction of a daisy-chain) which is not obvious will be explained when it is needed in the proof of Theorem 4.1.

Surgery, Pure Braids and Planar Open Book Decompositions

Closely associated with pob-decompositions of 3-manifolds are presentations by surgery on pure-braid links. A link $\mathcal{L} = (L_1, \ldots, L_n)$ in S^3 is said to be a pure braid if its components are once transverse to the pages (2-discs) of the standard fibration of the unknot (which is called the "axis" of the braid). We usually picture a pure braid as in figure 4-a. A good general reference for braids is [Bi].

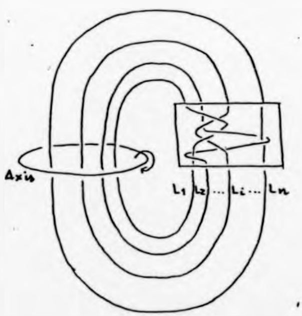


FIGURE 4-a

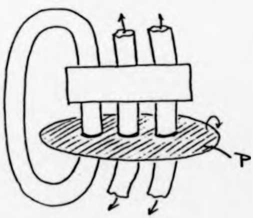


FIGURE 4-b

Consider a homeomorphism of the disc D^2, which restricts to the identity on the boundary and fixes n points $x_1, \ldots, x_n$ in $int(D^2)$. If we consider an isotopy H_t, $0 \le t \le 1$, fixed on ∂D^2 between the identity and h, and realize it in S^3 using the discs D_t, $0 \le t \le 1$, of the standard fibration of the unknot ($D_0 = D_1$), the tracks of $x_1, \ldots, x_n$ constitute a pure braid of n components. Any pure braid can be considered as arising in this way.

That any closed orientable 3-manifold can be obtained by surgery on a pure braid link is a direct consequence of Lickorish's proof [L1] (see [L2] or p.279 of [Rolf]) and it is easy to see how such a representation corresponds to a pob-decomposition of M^3: in figure 4-b the planar surface P with $\partial P = A_1 \cup l_1 \cup \ldots \cup l_n$ where A is the axis of the braid and the l_i's are longitudes in M^3 of the surgery tori, can be extended down to the cores of these tori thus determining a pob-decomposition.

Let $P \tilde{\times} I/h$ be a pob-decomposition of the closed orientable 3-manifold M^3. If we choose a complete system of arcs $c_1, \ldots, c_n$ for P (i.e. P cut open along the c_i's is a 2-cell) the homeomorphism $h: P \to P$ ($h|_{\partial P} = id$) is completely determined by the images $h(c_i)$ ($i = 1, \ldots, n$). We can then picture M by drawing in $P \times I$ (which is a handlebody of genus n) a complete system of arcs $c_1^-, \ldots, c_n^-$ at level $P \times 0$ and their images $c_1^+, \ldots, c_n^+$ at level $P \times 1$ (figure 5) to indicate the way the two levels are to be identified.

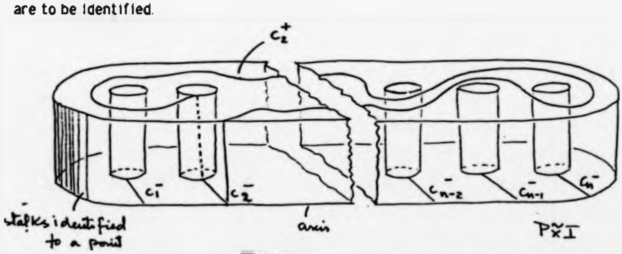


FIGURE 5

It is clear that the handlebody $H=P \times I$ with the set of curves $c_i=c_i^- \cup c_i^+$ $(i=1, \ldots ,n)$ determines a H-decomposition $M=H \cup H'$ for M where the curves c_i bound 2-discs in H' (this is essentially the decomposition $M=P \times [0,1/2] \cup P \times [1/2, 1]$) and that any n boundary components of P constitute a longitudinal system for both H and H'.

Following standard notation (see Chapter 1) we write $M^3= H(c_1,c_2, \ldots , c_n)$ to denote this H-decomposition and refer to $H(c_1, \ldots , c_n)$ as a H-diagram for M^3. This H-diagram is special in the sense of [B-P]: there exists an embedding of H in S^3 (the standard one) such that $H_1(c_1, \ldots , c_n)$, where $H_1= Cl(S^3-H)$, is a H-diagram for S^3.

If we consider such a standard embedding of $H=P \times I$ in S^3 (as in figure 5) then surgery in S^3 along the curves c_i $(i=1, \ldots ,n)$ (which constitute a pure braid) with framings given by parallel curves in ∂H will in fact yield M^3: by [R:lemma1] the result of that surgery is $H(c_1, \ldots , c_n) \cong H_1(c_1, \ldots , c_n)$ but as stated above $H_1(c_1, \ldots ,c_n)$ is clearly S^3

Conversely if we are given any surgery presentation $\mathcal{L}(s_1, \ldots ,s_n)$ with $\mathcal{L}=(L_1, \ldots , L_n)$ a pure braid link then there is a handlebody H, standardly embedded in S^3, such that the L_i's and respective surgery framings lie on ∂H, as before, defining a H-diagram for $\mathcal{L}(s_1, \ldots , s_n)$. To see this, consider the n solid tori $T_1, \ldots ,T_n$ where surgery is performed and the n disjoint annuli defining that surgery i.e. $\partial A_i=L_i \cup f_i$ where L_i is the core of T_i and $f_i=A_i \cap \partial T_i$ is the framing curve. If we take the union of the n solid tori $T_1, \ldots , T_n$ with $D \times I$ where D is a 2-disc bounded by the axis of the braid (as in figure 6) we obtain a handlebody H_1 and is clear there is an ambient isotopy of S^3 sending each of the annuli A_i into ∂H_1. The required handlebody H is now obtained by setting $H=Cl(S^3-H_1)$.

As illustrated in figure 6, we can see H is in fact a handlebody by <u>unbraiding</u> the whole picture (by an ambient isotopy of S^3) to standard position. Considering the braid, as explained above, as the track of n points in an isotopy H_t $(t \in I)$ of a self-homeomorphism of the 2-disc to the identity, this unbraiding is realized at the top plate $D \times \{1\}$ by the inverse isotopy H_{1-t} $(t \in I)$ (two stages are indicated by the dotted arrows).

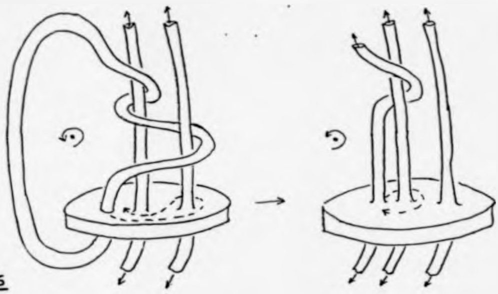


FIGURE 6

In figure 7 is shown the special H-diagram of S^3 that corresponds to the surgery.

FIGURE 7

Not only any 3-manifold can be represented as surgery in a pure braid in S^3, but there is also an effective procedure of passing from an arbitrary presentation to one in pure braid form by applying moves (a) and (b) of the Kirby calculus. That such a passage is possible is implicit in [B-P] where special presentations of 3-manifolds are constructed. However, in this work, we shall need a different approach. What we shall do is this: given an arbitrary link in S^3 then we can unknot it by changing crossings and each change of crossing can be realized by introducing a daisy-chain of 0-labelled components (figure 3-e). In this way we can obtain a surgery presentation where all the link components look perfectly round in some planar projection, if we use sufficiently long daisy-chains. These can then be deformed, preserving convexity, to encircle an axis for the resulting pure braid. The details of this are to be found in §3.

§2 . Unknotting in Homotopy-Spheres

This section is technical and contains well known facts which we include for completeness. Lemma 2.1 asserts that given a homotopy 3-sphere , H^3, S^3 can be obtained by surgery on a link in H^3 all components of which are unknotted (i.e. bound discs in H^3). This is a simple consequence of the fact that any curve in H^3 can be transformed into the unknot by a finite number of crossings with itself. After isotopy we can see those crossings as occurring in a trivial fashion in a set of disjoint balls, each of which can be realized by moves of the Kirby calculus, introducing only new unknotted curves.

Let $\mathcal{L}$ be a link in S^3 and $H^3=\mathcal{L}(s_1, \dots ,s_n)$ a homotopy 3-sphere. Using notation introduced in §1 we write $S^3=\mathcal{L}_H{}^3(s_1', \dots ,s_n')$ to refer to the dual representation of S^3 as surgery on the corresponding link $\mathcal{L}_H{}^3$ in H^3 (note that we do not necessarily have $s_i'=s_i$).

<u>Lemma 2.1</u> For any homotopy 3-sphere H^3 there is a link $\mathcal{L}_H{}^3 \subset H^3$ with all components unknotted such that $S^3=\mathcal{L}_H{}^3(\ )$.

We will introduce some material before proving the lemma. Let $c \subset H^3$ be a knot. By a "crossing" of c we mean the following way of locally changing c onto a new embedded curve c'. Let B^3 be a 3-cell intersecting c in two properly embedded arcs c_1, c_2 which are trivial in B^3, that is there are disjoint 2-discs D_1 and D_2, $c_i \subset \partial D_i$ and $D_i \cap (B^3 \cup c_1 \cup c_2) = \partial D_i$ $(i=1,2)$. The new curve c' is obtained by crossing c_1 and c_2 inside B^3, as shown in figure 8 and keeping c unaltered on the outside.

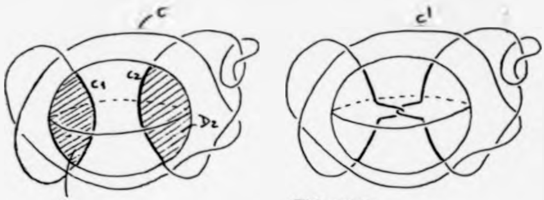


FIGURE 8

We have of course the same operation between any two components of a link in H^3.

Lemma 2.2. Let H^3 be a homotopy 3-sphere and $c \subset H^3$ a knot. Then c can be changed into the unknot by a finite number of crossings.

Proof: This is an application of standard techniques of PL topology (see [R-S]) and we will omit unnecessary details.

Let $h: D^2 \rightarrow H^3$ be a PL contraction of c, that is, $h(D^2)$ is a PL (singular) disc in H^3 and $h(\partial D^2) = c$. By general position we can assume that the singular set of h consists of transverse double curves, triple points and branch points. We can "pipe" the branch points over the edge of D^2 as follows. Choose a 2-disc C in D^2 meeting ∂D^2 in an arc and meeting the singular set in a number of small arcs (no two identified by h) one of which ends at a branch point (figure 9).

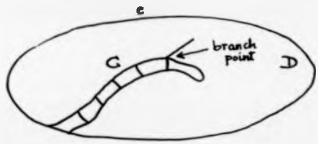


FIGURE 9

C is embedded by h and if we isotope c across C then at the end of the isotopy c bounds Cl(D-C) which has one fewer branch point Reversing this isotopy realizes the claimed "piping". A similar piping move can be used to remove all triple points. Thus we can assume that D^2 is in fact immersed by h with singular set consisting of disjoint double curves.

Now choose a triangulation T of D^2 such that each 2-simplex meets the double set (if at all) in a straight arc disjoint from the vertices.

Then each pair of 2-simplexes in $h(D^2)$ is either disjoint or meet as indicated in figure 10-a or 10-b.

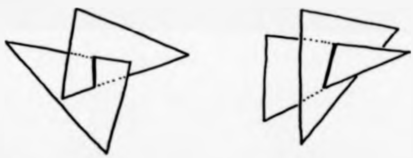


FIGURE 10 (a) (b)

Choose a 2-simplex $\Delta \in T$. We can construct an isotopy H_t ($t \in I$) between ∂D^2 and $\partial\Delta$ by successively collapsing T Δ across 2-simplexes in T-Δ (figure 11).

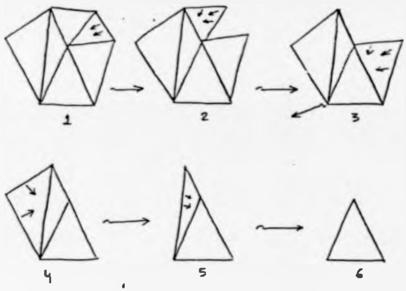


FIGURE 11

$H_t' = h \circ H_t$, ($t \in I$) is then a homotopy between c and the unknotted $h(\partial\Delta)$ (h is an embedding on simplexes of T), which is an embedding for all but a finite number of $t \in I$. The levels where H_t' is not an embedding occur when a linear collapse takes place in a 2-simplex that intersects the previous embedded image of c by H_t'. The change in the images of c before and after that linear collapse can be seen in an obvious way as the result of a finite number of crossings. The proof finishes by noting that having a finite number of simplexes in T, the total number of crossings is also finite.

Lemma 2.2 generalizes in the obvious way to an arbitrary link $\mathcal{L}_H{}^3$ in H^3 where $\mathcal{L}_H{}^3 = (L_1, \dots, L_n)$:

Lemma 2.3. $\mathcal{L}_H{}^3$ can be changed into the trivial link of n components (i.e. the one consisting of the boundaries of n disjoint embedded discs) by a finite number of crossings.

The method of proof is the same working simultaneously with n contractions $h_i : D^2 \dashrightarrow H^3$, $h_i(\partial D_i) = L_i$, $i = 1, \dots, n$.

Proof of lemma 2.1

Let $S^3 = \mathcal{L}_H{}^3(\ )$ where $\mathcal{L}_H{}^3 = (L_1, \ldots, L_n)$ is a link in the homotopy-sphere H^3. By Lemma 2.2 we can unknot and unlink all the components $L_1, \ldots, L_n$ by a finite number of crossings and assume that these crossings take place in disjoint balls $B_1, B_2, \ldots, B_m$. We only need to show that in the surgery picture each crossing can be realized locally (in each B_i) introducing only new unknotted components. For future convenience we use the variation (a') of move (a) of the Kirby calculus : instead of introducing an unknotted component labelled ±1 introduce the "Hopf link" with components labelled 0 (i.e. a daisy-chain of length 2) and separated from the rest of the link by a 2-sphere. The introduction of such a pair of unknotted components in each B_i and two applications of move (b) (Band-Sum move) as indicated in figure 12 produce the required crossings.

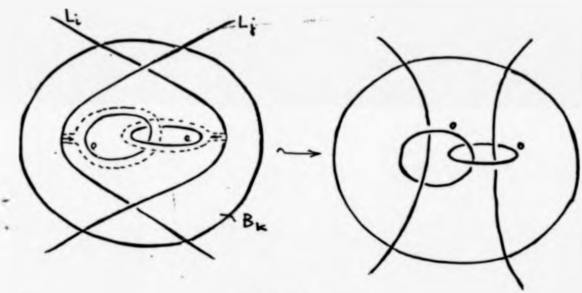


FIGURE 12

We observe that from the proof of lemma 2.1 it can be assumed that the unknotted components of the final link bound discs that either do not intersect or intersect in a set of clasps (figure 1) or ribbons (see figure 13-a bellow) and that there are no triple points (i.e. all clasps and ribbons are disjoint). As explained in figure 13-b we can "pipe" (as in the proof of lemma 2.2) a ribbon to the edge avoiding all the other singular set and thus change it into two new clasps.

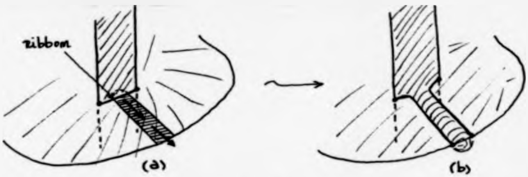


FIGURE 13

If we do this in turn for each for each ribbon we end up with a set of embedded discs which meet only in clasp form intersections. If we now consider a regular neighbourhood of the set of clasps, consisting of a finite number of disjoint 3-cells each one of which intersects the link in a trivial pair of arcs, and in each of those 3-cells we perform a crossing as before, we get the following extension of Lemma 2.1:

Lemma 2.4 For any homotopy 3-sphere H^3 there is a link $\mathcal{L}_{H^3} \subset H^3$ such that $S^3 = \mathcal{L}_{H^3}(\ )$ and such that the components of $\mathcal{L}_{H^3}$ bound discs in H^3 any two of which intersect at most once and, if so, in a clasp and moreover all the clasps are disjoint (i.e. there are no triple points).

§3 . Surgeries on Pure Braids

In this section we will show how, given a surgery of a 3-manifold $M^3 = \mathcal{L}(s_1, \dots, s_n)$ ($\mathcal{L}$ a link in S^3), it is possible to change it by moves of the Kirby calculus into another presentation $M^3 = \mathcal{L}'(s_1', \dots, s_n')$ such that $\mathcal{L}'$ is a pure braid.

The process we will develop is based on the following observation. In the local change of a surgery picture which corresponds to a crossing, as realized in the proof of lemma 2.1, we can substitute the Hopf-link by an arbitrarily long daisy chain.

Now given the link $\mathcal{L} = (L_1, \dots, L_n)$ in S^3 we can change it (Lemma 2.3) into the trivial link of n components $K = (K_1, \dots, K_n)$ by a number of crossings taking place in disjoint balls $B_1, \dots, B_m$ each a small regular neighbourhood of straight arc a_i $(i = 1, \dots, m)$ with end points in the link K. In a surgery picture with link $\mathcal{L}$ these crossings are realized by the insertion of m daisy-chains z_i along the arcs a_i (see figure 14).

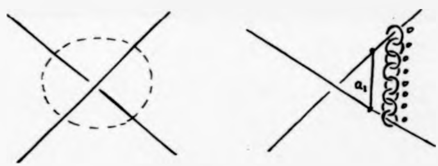


FIGURE 14

The effect in the new surgery picture of ambient isotoping K to a standard position (i.e. $K_1, \dots, K_n$ become disjoint circles in a vertical projection on some plane P) is to stretch the straight arcs a_i and the associated daisy-chain z_i ($i=1, ..., m$) to a position where they now might look "knotted" (imitating somehow the original link $\mathcal{L}$).

Suppose the daisy-chain z_i consists of t_i components $c_1{}^i$, ... , $c_{t_i}{}^i$. Let $D_1{}^i$, ... , $D_{t_i}{}^i$ be a system of 2-discs which support the daisy-chain z_i as shown in figure 15.

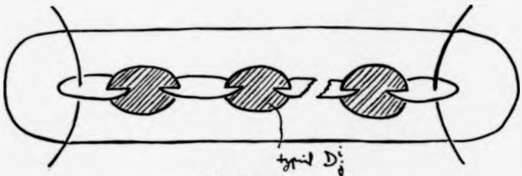


FIGURE 15

By increasing, if necessary, the number of components of each daisy-chain we can obviously assume that at the end of the ambient isotopy all the discs $D_r{}^i$ ($r=1, ... , t_i$) ($i=1, ... , m$) have planar projections (on some plane P) which are perfectly round (i.e. are closed balls for the euclidean metric on P, which can be assumed if wished to have the same radius). We will refer to daisy-chains of this form (relative to a fixed planar projection) as "perfect daisy-chains".

In figure 16 an example is given that illustrates what has just been said.

FIGURE 16

This process of globally unknotting a surgery presentation by the introduction of perfect daisy-chains gives almost immediately a pure braid (with a little extra control on each daisy-chain). For the details we proceed inductively, examining each perfect daisy-chain at a time.

We start by picturing the link $K=(K_1, \ldots, K_n)$ as a trivial pure braid on n strings as shown in figure 17.

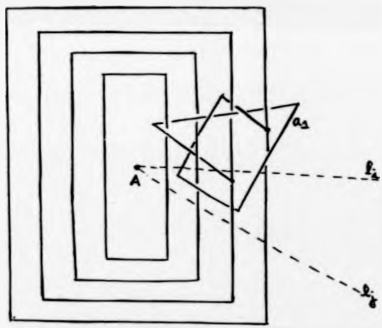


FIGURE 17

The axis of the braid is represented in this picture by the vertical line that projects into the centre dot A, and the fibering discs by vertical half planes that project into half lines starting at A.

Consider one of the perfect daisy-chains, z_1 say, which was built along the arc a_1. The idea now is to ambient isotope all the components of z_1, keeping the rest of the link (K and the other daisy-chains) fixed, so that at the end each one of them has a projection that bounds a convex region with the point A in the interior. This implies that they all wrap monotonically once around the axis A thus forming with $K_1, \ldots, K_n$ (which were kept unaltered) a pure braid. In order to visualize the required deformation of z_1, we consider in the planar projection sectors as indicated by the dotted lines l_i in figure 17 and work separately in each one of them. The sectors are chosen in the following way. We triangulate the braid K and the arc a_1 in such a way that all the vertices and under and over crossings project into distinct points not two of which lie in the same line through A (we are counting the end points of a_1 as crossings). We then draw the lines l_i so that each sector contains exactly either one vertice or one crossing. We can assume that at each point of intersection of a_1 with the l_i's the perfect daisy-chain z_1 is of the form shown in figure 18 with the component that intersects l_i crossing neighbouring components in a non alternating way.

FIGURE 18

We note that in certain situations it might not be possible to get this local picture by simple isotoping components of z_1. However it is clear that we can always do it by introducing some new components and elongating z_1, if necessary.

Let us now consider one of the sectors determined by the lines l_i. Figure 19 shows the type of picture that might occur.

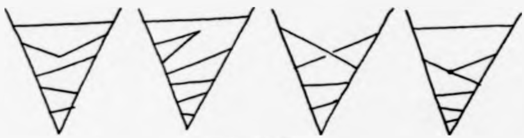


FIGURE 19

Starting with a subarc s of a_1 closest to A we can deform, as indicated in figure 20-a, the corresponding position of z_1 over everything that lies between s and A, keeping fixed the two components at the extremities. Moving outward we will similarly deform all the other subchains of z_1 that lie inside the sector. In the case of a subarc containing a vertex or an end point of a_1 we can always assume, to simplify, that the local pictures are as shown in figure 20-b. It should be clear now how to deform z_1 in the case of a crossing.

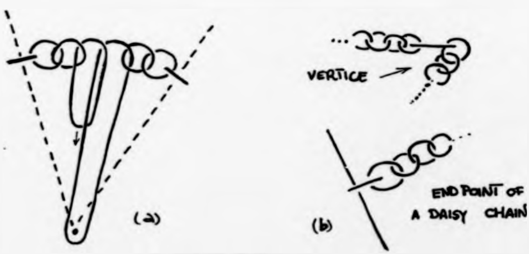


FIGURE 20

Proceeding in this way inside all the sectors we are left only with the components of z_1 that intersect the lines l_i. By the assumption made before on these components, we can now clearly deform them, one by one, keeping everything else fixed and passing under the rest of the link.

We conclude then that the union of the pure braid $K=(K_1, \ldots, K_n)$ with the perfect daisy-chain z_1 can be considered to be itself a pure braid We continue inductively by picturing this new braid in the form of picture 17 (but now no longer a trivial braid) and repeating the process with a perfect daisy-chain associated with the arc a_2. We only need to notice that in the first step of this induction the fact that K is trivial is not used in any way. This completes the process.

Observation : we should notice that in the above process of unknotting and unlinking a given link $\mathcal{L}$ in S^3 by the introduction of sufficiently long daisy-chains to obtain a pure braid, we can always assume that in the final link each component bounds a flat disc for the euclidean metric and therefore they all have convex projections on any plane (up to arbitrarily small isotopies).

§4 . A Characterization of Homotopy 3-Spheres

We are now in position to prove the main result of this chapter.

Theorem 4.1 A closed 3-manifold H^3 is a homotopy 3-sphere if and only if there exists a pob-decomposition $H^3 = P \times I/_h$ such that each component of boundary of P, $\partial P = b_0 \cup b_1 \cup \ldots \cup b_n$ bounds an embedded disc in H^3. furthermore it can be assumed that any two of these discs either do not intersect or intersect along clasps and that all the clasps are disjoint (i.e. there are no triple points).

Corollary 4.1.1 (Haken) - A closed 3-manifold H^3 is a homotopy 3-sphere iff there is a H-decomposition $H^3 = H \cup_h H'$ with a longitudinal system $(c_1, \ldots, c_n)$ for H such that each c_i bounds an embedded disc D_i in H^3. Moreover it can be assumed that any two of the D_i meet, if at all, in a set of clasps and there are no triple points.

Proof of corollary: as shown in S1 any set of n components of ∂P is a longitudinal system for the H-decomposition associated to the planar open book decomposition: $H^3 = H \cup_h H'$ with $H = P \times [0, 1/2]$ and $H' = P \times [1/2, 1]$.

To put the theorem (and the corollary) in perspective we will show, before giving its proof, that if the 2-discs are disjoint then H^3 is in fact S^3. As stated in the beginning of the chapter this gives a formulation of the Poincare conjecture in terms of longitudinal systems.

Theorem 4.2 Let $H^3 = H \cup_h H'$ be a H-decomposition such that H has a system of longitudes bounding disjoint embedded discs in H^3. Then H^3 is homeomorphic to S^3.

Proof : Let $(c_1, \ldots, c_n)$ be a system of longitudes for H bounding n disjoint embedded discs $D_1, \ldots, D_n$. for each $i = 1, \ldots, n$ let $T_i \subset H$ be the solid torus associated, by definition, with c_i (see figure 1). Without loss of generality we can assume that for each i, $D_i \cap T_i = c_i$ (choosing if necessary a different longitude in each T_i). The problem, as suggested by figure 21 is that the discs might still intersect H in the complement of those tori (which is a 3-cell called the "base": see figure 1) and so $c_1, \ldots, c_n$ do not give rise to a system of cancelling handles.

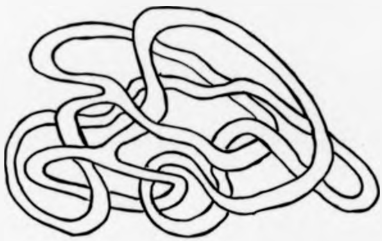


FIGURE 21

To finish the proof we will make use of a relation between H-decompositions and surgery. Suppose we do surgery on the link $\mathcal{L}_H{}^3=(c_1, \ldots, c_n)$ with framings given by parallel curves on ∂H. Because each c_i bounds a disc in the complement of T_i these surgeries are 0-surgeries and so we have $\mathcal{L}_H{}^3(0, \ldots, 0) = H^3 \# M_1 \# \ldots \# M_n$ with $M_i = S^2 \times S^1$ for each i. On the other hand [R: lemma 1] we have $\mathcal{L}_H{}^3(0, \ldots, 0) = H(c_1, \ldots, c_n) \# H'(c_1, \ldots, c_n)$ and since $H(c_1, \ldots, c_n)$ is obviously S^3 we conclude that $H^3 \# M_1 \# \ldots \# M_n$ has a H-decomposition of genus n. Therefore by the additivity of Heegaard genus [H1] we conclude that H^3 has genus 0 and so is S^3

Proof of Theorem 4.1

We start with a surgery presentation $S^3 = \mathcal{L}_H{}^3(\ )$ where $\mathcal{L}_H{}^3$ is a link in H^3 with all components unknotted (lemmas 2.1 and 2.4). We have the corresponding dual picture $H^3 = \mathcal{L}(\ )$ where $\mathcal{L}=(L_1, \ldots, L_n)$ is a link in S^3. As shown in §3 we can change this presentation into another one $H^3 = B(\ )$ where B is a pure braid, by executing m crossings of the link $\mathcal{L}$, as long as in doing so we use perfect daisy-chains. We will now see that in the corresponding picture $S^3 = B_H{}^3(\ )$, $B_H{}^3$ is still a link with all components unknotted. Consider the effect of a single crossing between L_i and L_k say (it is possible that i=k). We assume first for simplicity that the crossing is realized through the introduction of a daisy-chain of length two

Let $\mathcal{L}'$ be the resulting link with L_i' and L_k' the components corresponding to L_i and L_k respectively. We refer to figure 22 where the surgery tori for the pictures before and after the crossing are shown.

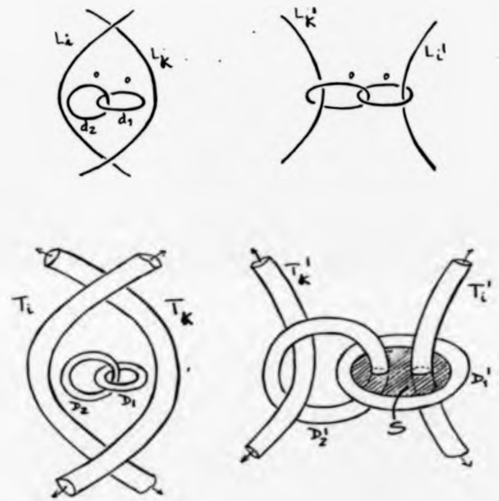


FIGURE 22

As established in the proof of lemma 2.1 (figure12) this crossing is realized by band summing L_i to d_1 and L_k to d_2. Because these band sums are, as explained in §1(figure3) the end results of isotopies there is a self-homeomorphism h : $\mathcal{L}(\ )$ --> $\mathcal{L}(\ )$ sending T_i to T_i' and T_k to T_k' and keeping all the other surgery tori T_r , r≠ i, k fixed (but not D_1 to D_1' or D_2 to D_2' : h(D_1) and h(D_2) are not represented in the picture). Therefore in the new picture both T_i' and T_k' are unknotted tori in H^3. The shaded surface S represents an annulus in H^3 which intersects the torus D_2' once in a meridional disc (whose boundary corresponds in the picture to the 0-framing) and that cobounds longitudes (in H^3) of D_1' and T_i'. D_1' and T_i' are therefore parallel. Analogously D_2' is parallel to L_k', across D_1', and so both D_1' and D_2' are unknotted in H^3. Their cores (which are unknotted curves) are the two new curves added to the original link $\mathcal{L}_{H^3}$ (up to the homeomorphism h) by this single crossing

We observe now how the use of a perfect daisy-chain of arbitrary length to realize a crossing in $\mathcal{L}(\ )$ reflects in the surgery picture back in H^3. Suppose that in figure 22 we substitute D_1' and D_2' by a long daisy-chain D_1', D_2', ... , D_{2m}' (figure 23).

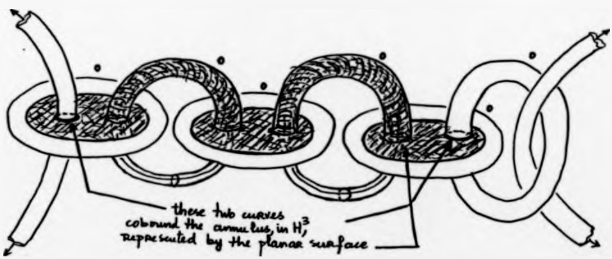


FIGURE 23

As before T_i' and T_k' are unknotted tori in H^3 and by considering planar surfaces of the type shown in figure 23 we see that each D_r' is parallel to either L_i' or L_k' according to whether r is even or odd respectively and therefore the 2m new components added to the link $\mathcal{L}_{H^3}$ in H^3 are all unknotted.

Although to have a picture like in figure 23 will be useful later on we need now to know what type of intersections may be created by the new discs (for $\mathcal{L}_{H^3}$ we had embedded discs intersecting only in a disjoint set of clasps - Lemma 2.4). For that it is perhaps better to see the previous changes using the remark in §1 about the way the Kirby moves reflect in the dual surgeries. We refer back to figure 22. Let K_i and K_k be the components of $\mathcal{L}_{H^3}$ that correspond to L_i and L_k respectively. The introduction of the Hopf-link with 0-framings $d_1 \cup d_2$ separated from $\mathcal{L}$ by a 2-sphere corresponds in an obvious way to the introduction in H^3 of a Hopf-link with 0-framings and also separated from the rest of the link by a 2-sphere : just consider in figure 22 an annulus (analogous to the shaded surface S) cobounding the 0-framing of D_2 and a meridional curve of D_1 ; that represents a disc in H^3 bounded by D_1 and intersecting D_2 once in a meridional disc and we have the equivalent picture for a disc in H^3 bounded by D_2 and intersecting once D_1. Let d_1', d_2' correspond to d_1 and d_2 respectively. By the remark in §1 band sum L_i to d_1 corresponds in H^3 to band sum d_1' to K_i and analogously for L_k, d_2, d_2' and K_k. The new link is therefore obtained by introducing two framing curves of K_i and K_k say f_i , f_k (which can be easily seen to bound discs still forming with each other and with all the previous ones a disjoint set of clasp intersections) and then band sum them to d_1' and d_2' respectively (see figure 24). This can perhaps be better visualized by deforming these framing curves (by an ambient isotopy of H^3 fixed on the rest of the link) along two tongues which are then clasped inside a small ball.

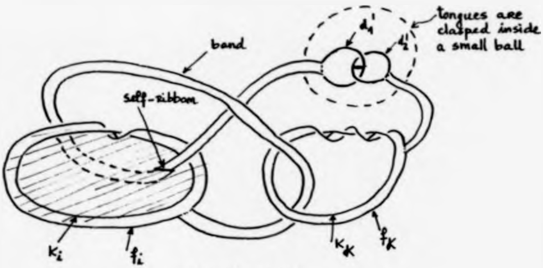


FIGURE 24

In the general case, the introduction in $\mathcal{L}(\ )$ of a daisy-chain with 2m components, separated from the rest of the link by a 2-sphere corresponds to introduction back in H^3 of 2m components all with 0-framings, separated from the rest of the link by a 2-sphere and forming a link pattern as shown in figure 25, which will be called an ear-ring

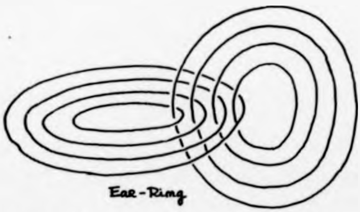


FIGURE 25

That this is so can be seen, as in the previous case of a daisy-chain of length two, by looking back at figure 23 and considering again planar surfaces of the type shown, one for each component of the daisy-chain (note that a daisy-chain with two components is also an ear-ring with two components). Analogously to the previous case the new link in H^3 after the crossing is obtained now by introducing m copies of the framing curve f_i, m copies of the framing curve f_k and band sum each one of them to a correspondent component of the ear-ring.

It is now clear that if we allow the new discs to self-intersect we can assume the singular set consists of a collection of disjoint clasps and ribbons. Note that, as suggested in figure 24, it might be necessary to consider self-intersections (which are of ribbon form) if we want to avoid closed curves of double points. We do that and will start working with these self-intersecting discs in the second part of the proof. The first part of the proof now follows by induction on the number of crossings needed to pass from $\mathfrak{A}$( ) to B( ).

We next transform each ribbon that might occur in the picture into two new clasps by piping to the edge as explained in Lemma 2.4 (figure 13-b). The new set of discs thus obtained has a singular set consisting of disjoint clasps some of which are self-intersections. We end the proof by the sequence of operations described in figure 26.

The first operation is a crossing of the link $B_H{}^3$ along one of those clasps by the introduction of a daisy-chain with two components. This creates two new clasps and corresponds in S^3 to adding to B two framing curves and perform a crossing between them.

The second operation consists of undoing this crossing by the introduction of a perfect daisy-chain. Since B with the two framing curves is still a pure braid we know by §3 that we can choose the perfect daisy-chain in such a way that the final link is still a pure braid. As explained above this operation reflects back in H^3 by the introduction of an ear-ring in the manner indicated in figure 26. Only new clasps type intersections are produced in this process and if we repeat it for all the other clasps in turn we get rid of all self-intersections and end up with a picture where any two discs intersect at most in a single clasp

At this point the planar open book decomposition of $H^3 = P \times I / h$ associated with the pure braid surgery in S^3 is such that all the boundary components of P are unknotted and bound discs as required with one possible exception, the curve that corresponds to the axis of the braid This is of course enough to guarantee Corollary-4.1.1

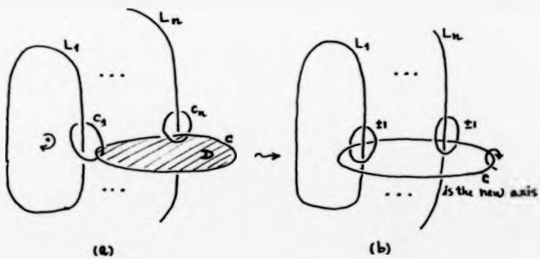


FIGURE 27

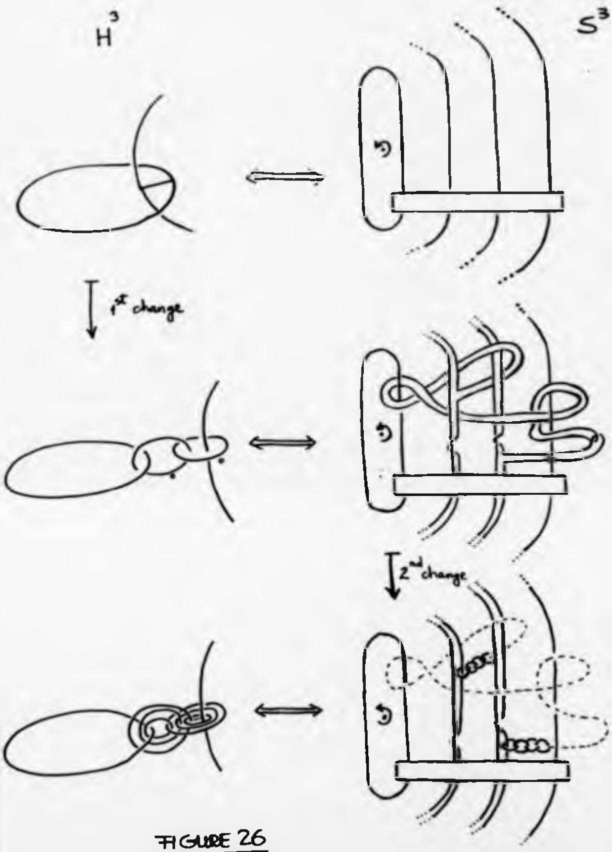


FIGURE 26

If we want the general situation we can now consider a disc D in the S^3 picture with $\partial D = c$ linked to each component L_i of the braid by a curve c_i as indicated in figure 27-a. Because each c_i is parallel in H^3 to a component of the link we can assume the c_i's bound discs D_i in H^3 which form with the previous discs a set intersecting in a disjoint set of clasps. We can also assume the disc D intersects all the other discs in clasps or ribbons and more generally by piping those ribbons to the edge that in fact we only have clasps. We end by performing ± 1 surgery on each c_i (move (c) of the Kirby calculus) thus crossing c with each component L_i as indicated in figure 27-b. c becomes in this the axis of the braid formed by the L_i's union with the c_i's and the proof is finished.

§5 . Final Observations and Definitions

5.1 . The Post Construction - from now on it will be more convenient to consider the process of associating a H-decomposition to a surgery on a pure braid in a slightly different way. As shown in figure 28, given a surgery picture $M^3 = \mathcal{L}(\ )$ with $\mathcal{L}$ a pure braid, we can consider a 3-cell at the bottom of the picture and disjoint from $\mathcal{L}$ and joint it to each surgery torus through the introduction of a "vertical" post. As in figure 6 we can unbraid the whole picture to a standard position to see this defines a H-decomposition for M^3.

5.2 . Pure Plats - more generally we can obviously do same to any surgery $M^3 = \mathcal{L}(\ )$ where $\mathcal{L}$ is simply a pure plat. A pure plat is a link in S^3 such that, in relation to some height function, each component has exactly one relative maximum and one relative minimum (figure 29).

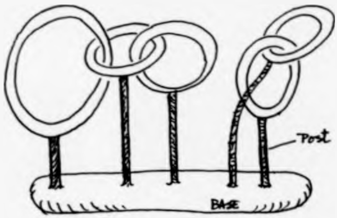


FIGURE 28

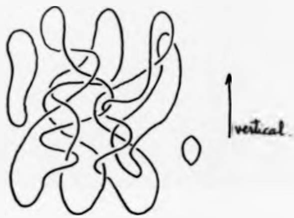


FIGURE 29

The existence of only one relative maximum and one relative minimum for each component is all that is necessary for the unbraiding to be possible and so we obtain in exactly the same way a H-decomposition for M^3.

5.3 General Post Construction - a variation of the post construction just described and that will be used later on consists of joining each component of the pure plat $\mathcal{L}$, from its minimum downward to the base or to some other component by a monotone (relative to the height function defining the plat) post in a cactus like formation as shown in figure 30.

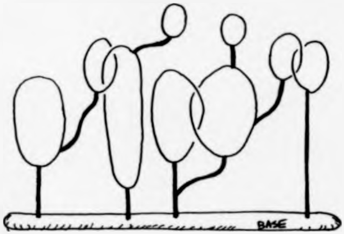


FIGURE 30

This picture can still be unbraided leading to a H-decomposition and from now on by "post-construction" we mean this more general construction.

In the work presented in the remaining chapters to use pure braids never seemed to bring any special advantage over the more general class of pure plats. Therefore we will work mainly with pure-plat surgeries for these are simpler and have obvious technical advantages - the pictures in Chapter 4 difficult as they are would become impossible to draw except for a professional designer if they were changed into pure braid form. But it is understood that at any moment we can always find a pure-braid picture with the same essential properties and that for this the post construction can always be done to yield the H-decomposition that corresponds to the natural pob-decomposition associated with the pure braid.

5.4. Definitions - a H-decomposition of a homotopy 3-sphere $H^3 = H \cup_h H'$ with a longitudinal system and a system of embedded discs as in Corollary 4.1.1 and associated by some post-construction to a surgery $H^3 = \mathcal{L}(\ )$ where $\mathcal{L}$ is a pure-plat (braid) in S^3 will be called a primitive H-decomposition of H^3, and $\mathcal{L}(\ )$ a primitive surgery presentation.

The surgery tori of $\mathcal{L}(\ )$ are the tori associated to the longitudinal system, the 3-cell union with the posts is the base and each disc of the system is seen in the

$\mathcal{L}(\ )$ picture as a planar surface with boundary curves lying on the boundaries of the surgery tori.

Note that a primitive H-decomposition may have base intersections (see Theorem 4.2) - they correspond in the associated primitive surgery presentation to intersections of the posts with those planar surfaces. We will see in the next chapter (Theorem 6.1) that it always possible to construct primitive H-decompositions without base intersections for any homotopy 3-sphere.

CONVENTION : In a primitive Heegaard decomposition with associated primitive surgery $H^3=\mathcal{L}(\ )$ we will always assume that the curves of the longitudinal system are precisely the 0-framings of $\mathcal{L}_{H^3}$, as we did in the proof of theorem 4.2.

We finally note that having a primitive surgery $H^3=\mathcal{L}(\ )$ we can easily built from this another one $H^3=\mathcal{L}'(\ )$ such that in $\mathcal{L}'_{H^3}(\ )$ all the surgery coefficients are either 0 or ± 1. All we need to do is to change all the framings in $\mathcal{L}_{H^3}(\ )$ to 0, ± 1 by the insertion of unknotted ± 1 components along meridional curves (move (c) of the Kirby calculus). This corresponds, as usual, back in S^3 to adding to $\mathcal{L}$ a certain number of framing curves and keeps us in the class of pure-plats (braids).

It should be clear that in this situation (all surgery coefficients of $\mathcal{L}'_{H^3}(\ )$ being 0 or ± 1) and with the previous convention, the planar surfaces in the S^3 picture, $\mathcal{L}'(\ )$, have boundaries that are either longitudes or meridional curves in the boundaries of the surgery tori (meridional curves corresponding exactly to the 0-framings, so in particular if we make all coefficients ± 1 in $\mathcal{L}'_{H^3}(\ )$ then the boundaries of the planar surfaces will all be longitudes).

We end this section with two questions of a general nature:

First - Give a simple process of effectively generating all primitive H-decompositions (this is related to the contents of Chapter 5)

Second - Characterize the pob-decompositions of S^3 with all but one of the binding components unknotted.

Chapter 3 - Characterizations of S^3

In this chapter we shall establish certain characterizations of S^3 in terms of primitive Heegaard decompositions (definition in §5 , see also the convention made there). In §6 we give the basic characterization (Theorem 6.1) which involves two conditions on the set of clasps and the assumption that there are no base intersections. In the subsequent sections we shall show how those hypothesis may be weakened while still giving a characterization of S^3

§6 . Trivial Ordered Clasps

Theorem 6.1 . Suppose $H^3 = H \cup_h H'$ is a primitive H-decomposition with $\{c_1, c_2, \ldots, c_n\}$ a longitudinal system for H and associated system of embedded discs $\{D_1, D_2, \ldots, D_n\}$, $c_i = \partial D_i$ for $i = 1, 2, \ldots, n$ such that:

(1) - There are no base intersections (i.e. for each i , $\mathrm{Int}D_i \cap H$ consists precisely of the meridional discs near clasps illustrated in figure 31)

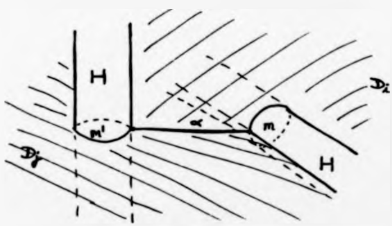


FIGURE 31

(figure 31 represents a typical neighbourhood of a clasp α . $\alpha \subset D_i \cap D_k$; m and m' are meridional discs , D_i meets H near α in m and the vertical line passing through the left-hand point of α and D_k meets H near α in m' and the horizontal line through the right-hand point of α)

(2) - The set of clasps $S=\{\alpha_1, \alpha_2, \dots, \alpha_t\}$ forms a trivial link in H', i.e. there are discs $\varphi_1, \varphi_2, \dots \varphi_t \subset H'$ such that $\alpha_i \subset \partial\varphi_i$, $(\partial\varphi_i - \alpha_i) \subset \partial H'$ $(=\partial H)$ (see figure 32)

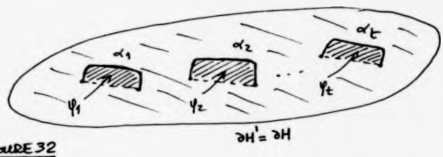


FIGURE 32

(3) - <u>The ordering condition</u>

There is a partial order on S such that for each disc D_i there is a meridional disc transverse to $c_i=\partial D_i$ (called the <u>cut-disc</u>) such that in each of the two subsets of S in D_i trapped between the base 3-ball and the cut-disc the partial order coincides with the order increasing away from base towards cut-disc (see figure 33)

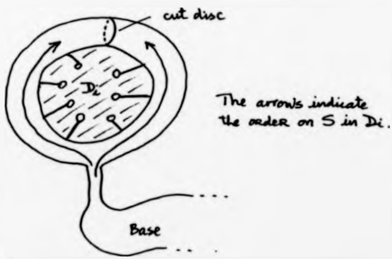


FIGURE 33

THEN H^3 is S^3

Proof. Consider $H_1 = H \cup$ (regular neighbourhoods of $\alpha_1, \alpha_2, \ldots, \alpha_t$) (see figure 31) and $H_1' = Cl(H^3 - H_1)$. H_1 is clearly a handlebody (of genus n+t) and that H_1' is a handlebody follows from the triviality of S as a link in H' (hypothesis (2)).

For each disc D_i consider the disc D_i' which is obtained from D_i by excising corresponding regular neighbourhoods of all the clasps in D_i together with associated meridional discs (figure 34).

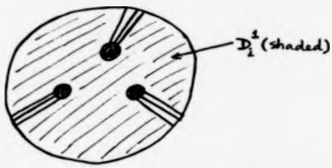


FIGURE 34

Then $D_1', D_2', \ldots, D_n'$ is a set of disjoint embedded discs in H_1'. To simplify notation we drop sub (super) scripts. We have $H^3 = H \cup_h H'$ (a H-decomposition of genus n+t) and n disjoint embedded discs in H' namely $D_1, \ldots, D_n$, whose boundaries are once transverse to the original meridional discs in H. The new meridional discs are illustrated in figure 35.

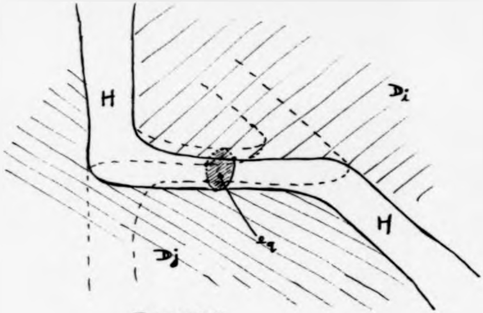


FIGURE 35

e_q is a new meridional disc for H which cuts once across the new 1-handle which is the regular neighbourhood of a typical clasp α_q. Note that ∂D_i , ∂D_j each meet ∂e_q in a "cancelling pair" of crossings of the type xx^{-1} or $x^{-1}x$. If we excise from φ_1, φ_2, . . . , φ_t corresponding regular neighbourhoods of $\alpha_1, \alpha_2, \ldots, \alpha_t$ (without changing the notation) then for each q=1, 2, ... , t, φ_q (not illustrated in figure 35) is a properly embedded disc in H' whose boundary is once transverse to ∂e_q and which misses ∂e_r for r≠q. Note that although the discs φ_q , q=1,2, ... ,t , are pairwise disjoint each one of them may intersect the discs D_i.

We now cancel the n original 1-handles of H using the disjoint embedded discs $D_1, D_2, \ldots, D_n$. The cancellation of a 1-handle h using such a "complementary" 2-disc D can be understood as cutting across a (original) meridional disc of h and collapsing D from the resulting free edge (figure 36)

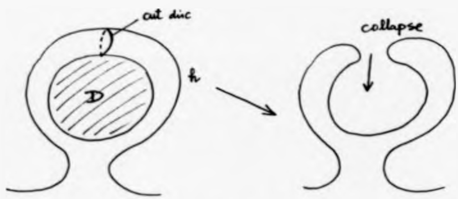


FIGURE 36

In the case of the original 1-handles of H, it matters which meridional disc is used, since after the addition of the new 1-handles (with new meridional discs e_q as in figure 35) the original meridional discs are not all isotopic. We choose to cut at the "cut-discs" given by hypothesis (3). We modify the discs $\varphi_1, \varphi_2, \ldots, \varphi_t$ after these cancellations by "dragging down" across the D_i any arcs of $\partial\varphi_j$ which are transverse to the cut-disc of D_i (figure 37).

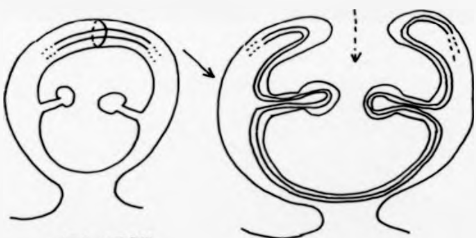


FIGURE 37

After these moves the $\{\varphi_j\}$ are no longer embedded discs (or disjoint) since an intersection of φ_j with D_i may result in double curves in the φ_j as suggested in figure 38.

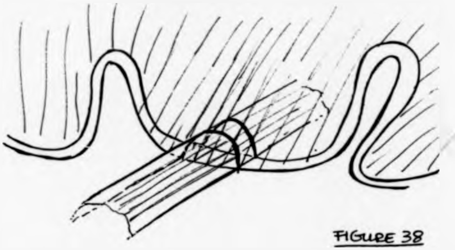


FIGURE 38

Moreover $\partial\varphi_j$ is no longer once transverse to ∂e_j since the new (immersed and no longer embedded) boundary curves follow along the boundaries of the D_i and hence there may be "cancelling pairs" of crossings introduced (figure 39) copying the cancelling pairs of crossings of ∂D_i with ∂e_j . However the original single point of intersection of $\partial\varphi_j$ with ∂e_j is still unchanged so that $\partial\varphi_j$ is transverse to ∂e_j apart from such cancelling pairs.

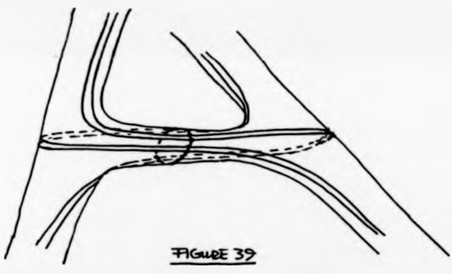


FIGURE 39

At this point we have a H-decomposition of H^3 of genus t − the t remaining 1-handles can be regarded as the thickened clasps $\{\alpha_j$, j=1, ..., t} with meridional discs $\{e_j$, j=1, ... ,t}. We now use the ordering hypothesis (3). Choose a maximal α, say $\alpha_q \subset D_i \cap D_j$. By maximality the cut discs of D_i and D_j are adjacent to α_q as illustrated in figure 40 But now we can regularly homotope the $\{\partial\varphi_j\}$ in ∂H near α_q so as to remove all the new cancelling pairs of intersections with ∂e_q . These homotopies take place through the cuts as indicated by the arrows in figure 40

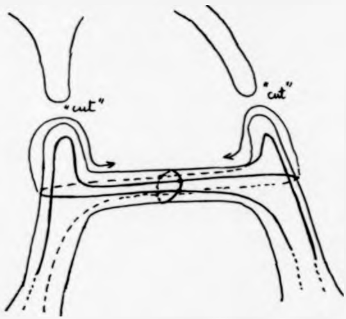


FIGURE 40

After these homotopies the $\{\alpha_j\}$ are immersed discs in H' such that $\{\partial\varphi_j\} \cap \partial e = \partial\varphi_j \cap \partial e_q$ = one point. By the Loop Theorem φ_q can be replaced by an embedded disc $\overline{\varphi_q} \subset H'$ whose boundary is also once transverse to ∂e_q

The precise statement of the Loop Theorem that we need is the one given in Hempel's book [He, p. 47]. The normal subgroup to choose is the subgroup comprising loops whose algebraic intersection number with ∂e_q is zero. The embedded disc $\overline{\varphi_q}$ given by the Loop Theorem must therefore meet e_q at least once but since $\overline{\varphi_q} \subset \varphi_q \cup$ (nbd. of singular set of φ_q) , it can only meet e_q once transversely, as required.

We can now cancel the 1-handle corresponding to α_q using $\overline{\varphi_q}$. Since no other $\partial\varphi_i$ crosses ∂e_q (after the regular homotopies) this cancellation leaves the other φ_i's unaltered – effectively the neighbourhood of α_q and φ_q vanish from the picture.

we can now repeat the process using a new maximal α_i and after t such steps H^3 has a H-decomposition of genus 0 and the theorem is proved.

S7 . Base Intersections

We will explain how hypothesis (1) of Theorem 6.1 can be dropped, allowing base intersections. Obviously the existence of any base intersections will prevent the cancelling of the discs D_i in Theorem 6.1 to be carried on.

We can always assume the base intersections form a system of disjoint and properly embedded discs in the base. It will be helpful in what follows to think of the base as the regular neighbourhood of a tree T in such a way that the base intersections are discs crossing T transversely (like the disc labelled q in figure 41) [it is easy to see that any set of disjoint properly embedded discs in the base can be seen as discs all transverse to some such tree].

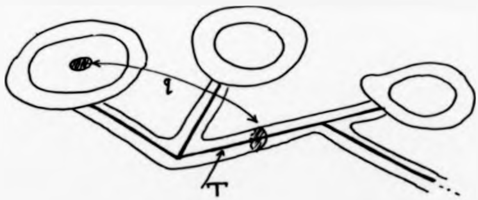


FIGURE 41

In short, we can and will assume that the base intersections are precisely the general position intersections with the handlebody H of an arbitrary spanning a longitudinal curve in addition to those entailed by intersections with other discs (as in figure 31). In the associated primitive surgery presentation $H^3 = \mathcal{C}(\ )$ the base intersections are just the transverse intersections of the posts with the planar surfaces

Definition . suppose we have a base intersection q in one of the discs D say. A tag for q is an arc in D from ∂q to ∂D , disjoint from all clasps in D and also from other intersections of D with H (i.e. base and meridional discs), (figure 42).

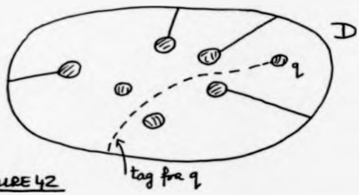


FIGURE 42

A system of tags $\{t_1, t_2, \ldots, t_s\}$ for the base intersections is a disjoint set of tags one for each base intersection

We shall now assume that we have such a system of tags such that hypothesis (2) of Theorem 6.1 can be strengthened as follows:

Hypothesis (2a) - the set of arcs $\{\alpha_1, \alpha_2, \ldots, \alpha_t, t_1, t_2, \ldots, t_s\}$ forms a trivial link in H'.

The proof of Theorem 6.1 is now adapted to this situation in a natural way by treating the tags in exactly the same way the clasps were treated: we add to H regular neighbourhoods of both clasps and tags and we remove from the D_i

corresponding regular neighbourhoods of clasps and tags with corresponding meridional discs and base intersections.

The difference now is that at the neighbourhood of a tag t_k (figure 43) only one of the discs D_i crosses the new meridional disc f_k (in fact the disc which originally contained t_k) and this crossing consists of a single cancelling pair of intersections.

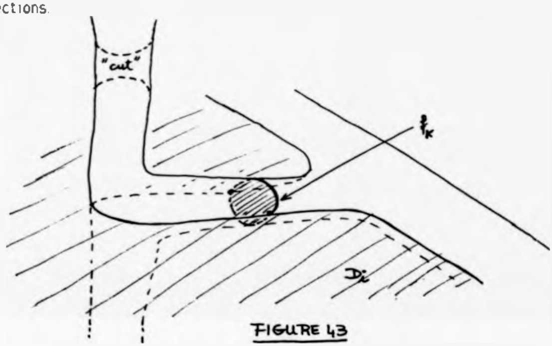


FIGURE 43

Thus after the original 1-handles are cancelled using the $\{D_i\}$ the extra intersections of the immersed discs $\{\varphi_j\}$ with ∂f_k all follow this cancelling pair and therefore in order to remove these intersections by regular homotopies as in the proof of Theorem 6.1, it is only necessary to have <u>one</u> adjacent "cut" (also illustrated dashed in figure 43).

This "cut" is across the tree and parallel to the base intersection tagged by t_k, and therefore we need that all the new 1-handles corresponding to clasps and tags to one side of this base intersection have already been cancelled but we do not need any special order for t_k inside the clasps and tags in D_i - we merely need that

t_k shall have been cancelled before the next clasp below (i.e. towards the base) is cancelled, (figure 44)

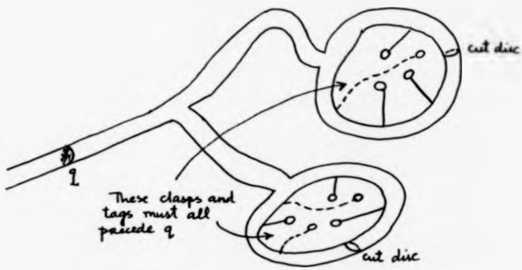


FIGURE 44

We have then proved the following

Theorem 7.1 : Let $H^3 = H \cup_h H'$ be a primitive H-decomposition such that

(i) – There is a system of tags for all the base intersections satisfying hypothesis (2a) above.

(ii) – The ordering condition:

There are cut-discs as in Theorem 6.1 and a base point $* \in T$ and a partial order for $S' = \{t_1, t_2, \ldots, t_k, \alpha_1, \alpha_2, \ldots, \alpha_l\}$ such that, along any arc starting at $*$, running along T to a copy of $S^1 \times D^2$ and round one side to the cut-disc the partial order restricts to the total order on base discs and clasps increasing from $*$ to cut-disc (base intersections are identified with their tags as far as order is concerned) and such that for each tag t lying further from * than a base disc or clasp c we have t > c (see figure 45).

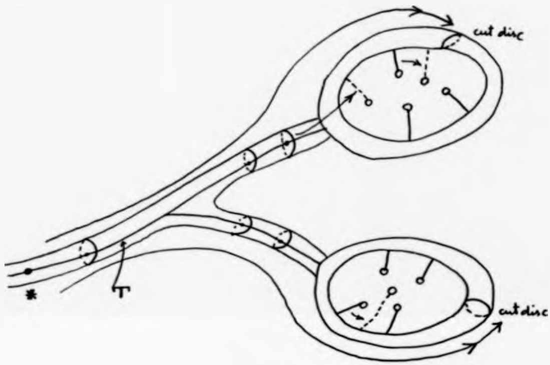


FIGURE 45

THEN H^3 is S^3.

§8 . Generalized Base-Intersections and Ribbons

Suppose we have a H-decomposition $H^3 = H \cup_h H'$, associated by some post construction to a surgery presentation $H^3 = \mathcal{L}(\ )$ ($\mathcal{L}$ a pure plat), with a system of longitudes for H bounding embedded discs in H^3 (with the convention of §5) that intersect in a disjoint set of <u>clasps and ribbons</u>. By changing any ribbon into two new clasps (pipping to the edge) we can obtain at once a primitive H-decomposition, but in Chapter 4 it will be convenient to consider those ribbon intersections - we shall call such a H-decomposition and corresponding surgery presentation <u>pre-primitive</u>.

In this section we extend Theorem 6.1 to this more general situation.

<u>Generalized Base Intersections</u>

In §7 we essentially regarded our standard handlebody H as the regular neighbourhood of a 1-complex τ which consists of the tree T with n circles attached to twigs (one for each copy of $S^1 \times D^2$), (figure 46).

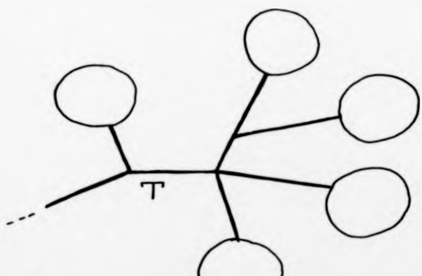


FIGURE 46

More generally we can regard a handlebody of genus n as the regular neighbourhood of a 1-complex τ which is formed by connecting n circles in a tree-like formation using a number of trees (figure 47)

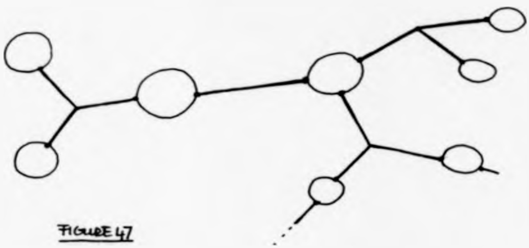


FIGURE 47

We call such a regular neighbourhood a "generalized standard handlebody" and there are obvious corresponding notions of longitudinal curves bounding a system of embedded discs (with the same convention of §5), meridional discs and base intersections as indicated in figure 48

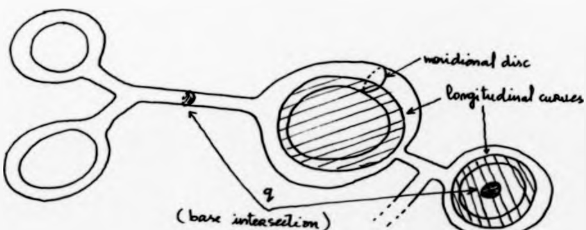


FIGURE 48

The proof of Theorem 7.1 goes through unaltered with H identified with a generalized standard handlebody (instead of a standard handlebody) provided the ordering condition is suitably restated

Generalized ordering condition

We need a base point $* \in T$ and "cut-points" $*_1, *_2, \dots, *_n$ (one in each circle - the transverse meridional discs being the "cut-discs") and a partial order as in Theorem 7.1 this time holding for each arc in T from $*$ to a $*_i$ avoiding other $*_j$'s (arcs indicated by a double arrow in figure 49)

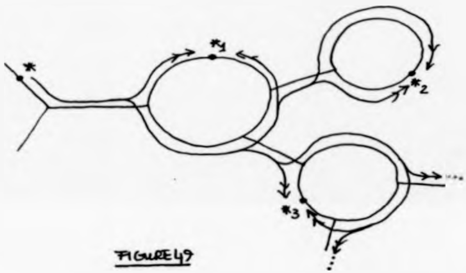


FIGURE 49

RIBBONS

Suppose now that we have a ribbon intersection, r , between discs D and D'. We can obviously assume r has a standard neighbourhood analogous to the one depicted for a clasp in figure 31, see figure 50.

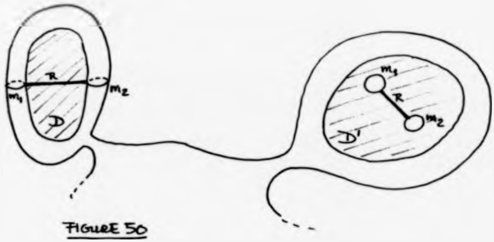


FIGURE 50

Suppose that we can thicken r to form a new 1-handle for H (increasing the genus of H by 1). Then, as indicated in figure 51, D breaks into two discs D^1 and D^2 and the intersection with D' can now be seen as a generalized base intersection.

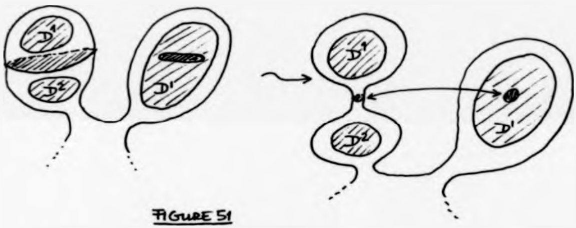


FIGURE 51

By a tag for r we mean a tag in D' for one of the meridional discs m_1, m_2 at its ends (missing other ribbons, tags, clasps) which thus becomes a tag for the new generalized base intersection arising from r (see figure 52).

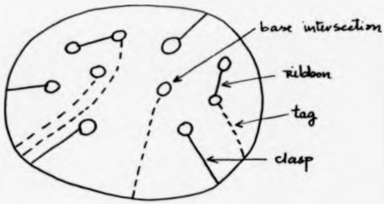


FIGURE 52

The following theorem can now be proved by trading the ribbons for generalized base intersections and then applying the proof of theorem 7.1

Theorem 8.1 : Let $H^3 = H \cup_h H'$ be a pre-primitive H-decomposition such that:

(1) There is a system of tags for the base intersections and ribbons such that the set of all ribbons, tags and clasps forms a trivial link in H'.

(2) There is a generalized ordering condition (as stated above and where ribbons are treated exactly like generalized base intersections).

THEN H^3 is S^3.

Remarks:

1. It is not necessary for all base intersections to be tagged in Theorem 7.1 (or ribbons in Theorem 8.1).

The same method of proof (Theorem 6.1) can be used if untagged base intersections satisfy a very flexible condition inside the given order: each untagged intersection (base or ribbon) inside a given D_i, say, must be ordered greater than every other intersection (tag!) and clasp in D_i. With this condition it is only necessary to note that in the proof of Theorems 6.1 and 7.1 the discs D_i don't need to be collapsed all at once but only in the order in which clasps or tags appear – a disc D_i can be kept unaltered until one of its clasps or tags comes into play. Therefore for an untagged intersection in D_i we can first cancel everything that is ordered greater than it and so remove it from D_i before this disc needs to be collapsed.

This observation will be used several times in the next chapter.

2. In our definition of primitive H-decomposition for a homotopy 3-sphere H^3 there is associated a primitive surgery presentation. In the next chapters we will always work with primitive surgery pictures but is is worth noting that Theorems 6.1, 7.1 and 8.1 are obviously valid for any H-decomposition $H^3 = H \cup_h H'$ having the same defining properties of a primitive one but not necessarily associated to a primitive surgery presentation.

Chapter 4 - Primitive Representations of Homotopy 3-Spheres

In this chapter we improve the general characterization of homotopy 3-spheres provided by Theorem 4.1 by proving first, in S9, that any homotopy 3-sphere has a primitive H-decomposition satisfying hypothesis (1) and (3) of Theorem 6.1 With this result (Theorem 9.1) as a starting point SS10 and 11 present techniques aimed at establishing the more general conditions of Theorems 7.1 and 8.1 , or some variations allowing untagged intersections (in the spirit of the remark at the end of the previous section) that could possible lead to a proof of the Poincaré conjecture.

S9 . Ordered Primitive H-decompositions

Theorem 9.1 : Any homotopy 3-sphere H^3 has a primitive H-decomposition $H^3 = H \cup_h H'$ such that.

(1) - There are no base intersections.

(2)- The clasps can be ordered as required in the ordering condition of Theorem 6.1 (hypothesis (3)).

Proof: Let $H^3 = \mathcal{L}(\ )$ be a primitive surgery picture associated with $H^3 = H \cup_h H'$, where $\mathcal{L}$ is a pure plat in S^3. we first arrange the ordering condition (ignoring for the time being base intersections). This is done by a sequence of Kirby moves and their duals in both $H^3 = \mathcal{L}(\ )$ and $S^3 = \mathcal{L}'_{H^3}(\ )$ as in the proof of Theorem 4.1.

The first move in $\mathcal{L}'_{H^3}(\ )$ consists of breaking each clasp into three as illustrated in figure 53 by means of a crossing along that clasp

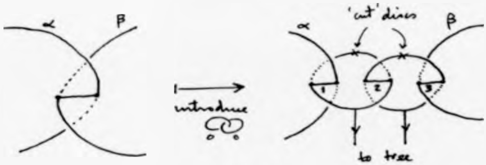


FIGURE 53

This results in the change in $\mathfrak{A}(\ )$ illustrated in figure 54 which may not be a pure plat

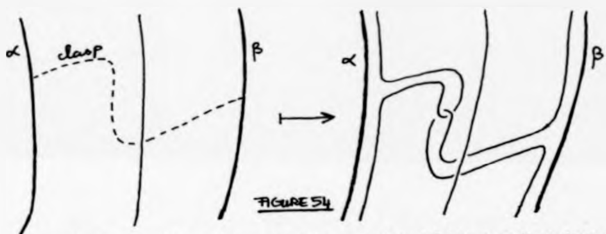


FIGURE 54

To restore pure plat we replace the tongues in figure 54 by a perfect daisy-chain (figure 55).

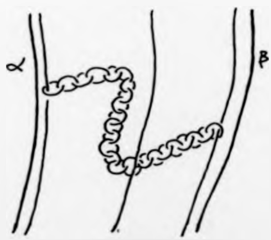


FIGURE 55

As is already known from the proof of Theorem 4.1 the introduction of this daisy-chain corresponds in $\mathcal{L}_H{}^3(\ )$ to the introduction of an ear-ring (figure 56).

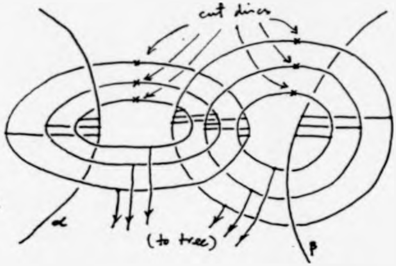


FIGURE 56

The handlebody H for $H^3 = H \cup_h H'$ is obtained as usual by posting each surgery component of the final pure plat in S^3 to a base. To simplify the picture we regard H (as in §7) as the regular neighbourhood of a 1-complex τ consisting of circles and a tree T connecting them. In figure 56 the "cut-discs" for the components of the ear-ring are indicated by the corresponding "cut-points" for τ and the posts by the lines with an arrow.

The new clasps obtained by the previous sequence of moves are indicated in figure 56 by horizontal lines. If, as pictured, each component of the ear-ring has its cut-disc at the top in figure 56 and is connected to the base at the bottom then the ordering of the central set of clasps which is increasing upwards in figure 56 is suitable for the ordering condition we pretend. The outermost clasps in α and β are such that each component of the ear-ring contains exactly one of them and because α and β bound disjoint discs they are all unrelated for the purpose of ordering and can therefore be ordered as required by any choice of "cut-discs" for α and β

We next explain how to trade base intersections for ribbons which are in turn traded into clasps and preserving the ordering condition. It is convenient to simplify notations to think of the 1-complex τ as consisting of circles attached to a tree T as shown in figure 57 : T is a wedge of arcs with base point at the wedge point.

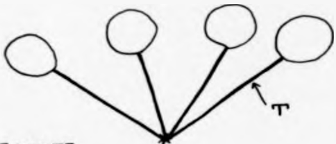


FIGURE 57

Choose an outermost base intersection (i.e one adjacent to a component of the surgery link) (figure 58).

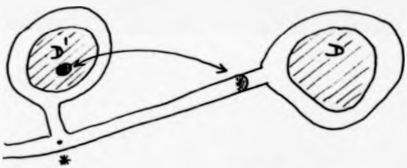


FIGURE 58

In figure 58 the base intersection is adjacent to the component bounding the disc D and lies in disc D^1. If $D=D^1$ we say the base intersection is of bad type. Assume first that $D\neq D^1$ (i.e. not bad type). Then we can trade the base intersection for a ribbon intersection of D^1 with D by pushing D along T to cross into D^1 (figure 59).

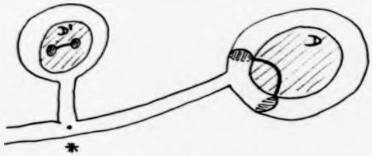


FIGURE 59

We next trade this ribbon for two clasps in the usual way by piping it to the edge of D^1 as indicated in figure 60

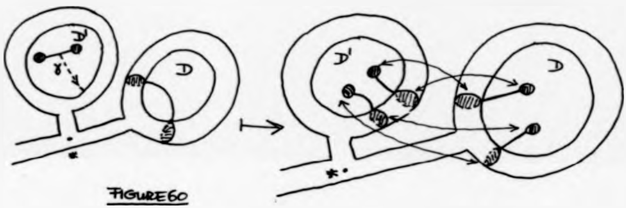


FIGURE 60

We choose the path γ in D^1, along which this piping takes place to end up adjacent to T and notice that in D the two clasps are also adjacent to T but on either side of the connection. Thus the new clasps can be fitted in at the bottom of the ordering in the obvious way

To complete the proof we show how to eliminate an outermost bad-type intersection (at the expense of several new base intersections not of bad type) and then by double induction on the numbers of base intersections of bad and not bad type, we can eliminate all base intersections.

We eliminate an outermost bad-type intersection by a sequence of Kirby moves in $S^3 = \mathcal{L}_{H^3}(\ )$ and $\mathcal{L}(\ )$ as we have done previously.

We first do the move in H^3 pictured in figure 61.

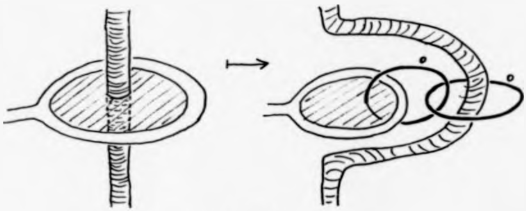


FIGURE 61

In S^3 this corresponds to a change of the type illustrated in figure 62.

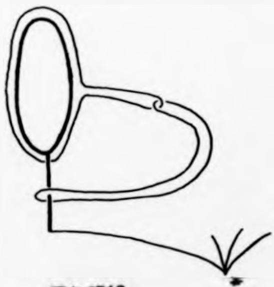


FIGURE62

Introducing as usual a daisy-chain to restore pure plat we have figure 63.

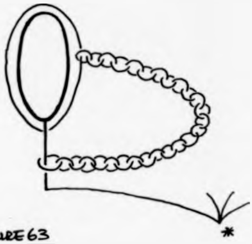


FIGURE 63

Each of the new components (which form an ear-ring in H^3) bounds an obvious disc in H^3 whose correspondent planar surface in S^3 can be seen as shown in figure 64 (the same as figure 23 in §4).

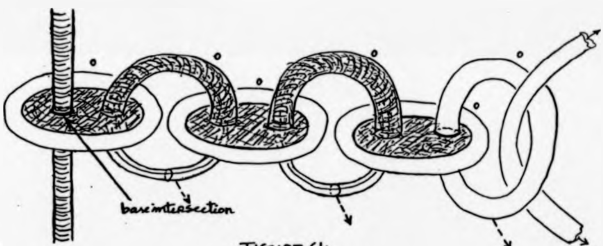


FIGURE 64

Thus the new components can be connected down to the base by posts which avoid the disc which each bounds (dashed lines in figure 65) and therefore the new base intersections created by those posts are never of bad-type.

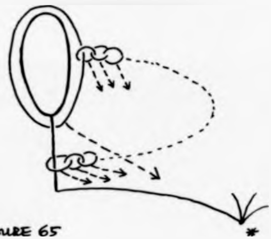


FIGURE 65

This completes the proof of the theorem.

It is convenient for the following sections to have an improved version of the previous Theorem.

Let $H^3 = H \cup_h H'$ be a primitive H-decomposition with associated primitive surgery $H^3 = \mathcal{L}(\ )$. We now show that we can always assume that in the dual surgery $S^3 = \mathcal{L}_H{}^3(s_1, \ldots, s_n)$ each $s_i = \pm 1$. As remarked in §5, in this situation, the planar surfaces in S^3 that correspond to the system of embedded discs in H^3 all have for boundary curves longitudes of the surgery tori of $\mathcal{L}$.

<u>Theorem 9.2</u>. Let H^3 be a homotopy 3-sphere. There exists a primitive H-decomposition $H^3 = H \cup_h H'$ with associated primitive surgery picture $H^3 = \mathcal{L}(\ )$ such that:

(1) - There are no base intersections.

(2) - The ordering condition holds.

(3) The dual surgery picture $S^3 = \mathcal{L}_H{}^3(s_1, \ldots, s_n)$ has framing numbers $s_i = \pm 1$, $i = 1, 2, \ldots, n$.

Proof. Start with a primitive H-decomposition satisfying the first two conditions (provided by Theorem 9.1). We arrange for all coefficients in $\mathcal{L}_H{}^3(\ )$ to be ±1 by introducing new components in H^3 (using move (c) of the Kirby calculus) of the type illustrated in figure 66.

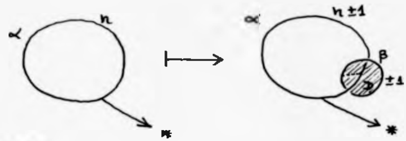


FIGURE 66

This results in one new clasp (dotted in figure 66) and clearly does not disturb the ordering condition. In S^3 the new component β is parallel to α and the disc D is represented by an annulus cobounding longitudes of α and β (figure 67).

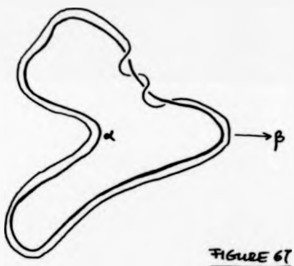


FIGURE 67

It is clear that if we choose the new component β adjacent to the tree T then we can post it to the base without introducing any new base intersections

S10 – Trivial Clasps

We show in this section how given a primitive H-decomposition of a homotopy 3-sphere $H^3 = H \cup_h H'$ satisfying the conditions of Theorem 9.1 (no base intersections and the ordering condition on the set of clasps) we obtain, at the expense of introducing base intersections and ribbons, another H-decomposition (pre-primitive) where the ordering condition for the set of clasps still holds, and the set of clasps form a trivial link in H'.

To "approach" the conditions of Theorems 7.1 and 8.1 we need convenient tag-systems for the base intersections and ribbons and that problem will be treated in S11.

The Natural Order :

Since from now on we will be working mainly with primitive or pre-primitive surgery presentations $H^3 = \mathcal{L}_{S^3}(\ )$ it is convenient to have a way of seeing the order on the clasps reflect in same simple manner in on those those pictures in S^3

We consider a primitive H-decomposition $H^3 = H \cup_h H'$ satisfying the three conditions of Theorem 9.2. Since each surgery coefficient in $\mathcal{L}_{H^3}(\ )$ is ±1 the preferred longitudes in H^3 (i.e. the curves $c_1, \ldots, c_n$ of the longitudinal system according to the convention in S5) are also longitudes in $\mathcal{L}(\ )$, therefore we can make the ordering within each disc D_i, $i=1, \ldots, n$ (figure 68) compatible with the "natural order" vertically up the plat $\mathcal{L}$ in S^3

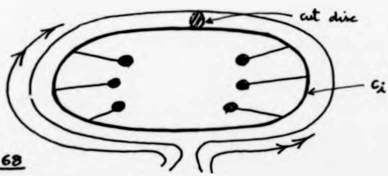


FIGURE 68

More precisely, embed the partial order of the ordering condition in a total order and choose corresponding levels in S^3 in the same order. Then by sliding the endpoints of the clasps around the longitudes c_i in S^3 we can assume that each clasp has its end points on the level corresponding to its position in the order (figure 69).

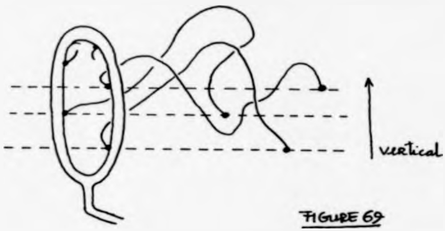


FIGURE 69

We will refer to this order on the clasps (or more generally, later on, also on base intersections or ribbons) which comes in this way from the order vertically up the plat as the natural order.

Trivial Clasps

Let $H^3 = \mathfrak{L}(\ )$ be a primitive surgery presentation, and $H^3 = H \cup_h H'$ an associated primitive H-decomposition where H is as usual obtained by "posting" the components of $\mathfrak{L}$. The basic idea is that any set of monotone arcs with end points in $\mathfrak{L}$ form a trivial in $H' = Cl(S^3 - H)$, where monotone means, of course, with respect to the height function (the vertical) defining $\mathfrak{L}$ as a pure plat and used for the "post construction". To see this, simply regard the arcs as extra strands of the braid and unbraid them at the same time as H is unbraided.

Therefore assuming that $H^3 = \mathfrak{L}(\ )$ satisfies the conditions of Theorem 9.1, we need to make all clasps monotone in such a way that the ordering condition is preserved. We assume further to simplify that $\mathcal{L}(\ )$ satisfies also the third condition of Theorem 9.2 and that the clasps are in natural order. We start by shifting the end points of each clasp a little to get them on different levels (by "a little" we mean that if clasp c is higher than clasp c' then the shifted end points of c will also be higher than the shifted end points of c') and then isotope the clasps to be monotone apart from a finite number of vertical crossings (as in figure 70).

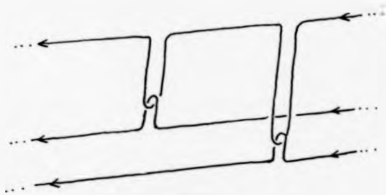


FIGURE 70

(Each crossing in figure 70 corresponds to an "unfavourable" crossing in some horizontal general position projection of the clasp set).

We now perform a sequence of Kirby moves and their duals in both $S^3 = \mathcal{L}_H^3(\ )$ and $H^3 = \mathcal{L}(\ )$:

FIRST. We introduce a number of new components in the S^3 picture labelled ± 1 so as to undo all those crossings (figure 71).

In H^3 this corresponds to introduce also a number of unknotted components labelled ± 1 (figure 72).

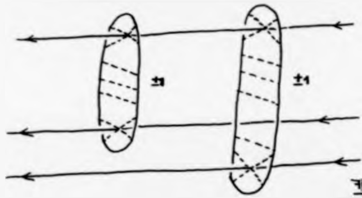


FIGURE 71

The new singular set comprises triple-point configurations at the top and bottom of each new component and a number of ribbons (where the new component pierces intervening sheets of the D_1). The new singular set is pictured dotted in figure 71. At this point all the original clasps were made monotone.

SECOND we now trade the triple points created for clasps and ribbons by means of the move in H^3 illustrated in figure 72.

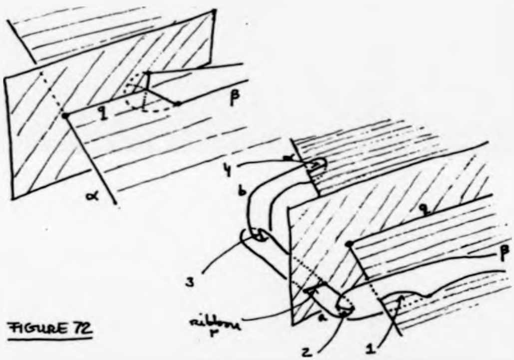


FIGURE 72

This move consists of executing one crossing (along the dotted line with an arrow) with the introduction of a daisy-chain with two components and isotoping the discs to the final position shown.

The triple point on the clasp q is deleted and there are four new clasps labelled 1,2,3,4 in figure 72 and one ribbon labelled r.

This move is performed for all the triple points in order, starting at the bottom of each clasp, trading the bottommost triple point downwards and then proceeding to the next one above and so on.

THIRD. We must see how the previous moves reflect in the S^3 picture. The crossing in H^3 between the new component β and the (original) component α which contains the lower end point of clasp q was done along a line parallel to the clasp q by sliding β and α over the 0-labelled components of a daisy-chain of length two. In S^3 this corresponds to the inverse slides and can be seen as introducing two framing curves for β and α and deforming them along to tongues parallel to clasp q which are then clasped. (as explained in the proof of Theorem 4.1).

It would be very difficult to draw the position of the discs (the original ones and the new ones) in the S^3 picture after these moves, but the new discs and the relevant part of the (original) disc bounded by α (correspondent to clasps 1 and 4) can be easily visualized by natural planar surfaces bounded by longitudes or meridional curves of the different components with tubes connecting some of those meridional curves (as in figure 64). To visualize the change in the original disc that contains the ribbon r is more difficult (a good exercise for nights of insomnia) but that is not needed in any way.

The pictures in S^3 we thus obtain are shown in figures 73 and 74.

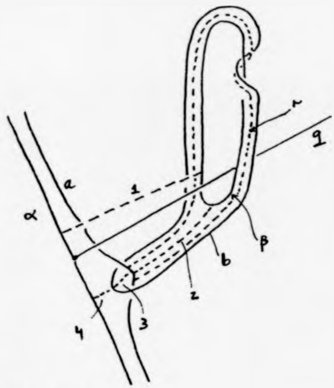


FIGURE 73

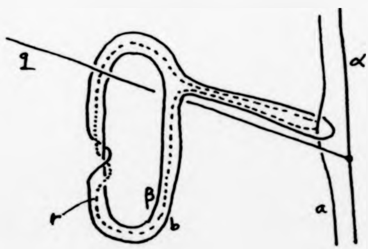


FIGURE 74

Figure 73 shows the case where the triple point is at the bottom of the new component β and figure 74 the case where the triple point is at the top. The labelling in figures 73 and 74 corresponds to the labelling in figure 72 and the omitted details in figure 74 are similar to those in figure 73.

In both pictures the new clasps 1, 2, 3, 4 are roughly parallel to parts of the clasp q and hence monotone in S^3. In figure 73 the new components can be taken to form part of the pure plat but in figure 74 the component b is not plat-like (it has two maxima and two minima).

FOURTH. We restore the pure plat in this situation by replacing the tongue by a daisy-chain of length two which corresponds in H^3 to parallel curves a', b' to a, b in the usual ear-ring arrangement (figures 75 and 76)

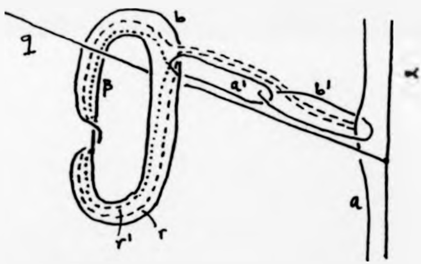


FIGURE 75

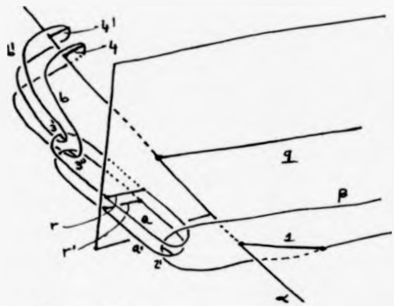


FIGURE 76

There are four new clasps 2', 3', 4', 3" which are again roughly parallel to q and hence monotone. There is also a new ribbon r' which in S^3 runs roughly parallel to r.

Control of the order:

At this point we have all clasps monotone and we shall analyze the ordering condition. We first consider the way the new components are joined to the base (by vertical or monotone posts) to extend the handlebody H. It is easy to see looking at the way the new discs are placed (in figures 73, 74 and 75) that we can join the new components to the base (in S11 we will use the more general post construction joining some components to other components to avoid some base intersections) from their minima downwards in such a way that.

(C-i) : all the base intersections created lie in the original discs D_i

(C-ii) : for components a and a' we can join from the side of the ribbons r and r' that contain clasps 2 and 2' respectively.

Looking at pictures 72 and 76 clasps 4, 4' and 1 are ordered according to the order of the clasps in the disc bounded by α. Component β has another clasp, say $\bar{1}$, lying on a original disc (which may or may not be the one bounded by α) and is therefore also included in the original order. In any case the top clasp of β can always be ordered greater than the bottom clasp : if the crossing corresponding to β was between different clasps say q_1 and q_2 then the top clasp of β will be parallel to q_1 and the bottom one parallel to q_2 with q_1 greater than q_2. Therefore in the order given by the original discs where q_1 and q_2 lie, the top clasp will be ordered greater than the bottom clasp. If β represents a crossing between the same original clasp q we can always assume (creating if necessary a extra ribbon intersection with the vertical sheet in figure 72 near the triple point) that in figures 76 and 72, β meets the clasp q from bellow, that is, in the order for α the new clasps 4 and 4' will be ordered greater than q. Because the triple points are removed in order starting from the bottom, in the end the top clasp of β will be ordered greater than the bottom one by the original order in α. We choose the cut-disc for β at the top.

Clasp 2 and 2' are then ordered according to the way they lie on the new disc bounded by β, relative to chosen cut-disc for β.

By construction the segment in a between the end points of clasps 3 and 3' does not contain the extremity of the joining path (post) for a. Analogously the segment in b between clasps 3 and 3" does not contain the extremity of the joining path for b. Therefore if we next order 3" greater than everything that has been ordered before then 3 and finally 3' we can choose cut-points for a and b so that the ordering condition is satisfied. (for a' and b' the choice of cut-disc is obvious since they contain only two clasps each)

<u>Types of Ribbons</u>:

The ribbons arising from the intersection of the new components β with intervening sheets of the original discs D_i (figure 71) and the ribbon r in the bottom pictures (figure 73) can all be seen as having one single maximum and no minima. Therefore they form, together with the monotone clasps a trivial set of arcs in H'. The proof that a set of monotone arcs forms a trivial link in H' also shows that a set of arcs each with one maximum (and no minima) form a trivial link: the arcs are unbraided from the bottom upwards and the two monotone segments leading to the maximum are unbraided independently. We will refer to these ribbons as the <u>trivial ribbons</u>.

However the ribbons r and r' in the top pictures (figure 75) have minima and therefore we don't have the situation where the set of <u>all</u> clasps and ribbons form a trivial link in H'. Nevertheless these <u>non-trivial ribbons</u> behave well in relation to the ordering condition established above: if we consider the meridional discs at the end points of ribbons r, r' naturally included in the order by their positions in a and a' respectively, we can assume they are ordered greater than any clasp on the (original) disc where they lie : in fact using condition (C-ii) above we can choose the cut-discs for a and a' in such a way that these meridional discs are ordered greater than clasps 2, 2' and therefore free from restrictions from the order in the original discs and the new discs bounded by β.

Note that having started with $\mathfrak{A}(\ )$ satisfying the third condition of Theorem 9.2 in the final picture in H^3, after all these moves, all coefficients are 0 and ±1.

§11. Tagging

In this section we study some possible ways of improving the picture of a homotopy 3-sphere obtained in §10 so as to "approach" the general conditions and method of proof of Theorems 7.1 and 8.1.

11.0 - Suppose in the conditions of §10 we have a system of tags for all base intersections and trivial ribbons such that conditions (1) and (2) of Theorem 8.1 are satisfied. As for the non-trivial ribbons we ignore them and keep the meridional discs at their ends untagged (figure 77). The trivial ribbons are, in the generalized ordering condition (2), ordered according to their positions from top to bottom in the new components β

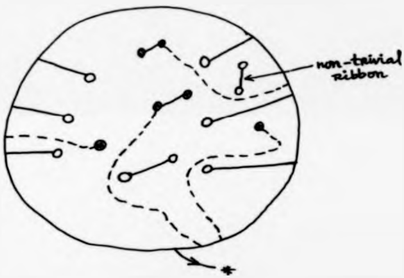


FIGURE 77

THEN $H^3 = S^3$.

Proof. In fact since the base intersections and trivial ribbons all lie in the original discs (condition (C-1) above) we can always assume that in the partial ordering the clasps 3', 3, 3" are maximal elements (in this order). Therefore by the proof of Theorem 8.1 (7.1 and 6.1) they can all be deleted first by collapsing only the four discs where they lie (the discs bounded by a, a', b, b') and leaving all the other discs unaltered. The untagged meridional discs at the end points of the non-trivial ribbons can then be isotoped off the original disc where they lie (because of condition (C-ii) above), before this one needs to be collapsed and so the process can be carried on exactly as in Theorem 8.1 .

A Project

The remaining of this section is a sketch of a project for getting either the strong condition of 11.0 on a system of tags or some similar picture where the proof of Theorem 8.1 would still work, if with some obvious adaptations

This is where the concept of natural order introduced at the beginning of S10 comes into play.

The first step is to observe that in the picture obtained in S10 we can avoid some of the base intersections by joining some of the new components to the original handlebody H in a different way

11.1 – Natural Joins Instead of joining all the new components to the base by vertical (or monotone) posts we will join some of them to the old ones by means of monotone paths from their minima downwards as shown in figure 78 (i.e. using the generalized "post construction" of S5).

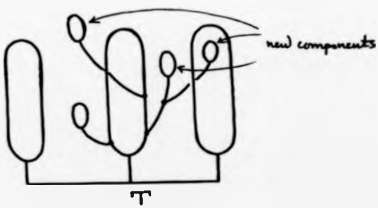


FIGURE 78

This means that for the purpose of getting the ordering condition we start considering the handlebody H so obtained as a "generalized standard handlebody" as defined in §8.

Looking at figures 72 (73) and 76 (75) we observe that for components β in figure 72 and b' in figure 76 there are natural joining paths to the component α, following the respective clasps and ending lower than these clasps in the order given by α. No base intersections are created by these monotone paths.

For component b (in both top and bottom pictures) there is a monotone path to the new component β parallel to the disc bounded by a. This entails a base intersection with the sheet containing ribbon r (see figure 79).

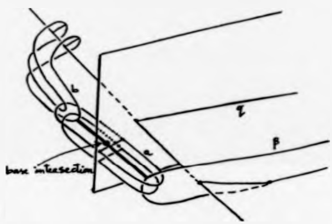


FIGURE 79

This choice of joining paths for the new components implies that for tagging purposes we only need to be concerned with the base intersections corresponding to components b (the ones placed next to the non-trivial ribbons) and with the trivial ribbons created by components β.

In fact by choosing the cut-point at the top of β conveniently we can assume that after deleting first clasps 3, 3' and 3" (as explained in 11.0) we can next delete clasps 2 and 2' without having to collapse any of the original discs where all base intersections and trivial ribbons lie. Analogously for the clasp 2 of the bottom picture : we conveniently choose the cut-point for the bottom disc obtained by trading the bottommost trivial ribbon in β for a generalized base intersection (if there are no ribbons created by β we just consider a fake one).

Therefore components a and a' disappear of the picture before any of the original discs need to be collapsed and we don't need to worry about the base intersections created by their posts to the base.

11.2 - Natural Tags : This second step is based on the following idea : from a base intersection or trivial ribbon needed to be tagged "drop" a line, in the disc D, where it lies down to a minimum on the boundary of that disc and avoiding all clasps and ribbons in D (we are of course referring to the S^3 picture, and the minima are relative to the vertical of the plat). Alternatively we can consider, instead of such a monotone path, a line that starting at the base intersection or trivial ribbon, goes first up and then down to a minimum as long as that minimum is lower in the plat than the base intersection or trivial ribbon. These lines are natural candidates for tags having the desired properties - because they are either monotone or have a single maximum they can be unbraided in the usual way to form with the set of clasps and trivial ribbons a trivial link in H' (condition (1) of Theorem 8.1) and using the natural order and the the fact that for each line its end point is lower in the plat than the respective base intersection or trivial ribbon these tags can be included in a generalized ordering condition (condition (2) of Theorem 8.1)

There are three basic problems with the possible existence of these "natural" tags:

First : Because in the S^3 picture a disc D is represented by a planar surface and it is in this planar surface that we have to draw the natural tags it might happen that a line does not end at the boundary of D but at a meridional disc at the end of a clasp or ribbon (figure 80).

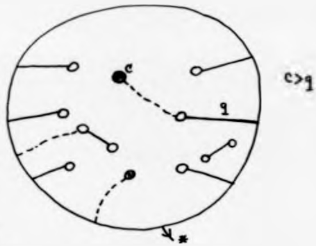


FIGURE 80

We will call this sort of lines pseudo-tags. Existence of pseudo-tags is not really a problem since the proof of Theorem 8.1 can obviously be adapted to cope with this new situation. In figure 80 since the base intersection c is ordered greater than the clasp q we can delete everything that is ordered greater than c before having to delete clasp q. The base intersection c will then have a free top through which we can do the necessary proper homotopies to delete pseudo-tag t. We can then isotope c off the disc D and we are in the normal situation when clasp q needs to be deleted.

More generally the proof of Theorem 8.1 can be adapted to the situation where we have trees of pseudo-tags (figure 81).

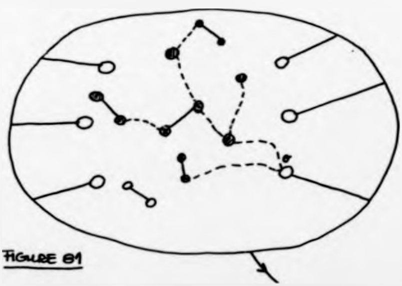


FIGURE 81

The relation between the ordering condition and the way the different base intersections and trivial ribbons are located in the tree is clear: for each intersection, c, there is a unique "injective" path along the tree to the root (labelled 0 in figure 81). All intersections, as well as the root, along that path have to be lower in the order than c.

Second , suppose we assume, in the beginning of § 10, that the planar surfaces have the property that all minima occur at boundary components. To see this is possible consider horizontal intersections (in general position) with the entire picture. What is seen is a collection of discs (the intersections with the thickened plat - the surgery tori) joined by arcs which can cross (at a point of the singular set). Obvious moves can be made to eliminate trivial free circles corresponding to minima not on the boundary. We omit the details.

The problem now is that when we isotope the clasps as in figure 70 to perform a sequence of crossings and make them monotone new minima not on the boundary are created. In figure 82 all sheets which are "trapped" between the clasp(s) will have minima underneath the point indicated by the arrow. Note that these "trapped" sheets are exactly the ones responsible for all the trivial ribbons created in the process of trivializing the clasps.

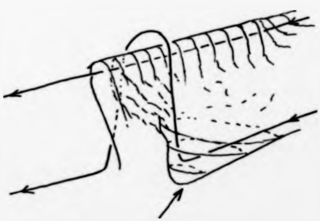


FIGURE 82

The second type of minima which may be created correspond to "fold lines" on the planar surfaces which are deformed by the isotopies.

A fold line can be briefly described (in differential terms) as a line of points with vertical tangent planes (figure 83).

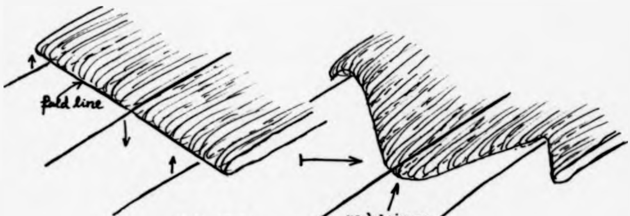


FIGURE 83

We now describe the sort of structure we should look for to overcome this second problem. To simplify we assume for the moment that all minima on the planar surfaces we will consider are away from the boundary (substitute each minimum at the boundary by an interior one by a slight deformation of the surface in its neighbourhood) with the exception of the lowest one.

By a tree of minima on a disc (planar surface) we mean a 1-complex tree disjoint from all clasps and ribbons, containing all minima of the planar surface obtained by excising a regular neighbourhood of the singular set, as vertices and with edges (the twigs) that are either monotone or have a single maximum in the S^3 picture. The root of the tree is at the lowest boundary minimum corresponding to the joining path for that disc (figure 84).

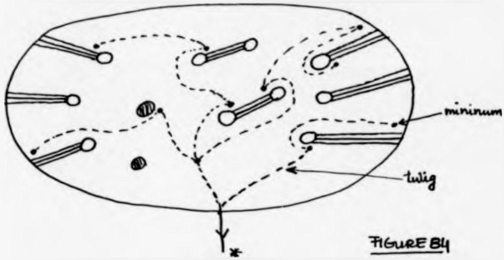


FIGURE 84

The tree of minima is natural if the order of the minima in the tree (as described above for trees of pseudo-tags) agrees with their natural order up the plat in the S^3 picture.

Now suppose that in the final picture of a homotopy 3-sphere H^3 obtained in §10 we have for each original disc a natural tree of minima.

THEN H^3 is S^3.

Proof : We start by posting each minimum down to the base. To simplify, instead of considering disjoint posts (one for each minimum) we can consider a single "tree of posts" containing all the minima and which does not contain any other base intersections apart from the intersections corresponding precisely to those minima. : consider the highest minimum in the plat and the corresponding post. Looking at the topmost base intersection in that post there is a monotone path down to a lower minimum in the disc where it lies (if on the way that path meets a monotone clasp or a ribbon or a twig of a tree we divert it following that curve to another minimum). We can then substitute the post to the base by a post to that lower minimum as indicated in figure 85.

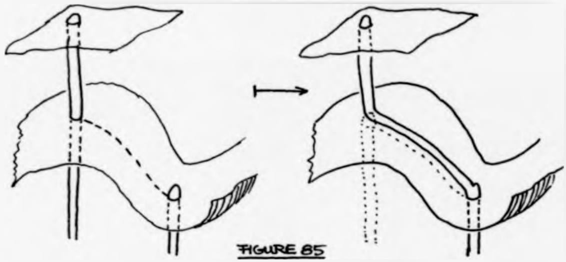


FIGURE 85

If we do that in the order decreasing down the plat for each minimum we get the required "tree of posts" (figure 86).

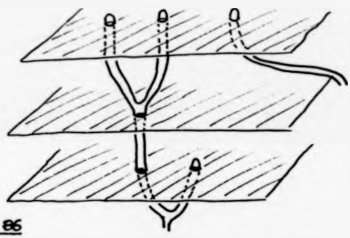


FIGURE 86

We can now for each base intersection and trivial ribbon consider a monotone path to one of the minima. This will be a pseudo-tag ending at a vertice of a tree. As explained before we only need those pseudo-tags for trivial ribbons and base intersections corresponding to the new components b (figure79). The proof of Theorem 8.1 can now be applied (with the adaptations indicated in 11.0 and 11.1) until we are left only with the trees of minima and the tree of posts. We can now proceed using the "naturality" of the trees of minima.

Consider a highest minimum in the S^3 picture. The tree of posts has a free top at this minimum and this minimum is a free vertice (i.e. it lies in only one edge) in the respective tree of minima (because the tree is natural). We can therefore do all the necessary regular homotopies to delete the twig of the tree that starts at this minimum. If we do the same in order for all the minima and twigs we end up with a 3-cell as before.

There is a generalization of what we have just described where we can keep minima on the boundaries of the planar surfaces, the set of minima of each planar surface may distribute through several trees and those trees may have roots at boundary minima other than the lowest one. What we require for the same method of proof to work using the natural order is that a base intersection or trivial ribbon having a pseudo-tag to a certain tree is ordered greater than the root of that tree. We omit the details.

Third: The last problem is related to the two previous ones: in the presence of minima of the types indicated in figures 82 and 83 find the way of gathering them into trees and for the relevant base intersections and trivial ribbons find pseudo-tags to trees of minima in such a way that the ordering condition is suitable for an adaptation of the proof of Theorem 8.1 (as suggested in the last paragraph).

The three main difficulties are:

a) The fold lines may be very complicated and it is not clear that a minimum in one of them (figure 83) has an arc with only one maximum leading to a lower minimum. We suggest it may be possible to prove we can start the whole process of §10 without any fold lines. This would involve careful isotopies of the planar surfaces and is where to work with pure braid primitive surgery presentations may be useful.

b) The same as before for the base intersections correspondent to the new components b. The discs where these base intersections lie become extremely complicated under the unknotting process and are not easy to visualize in the S^3 picture. It is perhaps better to avoid this problem by the use of the forced tagging as explained below in 11.3

c) It is not clear in the local picture of a crossing (figure 82) that the minima on the trapped sheets have pseudo-tags to trees of minima whose roots are lower in the natural order than the clasp trapping them.

We will not advance into this problem which is the central problem of the project but I would like to suggest the following:

It may be possible to get for each planar surface a single natural tree of minima with root at the lowest boundary minimum, by some general construction that would avoid having to do so much local analysis relative to the natural order.

A good starting point is to look for those trees of minima even before the trivialization of the clasps is done and, at the same time, study special situations where the proof in 11.2 would still work with the definition of trees of minima weekened to allow some intersections of the twigs with the singular set (and also note that for the proof in 11.2 we can exclude some of the minima from the trees as long as we guarantee that no pseudo-tags will run into those minima). See 11.4 bellow.

11.3 - Forced Tagging

Instead of looking for natural tags or pseudo-tags at the end of 11.1 we can first choose tags for all base intersections and trivial ribbons as in figure 77 without requiring that they form with the singular set a trivial in H'. We can of course draw in each of the original discs a system of tags in such a way that they fit the ordering relations between the different base intersections and trivial ribbons in that disc.

We next make all those tags monotone by the same process used to make clasps monotone. We isotope these tags to be monotone apart from crossings of the type illustrated in figure 82 but assuming that all these crossings are between tags.

To see we don't need to consider any crossings with the arcs of the singular set notice that we can regard all clasps and trivial ribbons as having a maximum (for a monotone arc an unnecessary maximum can obviously be created near the upper end point). Then lifting the maxima for the singular set and plat components all up to the same top level we can assume that the tags all lie in a trivial portion of the braid of components and singular set (we don't consider the non-trivial ribbons since we don't require that they form with the clasps, trivial ribbons and tags a trivial link in H'). The crossings in figure 82 are then the unfavourable crossings in some planar projection.

We now introduce further components to change the crossings of the tags, figure 87.

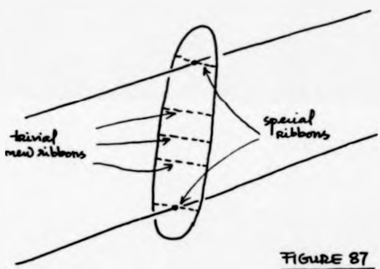


FIGURE 87

The new singular set consists only of ribbons (there are no triple points) and is dotted in figure 87. We now trade the ribbons called "special" in figure 87 for clasps by the same move used in S10 for the triple points. We will have pictures analogous to figures 72, 73 and 76, 75 where the vertical sheet and the ribbons r, r' should now be ignored.

We can now create all the natural joins for these new components and look for natural tags (pseudo-tags) for the new trivial ribbons and base intersections as in 11.1 and 11.2, but this time there are two simplifications which may be important:

First: the base intersections corresponding to the natural joins for components b (figure79) do not exist now.

Second: we don't have the problem as for the removal of triple points of the choice of side from which the new components β hit the tag lines.

We can of course repeat the process but there is no guarantee that the sequence will ever finish. As we said in the introduction a possible line for further developments is to search for the right notion of complexity that would guarantee that an iterated forced tagging would be finite.

11.4

To end this section we outline, very briefly, a sequence of constructions which is within the spirit of the suggestion at the end of 11.2 and that I believe is a right alternative in the project (it will be developed with "high resolution" in the near future).

1. Consider a primitive surgery picture of H^3 with the clasps in "natural" order as explained in the beginning of S10. Assume all minima occur at the boundaries of the planar surfaces and for each planar surface the lowest minimum is the one corresponding to the joining post to the base (and there are no base intersections).

2. Consider the planar surfaces with regular neighbourhoods of the clasps excised and for the discs so obtained substitute each minimum at the boundary by an interior minimum (with the exception of all the lowest minima), and consider them in natural order between two levels say a, b of the plat such that all the lowest minima are below b and all the end points of clasps are above a.

3. Consider tags for all those interior minima, ending adjacent to the joining posts and fitting in the natural order for those minima.

4. Isotope the tags to be monotone apart from a certain number of vertical crossings (as we did for clasps in S10). This time instead of undoing a crossing through a ±1 surgery we introduce (figure A below) a post with a new component at the top (a ringed post) considered with the trivial surgery.

This introduces a ribbon which is shown dotted in figure A, and some base intersections created by the posts. As suggested by figure E below there are discs that guarantee that the set of tags form a trivial link in the new handlebody obtained by adding all the ringed-posts.

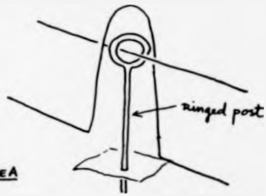


FIGURE A

5. Make the clasps monotone apart from a certain number of vertical crossings as shown in figure B and undo all these crossings by the introduction of a ±1 component. In the case the lower line is a tag we have a ribbon at the bottom instead of a triple point (shown dotted in figure C)

FIGURE B

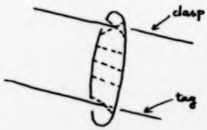


FIGURE C

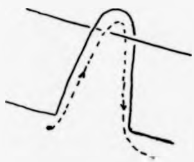
6. Remove all the triple points created, as in S10, using all the natural joins and the same control of the order

7. Assuming by some initial construction that there are no fold-lines check that the only new minima that are created correspond to boundary minima in the new components or are interior minima of the original planar surfaces that are trapped between two tag lines (arrow in figure A above).

8. Pipe each ribbon intersecting a tag line (figures A and C) along that tag line, changing it into two clasps.

9. Construct the trees of minima in the following way: for each of the new interior minima there is a path with a single maximum, (as indicated in figure D), leading to a lower minimum.

FIGURE D



Check that the trees so constructed are natural.

10. Check that for all the base intersections (the ones corresponding to step 4 and the ones corresponding to the natural joins of new components b in the removal of triple points, figure 79) and for all the trivial ribbons (created by the new components β in the removal of triple points) there are pseudo-tags to the trees constructed in 9.

In the case of a crossing between a clasp and a tag there is no natural join for the new component β, and so we post it to the base. For the base intersections created in this case, as well as for the base intersections created by the ringed-posts, we have to check that the pseudo-tag ends on a tree of minima whose root is lower than the tag in the case of figure B or the top-tag in the case of figure C.

11. The pseudo-tags for the trivial ribbons may have to intersect some clasps or ribbons of the singular set. As suggested at the end of 11.2 we have to check that happens in a way where the proof of 11.2 can still be carried on.

12. Check that all the clasps, tags and pseudo-tags form a trivial link in H' and that the ordering condition holds. Note that, as observed in step 4., the tags form a trivial link in H', with compressing discs as suggested in figure E below. These discs will have to be altered in the appropriate way to adapt to the other changes, but they can be used initially to check the naturality of the trees of minima and pseudo-tags

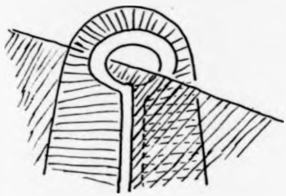


FIGURE E

Note: The ringed-post construction can be an alternative for the forced tagging of 11.3 (in particular if one wants to study the possibility of making that process finite).

S12 - Final Observations:

As mentioned in the introduction, it is stated in [H_4] that if a homotopy 3-sphere H^3 has a H-decomposition as in Corollary 4.1.1 and in addition the set of clasps form a trivial link in H' then H^3 is in fact S^3. In relation to the possibility of getting that special condition we observe:

1. Using the techniques of SS10 and 11 (and forgetting any ordering conditions) it is possible to obtain for any homotopy 3-sphere a pre-primitive H-decomposition such that in the corresponding surgery picture in S^3 the clasps are all monotone and the ribbons have <u>either one maximum or one minimum.</u> This of course doesn't solve the problem since in this situation the set of clasps and ribbons may be as far from forming a trivial link in H' as any other set of arcs looking more knotted.

2. The obstacle for getting this special condition lies in the existence of the non-trivial ribbons. In fact if we have a pre-primitive decomposition of H^3 with the set of clasps and ribbons forming a trivial link it is possible by certain tagging techniques to obtain the desired primitive decomposition.

It did not seem worth to me to give any details of these two points and as I stated already in the introduction I don't personally believe that it is possible to obtain that special condition.

Chapter 5 - Transversality Diagrams for Homotopy 3-Spheres

In this chapter we derive from primitive surgery pictures some easy results about H-diagrams for homotopy 3-spheres which in turn lead to results analogous to those in Chapter 1. We recall a well known fact about H-diagrams for orientable 3-manifols.

Cancelling handles : Let $H(c_1, \dots, c_n)$ be a H-diagram for a 3-manifold M^3. Let D be a properly embedded disc in H, parallel to ∂H and not necessarily disjoint from the curves c_i. Let B be the 3-cell in H whose boundary is $D \cup (\partial B \cap \partial H)$. Consider the standard decomposition of B: $S^1 \times D^2$ with a 2-handle attached along a longitude c_{n+1} We can choose c_{n+1} disjoint from $D \subset \partial(S^1 \times D^2)$ and also disjoint from $c_1, \dots, c_n$. Consider the handlebody $H_1 = Cl(H-B) \cup (S^1 \times D^2)$. Then $H_1(c_1, \dots, c_n, c_{n+1})$ is still a H-diagram for M^3 obtained from the previous one by the introduction of a pair of cancelling handles (namely a 1-handle corresponding to a meridional disc of $S_1 \times D^2$ and the 2-handle attached along c_{n+1}).

S13 . Some H-diagrams for Homotopy 3-Spheres

Theorem 13.1 : Let H^3 be a homotopy 3-sphere. H^3 has H-diagrams $H(c_1, \dots, c_n)$ and $H_1(c_1, \dots, c_n, s_1, \dots, s_k)$ with the following properties:

(1) - $H_1(c_1, \dots, c_n, s_1, \dots, s_k)$ is obtained from $H(c_1, \dots, c_n)$ by the introduction of k-pairs of cancelling handles.

(2) - There are n properly embedded mutually disjoint and non-separating discs $D_1, \dots, D_n$ in H_1 such that:

(i) for each $i=1, \ldots, n$ ∂D_i intersects c_i transversely once and is disjoint from c_j, $j \neq i$

(ii) for each $i=1, \ldots, n$ and $j=1, \ldots, k$ ∂D_i either does not intersect s_j or intersects it twice in opposite directions. Furthermore ∂D_i as a word in $s_1, \ldots, s_k$ reduces to the trivial word by a sequence of cancellations of the form $s_j s_j^{-1}$ or $s_j^{-1} s_j$. This means that designating the two points of intersection of ∂D_i with s_j by a_j, b_j (if they exist) then for every $j,h=1, \ldots, k$, a_h, b_h are not separated in ∂D_i by a_j, b_j.

Before proving the theorem we will prove the following corollary

Corollary 13.2. Let H^3 be a homotopy 3-sphere. There exists a handlebody H_1 standardly embedded in S^3 of genus $k+n$ such that

(1) There are k non-separating curves $d_1, \ldots, d_k$ in ∂H_1 bounding disjoint orientable surfaces in $Cl(S^3-H_1)$.

(2) There are n disjoint non-separating properly embedded discs $D_1, \ldots, D_n$ in H_1 with $\partial D_i \cap d_j = \emptyset$ $i=1, \ldots, n$ $j=1, \ldots, k$ and such that $H(d_1, \ldots, d_k)$ is a H-diagram for H^3 where H is the handlebody obtained by cutting H_1 along $D_1, \ldots, D_n$.

Proof. Let $H(c_1, \ldots, c_n)$ and $H_1(c_1, \ldots, c_n, s_1, \ldots, s_k)$ be H-diagrams for H^3 given by the previous theorem and $D_1, \ldots, D_n$ the discs for H_1 given by condition (2) of that theorem.

Consider H_1 standardly embedded in S^3 in such a way that $s_1, \ldots, s_k$ bound disjoint embedded discs $D_1', \ldots, D_k'$ in $Cl(S^3-H_1)$.

We can now consider a transversality diagram for H^3 as pictured in figure 88

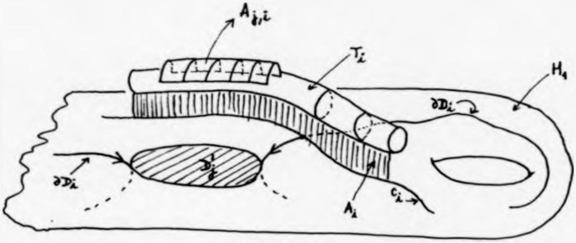


FIGURE 88

There are n "spare tubes" (see Chapter 1) $T_1, \ldots, T_n$ such that for each $i=1, \ldots, n$, T_i is parallel to c_i, that is, there exists an annulus A_i cobounding c_i and a longitude on ∂T_i. These annuli represent the inverse images by the associated degree-one map $S^3 \dashrightarrow H^3$ of the 2-handles whose attaching spheres are $c_1, \ldots, c_n$. For each j the 2-handle attached along s_j have inverse images D_j' union with annuli $A_{j,i}$ that cobound longitudes on ∂T_i and are disjoint from A_i, one for each i such that D_i intersects (twice in opposite directions) s_j. Because of condition (2)-(ii) of Theorem 13.1 we can assume all these annuli $A_{i,j}$ are disjoint.

We next perform changes in this transversality diagram as explained in Chapter 1 (figure 89)

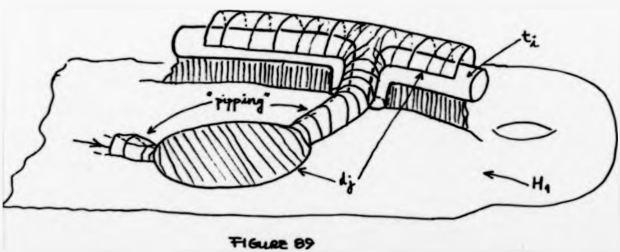


FIGURE 89

Each spare tube T_i gives rise to a new 1-handle t_i and the respective annulus A_i to a disc cancelling t_i. Each disc D_j' is "pipped" to all the annuli $A_{j,i}$ along convenient arcs of ∂D_i thus giving rise to an orientable surface in $Cl(S^3-H_1)$ with one boundary curve d_j. After cancelling all the 1-handles t_i, H_1 with the curves $d_1, \ldots, d_k$ and the discs $D_1, \ldots, D_n$ is the required handlebody.

Proof of Theorem 13.1:

Let $H^3 = \mathcal{L}(\ )$ be a primitive surgery picture for H^3 with $\mathcal{L}=(L_1, \ldots, L_n)$. we assume that the components of $\mathcal{L}$ can be posted down to a "base" without introducing any base intersections (see SS 5 and 9), to form a handlebody H. Then, as explained in Chapter 2, $H'(c_1, \ldots, c_n)$ is a H-diagram for H^3, where $H'=Cl(S^3-H)$ and for each $i=1, \ldots, n$, c_i is a framing curve for L_i. We next post each minimum of the planar surfaces down to the base, as well as each minimum of the set of clasps (a construction in [H3]). This breaks the clasps in a set of arcs each with only one maximum. For each of the base intersections thus created we consider a monotone arc in the disc where it lies to one of the minima already posted in that disc. We denote by $a_1, \ldots, a_k$ the set of all these arcs (resulting from the clasps and from the base intersections). Let H_1 be the handlebody obtained from H by thickening all the arcs $a_1, \ldots, a_k$ to obtain new 1-handles $t_1, \ldots, t_k$. Because the arcs $a_1, \ldots, a_k$ form a trivial link in H' the new 1-handles $t_1, \ldots, t_k$ form a system of cancelling handles. Letting $H_1'=Cl(S^3-H_1)$ it is therefore clear that $H_1'(c_1, \ldots, c_n, s_1, \ldots, s_k)$ is obtained from $H'(c_1, \ldots, c_n)$ by the introduction of k pairs of cancelling handles where for each $i=1, \ldots, k$ s_i denotes the boundary of a meridional disc of the 1-handle t_i. For each disc D_i $i=1, \ldots, n$ of the primitive surgery picture let D_i' be the complement in the correspondent planar surface of the set of posts and 1-handles $t_1, \ldots, t_n$ (cf. figure 34). D_i' is properly embedded in H_1' and $H'(c_1, \ldots, c_n)$,

$H_1'(c_1, \dots, c_n, s_1, \dots, s_k)$ with the system of discs $D_1', \dots, D_n'$ for H_1' are the H-diagrams for H^3 satisfying the required conditions.

Remarks

1. If in the proof of Theorem 13.1 we assume that in $H^3=\mathcal{L}(\ )$, $\mathcal{L}$ is a pure plat then the H-diagram $H^3=H(c_1, \dots, c_n)$ we obtain in that theorem is the usual diagram associated with a pob-decomposition of H^3 (figure 5).

2. Corollary 13.2 is analogous to Theorem 2 of Chapter1 . While this theorem is about an arbitrary H-diagram of a homotopy 3-sphere Corollary 13.2 states only the existence of a certain H-diagram. Nevertheless Theorem 13.1 in which the corollary is based covers the more restrictive class of all primitive H-diagrams and may therefore provide a more effective way of generating H-diagrams for all homotopy 3-spheres.

Question: Is there an effective way, using Theorem 13.1, of deciding given a pure plat surgery $M^3=\mathcal{L}(\ )$ if M^3 is a homotopy 3-sphere?

3. If in Corollary 13.2 the H-diagram $H(d_1, \dots, d_k)$ is such that H is standardly embedded in S^3 then it is a well known fact (and very easy to prove) that H^3 is in fact S^3 (cf. Theorem1 of Chapter1). It is clear from the proofs of Theorem 13.1 and Corollary 13.2 that the complement of H in S^3 is precisely a regular neighbourhood os the set of surgery tori of $\mathcal{L}(\ )$ union with all the posts and planar surfaces. This in turn is a handlebody iff a regular neighbourhood of the surgery tori union with the planar surfaces is a handlebody (the posts represent in an obvious way a set of 1-handles that can be introduced or removed from the picture).

This situation generalizes in the following way.

Let $H^3=\mathcal{L}(\ )$ be an arbitrary surgery presentation of the homotopy 3-sphere H^3 and $S^3=\mathcal{L}_{H^3}(\ )$ the dual presentation (no restrictions on the link $\mathcal{L}$). Let for each component L_i, $i=1, \ldots, n$, of $\mathcal{L}_{H^3}$ D_i be a singular disc in H^3 "bounding" L_i (no restrictions on the singular set). Finally let N be a regular neighbourhood in H^3 of the union of all the D_i's , and $\mathcal{L}_N(\ )$ be the manifold obtained from N by surgery along the link $\mathcal{L}_N$ (with the same surgery coefficients as $\mathcal{L}_{H^3}(\ )$).

Fact If both N and $\mathcal{L}_N(\ )$ are handlebodies then H^3 is S^3.

We omit the proof of this result which is simple and along the lines of the combined proofs of Theorem 13 1 and Corollary 13.2.

Fact: Let $H^3=\mathcal{L}(\ )$, $\mathcal{L}=(L_1, \ldots, L_n)$, be an arbitrary surgery presentation of the homotopy 3-sphere H^3 Then H^3 is homeomorphic to S^3 iff there is a k such that for the surgery $H^3=\mathcal{L}'(\ )$ obtained from $\mathcal{L}(\ )$ by k applications of move (a) of the Kirby calculus there are n+k discs in H^3 and corresponding regular neighbourhood N, as above, such that N and $\mathcal{L}'_N(\ )$ are both handlebodies

The proof which we also omit is a simple application of the Kirby moves and their duals .

These results are not very interesting in themselves (by comparison with our primitive surgery presentations a lot of information is lost!) but suggests a problem that might bring more useful information about homotopy 3-spheres . keeping these results in mind try to find new (more constructive) proofs that orientable 3-manifolds (and in particular homotopy 3-spheres) can be obtained by surgery in S^3.

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