Dynamic fracture analysis of plates loaded in tension and bending using the dual boundary element method

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Keywords: boundary element method (BEM); dual boundary element method; dynamic fracture mechanics; Mindlin plates; generalized plane stress.

Abstract

The purpose of this paper is to solve dynamic fracture problems of plates under both tension and bending using the boundary element method (BEM). The dynamic problems were solved in the Laplace-transform domain, which avoided the calculation of the domain integrals resulting from the inertial terms. The dual boundary element method, in which both displacement and traction boundary integral equations are utilized, was applied to the modelling of cracks. The dynamic fracture analysis of a plate under combined tension and bending loads was conducted using the BEM formulations for the generalized plane stress theory and Mindlin plate bending theory. Dynamic stress intensity factors were estimated based on the crack opening displacements.

1. Introduction

One of the most common damage types in structures is a crack. The growth of a crack can lead to structural failure. In the discipline of fracture mechanics, crack growth behaviour and its effect on residual strength are studied. The prime parameter used for characterisation and prediction of crack behaviour is the stress intensity factor [1]. Such factor is able to be determined experimentally [2], or obtained using analytical [3] and numerical methods [4]. The numerical simulations have gained increasing attention because it is not only suitable for complex structures and loading sequences, but also can reduce the costs of laboratory experiments.

At the beginning of the development of the computational fracture mechanics, some simple methods, such as superposition [5] and compounding techniques [6], were used. Nowadays, more advanced numeral methods are preferred, such as the finite element, boundary element and mesh-free methods. The boundary element method (BEM) has been favoured by many researchers because it can achieve more efficient and accurate modelling of high stress concentration [7].

If two co-planar crack surfaces are modelled directly using the displacement boundary integral equation, mathematical degeneration is caused [8]. In order to solve this problem, multi-domain formulations [9] were first used. However, artificial boundaries are required to be introduced in these formulations, which results in a large number of extra boundary elements. The so-called dual boundary element method (DBEM) [10] was proposed later to avoid introducing extra boundaries and to increase the computational efficiency. Since then, the DBEM has gained more and more popularity. The problem of the mathematical degeneration can be resolved using the DBEM because the upper crack surface is modelled using the displacement boundary integral equation, while the lower surface is modelled with the traction boundary integral equation.

Plate structures are commonly used in engineering, and thus solving the crack problems in plates using the BEM is of great interest. Although the three-dimensional DBEM [11] is general and applicable to a variety of engineering structures, it is not suitable for plates due to the inaccuracy in numerical integrations. This issue can be resolved using the DBEM formulated based on approximate theories.

Tension and bending are two basic types of loadings on plates. In the framework of linear elastic fracture mechanics, when a flat plate with a through-thickness crack is loaded in both tension and bending, the stress intensity factors can be obtained by superposition of the stress resultant intensity factors from plate bending and 2D generalized plane stress theories [12]. The DBEM formulation for the generalized plane stress theory [10] is well known and was proposed nearly three decades ago. When it comes to cracked plates under bending load, shear deformable theories (Reissner and Mindlin), rather than the classical theory (Kirchhoff), should be used because the latter theory is not accurate in representing stress concentration and not sufficient to satisfy three independent boundary conditions [12]. Based on the DBEM formulations for the generalized plane stress and Reissner plate bending theories, the static stress intensity factors for plates subjected to combined tension and bending loads have been estimated in Refs. [13, 14].

In this paper, the study in Ref. [13] is extended to dynamic crack problems. The elastodynamic equations are solved in the Laplace-transform domain, and hence inner discretization is not required to deal with the domain integrals from the inertial terms. Durbin's method [15] is used to carry out inverse Laplace transforms. The DBEM formulations and dynamic fundamental solutions for the 2D plane stress problems [16] and Mindlin bending problems [17, 18] are revisited. A mixed mode crack problem in a finite plate is solved using the proposed dual boundary element method.

2. The dual boundary integral equations in the Laplace transform domain

Throughout this paper, Greek subscripts (α,β,γ) vary from 1 to 2 and Roman subscripts (i,j,k) run from 1 to 3. The displacement boundary integral equation for the generalized plane stress is given by:

$$\begin{aligned} c_{\alpha\beta}(\mathbf{x}')\bar{u}_{\beta}(\mathbf{x}',s) = {} & \int_{\Gamma}\bar{U}_{\alpha\beta}(\mathbf{x}',\mathbf{x},s)\bar{t}_{\beta}(\mathbf{x},s)d\Gamma(\mathbf{x}) - \mathrm{CPV}\int_{\Gamma}\bar{T}_{\alpha\beta}(\mathbf{x}',\mathbf{x},s)\bar{u}_{\beta}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ & + \int_{\Omega}\bar{U}_{\alpha\beta}(\mathbf{x}',\mathbf{X},s)\bar{b}_{\beta}(\mathbf{X},s)d\Omega(\mathbf{X}) \end{aligned}, \qquad (1)$$

and for the Mindlin bending is as follows:

$$\begin{aligned} c_{ij}(\mathbf{x}')\bar{w}_{j}(\mathbf{x}',s) = {} & \int_{\Gamma}\bar{W}_{ij}(\mathbf{x}',\mathbf{x},s)\bar{p}_{j}(\mathbf{x},s)d\Gamma(\mathbf{x}) - \mathrm{CPV}\int_{\Gamma}\bar{P}_{ij}(\mathbf{x}',\mathbf{x},s)\bar{w}_{j}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ & + \int_{\Omega}\bar{W}_{i3}(\mathbf{x}',\mathbf{X},s)\bar{q}_{3}(\mathbf{X},s)d\Omega(\mathbf{X}) \end{aligned}, \qquad (2)$$

in which the variables designated by bar represent their Laplace transforms and s denotes the Laplace transform parameter; $\mathrm{CPV}\int_{\Gamma}$ denotes Cauchy principal value integral; $\mathbf{x}$ and $\mathbf{X}$ represent the field points on the boundary Γ and in the plate domain Ω, respectively; c_{ij} are the jump terms determined by the geometry at the source point $\mathbf{x}'$ on the boundary; $\bar{u}_{\beta}$ and $\bar{t}_{\beta}$ are in-plane displacement and tractions, respectively; $\bar{w}_{\alpha}$ are rotations and $\bar{w}_3$ is the out-of-plane displacement; $\bar{p}_{\alpha}$ denote bending moments and $\bar{p}_3$ represents shear traction; $\bar{b}_{\beta}$ and $\bar{q}_3$ are in-plane body forces and pressure load on the plate, respectively; $\bar{U}_{\alpha\beta}$, $\bar{T}_{\alpha\beta}$, $\bar{W}_{ij}$ and $\bar{P}_{ij}$ are Laplace transform fundamental solutions, which have been given in Refs. [16, 17].

The traction boundary integral equation for the generalized plane stress problems is:

$$\begin{aligned} \frac{1}{2}\bar{t}_{\beta}(\mathbf{x}',s) = n_{\alpha}(\mathbf{x}')\Big[& \mathrm{CPV}\int_{\Gamma}\bar{U}_{\gamma\alpha\beta}(\mathbf{x}',\mathbf{x},s)\bar{t}_{\gamma}(\mathbf{x},s)d\Gamma(\mathbf{x}) - \mathrm{HPV}\int_{\Gamma}\bar{T}_{\gamma\alpha\beta}(\mathbf{x}',\mathbf{x},s)\bar{u}_{\gamma}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ & + \int_{\Omega}\bar{U}_{\gamma\alpha\beta}(\mathbf{x}',\mathbf{X},s)\bar{b}_{\gamma}(\mathbf{X},s)d\Omega(\mathbf{X})\Big] \end{aligned}, \qquad (3)$$

and for the Mindlin bending is as follows:

$$\frac{1}{2}\bar{p}_{\alpha}(\mathbf{x}',s)=n_{\beta}\left(\mathbf{x}'\right)\Big[\mathrm{CPV}\int_{\Gamma}\bar{D}_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x},s)\bar{p}_{\gamma}(\mathbf{x},s)d\Gamma(\mathbf{x})+\int_{\Gamma}\bar{D}_{\alpha\beta 3}(\mathbf{x}',\mathbf{x},s)\bar{p}_{3}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ -\mathrm{HPV}\int_{\Gamma}\bar{S}_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x},s)\bar{w}_{\gamma}(\mathbf{x},s)d\Gamma(\mathbf{x})-\mathrm{CPV}\int_{\Gamma}\bar{S}_{\alpha\beta 3}(\mathbf{x}',\mathbf{x},s)\bar{w}_{3}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ +\int_{\Omega}\bar{D}_{\alpha\beta 3}(\mathbf{x}',\mathbf{X},s)\bar{q}_{3}(\mathbf{X},s)d\Omega(\mathbf{X})\Big] , \quad (4)$$

$$\frac{1}{2}\bar{p}_{3}(\mathbf{x}',s)=n_{\beta}\left(\mathbf{x}'\right)\Big[\int_{\Gamma}\bar{D}_{3\beta\gamma}(\mathbf{x}',\mathbf{x},s)\bar{p}_{\gamma}(\mathbf{x},s)d\Gamma(\mathbf{x})+\mathrm{CPV}\int_{\Gamma}\bar{D}_{3\beta 3}(\mathbf{x}',\mathbf{x},s)\bar{p}_{3}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ -\mathrm{CPV}\int_{\Gamma}\bar{S}_{3\beta\gamma}(\mathbf{x}',\mathbf{x},s)\bar{w}_{\gamma}(\mathbf{x},s)d\Gamma(\mathbf{x})-\mathrm{HPV}\int_{\Gamma}\bar{S}_{3\beta 3}(\mathbf{x}',\mathbf{x},s)\bar{w}_{3}(\mathbf{x},s)d\Gamma(\mathbf{x}) \\ +\int_{\Omega}\bar{D}_{3\beta 3}(\mathbf{x}',\mathbf{X},s)\bar{q}_{3}(\mathbf{X},s)d\Omega(\mathbf{X})\Big] , \quad (5)$$

where $\mathrm{HPV}\int_{\Gamma}$ denotes Hadamard principal value integral; n_{β} represent the components of a unit outward normal vector to the plate boundary; $\bar{U}_{\gamma\alpha\beta}$, $\bar{T}_{\gamma\alpha\beta}$, $\bar{D}_{k\beta j}$ and $\bar{S}_{k\beta j}$, fundamental solutions, can be found in Refs. [16, 17].

In this study, the displacement boundary integral equations (1) and (2) were used for the modelling of plate edges and the upper crack surface, while the lower crack surface was modelled with the traction boundary integral equations (3) - (5). The corresponding singular integrals in these boundary integral equations were calculated using the techniques introduced in Refs. [16, 17, 19].

In the discretization process, quadratic continuous elements and discontinuous element were adopted for plate edges and crack surfaces, respectively. The detailed information of these elements can be found in Ref. [10].

The inverse Laplace transforms were carried out using the formula given by Durbin [15]:

$$f(t)=2\frac{e^{at}}{T}\left\{\frac{1}{2}\mathrm{Re}\left[\bar{f}(a)\right]+\sum_{k=1}^{K}\mathrm{Re}\left[\bar{f}(s)\right]\cos k\frac{2\pi}{T}t-\sum_{k=1}^{K}\mathrm{Im}\left[\bar{f}(s)\right]\sin k\frac{2\pi}{T}t\right\}, \\ ,s=a+ik\frac{2\pi}{T}, \quad (6)$$

where s, a, T and K are the Laplace transform parameter, the real part of the Laplace transform parameter, the length of time and the maximum number of Laplace terms, respectively.

3. Stress intensity factors

Based on the generalized plane stress theory and Mindlin plate bending theory, two membrane crack modes and three bending crack modes can be described, respectively. Figure 1 illustrates these five crack modes. In these modes, stress resultant intensity factors ($K_{1\mathrm{m}}$, $K_{2\mathrm{m}}$, $K_{1\mathrm{b}}$, $K_{2\mathrm{b}}$ and $K_{3\mathrm{b}}$) are defined first, and the stress intensity factor (K_{I}, K_{II} and K_{III}) are able be obtained later using the relationship [13]:

$$K_{\mathrm{I}}=\frac{K_{1m}}{h}+K_{1b}\frac{12}{h^{3}}x_{3} \\ K_{\mathrm{II}}=\frac{K_{2m}}{h}+K_{2b}\frac{12}{h^{3}}x_{3} \quad , \\ K_{\mathrm{III}}=\frac{3}{2h}K_{3b}\left[1-\left(\frac{2x_{3}}{h}\right)^{2}\right] \quad (7)$$

where h denotes plate thickness; x_3 is out-of-plane coordinate and $x_3 = 0$ represents the mid-plane of the plate.

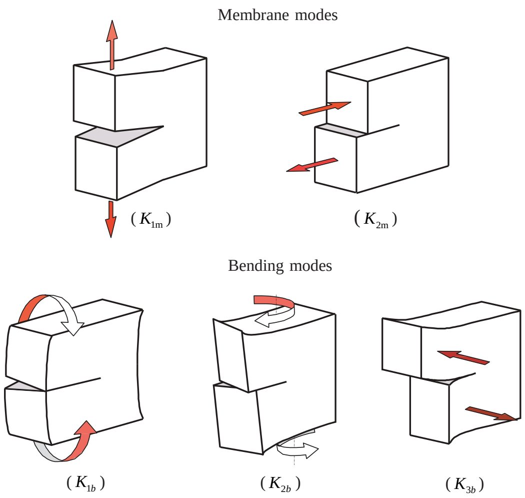

Figure 1. Crack modes in plates under membrane loading and Mindlin plates

In this paper, the method in Ref. [17] was adopted to estimate the stress resultant intensity factors. In order to represent the singular behaviour in the vicinity of the crack tip correctly, the crack-tip element was modified to be a discontinuous quarter-point element. After obtaining the crack opening displacements ($\Delta u_\alpha^{\mathrm{t}}$ and Δw_i^{t}) at the pair of nodes which are closest to the crack tip, the stress resultant intensity factors can be evaluated using the following expressions:

$$\begin{aligned} &K_{1\mathrm{m}} = \frac{3Eh}{4}\sqrt{\frac{2\pi}{l_e}}\Delta u_2^{\mathrm{t}}, K_{2\mathrm{m}} = \frac{3Eh}{4}\sqrt{\frac{2\pi}{l_e}}\Delta u_1^{\mathrm{t}}, \\ &K_{1\mathrm{b}} = \frac{Eh^3}{16}\sqrt{\frac{2\pi}{l_e}}\Delta w_2^{\mathrm{t}}, K_{2\mathrm{b}} = \frac{Eh^3}{16}\sqrt{\frac{2\pi}{l_e}}\Delta w_1^{\mathrm{t}}, K_{3\mathrm{b}} = \frac{5Eh}{6(1+\nu)}\sqrt{\frac{2\pi}{l_e}}\Delta w_3^{\mathrm{t}} \end{aligned}, \tag{8}$$

where E and ν denote Young's modulus and Poisson's ratio, respectively; l_e is the length of the crack-tip element.

4. Numerical examples

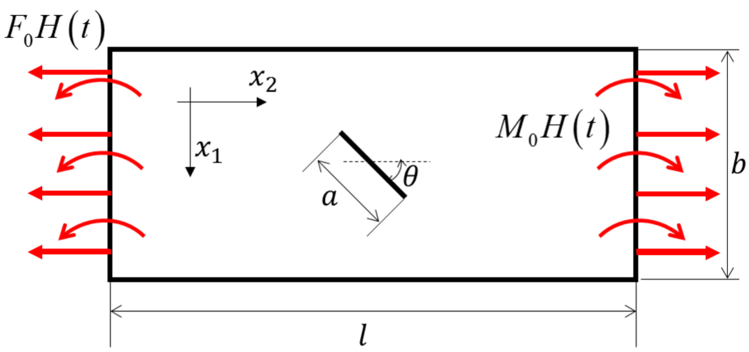

Figure 2. A rectangular plate with a central slant crack under tension and bending loads

A rectangular plate with a central slant crack, which is shown in Figure 2, is considered here. The plate has a length of l = 200 mm, a width of b = 100 mm and a thickness of h = 10 mm. The crack with a length of a = 20 mm makes an angle of $\theta = \pi / 4$ with the x_2 axis. The material properties of the plate are: $E = 69\,\text{Gpa}$, $\nu = 0.33$ and $\rho = 2700\,\text{kg/m}^3$. The tension force $F_0 H(t)$ and bending moment $M_0 H(t)$ are applied at the shorter edges of the plate, where F_0 and M_0 are the load amplitudes and $H(t)$ is the Heaviside function.

In the boundary element simulations, the crack surface, shorter edge and longer edge were discretized using 10, 20 and 40 elements, respectively. The inversion of Laplace transforms was conducted with 80 Laplace terms.

The normalized DSIFs $K_{1\text{m}}/K_{0\text{m}}$, $K_{2\text{m}}/K_{0\text{m}}$, $K_{1\text{b}}/K_{0\text{b}}$, $K_{2\text{b}}/K_{0\text{b}}$ and $K_{3\text{b}}/K_{03}$ are presented in Figure 3, where $K_{0\text{m}} = F_0\sqrt{\pi a/2}$, $K_{0\text{b}} = M_0\sqrt{a/2}$ and $K_{03} = M_0\sqrt{5a}\big/\left[(1+\nu)h\right]$. The time t is normalized with respect to a/c_2, where $c_2 = \sqrt{E\big/\left[2(1+\nu)\rho\right]}$ is the shear wave velocity.

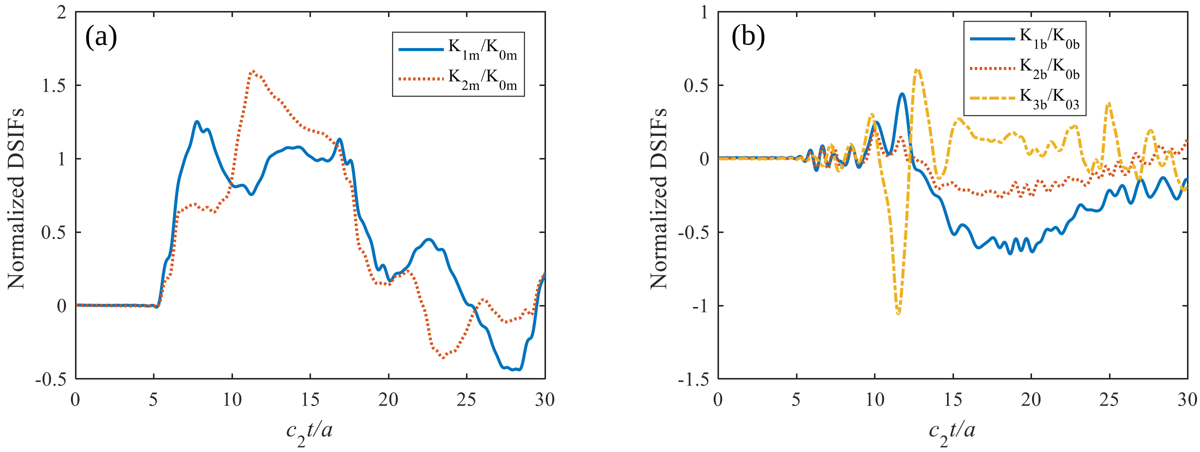

Figure 3. The normalized DSIFs for the rectangular plate with a central slant crack: (a) membrane modes, (b) bending modes

5. Conclusions

In this paper, the dual boundary element method has been applied to the dynamic fracture analysis of a cracked plate loaded by combined tension and bending. The dynamic problem was solved using a Laplace-transform method, which leaded to boundary-only BEM formulations without the domain discretization for the inertial terms. Crack opening displacements were used to compute the stress intensity factors. A mixed mode crack problem in a finite plate was solved using the proposed dual boundary element formulations. Finally, it is worth mentioning that in order to consider the plate thickness effect and a coupled out-of-plane fracture mode for plates under in-plane loads, the DBEM for a first-order plate theory (Kane-Mindlin theory) [20] is able to be used.

Acknowledgments

This research was supported by a grant provided by the China Scholarship Council (CSC).

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