Real Linear Substitutions with Equimodular Multipliers, and their Expression in terms of their Invariants.

Thesis submitted for the Degree of D. Sc. of the University of Glasgow by

David Gladstone Taylor,
M.A. (Glasgow and Cambridge);
sometime Scholar of St. John's College, Cambridge;
Lecturer in Pure Mathematics in the University College of South Wales and Monmouthshire.

Jan 6th 1915.

Introduction.

The first section of this paper is a preliminary discussion of linear substitutions in general and their invariants. The simple rules of combination of substitutions possessing the same full complement of linear invariants are noted, together with various classifications of such substitutions according to the nature of their invariants and multipliers. Nothing notably new occurs here, though the formula (6), which expresses the coefficients of a substitution in terms of its poles and multipliers, and which is copiously employed in what follows, does not appear to have been much used before. Some sections of a paper published in Proc. Edin. Math. Soc. Vol. XXX, (1911-12), are here reproduced.

Sections II, III deal with a particular type of substitutions coming under the general title of the paper, for which the relations between coefficients, invariants and multipliers are specially simple. Various geometrical applications are given for the cases of three and four variables, that to direction-cosines of a line being the most suggestive. In Section IV some of the results of Section II are generalized, and a complete determination is found of the coefficients of a real linear substitution in terms of a set of real linear and quadratic invariants, the multipliers being assumed equimodular. This finds an application in the substitution undergone by the direction-cosines $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ of a line when rotated through an angle $\underline{\theta}$ about any other line $(\underline{\alpha}, \underline{\beta}, \underline{\gamma})$; and the corresponding formulae for four dimensions are given. Sections II - IV are, to the best of my knowledge, new. A few of the paragraphs appeared in a previous paper (Proc. Edin. Math. Soc. Vol. XXXI (1912-13)).

Section V, the substance of which appeared in the same paper, deals with a substitution not included in the title, but very closely connected with those of Section II, and presenting some features worthy of notice.

The branch of algebra usually called Theory of Invariants is primarily concerned with quantics rather than with linear substitutions. It treats of such functions of the symbols appearing in certain quantics as remain unchanged, except for a constant multiplier, when the variables are subjected to an arbitrary linear substitution. Nothing in the results recalls the particular substitution employed, except the multiplier, which is a power of the modulus of the substitution; the essence of the results is that they are independent of the substitution adopted. The same is true of Boolian and orthogonal invariants. These ~~also~~ also belong to quantics; and though they come into evidence only when the variables are subjected to linear substitutions of certain restricted types, the results are still independent of the particular substitution employed.

The invariants which are the basis of the following discussion are, on the other hand, characteristic of the substitution itself, and not of any extraneous quantic. They consist of such functions of the variables and coefficients of the substitution as are reproduced unchanged, except for a constant multiplier, when the variables are subjected to the substitution. Such invariants exist of all degrees, but they can all be expressed

* See Burnside, Theory of Groups, 2nd edn., Ch. XIII;
L. E. Dickson, On the Structure of certain Linear Groups with Quadratic Invariants,
Proc. Lond. Math. Soc., (1st Series), Vol. XXX;
Burnside, Ibid. (2nd "), Vol. XII;
also Vols. VII, VIII etc.
cp. Hilton, Invariants of a Canonical Substitution,
Ibid., Vols X, XI.

in terms of those of degree unity. For a homogeneous substitution in n variables the greatest possible number of independent linear invariants is n. As a result of special relations among the coefficients the number may fall short of this maximum (cf. §10(2)). When the full complement exists, they may or may not be unique; but this last distinction is of small importance in the treatment, since the substitution can be completely determined from its multipliers and n independent linear invariants, whether unique or not.

Apart from the ordinary theory of Invariants, linear substitutions have been most frequently considered in recent years from the standpoint of the Theory of Finite Groups.* There is, for example, the question of the representation of any group of finite order as a group of linear substitutions, and the associated Hermitian invariant forms. With these investigations, which belong strictly to the theory of Groups, we are not here concerned. The limitation to substitutions of finite order, imposed by the connexion with finite group theory, is not adhered to in what follows. On the other hand, while a Hermitian invariant form is a joint possession of a substitution (supposed complex) and its conjugate, attention is chiefly directed in what follows upon real substitutions.

Though only occasional mention is made of substitutions of finite order, that is, such as reproduce the original variables, except for a constant multiplier, after a finite number of applications, the interest of substitutions with equimodular multipliers largely consists in this, that they include, and share many of the properties of, substitutions of finite order.

Section I.

Linear Substitutions in General.

§1. Let $\alpha, \beta, \ldots$ be symbols taking the values $(1, 2, \ldots n)$. Then we may denote by (l) the substitution in n variables

$$x'_\alpha = \sum_\beta l_{\alpha\beta} x_\beta. \qquad (1)$$

The application of (l) followed by (m) ~~leads~~ yields the substitution $(m)(l)$ given by

$$\begin{aligned} x''_\gamma &= \sum_\alpha m_{\gamma\alpha} x'_\alpha \\ &= \sum_\beta \left(\sum_\alpha m_{\gamma\alpha} l_{\alpha\beta}\right) x_\beta ; \end{aligned} \qquad (2)$$

thus as a rule (l), (m) are not permutable.

For the result of r repetitions of (l) we may use the notations $(l)^r$ and

$$x_\alpha^{(r)} = \sum_\beta l_{\alpha\beta}^{(r)} x_\beta.$$

§2. The Space Interpretation. The variables may be regarded as homogeneous "point" coordinates in space of $(n-1)$ dimensions. A set of values $x_{\alpha,\varepsilon}$ $(\alpha = 1, 2, \ldots n)$ will define a "point" P_ε, and a linear equation

$$L = \sum_\alpha p_\alpha x_\alpha = 0$$

may be said to specify an "$(n-2)$-plane", i.e., a locus of first degree ~~of~~ $(n-2)$ dimensions. Two sets of values of the x determine the same point when the ratios of corresponding values are equal; and similarly for the p. The point P' into which P is changed, or the locus L' into which L is changed by the substitution, may be called the transformed of P or L respectively.

§3. The Determinant of the Coefficients. If the determinant of the coefficients is equal to zero, the substitution is still valid, but the new or accented variables are no longer independent; the old variables cannot be expressed in terms of the new; in other words, the inverse of the substitution does not

* Hilton, _Finite Groups_, p. 20.

exist. In the sequel the determinant will be regarded as different from zero, so that the substitution always possesses an inverse.

§4. The Poles of the Substitution.*

If a point is unaltered by the substitution, its coordinates must satisfy for some value of k the equations

$$k x_\alpha = \sum_\beta l_{\alpha\beta} x_\beta, \quad (\alpha = 1, 2, --- n). \quad --- \quad (3)$$

Eliminating the x we obtain the characteristic equation for k:

$$\prod_{\varepsilon=1}^{n} (k_\varepsilon - k) = \begin{vmatrix} l_{11}-k, & l_{12}, & --- & l_{1n} \\ l_{21}, & l_{22}-k, & --- & l_{2n} \\ - & - & - & - \\ l_{n1}, & l_{n2}, & --- & l_{nn}-k \end{vmatrix} = 0. \quad --- (4)$$

All the roots being for the present assumed distinct, each root k_ε when substituted in (3) determines the coordinates of an invariant point or pole P_ε of the substitution; and k_ε is called the multiplier associated with the pole P_ε.

§5. The Determinant of the Poles: Expression of the Coefficients in terms of Poles and Multipliers.

Let us assume that the n points P_ε just obtained do not lie on an $(n-2)$-plane. Then, the coordinates of P_ε being denoted by $x_{\alpha,\varepsilon}$ $(\alpha = 1, 2, --- n)$, the determinant of the coordinates of the poles,

$$D = \begin{vmatrix} x_{11}, & x_{21}, & --- & x_{n1} \\ x_{12}, & x_{22}, & --- & x_{n2} \\ - & - & - & - \\ x_{1n}, & x_{2n}, & --- & x_{nn} \end{vmatrix}, \quad ---- \quad (5)$$

does not vanish.

Let $X_{\alpha\beta}$ as usual denote the cofactor of $x_{\alpha\beta}$ in D; then on substituting in (3) the coordinates of the P_ε and the appropriate values of k, we obtain n^2 equations of the form

$$l_{\alpha 1} x_{1\varepsilon} + l_{\alpha 2} x_{2\varepsilon} + \cdots + l_{\alpha n} x_{n\varepsilon} = k_\varepsilon x_{\alpha\varepsilon}.$$

Solving for the $\underline{l}$ the $\underline{n}$ equations obtained by keeping $\underline{\alpha}$ constant and making $\underline{\varepsilon}$ vary, we find

$$D\, l_{\alpha\beta} = \sum_\varepsilon k_\varepsilon x_{\alpha\varepsilon} X_{\beta\varepsilon}. \qquad (6)$$

This formula gives the ~~coords~~ coefficients of the substitution in terms of the multipliers $\underline{k_\varepsilon}$ and the coordinates of the poles. The right-hand member is linear in the multipliers, and both are linear in the coordinates of each pole. Thus a reduction of the multipliers in any constant ratio may be effected by reducing the coefficients in the same ratio, and *vice versa*. Any one coefficient or any one multiplier may be chosen at will.

§6. ## Substitutions Possessing the Same Poles.

(i) Let a second substitution have the same poles as the first, its coefficients and multipliers being distinguished by an accent. Then, $\underline{c}$, $\underline{c'}$ denoting any two constants, and $\underline{\lambda_{\alpha\beta}}$, $\underline{K_\varepsilon}$ being defined by

$$\lambda_{\alpha\beta} = c\, l_{\alpha\beta} + c' l'_{\alpha\beta}, \quad K_\varepsilon = c k_\varepsilon + c' k'_\varepsilon,$$

the substitution (λ) will have the same poles as $(\underline{l})$, $(\underline{l'})$, and for its multipliers the quantities $\underline{K_\varepsilon}$. This may be called *addition of substitutions*; it enables us to build up a substitution by means of two or more others possessing the same poles, but with simpler multipliers.

(ii) Consider the effect, on the coordinates of one of the common poles P_ε, of the substitution $(\underline{l'})(\underline{l})$, i.e., $(\underline{l})$ followed by $(\underline{l'})$. The first stage changes $\underline{x_{\alpha\varepsilon}}$ into $\underline{k_\varepsilon}\, \underline{x_{\alpha\varepsilon}}$, and the second changes this into $\underline{k'_\varepsilon}\, \underline{k_\varepsilon}\, \underline{x_{\alpha\varepsilon}}$. Thus the resultant substitution possesses the same poles as the components, and, for the multiplier associated with each pole, the product of the corresponding multipliers of the components. This may be called *multiplication*

* Strictly, it is ~~a multiplication~~ (Hilton, *Finite Groups*, p. 23), ~~a~~ a similarity substitution, reducing to the identical substitution when $k=1$.

of substitutions, and can at once be extended to any number of substitutions possessing the same poles. It is clear also that the resultant is independent of the order in which the components are applied; in other words, copolar substitutions are permutable.

(iii) As a special case, the effect of r repetitions of the substitution (l) on the coordinates of the pole P_ε is simply to multiply them by k_ε^r; and the coefficients of the resultant $(l)^r$ are given by

$$D\, l_{\alpha\beta}^{(r)} = \sum_\varepsilon k_\varepsilon^r\, x_{\alpha\varepsilon}\, X_{\beta\varepsilon}; \quad - - - - - \quad (4)$$

the coefficients of the r-times-repeated substitution are linear functions of the rth powers of the multipliers of the original substitution, with coefficients independent of r.

(iv). Suppose we are given n points P_ε not lying on an $(n-2)$-plane. If we give to all the multipliers the same value k, the substitution will reduce to

$$x'_\alpha = k x_\alpha, \quad (\alpha = 1, 2, --- n),$$

which, since it leaves each point unchanged in position, may be for present purposes called the identical substitution.* It follows that the inverse of the substitution (l) will be obtained by taking as multipliers $k_1^{-1}, k_2^{-1}, ---$; and similarly for $(l)^{-r}$, where r is a positive integer.

X

(v) Let $(l)^{p/q}$ denote the substitution formed with multipliers $k_\varepsilon^{p/q}$, any determinations of these multiform expressions being taken, and p, q denoting integers. The possible substitutions of this type are nq in number, of which n are effectively distinct. Each of them, on q repetitions, will yield $(l)^p$. We have thus established a complete index law, negative and fractional indices included, for a substitution

q^n, q^{n-1}?

possessing n independent poles.

§7. The Linear Invariants.

The equation
$$\xi_\varepsilon \equiv \sum_\alpha X_{\alpha\varepsilon} x_\alpha = 0 \qquad - - - - \qquad (8)$$
denotes the $(n-2)$-plane passing through the $(n-1)$ poles other than P_ε, and is therefore invariant. By giving to ε the values $1, 2, \dots n$ we thus obtain the n linear invariants of the substitution. Further, ξ_ε is reproduced with the same multiplier as the coordinates of P_ε. For, the effect of the substitution is to change ξ_ε into ξ'_ε where
$$\xi'_\varepsilon = \sum_{r=1}^{n} \left(\sum_s X_{s\varepsilon} l_{sr}\right) x_r ;$$
when we substitute for the l from (6), the coefficient of x_r becomes
$$D^{-1} \sum_{t=1}^{n} k_t \left(\sum_{s=1}^{n} x_{st} X_{s\varepsilon}\right) X_{rt} ;$$
and the expression in brackets here vanishes for $t \neq \varepsilon$, and is equal to D for $t = \varepsilon$; hence

$r\varepsilon]$?
$$\xi'_\varepsilon = k_\varepsilon \sum_\varepsilon X_r x_r = k_\varepsilon \xi_\varepsilon ;$$
which proves both that ξ_ε is invariant and that its multiplier is k_ε.

§8. Classification of Substitutions by means of Poles and Multipliers.

While many substitutions can be completely specified by giving their poles and multipliers, this is not true of all. Equalities among the roots of the characteristic equation may (though they do not of necessity do so) involve a reduction in the number of distinct poles; other data besides poles and multipliers will then be necessary for complete specification of the substitution.

The general substitution (1) contains n^2 constants, of which (n^2-1) are effectively independent. The knowledge of P', the transformed (§2) of any point P, involves $(n-1)$ relations between these constants; hence in general $(n+1)$ such pairs of related points are necessary and sufficient to determine the substitution. The poles, when distinct, constitute n pairs, each pole coinciding with its transformed; the knowledge of the multipliers is equivalent, as is seen from (1) and (6), to that of an $(n+1)$th pair. But when the number of effectively independent poles is reduced, the data yielded by them must be supplemented by a knowledge of other related point-pairs.

Suppose we are building up a substitution from its poles and multipliers. If the poles do not lie on an $(n-2)$-plane (or in other words if the linear invariants are linearly independent), and the multipliers all different, then by (6) the substitution is unique, and it possesses no other pole or linear invariant.

Suppose next ~~that~~ two of the multipliers are chosen equal, say $k_2 = k_1$. Then not only the coordinates of the corresponding poles P_1, P_2, but ~~all~~ also those of every point on the line (one-dimensional first-degree locus) joining them, will be reproduced with the same multiplier k_1. Hence every point on this line is invariant, and any two of them may be chosen to replace P_1, P_2 without altering the substitution. Similarly every $(n-2)$-plane of the form

$$\lambda\xi_1 + \mu\xi_2 \qquad (\lambda, \mu \text{ constants})$$

will be invariant with multiplier k_1, and any two of them may be chosen instead of ξ_1, ξ_2.

If m of the multipliers are chosen equal, the m corresponding poles will define likewise an $(m-1)$-plane, every point of which will be invariant; and any m points lying thereon, but not lying on an $(m-2)$-plane, might be chosen to replace the m given poles. Also any linear function of the corresponding invariants will be an invariant, and any m of them which are linearly independent would serve in place of the m given ones.

As an extreme case, when all the poles are distinct and not on an $(n-2)$-plane, and the multipliers are all equal, the substitution is identical (cf. § 6(iv)).

Thus, provided there are n distinct poles, not lying on an $(n-2)$-plane (or provided there are n linearly independent linear invariants), there is a unique substitution for a given set of multipliers, whatever equalities may exist among these latter.

§ 9. Approaching the question from the other side, let us assume that for a substitution of which the coefficients are given, the characteristic equation has two equal roots, say $k_1 = k_2$. The remaining poles $P_3, P_4, --- P_n$ will be determined uniquely as before; but instead of a unique determination of P_1, P_2, we shall have one of the following alternatives.

(i) Of the n equations obtained from (3) by writing k_1 instead of k, it may be that only $(n-2)$ are independent.* The coordinates of P_1 are then not

---

* The extension of the discussion to the case in which fewer than $(n-2)$ are independent is obvious.

determinate, but involve an arbitrary constant. Hence with the multiplier $\underline{k}_1$ will be associated, not one or two definite poles, but a *line* of poles. In this case any two points on the line may be taken for P_1, P_2, and the specification of the substitution in terms of poles and multipliers is unimpaired.

(ii) Of the $\underline{n}$ equations for the coordinates of P_1, $(\underline{n}-1)$ may be independent, and yield, as associated with the repeated root $\underline{k}_1$, only a single pole P_1 and a single linear invariant ξ_1. The pole P_2 has now become coincident with P_1, hence ξ_1 will pass through P_1. There are now no longer sufficient poles to determine the substitution; for its complete determination we shall require, in addition to the $(\underline{n}-1)$ poles and multipliers, the knowledge of the position of P', the transformed of some definite point P which is not a pole. The nature of the alternatives which arise as a consequence of further equalities among the roots may be easily gathered from the above simple case.

Examples are appended of the two types of substitution, those possessing and those not possessing the full complement of poles or linear invariants, for the case of three variables.

§10. Examples for the Case of Three Variables.

(i) In the substitution

$$\left.\begin{aligned} x_1' &= h_3 x_1 + h_2 x_2 + h_1 x_3 \\ x_2' &= h_1 x_1 + h_3 x_2 + h_2 x_3 \\ x_3' &= h_2 x_1 + h_1 x_2 + h_3 x_3 \end{aligned}\right\}, \qquad (9)$$

where h_1, h_2, h_3 are ~~arbitrary~~ any constants, the poles, invariants, multipliers, and coefficients in terms of multipliers, are as follows, ω denoting as usual an imaginary cube root of unity:

Suffix	P	ξ	k	3h
1	$(\omega^2, \omega, 1)$	$\omega x_1 + \omega^2 x_2 + x_3$	$\omega h_1 + \omega^2 h_2 + h_3$	$\omega^2 k_1 + \omega k_2 + k_3$
2	$(\omega, \omega^2, 1)$	$\omega^2 x_1 + \omega x_2 + x_3$	$\omega^2 h_1 + \omega h_2 + h_3$	$\omega k_1 + \omega^2 k_2 + k_3$
3	$(1, 1, 1)$	$x_1 + x_2 + x_3$	$h_1 + h_2 + h_3$	$k_1 + k_2 + k_3$

Here as a rule the multipliers as well as the poles are distinct, and the substitution is completely specified by multipliers and poles. Even when $k_1 = k_2$ (which involves $h_1 = h_2$), the poles P_1, P_2, though no longer definite, may be taken at random on the line

$$\xi_3 \equiv x_1 + x_2 + x_3 = 0,$$

every point of which is then invariant; and the substitution is still completely specified by multipliers and poles.

When all three roots are equal (i.e., $k_1 = k_2 = k_3$), then $h_1 = h_2 = 0$, and the substitution becomes, except for the multiplier h_3, identical.

Substitutions of this type, in three or more variables, are the subject of Sections II, III.

(ii). The substitution

$$\left.\begin{aligned} x_1' &= k_1 x_1 \\ x_2' &= l_{21} x_1 + k_2 x_2 \\ x_3' &= l_{31} x_1 + l_{32} x_2 + k_3 x_3 \end{aligned}\right\}, \quad - \; - \; - \; - \qquad (10)$$

evidently has the multipliers k_1, k_2, k_3.

First, let these multipliers be all different.

Then there are three distinct poles, in terms of which, combined with the multipliers, the substitution can be completely specified, namely:

$$\begin{array}{llll} P_1 \{ & (k_2-k_1)(k_3-k_1), & -l_{21}(k_3-k_1), & l_{21}l_{32} - l_{31}(k_2-k_1) \}, \\ P_2 \{ & 0, & -(k_3-k_1), & l_{32} \}, \\ P_3 \{ & 0, & 0, & 1 \}. \end{array}$$

Second, let $k_2 = k_1$. Then P_2 coincides with P_1 at a definite point, and the only poles and invariants are

$$P_1 \quad (0, k_3-k_1, -l_{32}), \qquad \xi_1 \equiv x_1 = 0,$$

$$P_3 \quad (0, \quad 0, \quad 1),$$

$$\xi_3 \equiv \{l_{32}l_{21} - (k_3-k_1)l_{31}\}x_1 + (k_3-k_1)l_{32}x_2 + (k_3-k_1)^2 x_3 = 0;$$

where obviously P_3 lies on ξ_1, and P_1 is the intersection of ξ_1, ξ_3. In this case the expression of the coefficients in terms of poles and multipliers breaks down.

Third, let $k_3 = k_2 = k_1$. We are now reduced to a single pole, and a single invariant line passing through it, namely

$$P_1 (0, 0, 1), \qquad \xi_1 \equiv x_1 = 0,$$

which again are insufficient to determine the substitution.

Thus a substitution may have all the roots of its characteristic equation equal to unity, and yet not be identical; but this cannot happen when there are n effectively independent poles.

In what follows attention is confined to substitutions possessing n independent poles, and therefore admitting of complete specification in terms of poles and multipliers.

§ 11. <u>Some Simple Types.</u> in n variables

Substitutions may be classed according to the system of equalities among their multipliers. If s_1 of the multipliers are equal to k_1, s_2 to k_2, and so on, we may denote the type by the symbol $(s_1, s_2, \dots)$, the order of the numbers within brackets being immaterial. The number of types is equal to the number of partitions of n in integers.

Suppose that s of the multipliers (say the first s) have the value k_1, and the remainder the value k_2, the type is $(s, n-s)$, and we have

$$\left.\begin{aligned} l_{\alpha\beta} &= (k_1 - k_2) \sum_{\varepsilon=1}^{s} x_{\alpha\varepsilon} X_{\beta\varepsilon} / D, \quad \beta \neq \alpha \\ l_{\alpha\alpha} &= (k_1 - k_2) \sum_{\varepsilon=1}^{s} x_{\alpha\varepsilon} X_{\alpha\varepsilon} / D + k_2 \end{aligned}\right\}; \qquad (11)$$

in particular if $s = 1$, and if we write

$$\lambda_\alpha = x_{\alpha 1}, \quad \mu_\alpha = X_{\alpha 1}/D,$$

then

$$\left.\begin{aligned} l_{\alpha\beta} &= (k_1 - k_2)\lambda_\alpha \mu_\beta \\ l_{\alpha\alpha} &= (k_1 - k_2)\lambda_\alpha \mu_\alpha + k_2 \end{aligned}\right\}, \qquad (12)$$

where $\sum_\alpha \lambda_\alpha \mu_\alpha = 1$;

and the substitution takes the form

$$x'_\alpha = k_2 x_\alpha + (k_1 - k_2)\lambda_\alpha \sum_\beta \mu_\beta x_\beta, \quad (\alpha, \beta = 1, 2, \dots n). \qquad (13)$$

Similarly the general formula in the case of the type $(2, n-2)$ is

$$x'_\alpha = k_2 x_\alpha + (k_1 - k_2)\left(\lambda_\alpha \sum_\beta \mu_\beta x_\beta + \lambda'_\alpha \sum_\beta \mu'_\beta x_\beta\right),$$

where $\sum_\alpha \lambda_\alpha \mu_\alpha = \sum_\alpha \lambda'_\alpha \mu'_\alpha = 1$,

and so in other cases.

It is to be noted ~~that~~ the coordinates of the poles associated with the repeated root only appear in the expressions for the $\underline{X}$ and for $\underline{D}$, and that the number of arbitrary constants is much reduced in (12), (13). This was to be expected, since the equality of ($\underline{n}-1$) multipliers in the case of (12) confers polarity on every point of the ($\underline{n}-2$)-plane defined by the associated poles, so that the invariant points are determined when we know the single isolated pole P_1 and the invariant ($\underline{n}-2$)-plane ξ_1; the latter being invariant in the special sense that every point on it is so. Similarly in other cases.

We may represent (12) pictorially by the determinant of its coefficients, taking out the factor ($\underline{k}_1-\underline{k}_2$) from each row, and writing $\underline{k}_2/(\underline{k}_1-\underline{k}_2) = \underline{\sigma}$, as follows:

$$(k_1-k_2)^n \begin{vmatrix} \lambda_1\mu_1+\sigma, & \lambda_1\mu_2, & --- & \lambda_1\mu_n \\ \lambda_2\mu_1, & \lambda_2\mu_2+\sigma, & & \lambda_2\mu_n \\ -- & -- & -- & -- \\ \lambda_n\mu_1, & \lambda_n\mu_2, & --- & \lambda_n\mu_n+\sigma \end{vmatrix} \cdot \quad --- (14)$$

This substitution is, as just stated, completely given when we know, in addition to the multipliers $\underline{k}_1, \underline{k}_2$, the pole and linear invariant ($\underline{P}_1, \underline{\xi}_1$) corresponding to the non-repeated root $\underline{k}_1$. If ξ_1 is so related to P_1 that its equation is of the form

$$\xi_1 = \frac{x_1}{x_{11}} + \frac{x_2}{x_{21}} + --- + \frac{x_n}{x_{n1}} = 0,$$

we may, by analogy with the case of three variables, call $\underline{P}_1, \underline{\xi}_1$ harmonic pole and polar with respect to the frame of reference. We then have

$$\lambda_1\mu_1 = \lambda_2\mu_2 = --- = \lambda_n\mu_n = 1/n \text{ say,}$$

and (14) takes the form

$$\left(\frac{k_1-k_2}{n}\right)^n \begin{vmatrix} 1+n\sigma, & \lambda_1\lambda_2^{-1}, & --- & \lambda_1\lambda_n^{-1} \\ \lambda_2\lambda_1^{-1}, & 1+n\sigma, & --- & \lambda_2\lambda_n^{-1} \\ --- & --- & --- & --- \\ \lambda_n\lambda_1^{-1}, & \lambda_n\lambda_2^{-1}, & --- & 1+n\sigma \end{vmatrix}. \quad ---(15)$$

§ 12. <u>Substitutions of Finite Order.</u>

The conditions that a linear substitution in <u>n</u> variables be of finite order <u>r</u> are (Burnside, <u>Theory of Groups</u>, 2nd <u>edn.</u>, § 193):

(i) it must possess <u>n</u> effectively independent poles (or linear invariants); in other words, it must admit of complete specification in terms of poles and multipliers;

(ii) the mutual ratios of its multipliers must be <u>r</u>th roots of unity.

If we assume the first of these conditions, the second follows at once by (4). Many examples of substitutions of finite order will occur later; the formulae of last article also will yield such, provided <u>$k_1 : k_2$</u> is a root of unity, other than unity itself. Two simple examples may be noted, both of order two, which will come up later. The factor outside is so chosen in each that the multiplier, with which the variables are reproduced after two applications, is equal to unity:

$$\left(\frac{2}{3}\right)^3 \begin{vmatrix} -\frac{1}{2}, & 1, & 1 \\ 1, & -\frac{1}{2}, & 1 \\ 1, & 1, & -\frac{1}{2} \end{vmatrix}, \quad \left(\frac{1}{2}\right)^4 \begin{vmatrix} -1, & 1, & 1, & 1 \\ 1, & -1, & 1, & 1 \\ 1, & 1, & -1, & 1 \\ 1, & 1, & 1, & -1 \end{vmatrix}.$$

(<u>Cp.</u> §§ 25, 32).

§ 13. Invariants of Higher Degrees.

It is easy to see that the only invariants of degrees above the first are combinations of the linear invariants; thus a specimen of an invariant of degree p will be

$$\prod_{s=1}^{p} \xi_s, \quad \text{multiplier} \prod_{s=1}^{p} k_s.$$

Just as, in the case of linear invariants, equalities between multipliers endow with invariancy every linear function of the corresponding invariants, so if any two or more functions of the linear invariants possess the same multiplier, every linear function of them will also be an invariant. The number of linearly independent invariants of degree p is ${}_nH_p$*; but the system is not necessarily unique, even when the k_ϵ are all different. In the case discussed below, in which the multipliers consist of pairs of conjugate imaginaries with a common modulus, we obtain a set of quadratic invariants with common multiplier, which are of great importance in the treatment. The case is similar for all real substitutions, except that the multipliers ~~are not necessarily equal.~~ of the quadratic invariants are not as a rule equal.

---

* Proc. Edin. Math. Soc. Vol XXX (1911–12), p. 126.

Section II.

Linear Substitutions whose Poles are the Unit Points.

§ 14. This section treats of a class of substitutions for which the roots of the characteristic equation can be written down as linear functions of the coefficients of the substitution, whatever the number of variables. In Section IV some of the methods and results will be extended to more general cases. The case of real coefficients and equimodular multipliers is the one mainly kept in view.

§ 15. The Unit Points. In any system of n homogeneous point-coordinates there are n special sets of values which may be said to define the n unit points of this system. They are

$$P_\varepsilon \quad (x_{1\varepsilon}, x_{2\varepsilon}, --- x_{n\varepsilon}), \qquad (\varepsilon = 1, 2, --- n),$$

where

$$\rho^{\varepsilon} x_{1\varepsilon} = \rho^{2\varepsilon} x_{2\varepsilon} = \quad --- \quad = \rho^{(n-1)\varepsilon} x_{n-1,\varepsilon} = x_{n\varepsilon}, \quad --- (16)$$

ρ being any primitive nth root of unity. We shall as a rule assume $\rho = \exp(2\pi i/n)$, since any other admissible value would yield the same points, only in a different order. The $(n-2)$-plane passing through all of these points except P_ε is, as can at once be verified,

$$\xi_\varepsilon \equiv \sum_{s=1}^{n} \rho^{s\varepsilon} x_s = 0, \quad (\varepsilon = 1, 2, -- n). \quad --- (17)$$

§ 16. Consider the substitution

$$\left.\begin{array}{l} x_1' = h_n x_1 + h_{n-1} x_2 + --- + h_1 x_n \\ x_2' = h_1 x_1 + h_n x_2 + --- + h_2 x_n \\ \text{— — — — — —} \\ x_n' = h_{n-1} x_1 + h_{n-2} x_2 + --- + h_n x_n \end{array}\right\}, \quad --- \quad (18)$$

which may be described by saying that each row of its determinant consists of the same elements in the same cyclic order, with the same element h_n always in the leading diagonal. The value of the determinant is the product of n linear factors, which may be written as follows:*

$$k_\varepsilon = \sum_{s=1}^{n} \rho^{s\varepsilon} h_s, \quad (\varepsilon = 1, 2, \cdots n), \quad \text{---} \quad (19)$$

ρ being any primitive nth root of unity. Indeed it becomes clear that k_ε is a factor if we multiply the respective rows after the first by $\rho^\varepsilon, \rho^{2\varepsilon}, \cdots \rho^{(n-1)\varepsilon}$, and the respective columns after the first by the same quantities in reverse order, and then add all the rows (or columns) together. It is clear also that the above k_ε are the roots of the characteristic equation of the substitution; and it is easily verified that the pole and linear invariant with which k_ε is associated as multiplier are the unit-point P_ε and the $(n-2)$-plane ξ_ε of last article. Now equations (19) can be solved uniquely for the coefficients h_s in terms of the multipliers k_ε; it follows that <u>every linear substitution possessing the unit-points as poles is of the form (18)</u>.

The unit-points being all different, every substitution of this type can be completely specified in terms of its poles and multipliers; and this is true even if equalities exist among the latter.

§14. The actual expressions for the coefficients in terms of the multipliers are given by

$$n h_s = \sum_{\varepsilon=1}^{n} \rho^{-s\varepsilon} k_\varepsilon, \quad (s = 1, 2, \cdots n); \quad \text{---} \quad (20)$$

these may be obtained as follows:

---

* Scott, *Determinants*, p. 81.

The determinant

$$\Delta = \begin{vmatrix} \sigma_1, & \sigma_1^2, & --- & \sigma_1^{n-1}, & 1 \\ \sigma_2, & \sigma_2^2, & --- & \sigma_2^{n-1}, & 1 \\ - & - & - & - & - \\ \sigma_n, & \sigma_n^2, & --- & \sigma_n^{n-1}, & 1 \end{vmatrix},$$

in which the σ denote any quantities whatever, is the product of the $\frac{1}{2}n(n-1)$ linear factors

$$\Delta = \Pi(\sigma_s - \sigma_r), \quad \begin{pmatrix} r = 1, 2, --- s-1 \\ s = 2, 3, --- n \end{pmatrix}.$$

Let $f(\sigma_p)$ denote the product of those factors, $(n-1)$ in number, which involve σ_p; let (p,q) denote the coefficient of σ_p^q in $f(\sigma_p)$; and let $X_{p,q}$ denote the cofactor of the element in the pth row and qth column of Δ. Then $X_{p,q}$ is also the coefficient of σ_p^q in the expansion of Δ; and, since every factor of Δ which does not contain σ_p is also of necessity a factor of $X_{p,q}$, we have

$$X_{p,q}/\Delta = (p,q)/f(\sigma_p). \quad ---- (21)$$

Now writing $\sigma_\varepsilon = \rho^\varepsilon$ for all values of ε from 1 to n except p, we have

$$f(\sigma_p) = (-)^{n-p} \prod_\varepsilon (\sigma_p - \rho^\varepsilon);$$

whence if ρ is a primitive nth root of unity,

$$f(\sigma_p) = (-)^{n-p}\{\sigma_p^{n-1} + \rho^p \sigma_p^{n-2} + --- + \rho^{(n-q-1)p}\sigma_p^q + \cdots + \rho^{(n-1)p}\},$$

yielding

$$f(\rho^p) = (-)^{n-p} n \rho^{(n-1)p}, \quad (p,q) = (-)^{n-p}\rho^{(n-q-1)p},$$

and therefore by (21)

$$X_{p,q}/\Delta = \frac{1}{n}\rho^{-pq}.$$

But equations (19) when solved for the h yield

$$\Delta h_s = \sum_{\varepsilon=1}^{n} X_{\varepsilon s} k_\varepsilon,$$

hence

$$n h_s = \sum_{\varepsilon=1}^{n} \rho^{-s\varepsilon} k_\varepsilon, \quad (s = 1, 2, --- n),$$

as was to be proved.

§ 18. Coefficients and Multipliers.

(i) In general the coefficients, multipliers and variables of (18) are alike complex. When the multipliers are all different, only one of the linear invariants, namely ξ_n, has real coefficients when n is odd; and only two, namely $\xi_n, \xi_{\frac{1}{2}n}$ when n is even (see (14)). When equalities occur among the multipliers, which may take place as ~~the~~ a result of certain relations among the coefficients, additional linear invariants with real coefficients may appear.

(ii) Since substitutions with zero determinant are excluded, none of the multipliers can vanish. They cannot all be equal, else the substitution becomes identical. By reducing the coefficients in a constant ratio, any one of the multipliers, say k_n, can be made equal to unity.

(iii) We have seen (§ 8) that when any two of the multipliers, say $k_\varepsilon, k_{\varepsilon'}$ are equal, any linear function of the corresponding invariants, say $\lambda\xi_\varepsilon + \mu\xi_{\varepsilon'}$, where λ, μ are constants, is also invariant with the same multiplier. In particular let $k_\varepsilon = k_{n-\varepsilon}$; then since, for all values of s from 1 to n,

$$\rho^{s\varepsilon} + \rho^{s(n-\varepsilon)}, \quad i\left(\rho^{s\varepsilon} - \rho^{s(n-\varepsilon)}\right)$$

are real, it is clear from (14) ~~that~~ there will be two additional linear invariants with real coefficients; namely

$$\xi_\varepsilon + \xi_{n-\varepsilon}, \quad i\left(\xi_\varepsilon - \xi_{n-\varepsilon}\right).$$

(iv) If the coefficients h_s are all real, then by (19) k_n is real, and $k_\varepsilon, k_{n-\varepsilon}$ are conjugate imaginaries for $\varepsilon = 1, 2, ---$; and conversely. This implies ~~that~~

when $\underline{n}$ is even, $\underline{k}_{\frac{1}{2}n}$ is real. As a special case $\underline{k}_\varepsilon, \underline{k}_{n-\varepsilon}$ may be real and equal.

Similarly if the multipliers $\underline{k}_\varepsilon$ are all real, then by (20) $\underline{h}_n$ is real, and $\underline{h}_\varepsilon, \underline{h}_{n-\varepsilon}$ are conjugate imaginaries.

~~(VI)~~

(V). Confining attention to the former case, the imaginary linear invariants can then be combined into <u>quadratic invariants with real coefficients</u>,* namely

$$\xi_\varepsilon \xi_{n-\varepsilon}, \quad \text{multiplier } |k_\varepsilon|^2, \ (\varepsilon = 1, 2, \cdots p),$$

where $p = \frac{1}{2}(\underline{n}-1)$ or $(\frac{1}{2}\underline{n}-1)$ according as $\underline{n}$ is ~~even~~ odd or even. These real quadratic invariants are characteristic of the case in which the coefficients of the substitution are real.

(vi) When, in addition to the coefficients being real, the multipliers have all the same modulus, certain relations subsist between the coefficients (see below, § 20), and the quadratic invariants have all the same multiplier. This is the case of <u>equimodular multipliers</u> discussed below.

(vii) Suppose that, in addition to the foregoing conditions, the amplitudes of the multipliers are all of the form $2s\pi/\underline{r}$ ($\underline{s}, \underline{r}$ integers), then the mutual ratios of the multipliers will be $\underline{r}$th roots of unity, and the substitution will be of <u>finite</u> <u>order $\underline{r}$</u>.

§ 19. <u>Notation for Quadratic Functions.</u> Placing the coefficients $\underline{h}_1, \underline{h}_2, \cdots \underline{h}_n$ in order round a circle, multiply each by that coefficient which

---

* See above, § 13.

is $\underline{s}$ places in advance of it, and denote the sum of such products by $\underline{H}_s$, thus:

$$H_s = h_1 h_{s+1} + h_2 h_{s+2} + \cdots + h_n h_s; \quad --- \quad (22)$$

and let $\underline{H}_0$ or $\underline{H}_n$ be written indifferently for the quantity $\sum_s \underline{h}_s^2$. Let $\underline{t}$ stand for $\frac{1}{2}(\underline{n}-1)$ or $\frac{1}{2}\underline{n}$ according as $\underline{n}$ is odd or even. Then there will be $(\underline{t}+1)$ distinct functions $\underline{H}_s$, defined by $\underline{s} = 0, 1, \cdots \underline{t}$. Each will consist of a sum of $\underline{n}$ products; except $\underline{H}_{\frac{1}{2}n}$ in the case of $\underline{n}$ even, which will contain only half the number: thus for $\underline{n} = 4$,

$$H_2 = h_1 h_3 + h_2 h_4,$$

containing but two products.

This notation, explained for the coefficients $\underline{h}$, ~~will~~ may be used also for the variables $\underline{x}$ and multipliers $\underline{k}$.

§20. **Relations between the Coefficients.**

We shall prove that when the coefficients are real and the multipliers equimodular, $\underline{H}_0$ is equal to the square of the common modulus, and the other functions $\underline{H}_s$ vanish.

Let the common modulus be $\underline{\kappa}$; then equating $\underline{\kappa}^2$ to $|\underline{k}_\varepsilon|^2$ as obtained from (19), we obtain the $(\underline{t}+1)$ equations

$$(H_0 - \kappa^2) + 2H_1 \cos\frac{2\varepsilon\pi}{n} + \cdots + 2H_t \cos\frac{2t\varepsilon\pi}{n} = 0, \quad (\varepsilon = 0, 1, \cdots \underline{t}). \quad --- \quad (23)$$

Now it is easily shown that the determinant of the coefficients of these equations does not vanish; hence

$$H_0 = \kappa^2, \quad H_1 = H_2 = \cdots = H_t = 0. \quad ---- \quad (24)$$

Since $\underline{H}_0$, $\underline{H}_n$ are identical, and $\underline{k}_n$ is real, we have

$$k_n^2 = H_n = (\Sigma h)^2.$$

§21. <u>The Quadratic Invariants.</u> Assuming only, to begin with, that the coefficients $\underline{h}_s$ are real, we have by §18 (i), (v), the following quadratic invariants with real coefficients:

$$\xi_\varepsilon \xi_{n-\varepsilon}, \quad \text{multiplier } |k_\varepsilon|^2, \quad (\varepsilon = 1, 2, --- t)$$
$$\xi_n^2, \quad \text{"} \quad k_n^2.$$

Noting the similarity in form between (17) and (19), we have on the analogy of (23)

$$\xi_\varepsilon \xi_{n-\varepsilon} = X_0 + 2X_1 \cos\frac{2\varepsilon\pi}{n} + --- + 2X_t \cos\frac{2t\varepsilon\pi}{n}, \quad (\varepsilon = 0, 1, --- t), \qquad ---- (25)$$

with $X_0 = \Sigma x^2$, etc., as for H_0, H_1, etc., §19

where the case $\underline{\varepsilon} = 0$ corresponds to $\underline{\xi}_n^2$ above.

These expressions constitute the $(\underline{t}+1)$ quadratic invariants with real coefficients which exist independently of any equalities among the multipliers $\underline{k}_\varepsilon$ or their moduli.

Now let the multipliers be equimodular. Then these invariants all possess the same modulus $\underline{H}_0$; hence every linear function of them is also an invariant with the same modulus. It follows that, <u>when the coefficients are real and the multipliers equimodular, the $(\underline{t}+1)$ quadratic forms $\underline{X}_0, \underline{X}_1, -- \underline{X}_t$ are invariants with common multiplier H_0.</u>

§22. <u>Permutations.</u> Consider the set of multipliers

$$k_\varepsilon = \rho^{p\varepsilon}, \quad (\varepsilon = 1, 2, --- n), \qquad ----- (26)$$

where $\underline{p}$ is an integer. By (20) these yield

$$n h_s = \sum_\varepsilon \rho^{(p-s)\varepsilon};$$

hence all the $\underline{h}$ vanish except $\underline{h}_p$, which is equal to unity. Hence the substitution takes the form

$$x'_r = x_{n-p+r}, \quad (r = 1, 2, --- n),$$

a cyclic permutation. Conversely every cyclic permutation of the letters $\underline{x}$ corresponds to a set of multipliers of the form (26).

§ 23. Two Sets of Variables.

Let x_r, y_r, h_r, l_r ($r = 1, 2, \cdots n$) be any four sets of quantities; we shall use the notation

$$\left.\begin{aligned} U_r &= \sum_{s=1}^{n} x_s y_{r+s}, & V_r &= \sum_{s=1}^{n} x_s y_{r-s} \\ F_r &= \sum_{s=1}^{n} h_s l_{r+s}, & G_r &= \sum_{s=1}^{n} h_s l_{r-s} \\ H_r &= \sum_{s=1}^{n} h_s h_{r+s}, & H'_r &= \sum_{s=1}^{n} h_s h_{r-s} \end{aligned}\right\}, \quad --- \quad (27)$$

with the proviso that every suffix exceeding n shall be reduced by n, and every negative suffix increased by n.

Let the x be subjected to the substitution (18) with the h for coefficients, and the y to one of the same form, with the l for coefficients; and let accents denote the new values of U_r, V_r. Then

$$U'_r = \sum_s F_{r-s} U_s, \quad V'_r = \sum_s G_{r-s} V_s; \quad --- \quad (28)$$

thus the U, V are also transformed by substitutions of the form (18), but with F_r, G_r respectively in place of h_r. The substitutions which transform the x, y, U, V have therefore all the same poles; and if the multipliers corresponding to the pole P_ε in the four be respectively denoted by k_ε, k'_ε, K_ε, K'_ε, then

$$\begin{aligned} K_\varepsilon &= \sum_r \rho^{r\varepsilon} F_r \\ &= \left(\sum \rho^{-r\varepsilon} h_r\right)\left(\sum \rho^{r\varepsilon} l_r\right) \\ &= k'_\varepsilon \sum \rho^{-r\varepsilon} h_r; \\ K'_\varepsilon &= \sum_r \rho^{r\varepsilon} G_r \\ &= \left(\sum \rho^{r\varepsilon} h_r\right)\left(\sum \rho^{r\varepsilon} l_r\right) \\ &= k_\varepsilon k'_\varepsilon; \end{aligned}$$

thus the substitution transforming the V is simply equal to the product of those transforming the $\underline{x}, \underline{y}$.

If we write $\underline{l}_r = \underline{h}_r$ $(\underline{r} = 1, 2, \cdots \underline{n})$, then $\underline{F}_r, \underline{G}_r$ reduce respectively to $\underline{H}_r, \underline{H}'_r$; if we put $\underline{l}_r = \underline{h}_{r+p}$, they reduce respectively to $\underline{H}_{r+p}, \underline{H}'_{r+p}$, yielding substitutions for the $\underline{U}, \underline{V}$ which are cyclic permutations of those of the previous case. If we write $\underline{l}_r = \underline{h}_{n-r}$, then $\underline{F}_r, \underline{G}_r$ reduce to $\underline{H}'_{n-r}, \underline{H}_{n-r}$. Finally, writing $\underline{y}_r = \underline{x}_r$, we find that the quantities $\underline{X}_r$ (§§19, 21) are transformed by a substitution of form (18), with the quantities $\underline{H}_r$ for coefficients.

Assuming $\underline{l}_r = \underline{h}_r$, but $\underline{y}_r \neq \underline{x}_r$, let the $\underline{h}$ satisfy conditions (24). Then clearly all the quantities $\underline{U}_r$ are invariant, with the same multiplier as the $\underline{X}_r, \underline{Y}_r$.

Section III.

Applications to the Cases $n = 3, 4$.

Case of $n = 3$

§24. The poles, invariants etc. of the substitution of form (18) for three variables have been already given (§10(i)). If we assume the coefficients real, then by §18(iv) we may put

$$k_1 = \kappa e^{i\theta}, \quad k_2 = \kappa e^{-i\theta}, \quad (\kappa, \theta, k_3 \text{ real});$$

there are then two real quadratic invariants, viz.

$$\xi_1 \xi_2 = X_0 - X_1, \text{ multiplier } \kappa^2,$$
$$\xi_3^2 = X_0 + 2X_1, \quad \text{"} \quad k_3^2.$$

If in addition we assume the multipliers equimodular, then $\kappa = k_3$, and by (24)

$$H_1 \equiv h_2 h_3 + h_3 h_1 + h_1 h_2 = 0, \quad - - - \quad (30)$$

while X_0, X_1 become invariants with common multiplier k_3^2. The coefficients can then be expressed in terms of k_3, θ as follows:

$$h_s = \tfrac{1}{3} k_3 \left(2 \cos \overline{\theta - 2s\pi/3} + 1\right), \quad (s = 1, 2, 3), \; --- (31)$$

and we may without loss assume $k_3 = 1$ if desirable. The substitution will now be of finite order r provided θ is of the form $2p\pi/r$, where p is integral.

§25. **Medians of a Triangle.**

Let α', β', γ' denote the lengths of the medians of a triangle whose sides are of lengths a, b, c; and let α'', β'', γ'' be the medians of the triangle whose sides are α', β', γ'; then from the well-known relations

$$\alpha'^2 = -\tfrac{1}{4} a^2 + \tfrac{1}{2} b^2 + \tfrac{1}{2} c^2, \text{ etc.,}$$

we find ~~that~~

(i) the squares of the medians are connected with those of the sides by relations which constitute a linear substitution, identical, except for a fixed multiplier, with the first of the two mentioned in §12, viz.

$$\left.\begin{aligned} x_1' &= -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3 \\ x_2' &= \tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3 \\ x_3' &= \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3 \end{aligned}\right\},$$

and hence of order two;
(ii) the variables are therefore reproduced in fixed proportions after two repetitions, thus:

$$\frac{\alpha''^2}{a^2} = \frac{\beta''^2}{b^2} = \frac{\gamma''^2}{c^2} = \frac{9}{16},$$

so that the triangle with sides equal to the medians of the second is similar to the first;
(iii) the invariants being

$x_1 + x_2 + x_3$, multiplier $+1$,

$x_2 - x_3$, $x_3 - x_1$, $x_1 - x_2$, " -1,

(any two of the last three being selected), we have for the triangles

$$\frac{\alpha'^2 + \beta'^2 + \gamma'^2}{a^2 + b^2 + c^2} = \frac{3}{4},$$

$$\frac{\beta'^2 - \gamma'^2}{b^2 - c^2} = \frac{\gamma'^2 - \alpha'^2}{c^2 - a^2} = \frac{\alpha'^2 - \beta'^2}{a^2 - b^2} = -\frac{3}{4},$$

while the areas satisfy

$$\Delta' / \Delta = 9/16.$$

§ 26. These results may be generalized. Let the sides BC, CA, AB be divided at A', B', C' in any the same ratio, which we may denote by $(1+\underline{l})/(1-\underline{l})$; let the sides of triangle $A'B'C'$ be denoted by $\underline{a}'$, $\underline{b}'$, $\underline{c}'$, the lengths AA', BB', CC' by $\underline{\alpha}'$, $\underline{\beta}'$, $\underline{\gamma}'$. Then easily

$$\left.\begin{aligned} a'^2 &= h_3 a^2 + h_2 b^2 + h_1 c^2, & \alpha'^2 &= \eta_3 a^2 + \eta_2 b^2 + \eta_1 c^2 \\ b'^2 &= h_1 a^2 + h_3 b^2 + h_2 c^2, & \beta'^2 &= \eta_1 a^2 + \eta_3 b^2 + \eta_2 c^2 \\ c'^2 &= h_2 a^2 + h_1 b^2 + h_3 c^2, & \gamma'^2 &= \eta_2 a^2 + \eta_1 b^2 + \eta_3 c^2 \end{aligned}\right\},$$

where

$$\left.\begin{array}{ll} h_1 = \frac{1}{2}\, l(1+l), & \eta_1 = -\frac{1}{2}(1-l) \\ h_2 = -\frac{1}{2}\, l(1-l), & \eta_2 = \frac{1}{2}(1+l) \\ h_3 = \frac{1}{4}(1-l^2), & \eta_3 = -\frac{1}{4}(1-l^2) \end{array}\right\}.$$

Hence the squares of a', b', c', and ~~also~~ of α', β', γ' are connected with those of a, b, c by substitutions of the form (18), with coefficients satisfying (30), and therefore <u>possessing equimodular multipliers.</u>

Confining attention to the former of the two substitutions, we have the following results. ⊕

(i) Putting $k_1 = k_3 e^{i\theta}$, $k_2 = k_3 e^{-i\theta}$, we have

$$k_3 = h_1 + h_2 + h_3 = \frac{1}{4}(1+3l^2),$$
$$\cos\theta = (k_1+k_2)/2k_3 = (1-3l^2)/(1+3l^2), \quad \tan\tfrac{1}{2}\theta = l\sqrt{3}.$$

(ii) The quadratic invariants are

$$X_0 = a^4+b^4+c^4, \quad X_1 = b^2c^2+c^2a^2+a^2b^2,$$

the multiplier being

$$H_0 = \frac{1}{16}(1+3l^2)^2 = \frac{1}{16}\sec^4\tfrac{1}{2}\theta\,;$$

thus the areas are in the ratio

$$\Delta'/\Delta = \frac{1}{4}\sec^2\tfrac{1}{2}\theta.$$

(iii) Let the sides of $A'B'C'$ be divided at A'', B'', C'' in the ratio $(1+l')/(1-l')$, those of $A''B''C''$ in the ratio $(1+l'')/(1-l'')$, and so on, $l', l'', \cdots l^{(r)}$ being any real numbers; and let us put

$$l^{(r)}\sqrt{3} = \tan\tfrac{1}{2}\theta^{(r)}, \quad \frac{1}{4}\sec^2\tfrac{1}{2}\theta^{(r)} = k_3^{(r)},$$

$$\sum_r \theta^{(r)} = \Theta, \quad \prod_r k_3^{(r)} = K_3, \quad K_3 e^{\pm i\Theta} = K_1, K_2\,;$$

then the squares of the sides of the final triangle $A^{(r)}B^{(r)}C^{(r)}$ will be connected with those of ABC by a substitution of form (18), with the equimodular multipliers K_1, K_2, K_3.

(iv) The shape, size and position of the final triangle are independent of the order in which the operations, which we may here denote by ($\underline{l}$), ($\underline{l}'$), --- ($\underline{l}^{(r)}$), are performed.

(v) All triangles of the set have the same centroid.

(vi) When $\sum_r \theta^{(r)} \equiv \Theta$ is a multiple of 2π, the final triangle is similar to the first, and corresponding sides are parallel.

(vii) Let $\underline{l}^{(r)} = \underline{l}^{(r-1)} = \cdots = \underline{l}$; then the triangle obtained by $\underline{r}$ repetitions of the operation ($\underline{l}$) will be similar and similarly placed to ABC, provided $\underline{l}$ is of the form

$$l = \frac{1}{\sqrt{3}} \tan p\pi/r.$$

(viii) It is always possible to find one or more values of $\underline{l}$ such that a given number ~~of~~ $\underline{r}$ of applications of the substitution ($\underline{l}$) upon the squares of the sides will change a triangle of any one given shape into one of any other given shape.

(ix). The relation between the two substitutions at the beginning of this article is very close. If in the expressions for the η we replace $\underline{l}$ by $-1/\underline{l}$ and multiply by $\underline{l}^2$, we obtain the expressions for the $\underline{h}$. This is clear also from the geometry of the figure.

§ 27. Homogeneous Coordinates in Two Dimensions.

In this and the following article two geometrical concepts are developed, which are applied to the discussion of our substitutions in § 29.

Triangles in Multiple Perspective.*

The triangle $X_1 X_2 X_3$ is in perspective with $Y_\alpha Y_\beta Y_\gamma$, the order of the vertices being significant, when $X_1 Y_\alpha$, $X_2 Y_\beta$, $X_3 Y_\gamma$ meet in a point. Thus $X_1 X_2 X_3$ may be in perspective with another triangle in ~~either~~ any one of six different ways, according to the order of the vertices. Let the vertices of $Y_1 Y_2 Y_3$, referred to $X_1 X_2 X_3$ as triangle of reference in any system of point-coordinates, be

$$Y_r \quad (x_{r1}, x_{r2}, x_{r3}) \quad , \quad (r = 1, 2, 3);$$

then the six cases, with their conditions, fall into two sets of three as follows.

$X_1 X_2 X_3$ is in perspective with

$$\left.\begin{array}{lll} \text{(i)} & Y_1 Y_2 Y_3 \quad \text{provided} & x_{12} x_{23} x_{31} = x_{13} x_{21} x_{32} \\ \text{(ii)} & Y_2 Y_3 Y_1 \quad \text{"} & x_{11} x_{22} x_{33} = x_{12} x_{23} x_{31} \\ \text{(iii)} & Y_3 Y_1 Y_2 \quad \text{"} & x_{13} x_{21} x_{32} = x_{11} x_{22} x_{33} \end{array}\right\} (A)$$

$$\left.\begin{array}{lll} \text{(iv)} & Y_1 Y_3 Y_2 \quad \text{"} & x_{22} x_{31} x_{13} = x_{33} x_{12} x_{21} \\ \text{(v)} & Y_3 Y_2 Y_1 \quad \text{"} & x_{33} x_{12} x_{21} = x_{11} x_{23} x_{32} \\ \text{(vi)} & Y_2 Y_1 Y_3 \quad \text{"} & x_{11} x_{23} x_{32} = x_{22} x_{31} x_{13} \end{array}\right\} . (B)$$

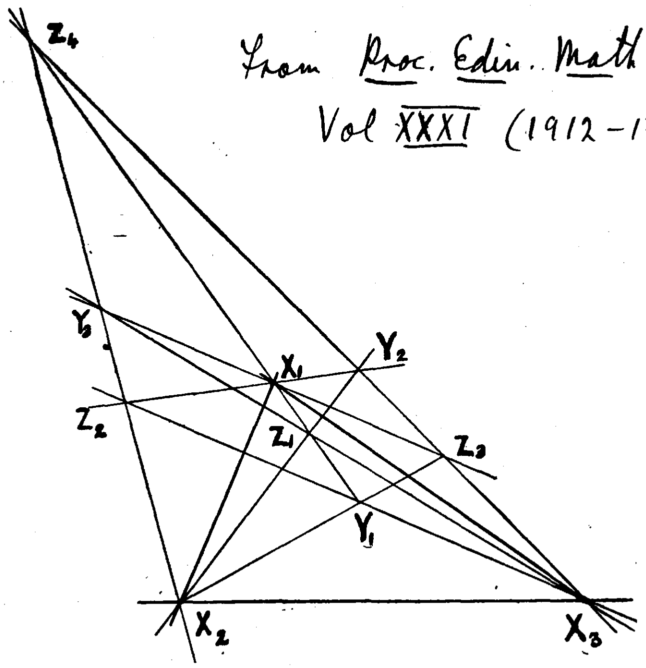
The corresponding poles of perspective will be called Z_r ($r = 1, 2, \ldots 6$). Sets (A), (B) are distinguished

---

* In Homologous Triangles, by Thomas Muir, M.A. [Mess. Math. 2 (1873)], "triply homologous" triangles are discussed, the vertices being taken in the same cyclic order.

Edmund Hess [Perspektive Dreiecke und Tetraeder, Math. Annalen 28 (1887)] introduces sextuply perspective triangles, but with a different object from the present.

From Proc. Edin. Math. Soc.
Vol XXXI (1912–13).



Triangles $X_1X_2X_3$, $Y_1Y_2Y_3$ *in quadruple perspective.*

by the cyclic order of the vertices $Y_1 Y_2 Y_3$. In each set any two of the conditions involve the third.

(α) Triple Perspective. This occurs when any two conditions of one set are fulfilled. Two of the vertices, say Y_2, Y_3, may be taken at random, and the third is easily obtained. The three poles of perspective are the vertices of a third triangle in triple perspective with each of the other two, the poles for each pair being the vertices of the remaining one.

(β) Quadruple Perspective occurs when three conditions, not all of one set, are fulfilled. The construction of such a case is a simple exercise in homogeneous coordinates. One vertex, say Y_3, may be taken at random, and the system of coordinates so chosen that it becomes the point $(1,1,1)$. Then if e.g. cases (i) – (iv) hold, we have for the coordinates of the other vertices

$$x_{12}x_{23} = x_{13}x_{21} = x_{11}x_{22}, \qquad x_{13}x_{22} = x_{12}x_{21},$$

whence

$$\frac{x_{12}}{x_{13}} = \frac{x_{21}}{x_{23}} = \frac{x_{22}}{x_{21}} = \frac{x_{13}}{x_{11}} = \alpha, \text{ say},$$

and the other vertices are $Y_1(1, \alpha^2; \alpha)$, $Y_2(\alpha, \alpha^2; 1)$, where α is arbitrary.

The figure is given for an arbitrary position of Y_3, and $\alpha = -2$. The collinearity of Z_1, Z_2, Z_3 is a coincidence, due to the value of α chosen; for the coordinates of the poles are easily found to be

$$Z_1(\alpha, \alpha, 1), \quad Z_2(1, \alpha^2, 1), \quad Z_3(1, \alpha, \alpha), \quad Z_4(1, \alpha, 1),$$

and the determinant of the coordinates of the first three is $\alpha(\alpha-1)^2(\alpha+2)$, vanishing for $\alpha = -2$.

(γ) Sextuple Perspective occurs on the fulfilment of two conditions from each set. One vertex Y_3 can still be taken at random, and called the point $(1, 1, 1)$. For the others we have

$$\frac{x_{11}}{x_{12}} = \frac{x_{12}}{x_{13}} = \frac{x_{13}}{x_{11}} = \frac{x_{22}}{x_{21}} = \frac{x_{23}}{x_{22}} = \frac{x_{21}}{x_{23}}$$

$$= \text{(clearly) } 1, \ \omega, \text{ or } \omega^2.$$

Unity is rejected as reducing the triangle $Y_1 Y_2 Y_3$ to a point. With the value ω we obtain $Y_1(\omega^2, \omega, 1)$, $Y_2(\omega, \omega^2, 1)$, $Y_3(1, 1, 1)$, the unit-points for this system of coordinates. The value ω^2 would simply interchange Y_1, Y_2. Hence there is only one triangle in sextuple perspective with the triangle of reference, and having a vertex at a given point. If that point is the real unit-point $(1, 1, 1)$, the other vertices are at the imaginary unit-points.

Thus in the case of sextuple perspective $Y_1 Y_2 Y_3$ coincides with $P_1 P_2 P_3$ (§10). The six poles in order are

(i) $Q_3\ (1, 1, \omega^2)$, (iv) $R_3\ (1, 1, \omega)$,
(ii) $Q_2\ (1, \omega^2, 1)$, (v) $R_2\ (1, \omega, 1)$,
(iii) $Q_1\ (\omega^2, 1, 1)$, (vi) $R_1\ (\omega, 1, 1)$.

Now in passing from one system of point-coordinates to another, with the same triangle of reference, we simply alter the coordinates of each point in definite ratios, so that, e.g., if the old coordinates of a point are (x_1, x_2, x_3), its new ones are $(\alpha x_1, \beta x_2, \gamma x_3)$, where α, β, γ are the same for all points. Thus the triangle $T_1(\omega^2\alpha, \omega\beta, \gamma)$ $T_2(\omega\alpha, \omega^2\beta, \gamma)$ $T_3(\alpha, \beta, \gamma)$, whatever the system of coordinates, is in sextuple perspective with the triangle of reference;

and $T_1 T_2 T_3$ may represent any triangle in this relation, if proper values are given to $\alpha : \beta : \gamma$. Now the two sets of multipliers $\alpha : \beta : \gamma = \omega^2 : 1 : 1$ and $\omega : 1 : 1$ respectively change the points

$$P_1\ P_2\ P_3\ \ Q_1\ Q_2\ Q_3\ \ R_1\ R_2\ R_3$$

into $$Q_3\ Q_2\ Q_1\ \ R_1\ R_3\ R_2\ \ P_3\ P_1\ P_2,$$

and $$R_2\ R_3\ R_1\ \ P_3\ P_2\ P_1\ \ Q_1\ Q_3\ Q_2.$$

Hence each of the triangles $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ is in sextuple perspective with the triangle of reference, and the poles of perspective in each case are the vertices of the other two.

§28. Conjugate Triangles.

A point and a line are harmonic pole and polar with respect to a triangle when the line, and the joins of the point to the vertices, divide the three sides harmonically. If each vertex and opposite side of one triangle are harmonic pole and polar with respect to another, the former may be called self-polar with respect to the latter; and, as it follows from the conditions obtained below that the relation is reciprocal, the two triangles may be called conjugate.

To find the conditions that the three points

$$(x_{\alpha 1}, x_{\alpha 2}, x_{\alpha 3}), \quad (\alpha = 1, 2, 3)$$

shall form a triangle self-polar with respect to the triangle of reference.

Denoting by $X_{\alpha\beta}$ the cofactor of $x_{\alpha\beta}$ in the determinant Δ of the coefficients, and expressing that each point is the harmonic pole of the join of the other two, we obtain the nine relations

$$x_{\alpha\beta} X_{\alpha\beta} = \text{const.} = \tfrac{1}{3}\Delta, \quad (\alpha, \beta = 1, 2, 3).$$

The two relations
$$x_{11} X_{11} = x_{12} X_{12}, \qquad x_{33} X_{33} = x_{23} X_{23},$$
may be written
$$x_{33}(x_{11} x_{22} + x_{12} x_{21}) = x_{23}(x_{12} x_{31} + x_{11} x_{32}),$$
$$x_{33}(x_{11} x_{22} - x_{12} x_{21}) = x_{23}(x_{12} x_{31} - x_{11} x_{32}),$$
whence, and by symmetry,
$$x_{11} x_{22} x_{33} = x_{23} x_{31} x_{12} = x_{32} x_{13} x_{21},$$
$$x_{11} x_{23} x_{32} = x_{22} x_{31} x_{13} = x_{33} x_{12} x_{21}.$$
These conditions, which satisfy all the nine relations, are exactly those for sextuple perspective (§ 27). Hence triangles in sextuple perspective are conjugate, and *vice versa*.

§ 29. In the substitution (18) for $n = 3$ (see (9), § 10) let us assume (x_1, x_2, x_3) to be the coordinates of a point, the triangle of reference being named $X_1 X_2 X_3$. Then the effect of the substitution is to change each point (x_1, x_2, x_3) into the position (x_1', x_2', x_3'). The triangle $X_1 X_2 X_3$ is thus itself shifted into $Y_1(h_3, h_1, h_2)$ $Y_2(h_2, h_3, h_1)$ $Y_3(h_1, h_2, h_3)$. These coordinates satisfy equations (A) of § 27; hence every triangle into which $X_1 X_2 X_3$ is changed by (9) is in triple perspective with $X_1 X_2 X_3$ itself.

The poles of the perspectives (p. 32) are found to be
$$Z_1(h_3^{-1}, h_2^{-1}, h_1^{-1}),\ Z_2(h_1^{-1}, h_3^{-1}, h_2^{-1}),\ Z_3(h_2^{-1}, h_1^{-1}, h_3^{-1}),$$
satisfying equations (B) of § 27, and therefore forming a triangle in triple perspective with $X_1 X_2 X_3$, but in reversed cyclic order.

The condition that $Y_1 Y_2 Y_3$ should satisfy equation (iv) as well as (i)–(iii) of § 27 is
$$h_2^3 = h_1^3;$$
thus the equality of two coefficients in the substitution involves quadruple perspective

between $X_1 X_2 X_3$ and every triangle obtained from it by the substitution; and the same would hold if two coefficients had the ratio ω or ω^2; though the other triangle would then be imaginary. Whatever the system of point-coordinates and the value of α, the triangle $(\alpha, 1, 1)$, $(1, 1, \alpha)$, $(1, \alpha, 1)$ is in quadruple perspective in ways (i) – (iv) with $X_1 X_2 X_3$.

The triangle whose vertices are the unit-points P_1, P_2, P_3 (§10) is, as we have seen, in sextuple perspective with, and conjugate to, the triangle of reference; and not only so, but, since perspective and harmonic properties are unaltered by a linear transformation, $P_1 P_2 P_3$ is in sextuple perspective with, and conjugate to, every triangle obtainable from the triangle of reference by a substitution of the form (9).

§30. Direction - Cosines.

In the real substitution of form (18) for three variables, if we put $\underline{k} = \underline{k}_3 = 1$, we have (see §§20,24)

$$h_1^2 + h_2^2 + h_3^2 = 1, \quad h_2 h_3 + h_3 h_1 + h_1 h_2 = 0 ;$$

hence $(\underline{h}_1, \underline{h}_2, \underline{h}_3)$ may be regarded as direction-cosines of a generator of the cone

$$yz + zx + xy = 0 ;$$

indeed, the three sets of direction-cosines $(\underline{h}_1, \underline{h}_2, \underline{h}_3)$, $(\underline{h}_2, \underline{h}_3, \underline{h}_1)$, $(\underline{h}_3, \underline{h}_1, \underline{h}_2)$ define three mutually rectangular generators of this cone. If $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$, $(\underline{y}_1, \underline{y}_2, \underline{y}_3)$ are the direction-cosines of any other two lines, then the application of our substitution to them will leave invariant the following quantities:

$$\sum_r x_r^2, \quad \sum_r y_r^2, \quad \sum_r x_r y_r, \quad \sum_r x_r, \quad \sum_r y_r .$$

Thus the transformed variables will also represent direction-cosines of two lines, containing the same angle as the two original lines, and each making the same angle as its original with the line of symmetry $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. The substitution therefore represents a rotation through a definite angle about the line of symmetry.

It is easily shown, with the usual sign convention, that the amount of the rotation is $+\theta$, where θ is the amplitude of the multiplier k_1. The rotation changes the z-axis into the line (h_1, h_2, h_3), and similarly for the other axes.

Developments of this application will appear in Section IV.

§ 31. Case of $n = 4$.

The poles and linear invariants in this case are

$$\left.\begin{array}{ll} P_1(-i,-1,i,1), & \xi_1 = ix_1 - x_2 - ix_3 + x_4 \\ P_2(-1,1,-1,1), & \xi_2 = -x_1 + x_2 - x_3 + x_4 \\ P_3(i,-1,-i,1), & \xi_3 = -ix_1 - x_2 + ix_3 + x_4 \\ P_4(1,1,1,1), & \xi_4 = x_1 + x_2 + x_3 + x_4 \end{array}\right\}; \quad --- (32)$$

multipliers and coefficients satisfy

$$\left.\begin{array}{ll} k_1 = ih_1 - h_2 - ih_3 + h_4, & 4h_1 = -ik_1 - k_2 + ik_3 + k_4 \\ k_2 = -h_1 + h_2 - h_3 + h_4, & 4h_2 = -k_1 + k_2 - k_3 + k_4 \\ k_3 = -ih_1 - h_2 + ih_3 + h_4, & 4h_3 = ik_1 - k_2 - ik_3 + k_4 \\ k_4 = h_1 + h_2 + h_3 + h_4, & 4h_4 = k_1 + k_2 + k_3 + k_4 \end{array}\right\}, \quad --- (33)$$

whence also

$$\left.\begin{array}{ll} k_1 + k_3 = -2(h_2 - h_4), & k_1 - k_3 = 2i(h_1 - h_3) \\ k_2 + k_4 = 2(h_2 + h_4), & k_2 - k_4 = -2(h_1 + h_3) \end{array}\right\} --- (34)$$

(i) If we make only the condition that the coefficients shall be real, we have

$$k_1 = \kappa e^{i\theta}, \quad k_2 = \kappa e^{-i\theta}, \qquad (k_2,\ k_4,\ \kappa,\ \theta \text{ real}).$$

and there are four real quadratic invariants, viz.:

$$\left.\begin{aligned}
\xi_4^2 &\equiv X_0 + 2X_1 + 2X_2 = (x_1+x_2+x_3+x_4)^2, && \text{mult. } \underline{k}_4^2 \\
\xi_1\xi_3 &\equiv X_0 \qquad - 2X_2 = (x_1-x_3)^2+(x_2-x_4)^2, && \quad " \quad \kappa^2 \\
\xi_2^2 &\equiv X_0 - 2X_1 + 2X_2 = (x_1-x_2+x_3-x_4)^2, && \quad " \quad \underline{k}_2^2 \\
\xi_2\xi_4 &\equiv (x_1+x_3)^2-(x_2+x_4)^2, && \quad " \quad \underline{k}_2\underline{k}_4
\end{aligned}\right\} \cdots\cdots (35)$$

(ii) If the multipliers are also equimodular, then by (24)

$$H_1 \equiv h_1h_2+h_2h_3+h_3h_4+h_4h_1 = (h_1+h_3)(h_2+h_4) = 0,$$
$$H_2 \equiv h_1h_3 + h_2h_4 = 0.$$

We must distinguish two cases.

(A). Let $\underline{h}_2 + \underline{h}_4 = 0$, then $\underline{k}_2 = -\underline{k}_4$ ~~(34)~~ (see (34)).

~~Putting $\kappa = k$~~ Putting $\kappa = \underline{k}_4$, we have

$$h_1 = \tfrac{1}{2}k_4(1+\sin\theta),$$
$$h_2 = -\tfrac{1}{2}k_4\cos\theta,$$
$$h_3 = \tfrac{1}{2}k_4(1-\sin\theta),$$
$$h_4 = \tfrac{1}{2}k_4\cos\theta.$$

$\underline{X}_0, \underline{X}_1, \underline{X}_2$ are invariants, multiplier $\underline{k}_4^2$,
$(\underline{x}_1+\underline{x}_3)^2-(\underline{x}_2+\underline{x}_4)^2$ is invariant, " $-\underline{k}_4^2$.

(B). Let $\underline{h}_1+\underline{h}_3 = 0$, then $\underline{k}_2 = \underline{k}_4$,

$$h_1 = \tfrac{1}{2}k_4\sin\theta,$$
$$h_2 = \tfrac{1}{2}k_4(1-\cos\theta),$$
$$h_3 = -\tfrac{1}{2}k_4\sin\theta,$$
$$h_4 = \tfrac{1}{2}k_4(1+\cos\theta).$$

There are now, in virtue of two multipliers being equal, <u>five</u> real quadratic invariants with the same multiplier $\underline{k}_4^2$, viz.,

$$X_0, \quad X_1, \quad X_2, \quad (x_1+x_3)^2, \quad (x_2+x_4)^2.$$

Indeed, $\underline{x}_1+\underline{x}_3$, $\underline{x}_2+\underline{x}_4$ are now real linear invariants with multiplier $\underline{k}_4$.

Two applications will now be given; the first, curious rather than important, to the sides of any cyclic quadrilateral; the second, to the operation in four dimensions which is analogous to a rotation about the line of symmetry in three dimensions.

§32. Cyclic Quadrilaterals. *

Given the sides of a convex quadrilateral, its area is a maximum when it is cyclic. In whatever order the sides are taken, we obtain the same maximum area S and the same circum-radius R, given by the well-known formulae

$$S^2 = (s-a.\,s-b.\,s-c.\,s-d), \quad 16R^2S^2 = (ad+bc.\,bd+ca.\,cd+ab).$$

If we write

$$\frac{(bd+ca.\,cd+ab)}{ad+bc} = x^2, \quad \frac{(cd+ab.\,ad+bc)}{bd+ca} = y^2, \quad \frac{(ad+bc.\,bd+ca)}{cd+ab} = z^2,$$

then two of the quantities x, y, z, according to the order of the sides, are ~~the~~ equal to the diagonals of the cyclic quadrilateral, while the square of the third, multiplied by the "power" of O the point of intersection of the diagonals, is equal to the product of the four sides. Thus taking the sides in the order a, b, c, d, and lettering them AB, BC, etc., we have

$$AC = z, \quad BD = x, \quad AO.\,OC.\,y^2 = abcd.$$

For the third or external diagonal EF we have

$$EF^2 = (cd+ab)(ad+bc)\left\{\frac{ac}{(a^2-c^2)^2} + \frac{bd}{(b^2-d^2)^2}\right\}.$$

Consider the cyclic quadrilaterals the lengths of whose sides have been formed from those of a given one by a linear substitution of type (A) or (B) of last article. In either case, assuming $k_4 = 1$, the following elements of the quadrilateral remain unchanged:

---

* For the results here assumed see Hobson, Plane Trigonometry.

(i) the perimeter,
(ii) the sum of squares of sides,
(iii) the product of diagonals,
(iv) the product of sums of opposite sides, viz.

$$(\underline{a}+\underline{c})(\underline{b}+\underline{d}).$$

Next let us find the conditions that the new quadrilateral shall have its diagonals <u>equal</u> to those of the old. From the substitution itself it is easily proved that the quantities $\underline{a}\,\underline{d}+\underline{b}\,\underline{c}$, $\underline{c}\,\underline{d}+\underline{a}\,\underline{b}$ will remain unchanged in value provided

$$h_1h_2+h_3h_4 = h_2h_3+h_1h_4 = \left.\begin{matrix} h_1^2+h_3^2+2h_2h_4 = 0 \\ h_2^2+h_4^2+2h_1h_3 = 1 \end{matrix}\right\},$$

and that they will have their values simply interchanged provided

$$h_1h_2+h_3h_4 = h_2h_3+h_1h_4 = \left.\begin{matrix} h_2^2+h_4^2+2h_1h_3 = 0 \\ h_1^2+h_3^2+2h_2h_4 = 1 \end{matrix}\right\}.$$

These conditions virtually lead to a simple quadrilateral, whose sides, in terms of those of the given one, are

$$a' = s-a,\quad b' = s-b,\quad c' = s-c,\quad d' = s-d.$$

The given and final quadrilaterals have then the same perimeter, the same diagonals, the same sum of squares of sides. The sum of any two sides of one is equal to the sum of the <u>remaining</u> two sides of the other. The third or external diagonals are also equal. The area of each is equal to the square root of the product of the sides of the other. The radii of their circum-circles are inversely proportional to their areas. The distance from intersection of diagonals to circum-centre is the same fraction of the radius in each.

If $\underline{a}+\underline{c}=\underline{b}+\underline{d}$, i.e., if the given quadrilateral can have a circle inscribed in it, then $a' = c$, $b' = d$, $c' = a$, $d' = b$, and the quadrilaterals become identical.

§33. "Rotation" in Four Dimensions.

Two sets of quantities x_r, y_r $(r = 1, 2, 3, 4)$ satisfying the condition

$$\Sigma x^2 = \Sigma y^2 = 1$$

may be called the direction-cosines of two lines in space of four dimensions. Since

$$\Sigma x^2 \cdot \Sigma y^2 - \left(\sum_r x_r y_r\right)^2 = \sum_{r,s} (x_r y_s - x_s y_r)^2,$$

the summation on the right extending to six terms, therefore $\sum_r x_r y_r$ is always numerically less than unity. It may be called the cosine of the angle between the lines: two lines may be called parallel when their cosine is unity, and at right angles when it is zero.

Putting $k_4 = 1$ in §31 we see that, corresponding to any value of the parameter θ, there are two distinct real equimodular linear substitutions of form (18) in four variables; that is, the operation in four dimensions corresponding to a rotation θ about the line of symmetry in three dimensions is ambiguous. These alternative operations may for the present purpose be spoken of as rotation θ in mode (A) or in mode (B) according as form (A) or (B) of §31 is in question.

Since the squares of the multipliers in (A), (B) are respectively equal, two repetitions of each yield the same result, or, as we may symbolize it, $(A)^2 = (B)^2$; and the result is a rotation 2θ in mode (B). From the products of corresponding multipliers it follows also that successive application of (A), (B) in either order yields a rotation 2θ in mode (A).

The two modes are distinguished by the fact that $x_1 + x_3$, $x_2 + x_4$ are invariant for (B) but not for (A): a rotation in mode (B) about the line of symmetry leaves ~~the cos~~ unchanged the cosines of the angles which the line makes with the lines $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$, $\left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$; but not so for mode (A).

§ 34. Higher Values of n.

A few general statements can now be made on the cases of higher values of n.

$n = 5$.

Here, for real coefficients, the first four multipliers must have the forms

$$k_1 = \kappa_1 e^{i\theta_1}, \quad k_2 = \kappa_2 e^{i\theta_2}, \quad k_3 = \kappa_2 e^{-i\theta_2}, \quad k_4 = \kappa_1 e^{-i\theta_1}.$$

The real quadratic invariants are

$$X_0 + 2X_1 + 2X_2, \qquad \text{multiplier } k_5^2,$$
$$X_0 + 2X_1 \cos\tfrac{2\pi}{5} + 2X_2 \cos\tfrac{4\pi}{5}, \qquad \text{"} \quad \kappa_1^2,$$
$$X_0 + 2X_1 \cos\tfrac{4\pi}{5} + 2X_2 \cos\tfrac{2\pi}{5}, \qquad \text{"} \quad \kappa_2^2.$$

If in addition the multipliers are equimodular, so that $\kappa_1 = \kappa_2 = k_5$, then X_0, X_1, X_2 themselves become invariant with common multiplier k_5^2, and the coefficients can be written down by (20) in explicitly real form in terms of k_5, θ_1, θ_2.

Thus, as to the analogue in five dimensions of a rotation about the line of symmetry in three dimensions, two parameters θ_1, θ_2 are necessary for its specification; but for given values of these parameters the operation is determined without ambiguity.

<u>$n = 6$.</u>

If the coefficients are real and the multipliers equimodular, then by (24)

$$H_1 = H_2 = H_3 = 0,$$

whence

$$(h_1 + h_3 + h_5)(h_2 + h_4 + h_6) = H_1 + H_3 = 0,$$

yielding, as with $\underline{n} = 4$, an ambiguous case. Here, as with $\underline{n} = 5$, a "rotation" about the axis of symmetry involves two independent parameters; but in this case, when the values of the parameters are given, there are still <u>two</u> operations satisfying the conditions.

(A) Assume $\underline{h}_2 + \underline{h}_4 + \underline{h}_6 = 0$; then $\underline{k}_3 = -\underline{k}_6$, and the real quadratic invariants are

$X_0, X_1, X_2, X_3,$ mult. $\underline{k}_6^2$

$(x_1 + x_3 + x_5)^2 - (x_2 + x_4 + x_6)^2,$ " $-\underline{k}_6^2$.

(B) Assume $\underline{h}_1 + \underline{h}_3 + \underline{h}_5 = 0$; then $\underline{k}_3 = \underline{k}_6$, and the real quadratic invariants are, all with the common multiplier $\underline{k}_6^2$,

$$X_0, X_1, X_2, X_3, (x_1 + x_3 + x_5)^2, (x_2 + x_4 + x_6)^2.$$

We can now generalize as follows.

In the case of $\underline{n}$ real variables, with equimodular multipliers, the coefficients satisfy the relations

$$H_0 = k_n^2, \quad H_1 = H_2 = \cdots = H_t = 0,$$

and the quadratic functions

$$X_0, X_1, X_2, \cdots X_t$$

are invariant, multiplier $\underline{H}_0$; where $\underline{t} = \frac{1}{2}(n-1)$ or $\frac{1}{2}\underline{n}$ according as $\underline{n}$ is odd or even.

When $\underline{n}$ is odd, the number of angular parameters $\underline{\theta}$ required to specify the substitution is $\frac{1}{2}(n-1)$, and, these being given, the substitution is unique.

When $\underline{n}$ is even, the number required is $(\frac{1}{2}\underline{n}-1)$; but, these being given, the substitution is ambiguous. In the second or $(\underline{B})$ mode in such cases, the functions

$$(\Sigma x_{2r+1})^2, \quad (\Sigma x_{2r})^2$$

are also invariants with multiplier $\underline{H}_0$; in the first or (A) mode, the difference of these functions is an invariant with multiplier $-\underline{H}_0$.

Section IV.

Construction of Real Linear Substitutions ~~from their~~ of Equimodular Type from their Real Linear and Quadratic Invariants

§35. In Sections II, III results have been obtained for a specially simple type of linear substitutions. Some of these results will now be extended to more general cases.

The coefficients of the characteristic equation being real functions of the coefficients of the substitution, the imaginary multipliers of a real substitution must occur in conjugate pairs. It follows that the imaginary linear invariants can be also arranged in pairs, such that the product to the two members of a pair is a real quadratic invariant (cp. §13).

Thus the invariants of a real substitution can always be given as one or two* real linear, and $\frac{1}{2}(\underline{n}-1)$ or $(\frac{1}{2}\underline{n}-1)$ real quadratic invariants, according as $\underline{n}$ is odd or even. As a rule each quadratic invariant must be the product of two conjugate imaginary linear invariants; but this will not be necessary when equalities exist among the multipliers or their moduli. In particular when all the multipliers are equimodular, the quadratic functions, whose linear factors constitute the imaginary linear invariants, need not be the given quadratic invariants, but linear combinations of these and of the square of the real linear invariant, (or of the squares and products of the real linear invariants, should such exist).

---

* For a case with no real linear invariants, but $\frac{1}{2}\underline{n}$ real quadratic invariants, see below, §46.

§ 36. $\underline{n = 3.}$

~~Hence~~ In the case of a real substitution in three variables with equimodular multipliers, the most general ~~assumption~~ assumption for the linear invariants is

$$\left.\begin{array}{llll} \xi_1 = (p_1+iq_1)x_1 + (p_2+iq_2)x_2 + (p_3+iq_3)x_3, & \underline{\text{mult.}} & e^{i\theta} \\ \xi_2 = (p_1-iq_1)x_1 + (p_2-iq_2)x_2 + (p_3-iq_3)x_3, & \text{"} & e^{-i\theta} \\ \xi_3 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, & \text{"} & 1 \end{array}\right\} \quad \text{---}(36)$$

where the $\underline{p}, \underline{q}, \underline{\alpha}$ are real.

The coordinates of the poles, being proportional to the minors of the determinant of the coefficients of (36), are given by

$$\left.\begin{array}{ll} P_1: & \dfrac{x_{11}}{(p_2\alpha_3)-i(q_2\alpha_3)} = \dfrac{x_{21}}{(p_3\alpha_1)-i(q_3\alpha_1)} = \dfrac{x_{31}}{(p_1\alpha_2)-i(q_1\alpha_2)} \\ P_2: & \dfrac{x_{12}}{-(p_2\alpha_3)-i(q_2\alpha_3)} = \dfrac{x_{22}}{-(p_3\alpha_1)-i(q_3\alpha_1)} = \dfrac{x_{32}}{-(p_1\alpha_2)-i(q_1\alpha_2)} \\ P_3: & \dfrac{x_{13}}{-2i(p_2q_3)} = \dfrac{x_{23}}{-2i(p_3q_1)} = \dfrac{x_{33}}{-2i(p_1q_2)} \end{array}\right\}, \quad \text{---}(37)$$

where

$$(p_r q_s) \equiv p_r q_s - p_s q_r, \quad \text{etc.}$$

If we write

$$D_0 \equiv \begin{vmatrix} p_1, & p_2, & p_3 \\ q_1, & q_2, & q_3 \\ \alpha_1, & \alpha_2, & \alpha_3 \end{vmatrix},$$

then the determinant of the coefficients in (36) is equal to $-2\underline{i}\,D_0$, and that of the coordinates in (37) is say

$$D = -4D_0^2.$$

Now applying formula (6),

$$D\,l_{rs} = (x_{r1}X_{s1} + x_{r2}X_{s2})\cos\theta + (x_{r1}X_{s1} - x_{r2}X_{s2})\,i\sin\theta + x_{r3}X_{s3};$$

so that for $\underline{r} \neq \underline{s}$,

$$D\,l_{rs} = x_{r3}X_{s3}(1-\cos\theta) + i(x_{r1}X_{s1} - x_{r2}X_{s2})\sin\theta,$$

while

$$D\,l_{rr} = D\cos\theta + x_{r3}X_{r3}(1-\cos\theta) + i(x_{r1}X_{r1} - x_{r2}X_{r2})\sin\theta.$$

For the $\underline{x}$ we may take the actual denominators in (37); and then, for each $\underline{X}$, we must take $-2\underline{i}\underline{D}_0$ times the corresponding coefficient in (36). Putting for short

$$p_r p_s + q_r q_s = \delta_{rs}, \quad p_r q_s - p_s q_r = \varepsilon_{rs},$$

we have the following formulae for the coefficients of the substitution:

$$\left.\begin{aligned} D_0(l_{11} - \cos\theta) &= \varepsilon_{23}\,\alpha_1(1-\cos\theta) + (\delta_{13}\alpha_2 - \delta_{12}\alpha_3)\sin\theta \\ D_0\, l_{12} &= \varepsilon_{23}\,\alpha_2(1-\cos\theta) + (\delta_{23}\alpha_2 - \delta_{22}\alpha_3)\sin\theta \\ D_0\, l_{13} &= \varepsilon_{23}\,\alpha_3(1-\cos\theta) + (\delta_{33}\alpha_2 - \delta_{32}\alpha_3)\sin\theta \end{aligned}\right\}, \quad \cdots(38)$$

and the six others derived from these by cyclic interchange of the suffixes 1, 2, 3.

§ 37. As an example, let us suppose ~~that~~ in addition to the linear invariant $\Sigma \underline{\alpha}\,\underline{x}$, we are given the quadratic $\Sigma \underline{x}^2$ (see § 35), each with multiplier* unity. Then since $\Sigma \underline{x}^2$ must be a function of the linear invariants, there must exist a relation of the form

$$(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)^2 + \lambda(x_1^2 + x_2^2 + x_3^2) = \mu\{(p_1 x_1 + p_2 x_2 + p_3 x_3)^2 \cancel{+(q_1x_1+q_2x_2+q_3x_3)^2}$$
$$+ (q_1 x_1 + q_2 x_2 + q_3 x_3)^2\}.$$

The condition for the breaking-up of the left-hand member into factors yields

$$\lambda = -(\alpha_1^2 + \alpha_2^2 + \alpha_3^2),$$

and we then have

$$\left.\begin{aligned} \alpha_2^2 + \alpha_3^2 &= -\mu(p_1^2 + q_1^2), & \alpha_2\alpha_3 &= \mu(p_2 p_3 + q_2 q_3) \\ \alpha_3^2 + \alpha_1^2 &= -\mu(p_2^2 + q_2^2), & \alpha_3\alpha_1 &= \mu(p_3 p_1 + q_3 q_1) \\ \alpha_1^2 + \alpha_2^2 &= -\mu(p_3^2 + q_3^2), & \alpha_1\alpha_2 &= \mu(p_1 p_2 + q_1 q_2) \end{aligned}\right\} \cdots(39)$$

We may without loss assume

$$p_3 = -(\alpha_1^2 + \alpha_2^2), \quad q_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1;^{*}$$

---

* If $\underline{\alpha}_1^2 + \underline{\alpha}_2^2 + \underline{\alpha}_3^2 = \eta^2$, then we have only to multiply by $\underline{\eta}$ the values of $\underline{q}_1$, $\underline{q}_2$ obtained above. See later, § 38

then

$$\mu = p_3^{-1}, \quad p_1 = \alpha_3\alpha_1, \quad p_2 = \alpha_2\alpha_3, \quad q_1 = \mp\alpha_2, \quad q_2 = \pm\alpha_1;$$

for definiteness we may take the upper signs, as it is only a question of the order of the invariants $\underline{\xi}_1, \underline{\xi}_2$.

Thus

$$\left.\begin{aligned} \xi_1 &= (\alpha_3\alpha_1 - i\alpha_2)x_1 + (\alpha_2\alpha_3 + i\alpha_1)x_2 - (\alpha_1^2+\alpha_2^2)x_3 \\ \xi_2 &= (\alpha_3\alpha_1 + i\alpha_2)x_1 + (\alpha_2\alpha_3 - i\alpha_1)x_2 - (\alpha_1^2+\alpha_2^2)x_3 \end{aligned}\right\}, \quad --- (40)$$

and the poles are

$$\left.\begin{aligned} &P_1 \quad (\alpha_2 - i\alpha_3\alpha_1, \quad -\alpha_1 - i\alpha_2\alpha_3, \quad i(\alpha_1^2+\alpha_2^2)) \\ &P_2 \quad (-\alpha_2 - i\alpha_3\alpha_1, \quad \alpha_1 - i\alpha_2\alpha_3, \quad i(\alpha_1^2+\alpha_2^2)) \\ &P_3 \quad (\alpha_1, \quad \alpha_2, \quad \alpha_3) \end{aligned}\right\}, \quad --- (41)$$

where the common factor $-2i(\underline{\alpha}_1^2+\underline{\alpha}_2^2)$ has been discarded from the coordinates of P_3. The coefficients of the substitution can now be written down:

$$\left.\begin{aligned} l_{11} &= \cos\theta + \alpha_1^2(1-\cos\theta) \\ l_{12} &= \alpha_1\alpha_2(1-\cos\theta) - \alpha_3\sin\theta \\ l_{13} &= \alpha_1\alpha_3(1-\cos\theta) + \alpha_2\sin\theta \end{aligned}\right\}, \quad --- (42)$$

together with the six others obtained by cyclic interchange of the suffixes.

As was to be expected, these are the coefficients of the substitution which the direction-cosines of any line undergo when it is rotated through any angle $\underline{\theta}$ about an axis whose direction-cosines are $\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_3$. They agree with the formulae in Whittaker, Anal. Dyn. § 7. They reduce to those of (31), § 24 above on putting $\underline{\alpha}_1 = \underline{\alpha}_2 = \underline{\alpha}_3 = 1/\sqrt{3}$, $\underline{k}_3 = 1$.

Thus the general ~~rotation~~-substitution in three dimensions has been deduced purely from the knowledge of its multipliers and invariants $(\Sigma\underline{\alpha}\underline{x}, \Sigma\underline{x}^2)$; and its imaginary linear invariants and its poles are given in terms of the direction-cosines of the axis and the amount of the rotation by (40), (41).

§38. We may assume more generally ~~that~~ the function

$$S \equiv ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2,$$

in which the coefficients are real, is an invariant with multiplier unity, in addition to the real linear invariant $\xi_3 \equiv \Sigma \alpha x$. As before a relation must exist of the form

$$\xi_3^2 + \lambda S = \mu \xi_1 \xi_2.$$

The value of λ, determined as before, is given by

$$\Delta\lambda = -(A\alpha_1^2 + B\alpha_2^2 + C\alpha_3^2 + 2F\alpha_2\alpha_3 + 2G\alpha_3\alpha_1 + 2H\alpha_1\alpha_2) = -E \text{ say},$$

the usual notation being employed for the discriminant of S and its minors. Writing $\Delta\mu = \mu'$, we then have

$$\left.\begin{aligned} \Delta\alpha_1^2 - Ea &= \mu'(p_1^2+q_1^2), & \Delta\alpha_2\alpha_3 - Ef &= \mu'(p_2p_3+q_2q_3) \\ \Delta\alpha_2^2 - Eb &= \mu'(p_2^2+q_2^2), & \Delta\alpha_3\alpha_1 - Eg &= \mu'(p_3p_1+q_3q_1) \\ \Delta\alpha_3^2 - Ec &= \mu'(p_3^2+q_3^2), & \Delta\alpha_1\alpha_2 - Eh &= \mu'(p_1p_2+q_1q_2) \end{aligned}\right\} \cdots (43)$$

Assuming, as we evidently may ~~without~~ loss,

$$q_3 = 0, \quad \mu' p_3 = 1,$$

we have

$$\left.\begin{aligned} p_3 &= \Delta\alpha_3^2 - Ec \\ p_1 &= \Delta\alpha_3\alpha_1 - Eg, \quad & q_1 &= \mp(H\alpha_1 + B\alpha_2 + F\alpha_3)\sqrt{E} \\ p_2 &= \Delta\alpha_3\alpha_2 - Ef, \quad & q_2 &= \pm(A\alpha_1 + H\alpha_2 + G\alpha_3)\sqrt{E} \end{aligned}\right\},$$

where $\underline{E}$ is obviously bound to be positive, and the upper signs may be taken for definiteness.

We now have

$$D_0 = -p_3 E^{3/2},$$

$$\varepsilon_{23} = -p_3\sqrt{E}.(A\alpha_1 + H\alpha_2 + G\alpha_3) = -p_3\sqrt{E}.E_{\alpha_1} \quad \text{say},$$

$$\varepsilon_{31} = -p_3\sqrt{E}.(H\alpha_1 + B\alpha_2 + F\alpha_3) = -p_3\sqrt{E}.E_{\alpha_2} \quad '' ,$$

$$\varepsilon_{12} = -p_3\sqrt{E}.(G\alpha_1 + F\alpha_2 + C\alpha_3) = -p_3\sqrt{E}.E_{\alpha_3} \quad '' ;$$

whence

$$\left.\begin{aligned} E(l_{11} - \cos\theta) &= \alpha_1 E_{\alpha_1}(1-\cos\theta) + (g\alpha_2 - h\alpha_3)\sqrt{E}.\sin\theta \\ E\,l_{12} &= \alpha_2 E_{\alpha_1}(1-\cos\theta) + (f\alpha_2 - b\alpha_3)\sqrt{E}.\sin\theta \\ E\,l_{13} &= \alpha_3 E_{\alpha_1}(1-\cos\theta) + (c\alpha_2 - f\alpha_3)\sqrt{E}.\sin\theta \end{aligned}\right\} \cdots (44)$$

together with the six others derived from these by simultaneous cyclic interchange of the suffixes 1, 2, 3, and the two sets of symbols $\underline{a}, \underline{b}, \underline{c}$; $\underline{f}, \underline{g}, \underline{h}$.

The process here used is analogous to that which arises in finding, in homogeneous plane point-coordinates, the (imaginary) tangents to the conic $\underline{S}$ at its points of intersection with the line $\underline{\xi}_3$; or in finding, in three-dimensional rectangular coordinates, the tangent planes to the cone $\underline{S}$ which touch along the generators in which the cone is intersected by the plane $\underline{\xi}_3$. The condition that $\underline{E}$ must be positive ensures that the points of intersection in the first case, and the lines of intersection in the second, shall be imaginary, and therefore also the tangent lines and planes.

The substitution may be interpreted geometrically as follows. Let

$$S = \text{const.}, \quad \xi_3 = \text{const.},$$

denote respectively a conicoid and a plane in trirectangular ~~three dimensional~~ coordinates;

let Q be any point on their curve of intersection, and ON the normal to the plane from the origin. Let Q move along the curve of intersection till the plane ONQ has rotated through an angle θ about ON; then the direction-cosines of the new position of OQ are obtained from those of the old by the substitution whose coefficients l_{rs} are defined in (44).

It would not impair the generality of the results to assume $E = 1$; we would then have

$$\alpha_1 E_{\alpha_1} + \alpha_2 E_{\alpha_2} + \alpha_3 E_{\alpha_3} = E = 1.$$

§ 39. $n = 4$.

Generalizing the case of §31, let us take as the invariants of Θ a real substitution in four variables with equimodular multipliers the following:

$$\left.\begin{aligned} \xi_1 &= \sum_{r=1}^{4} (p_r + i q_r)\, x_r, & &\text{mult. } e^{i\theta} \\ \xi_2 &= \sum \beta_r x_r, & &\quad " \quad \eta \\ \xi_3 &= \sum (p_r - i q_r)\, x_r, & &\quad " \quad e^{-i\theta} \\ \xi_4 &= \sum \alpha_r x_r, & &\quad " \quad 1 \end{aligned}\right\}, \quad \text{- - - - (45)}$$

where the α, β, p, q are real, and η denotes -1 or $+1$ according as mode (A) or (B) is under consideration. ~~The determinant of the~~

The coordinates of the poles, and the formulae for the coefficients of the substitution, can be obtained exactly as in the case of $n = 3$. Thus taking as coordinates the actual first minors of the coefficients of (45), and writing

$$(p_2\beta_3\alpha_4) \equiv \begin{vmatrix} p_2, & p_3, & p_4 \\ \beta_2, & \beta_3, & \beta_4 \\ \alpha_2, & \alpha_3, & \alpha_4 \end{vmatrix}, \text{ etc.,}$$

the poles are

P_1 $\{-(p_2\beta_3\alpha_4)+i(q_2\beta_3\alpha_4),\ (p_1\beta_3\alpha_4)-i(q_1\beta_3\alpha_4),\ -(p_1\beta_2\alpha_4)+i(q_1\beta_2\alpha_4),\ (p_1\beta_2\alpha_3)-i(q_1\beta_2\alpha_3)\}$

P_2 $\{\ 2i(p_2q_3\alpha_4),\quad -2i(p_1q_3\alpha_4),\quad 2i(p_1q_2\alpha_4),\quad -2i(p_1q_2\alpha_3)\}$

P_3 $\{(p_2\beta_3\alpha_4)+i(q_2\beta_3\alpha_4),\ -(p_1\beta_3\alpha_4)-i(q_1\beta_3\alpha_4),\ (p_1\beta_2\alpha_4)+i(q_1\beta_2\alpha_4),\ -(p_1\beta_2\alpha_3)-i(q_1\beta_2\alpha_3)\}$

P_4 $\{-2i(p_2q_3\beta_4),\quad 2i(p_1q_3\beta_4),\quad -2i(p_1q_2\beta_4),\quad 2i(p_1q_2\beta_3)\}$.

Write
$$D_0 = \begin{vmatrix} p_1, & p_2, & p_3, & p_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ q_1, & q_2, & q_3, & q_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \end{vmatrix};$$

then the determinant of the coefficients of (45) is $-2i\underline{D}_0$, and that of the coordinates just set down is

$$D = (-2iD_0)^3 = 8iD_0^3.$$

If we denote by X_{rs} as usual the appropriate first minor of $\underline{D}$, then it is clearly equal to $-4\underline{D}_0^2$ times the corresponding coefficient in (45). Now applying formula (6), we have

$$Dl_{rs} = (x_{r1}X_{s1}+x_{r3}X_{s3})\cos\theta + (x_{r1}X_{s1}-x_{r3}X_{s3})\,i\sin\theta + \eta\, x_{r2}X_{s2} + x_{r4}X_{s4}.$$

Denoting the coefficients of (45) for convenience by $\underline{c}_{rs}$ $(\underline{r}, \underline{s} = 1, 2, 3, 4)$, we can put

$$\left.\begin{aligned} 2D_0 l_{rs} &= A_{rs}\cos\theta + B_{rs}\sin\theta + C_{rs}, && \text{mode } (A) \\ 2D_0 l_{rs} &= A_{rs}\cos\theta + B_{rs}\sin\theta + D_{rs}, && \text{mode } (B) \end{aligned}\right\} \quad \text{---(46)}$$

where

$$\left.\begin{aligned} A_{rs} &= i(x_{r1}c_{s1}+x_{r3}c_{s3}), & C_{rs} &= -i(x_{r2}c_{s2}-x_{r4}c_{s4}) \\ B_{rs} &= -(x_{r1}c_{s1}-x_{r3}c_{s3}), & D_{rs} &= i(x_{r2}c_{s2}+x_{r4}c_{s4}) \end{aligned}\right\} \text{----(47)}$$

It would be tedious to express the 64 coefficients at length in terms of the $\underline{\alpha}, \underline{\beta}, \underline{p}, \underline{q}$; but they will be found for the special case which is analogous to ~~a~~ rotation about any axis in three dimensions (see below, § 41).

Mode (A).

§40. An important divergence from the case of $n=3$ arises from the existence, in the present case, of two real linear ~~linear~~ invariants. For there are now three independent real quadratic invariants with multiplier $+1$, viz.,

$$\xi_2^2, \quad \xi_4^2, \quad \xi_1 \xi_3.$$

If therefore on the analogy of § 34 we assume as given, in addition to

$$\xi_2 \equiv \Sigma \beta_r x_r, \qquad \xi_4 \equiv \Sigma \alpha_r x_r,$$

the function Σx_r^2 as an invariant, the relation from which we must find the p, q of ξ_1, ξ_3 is of the form

$$(\Sigma \alpha x)^2 + \nu (\Sigma \beta x)^2 + \lambda \Sigma x^2 = \mu \{(\Sigma p x)^2 + (\Sigma q x)^2\} \cdots (48)$$

(Now Sylvester's Law of Inertia states ~~that~~ when a quadratic form is expressed as a sum
Omit? (cf. p 59) of squares, the number of positive squares and the number of negative squares are both
? fixed. Hence in (48) λ, μ must be negative and ν positive.) We may (by reducing the β, if necessary, in a ~~constant~~ ratio) assume $\nu = +1$. ~~Then~~ The condition ~~that~~ the left-hand member break, like the right-hand, into linear factors is (Salmon–Rogers, Analytic Geometry of Three Dimensions, (1912) § 79) ~~that~~ not only its discriminant, but also every first minor thereof, shall vanish. The discriminant is a biquadratic in λ, of which the absolute term and the coefficient of the first power of λ are easily seen to be zero. Rejecting the double zero ~~root~~ we are left with a quadratic. But* this quadratic must have

---

* See e.g. Routh, Advanced Rigid Dynamics, § 269.

equal roots, so that there is only one suitable value of λ.

Writing

$$\alpha_r \alpha_s + \beta_r \beta_s = Y_{rs},$$

one of the first minors is

$$\begin{vmatrix} Y_{21}, & Y_{22}+\lambda, & Y_{23} \\ Y_{31}, & Y_{32}, & Y_{33}+\lambda \\ Y_{41}, & Y_{42}, & Y_{43} \end{vmatrix};$$

this, equated to zero, yields the zero root and another, the one we are in search of, viz.,

$$\lambda = -Y_{22} - Y_{33} + \frac{Y_{31}Y_{43} + Y_{21}Y_{42}}{Y_{41}},$$

$$= -(Y_{11}+Y_{22}+Y_{33}+Y_{44}) + \frac{\alpha_1\alpha_4 \Sigma\alpha^2 + (\alpha_1\beta_4+\alpha_4\beta_1)\Sigma\alpha\beta + \beta_1\beta_4\Sigma\beta^2}{\alpha_1\alpha_4+\beta_1\beta_4}.$$

Since a similar relation must hold for every other pair of suffixes as well as for 1, 4, we must have

$$\Sigma\alpha^2 = \Sigma\beta^2, \quad \Sigma\alpha\beta = 0; \quad \text{------} \quad (49)$$

and then

$$-\lambda = \Sigma\alpha^2 = \Sigma\beta^2.$$

On equating coefficients in (48) we now have ten equations of which the two following are typical:

$$\left.\begin{aligned} \alpha_1^2+\beta_1^2+\lambda &= \mu(p_1^2+q_1^2) \\ \alpha_2\alpha_3+\beta_2\beta_3 &= \mu(p_2p_3+q_2q_3) \end{aligned}\right\}. \quad \text{------} \quad (50)$$

Let us assume as before

$$q_4 = 0, \quad \mu p_4 = 1;$$

then

$$\left.\begin{aligned} \mu^{-1} = p_4 = \alpha_4^2+\beta_4^2+\lambda &= -(\alpha_1^2+\alpha_2^2+\alpha_3^2-\beta_4^2) \\ p_1 &= \alpha_1\alpha_4+\beta_1\beta_4 \\ p_2 &= \alpha_2\alpha_4+\beta_2\beta_4 \\ p_3 &= \alpha_3\alpha_4+\beta_3\beta_4 \end{aligned}\right\};$$

also

$$\begin{aligned} q_1^2 &= (\alpha_1^2+\beta_1^2+\lambda)(\alpha_4^2+\beta_4^2+\lambda) - (\alpha_1\alpha_4+\beta_1\beta_4)^2, \\ &= (-\beta_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)(\alpha_1^2+\alpha_2^2+\alpha_3^2-\beta_4^2) - (\alpha_1\alpha_4+\beta_1\beta_4)^2, \\ &= (\alpha_2^2+\alpha_3^2)^2 + (\alpha_2^2+\alpha_3^2)(\alpha_1^2+\alpha_4^2-\beta_1^2-\beta_4^2) - (\alpha_1\beta_1+\alpha_4\beta_4)^2, \\ &= (\alpha_2^2+\alpha_3^2)(\beta_2^2+\beta_3^2) - (\alpha_2\beta_2+\alpha_3\beta_3)^2 \quad \text{by } (\cancel{50}\ 49), \\ &= (\alpha_2\beta_3-\alpha_3\beta_2)^2; \end{aligned}$$

whence by symmetry we can put

$$\left.\begin{aligned} q_1 &= \alpha_2\beta_3-\alpha_3\beta_2 \\ q_2 &= \alpha_3\beta_1-\alpha_1\beta_3 \\ q_3 &= \alpha_1\beta_2-\alpha_2\beta_1 \end{aligned}\right\},$$

and it is easily verified that these values satisfy all the conditions.

§ 4~~2~~1. For the further discussion of this case we may assume

$$-\lambda \equiv \Sigma\alpha^2 \equiv \Sigma\beta^2 = 1, \text{ whence } p_4 = \alpha_4^2+\beta_4^2-1.$$

Referring back to the coordinates of the poles P_r (§ 39) we have

$$(p_2\beta_3\alpha_4) = \begin{vmatrix} p_2, & p_3, & p_4 \\ \beta_2, & \beta_3, & \beta_4 \\ \alpha_2, & \alpha_3, & \alpha_4 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_2\alpha_4+\beta_2\beta_4, & \alpha_3\alpha_4+\beta_3\beta_4, & \alpha_4^2+\beta_4^2-1 \\ \beta_2, & \beta_3, & \beta_4 \\ \alpha_2, & \alpha_3, & \alpha_4 \end{vmatrix}$$

$$= \begin{vmatrix} 0, & 0, & -1 \\ \beta_2, & \beta_3, & \beta_4 \\ \alpha_2, & \alpha_3, & \alpha_4 \end{vmatrix} = q_1,$$

$$(q_2\beta_3\alpha_4) = -(\alpha_1\alpha_4+\beta_1\beta_4) = -p_1, \text{ etc.},$$

hence the coordinates take the simple forms

$$a_{rs} = -(\alpha_r\alpha_s + \beta_r\beta_s), \qquad r \neq s$$
$$a_{rr} = -(\alpha_r^2 + \beta_r^2 - 1)$$

$$P_1: \quad \frac{x_{11}}{-i(p_1 - iq_1)} = \frac{x_{21}}{-i(p_2 - iq_2)} = \frac{x_{31}}{-i(p_3 - iq_3)} = \frac{x_{41}}{-i(p_4 - iq_4)},$$

$$P_2: \quad \frac{x_{12}}{2ip_4\beta_1} = \frac{x_{22}}{2ip_4\beta_2} = \frac{x_{32}}{2ip_4\beta_3} = \frac{x_{42}}{2ip_3\beta_4},$$

$$P_3: \quad \frac{x_{13}}{-i(p_1 + iq_1)} = \frac{x_{23}}{-i(p_2 + iq_2)} = \frac{x_{33}}{-i(p_3 + iq_3)} = \frac{x_{43}}{-i(p_4 + iq_4)},$$

$$P_4: \quad \frac{x_{14}}{2ip_4\alpha_1} = \frac{x_{24}}{2ip_4\alpha_2} = \frac{x_{34}}{2ip_4\alpha_3} = \frac{x_{44}}{2ip_4\alpha_4}.$$

also
$$\begin{aligned} D_0 &= (\text{clearly})\ q_1^2 + q_2^2 + q_3^2 \\ &= 1 - \alpha_4^2 - \beta_4^2 \\ &= -p_4\,; \end{aligned}$$

hence finally the coefficients of the substitution in mode (A) for which $\Sigma\underline{\alpha}\underline{x}$, $\Sigma\underline{\beta}\underline{x}$, $\Sigma\underline{x}^2$ are invariant, are given by

$$l_{rs} = a_{rs}\cos\theta + b_{rs}\sin\theta + c_{rs}, \quad --- \quad (51)$$

where the $\underline{a}$, $\underline{b}$, $\underline{c}$ are given in the three schemes below, the arrangement in respect of suffixes being the same as in the table of coordinates of poles just above:

$\underline{a}$:
$$\begin{array}{llll} -(\alpha_1^2+\beta_1^2-1), & -(\alpha_1\alpha_2+\beta_1\beta_2), & -(\alpha_1\alpha_3+\beta_1\beta_3), & -(\alpha_1\alpha_4+\beta_1\beta_4), \\ -(\alpha_1\alpha_2+\beta_1\beta_2), & -(\alpha_2^2+\beta_2^2-1), & -(\alpha_2\alpha_3+\beta_2\beta_3), & -(\alpha_2\alpha_4+\beta_2\beta_4), \\ -(\alpha_1\alpha_3+\beta_1\beta_3), & -(\alpha_2\alpha_3+\beta_2\beta_3), & -(\alpha_3^2+\beta_3^2-1), & -(\alpha_3\alpha_4+\beta_3\beta_4), \\ -(\alpha_1\alpha_4+\beta_1\beta_4), & -(\alpha_2\alpha_4+\beta_2\beta_4), & -(\alpha_3\alpha_4+\beta_3\beta_4), & -(\alpha_4^2+\beta_4^2-1)\,; \end{array}$$

$\underline{b}$:
$$\begin{array}{llll} 0, & -(\alpha_3\beta_4), & (\alpha_2\beta_4), & -(\alpha_2\beta_3), \\ (\alpha_3\beta_4), & 0, & -(\alpha_1\beta_4), & -(\alpha_3\beta_1), \\ -(\alpha_2\beta_4), & (\alpha_1\beta_4), & 0, & -(\alpha_1\beta_2), \\ (\alpha_2\beta_3), & (\alpha_3\beta_1), & (\alpha_1\beta_2), & 0\,; \end{array}$$

where $(\alpha_r\beta_s) \equiv \alpha_r\beta_s - \alpha_s\beta_r$;

$$C_{rs} = \alpha_r \alpha_s - \beta_r \beta_s$$

$$\underline{c}: \quad \begin{matrix} \alpha_1^2-\beta_1^2, & \alpha_1\alpha_2-\beta_1\beta_2, & \alpha_1\alpha_3-\beta_1\beta_3, & \alpha_1\alpha_4-\beta_1\beta_4, \\ \alpha_1\alpha_2-\beta_1\beta_2, & \alpha_2^2-\beta_2^2, & \alpha_2\alpha_3-\beta_2\beta_3, & \alpha_2\alpha_4-\beta_2\beta_4, \\ \alpha_1\alpha_3-\beta_1\beta_3, & \alpha_2\alpha_3-\beta_2\beta_3, & \alpha_3^2-\beta_3^2, & \alpha_3\alpha_4-\beta_3\beta_4, \\ \alpha_1\alpha_4-\beta_1\beta_4, & \alpha_2\alpha_4-\beta_2\beta_4, & \alpha_3\alpha_4-\beta_3\beta_4, & \alpha_4^2-\beta_4^2. \end{matrix}$$

It can be easily verified that these values agree with those given in § 31 (A) for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\beta_1 = +\beta_2 = -\beta_3 = +\beta_4 = \tfrac{1}{2}.$$

in mode A

We have thus completed the determination of the substitution analogous to a rotation $\underline{\theta}$ about a given axis in three dimensions, in mode (A).

trs/

Mode (B).

If, for the case of mode (B), we were to make the same assumption as in (48), we should find that the alteration needed in going from (A) to (B), equivalent to the change from $\underline{C}_{rs}$ to $\underline{D}_{rs}$ in (47), amounts simply to changing the minus signs in the $\underline{c}$ above into plus. (It will have been noted that the formulae for the $\underline{h}$ in modes (A), (B) (§ 31) differ simply ~~by~~ in the constant terms).

But it appears ~~that~~ the assumption (48) is not sufficiently general for mode (B), since a term of the form

$$\eta(\Sigma\alpha x)(\Sigma\beta x), \qquad \eta \text{ const.},$$

might also appear on the left.

To see whether this would introduce any more generality into the result, it will be sufficient to replace the $\underline{\alpha}_r, \underline{\beta}_r$ in § 40 by

$$\alpha_r' = l\alpha_r + l'\beta_r, \quad \beta_r' = m\alpha_r + m'\beta_r, \quad (r=1,2,3,4),$$

where we may obviously assume

$$-\lambda = \Sigma\alpha_r'^2 = \Sigma\beta_r'^2, \quad \Sigma\alpha_r'\beta_r' = 0.$$

These relations yield

$$l^2+l'^2 = m^2+m'^2 = 1, \qquad ll'+mm' = 0,$$

whence say

$$l = \cos\xi \quad , \quad l' = \sin\xi,$$
$$m = -\sin\xi \quad , \quad m' = \cos\xi,$$
$$l^2+m^2 = l'^2+m'^2 = 1 \quad , \quad lm'-l'm = 1.$$

Now denoting the new coefficients of the imaginary invariants by accented letters, p'_r, q'_r, we have

$$p'_4 = (l\alpha_4+l'\beta_4)^2 + (m\alpha_4+m'\beta_4)^2 + \lambda$$
$$= \alpha_4^2+\beta_4^2+\lambda \quad = p_4,$$
$$p'_1 = p_1 \ , \quad \text{etc.},$$
$$q'_1 = q_1 \ , \quad \text{etc.},$$

so ~~that~~ no generality is lost in the case of mode (B) by omission of the product term from the left of (48). A similar proof may be devised for the more general case treated below (§45).

The substitution defined by (51) is as closely related to the axis $(\beta_1, \beta_2, \beta_3, \beta_4)$ as to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$; it requires indeed two axes for its determination, the only restriction being that they must be "at right angles" (§33).

If we interchange the α, β in the above formulae, the coefficients a, b, c in (51) are changed at most in sign; the a alone remaining unchanged in mode (A), and the b alone being changed in mode (B). Thus in mode (B) a rotation θ about the α-axis is equivalent to a rotation $-\theta$ about the β-axis; and in mode (A) a rotation $(\pi-\theta)$ about the β-axis yields the direction <u>opposite</u> to that obtained by a rotation θ about the α-axis.

§42. Suppose that in place of $\Sigma \underline{x}^2$ we are given as quadratic invariant the general quadratic form

$$S = (a, b, c, d, f, g, h, l, m, n \between x_1, x_2, x_3, x_4)^2 \quad \text{---} (52)$$

It is known (Bromwich, <u>Quadratic Forms</u>, §8) ~~that~~ this can be written

$$S = \frac{\xi_1^2}{a} + \frac{\xi_2^2}{aC} + \frac{\xi_3^2}{C\Delta} + \frac{\xi_4^2}{\Delta D},$$

where the $\underline{\xi}$ are linear functions of the $\underline{x}$, $\underline{D}$ is the discriminant of $\underline{S}$, and $\underline{\Delta}$, $\underline{C}$ are obtained in succession therefrom by removals of the last row and column.

Now in the case of an odd number of variables, if the discriminant of the form were negative, we could consider instead the same form with the <u>minus</u> sign prefixed, whose discriminant is positive; and we need not then have separate discussions for forms of positive and negative discriminant. But when, as here, the number of variables is even, the discriminants of $\underline{S}$, $-\underline{S}$ are identical, and we have two cases, according as $\underline{D}$ is positive or negative.

I. <u>D positive</u>.

Here $\underline{S}$ must either be a sum* of four positive squares, or of two positive and two negative, or of four negative, according to the signs of $\underline{a}$, $\underline{C}$, $\underline{\Delta}$. We may therefore make an assumption of the form

?

(wants explaining: cf. p 53)

$$(\Sigma \alpha x)^2 + (\Sigma \beta x)^2 + \lambda S = \mu \{(\Sigma p x)^2 + (\Sigma q x)^2\}. \quad \text{---} (53)$$

II. <u>D negative</u>.

Here $\underline{S}$ is either a sum of three positive and one negative squares, or of one positive and

---

* Sylvester's <u>Law of Inertia</u>: see above, §40.

three negative, and our assumption must be of the form

$$(\Sigma\alpha x)^2 - (\Sigma\beta x)^2 + \lambda S = \mu\{(\Sigma p x)^2 + (\Sigma q x)^2\}.$$

Results for the second case may however be deduced from those for the first by affecting each β ~~that~~ occurs with the factor i.

§43. We can follow the method of the less general case (§40) up to a certain point. Let us put

$$\phi_1 \equiv \begin{vmatrix} a, & h, & g, & l, & \alpha_1 \\ h, & b, & f, & m, & \alpha_2 \\ g, & f, & c, & n, & \alpha_3 \\ l, & m, & n, & d, & \alpha_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & 0 \end{vmatrix};$$

let ϕ_2 denote the corresponding expression when the α in the last row and column are replaced by the β with same suffixes, and ϕ_{12} the expression when the α of the last row, but not of the last column, are replaced by the β.

The discriminant of the left-hand member of (53) is, as before, a biquadratic in λ, with two roots zero; and the residuary quadratic is

$$D\lambda^2 + (\phi_1 + \phi_2)\lambda + \Psi = 0,$$

where $$D\Psi = \phi_1\phi_2 - \phi_{12}^2.$$

Since this quadratic must as before have equal roots, hence

$$\phi_1 = \phi_2, \quad \phi_{12} = 0, \quad \text{-----} \quad (54)$$

and

$$-D\lambda = \phi_1 = \phi_2.$$

§ 44. Using $\underline{\varphi}$ to denote either φ_1 or φ_2, we have for the $\underline{p}$, $\underline{q}$ ten relations, typified by the three

$$\left.\begin{aligned} D(\alpha_1^2+\beta_1^2) - \varphi a &= \mu D(p_1^2+q_1^2) \\ D(\alpha_2\alpha_3+\beta_2\beta_3) - \varphi f &= \mu D(p_2p_3+q_2q_3) \\ D(\alpha_1\alpha_4+\beta_1\beta_4) - \varphi l &= \mu D(p_1p_4+q_1q_4) \end{aligned}\right\} \quad (55)$$

Assuming $\underline{q_4}=0$, $\underline{\mu}\,\underline{D}\,\underline{p_4}=1$, we at once obtain as before (see §40) $\underline{p_1}, \underline{p_2}, \underline{p_3}, \underline{p_4}$ as rational quadratic functions of the $\underline{\alpha}, \underline{\beta}$; and for $\underline{q_1}, \underline{q_2}, \underline{q_3}$ equations such as

$$\left.\begin{aligned} p_1^2+q_1^2 &= \{D(\alpha_1^2+\beta_1^2)-\varphi a\}\{D(\alpha_4^2+\beta_4^2)-\varphi d\} \\ p_2p_3+q_2q_3 &= \{D(\alpha_2\alpha_3+\beta_2\beta_3)-\varphi f\}\{D(\alpha_4^2+\beta_4^2)-\varphi d\} \end{aligned}\right\} \quad (56)$$

It is evident from the relation

$$\varphi^2 = \varphi_1\varphi_2 - \varphi_{12}^2,$$

and the forms of the expressions on the right-hand of this identity,* that φ^2 contains the factor $\underline{D}$. It therefore follows from (56) that $\underline{q_1}, \underline{q_2}, \underline{q_3}$ are each of the form

$$D^{\frac{1}{2}} \text{ (rational quadratic function of the } \underline{\alpha}, \underline{\beta}).$$

The labour, however, of obtaining these expressions from the above identities would be great, and can be avoided by turning to the geometrical interpretation of the analytical process we are here carrying out; which will also throw new light on the meaning of the coefficients $\underline{p}, \underline{q}$.

---

* See last article.

§45. In the phraseology of the theory of the six coordinates (Plücker coordinates) of a line (see Salmon–Rogers, Analytic Geometry of Three Dimensions, §§ 53 ff*), let $\underline{s}_r$, $\underline{\sigma}_r$ ($r = 1, 2, \cdots 6$) denote respectively the ray-coordinates and axial-coordinates of the line of intersection of the planes

$$\Sigma \alpha x = 0, \quad \Sigma \beta x = 0;$$

let $\underline{s}'_r$, σ'_r denote ~~respectively~~ the corresponding coordinates of the line of intersection of the planes

$$\Sigma p x = 0, \quad \Sigma q x = 0;$$

and let these lines be called respectively (1), (2). Then with the abbreviations

$$(\alpha_r \beta_s) \equiv (\alpha_r \beta_s - \alpha_s \beta_r), \text{ etc.,}$$

we have

$$\left.\begin{aligned} &(\alpha_2 \beta_3) = \sigma_1 = s_4, \quad (\alpha_3 \beta_1) = \sigma_2 = s_5, \quad (\alpha_1 \beta_2) = \sigma_3 = s_6 \\ &(\alpha_1 \beta_4) = \sigma_4 = s_1, \quad (\alpha_2 \beta_4) = \sigma_5 = s_2, \quad (\alpha_3 \beta_4) = \sigma_6 = s_3 \end{aligned}\right\}, \quad \text{--- (57)}$$

$$\left.\begin{aligned} &(p_2 q_3) = \sigma'_1 = s'_4, \quad \text{etc.} \\ &(p_1 q_4) = \sigma'_4 = s'_1, \quad \text{etc.} \end{aligned}\right\}. \quad \text{--- (58)}$$

Now it is clear from (53) ~~that~~ the planes $\Sigma \underline{\alpha} \underline{x}$, $\Sigma \underline{\beta} \underline{x}$, $\Sigma \underline{p} \underline{x}$, $\Sigma \underline{q} \underline{x}$ form a tetrahedron self-conjugate with respect to the conicoid $\underline{S}$, and therefore ~~that~~ line (2) is the polar line of line (1). But the coordinates of the polar of a given line with respect to a conicoid can be obtained as follows.

Let a_{rs} ($r, s = 1, 2, \cdots 6$) denote (see Salmon–Rogers, § 80 (c)*) the second minors of the discriminant of $\underline{S}$, and let

---

* I have altered the notation for the line-coordinates.

$$\Psi \equiv (a_{11}, \cdots a_{56} \between s_1, s_2, \cdots s_6)^2 ;$$

then, unaccented letters denoting the coordinates of the given line, and accented letters those of the required line,

$$2\sigma'_r = \frac{\partial \Psi}{\partial s_r}, \quad (r = 1, 2, \cdots 6). \quad --- \quad (59)$$

But with the *single* assumption $\underline{q_4} = 0$, we have

$$-\sigma'_4 = p_4 q_1, \quad -\sigma'_5 = p_4 q_2, \quad -\sigma'_6 = p_4 q_3 ;$$

or rather, since in the six-coordinate theory the ratios alone of σ are in question,

$$\frac{q_1}{\sigma'_4} = \frac{q_2}{\sigma'_5} = \frac{q_3}{\sigma'_6}.$$

The q are thus determined except for a constant factor, which must, by consideration of the degrees of the $\underline{p}$, $\underline{q}$ in the *coefficients* of $\underline{S}$, be a numerical multiple of $\sqrt{\underline{D}}$. We thus have finally the following, which can be verified in a variety of simple cases:

$$p_1 = D(\alpha_1\alpha_4 + \beta_1\beta_4) - \varphi\, l,$$
$$p_2 = D(\alpha_2\alpha_4 + \beta_2\beta_4) - \varphi\, m,$$
$$p_3 = D(\alpha_3\alpha_4 + \beta_3\beta_4) - \varphi\, n,$$
$$p_4 = D(\alpha_4^2 + \beta_4^2) - \varphi\, d,$$

$$q_1 = \sqrt{D}. \begin{vmatrix} a, & h, & g, & l \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ l, & m, & n, & d \end{vmatrix}, \qquad q_2 = -\sqrt{D}. \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ h, & b, & f, & m \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ l, & m, & n, & d \end{vmatrix},$$

$$q_3 = \sqrt{D}. \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix}.$$

§46. Case of Two Quadratic Invariants Given.

We can now indicate the answer to another question, viz.,

Under what conditions will two real quadratic invariants serve to define a real equimodular substitution in four variables? If $\underline{S}$, $\underline{S}'$ are the given invariants, it is necessary in order to obtain the requisite four linear invariants, ~~that~~ $\underline{S}+\underline{\lambda}\underline{S}'$ shall break up into two planes (real or imaginary) for two distinct values of $\underline{\lambda}$. For each of these values, therefore, the discriminant of $\underline{S}+\underline{\lambda}\underline{S}'$ and all its first minors must vanish.

Now in the reduced forms (see Bromwich, Quadratic Forms §18)* the only case satisfying these conditions is that numbered 9 on Lindemann's list, with the symbol $[(11)(11)]$; namely ~~that~~ in which $\underline{S}$, $\underline{S}'$ have four generators in common, ~~and~~ forming a skew quadrilateral, at ~~which~~ the vertices of which the conicoids touch. Denoting by $\underline{\xi_r}$ certain linear functions of the $\underline{x}$, we can then write ~~in the notation of Bromwich~~ (§42)

$$S = k_1\xi_1^2 + k_2\xi_2^2 + k_3\xi_3^2 + k_4\xi_4^2,$$
$$S' = c_1(k_1\xi_1^2 + k_2\xi_2^2) + c_3(k_3\xi_3^2 + k_4\xi_4^2);$$

the one pair of linear invariants will then be linear functions of $\underline{\xi_1}, \underline{\xi_2}$; the other, of $\underline{\xi_3}, \underline{\xi_4}$. Of these invariants there will be four, two, or none real according to the signs of the coefficients $\underline{k_r}$ (which of course do not here represent the roots of the characteristic equation, but are as above §42).

~~We can in fact write~~

Thus, assuming $\underline{S}$, $\underline{S}'$ to satisfy the

---

* See also Bell, Coordinate Geometry of Three Dimensions, §§166-8; Salmon-Rogers, §202.

above conditions, and reducing them to sums of squares by the method of Bromwich (*ibid.* *Art.* 17), we immediately obtain the linear invariants, from which the coefficients of the substitution can be obtained by the method of § 39. The variety of possible cases seems to preclude the possibility of a general formula like that of the last article. The only novelty which appears is the case in which all four linear invariants, and therefore also the multipliers, are imaginary; they consist, of course, of conjugate pairs; and the treatment offers no special difficulties.

Section V.

On a Certain Substitution of Order Four, whose Invariants are all Real.

§ 47. If equations (17), viz.,

$$\xi_\varepsilon = \sum_{s=1}^{n} \rho^{s\varepsilon} x_s \,, \quad (\varepsilon = 1, 2, --- n),$$

are regarded as defining a substitution, ~~that substitution~~ will, in the interpretation of §2, transform the frame of reference into the frame determined by the unit-points. It is remarkable that, as seen by (19), it is also the substitution connecting the coefficients of the substitution of Section II with its multipliers. It has thus a double connexion with the theory of that Section; its investigation will also afford examples of the method of treatment of substitutions adopted in this paper.

It will be convenient to consider the substitution in a slightly modified form, in which its determinant is unity, viz.,

$$x'_\alpha = n^{-\frac{1}{2}} \sum_{\beta=1}^{n} \rho^{\alpha\beta} x_\beta \,, \quad (\alpha = 1, 2, --- n), \quad --- (60)$$

where $\underline{n}^{-\frac{1}{2}}$ is assumed to be taken with the positive sign. We shall denote this substitution by $\underline{S}(\rho)$, the left-hand member of its characteristic equation by $\underline{\Delta}_n(\rho, \underline{k})$, and its determinant by $\Delta_n(\rho, \underline{0})$ or simply $\Delta_n(\underline{\rho})$. Thus writing $\underline{k} = \underline{n}^{\frac{1}{2}}\underline{K}$, we have

$$\Delta_n(\rho, k) = (n^{-\frac{1}{2}})^n \begin{vmatrix} \rho - K, & \rho^2, & --- & \rho^{n-1}, & 1 \\ \rho^2, & \rho^4 - K, & --- & \rho^{2(n-1)}, & 1 \\ - & - & - & - & - \\ \rho^{n-1}, & \rho^{2(n-1)}, & --- & \rho^{(n-1)^2} - K, & 1 \\ 1, & 1, & --- & 1, & 1 - K \end{vmatrix}.$$

To obtain the substitution inverse to $S(\rho)$, multiply both sides of (60) by $\rho^{\gamma\alpha}$, and add for the n values of α. The coefficient of x_β on the right is

$$n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} \rho^{\alpha(\beta+\gamma)},$$

which differs from zero only in the case of $\beta = n-\gamma$, and then takes the value $n^{\frac{1}{2}}$. Hence

$S^{-1}(\rho)$. $$x_{n-\gamma} = n^{-\frac{1}{2}} \sum_{\alpha=1}^{n} \rho^{\gamma\alpha} x'_\alpha, \; (\gamma = 1, 2, \dots n), \quad \text{---} (61)$$

where x_0 is regarded as equivalent to x_n.

The only difference between (61) and (60) is that on the left of (61) the order of the first $(n-1)$ variables has been reversed.

Now let x''_γ $(\gamma = 1, 2, \dots n)$ be the result of two applications of $S(\rho)$. Then

$S^2(\rho)$. $$x''_\gamma = n^{-\frac{1}{2}} \sum_{\beta=1}^{n} \rho^{\gamma\beta} x'_\beta \quad \text{by (60)},$$

$$= x_{n-\gamma} \quad \text{by (61)};$$

the original variables being restored, but, with the exception of the last, which remains unchanged, reversed in order. It follows that $S^4(\rho)$ is equivalent to the identical substitution, and that $S(\rho)$ is of order four. Further, since the multiplier with which the variables are restored is unity, therefore

$$\Delta_n^4(\rho) = 1,$$

$$\Delta_n(\rho) = \exp\left(\tfrac{1}{2} p\pi i\right), \quad \text{- - - -} \quad (62)$$

p having one of the values 0, 1, 2, 3.

§ 48. $S(\rho)$ being of order four, the roots of the characteristic equation must be fourth roots of unity (§ 12), whence

$$\Delta_n(\rho, k) \equiv (-)^n (k-1)^q (k+1)^r (k-i)^s (k+i)^t,$$

where $\quad q + r + s + t = n.$ — — — — — — — (63)

In finding these indices, we shall confine ourselves to a specific value of $\underline{\rho}$ (§15),

$$\rho = \underline{\exp}\,(2\pi i/n).$$

The more general case need not detain us, for other values of $\underline{\rho}$ lead to the same quantities x'_α in (60), only in a different order.

By multiplication of determinants we find for

$$\left.\begin{array}{lll} n = 2m+1, & \Delta_n(\rho, k)\Delta_n(\rho, -k) = & -(k^2-1)^{m+1}(k^2+1)^m \\ n = 2m, & \qquad " \qquad\quad = & (k^2-1)^{m+1}(k^2+1)^{m-1} \end{array}\right\}. \text{ — — — (64)}$$

By a theorem of Gauss (see Mathews, Theory of Numbers, §184), writing

$$S = \sum_{s=0}^{n-1} \underline{\exp}\,(2s^2\pi i/n),$$

we have for

$$\begin{array}{llll} n \equiv 0 \ (\text{mod } 4) & S & = & (1+i)\sqrt{n} \\ \quad\equiv 1 & & = & \sqrt{n} \\ \quad\equiv 2 & & = & 0 \\ \quad\equiv 3 & & = & i\sqrt{n}. \end{array}$$

Now the coefficient of $\underline{k}^{n-1}$ in $\underline{\Delta}_n(\underline{\rho}, \underline{k})$ is

$$(-)^{n-1} n^{-\frac{1}{2}} S;$$

and with $\underline{n} = 2\underline{m} + 1$, $\underline{\Delta}_n(\underline{\rho}, \underline{k})$ must by (64) have the form

$$\Delta_n(\rho, k) = -(k-1)^\alpha (k+1)^{m+1-\alpha} (k-i)^\beta (k+i)^{m-\beta}.$$

Hence, picking out the coefficient of the second-highest power of $\underline{k}$, we have for

$$n \equiv 1 \ (\text{mod } 4), \quad (2\alpha - m - 1) + i(2\beta - m) = 1,$$

yielding $\qquad \alpha = \frac{1}{2}m + 1, \quad \beta = \frac{1}{2}m;$

and for

$$n \equiv 3 \ (\text{mod } 4), \quad (2\alpha - m - 1) + i(2\beta - m) = i,$$

whence $\qquad \alpha = \frac{1}{2}(m+1) = \beta.$

Again, with $n = 2m$, $\Delta_n(\rho, k)$ is of form

$$\Delta_n(\rho, k) = (k-1)^{\alpha}(k+1)^{m+1-\alpha}(k-i)^{\beta}(k+i)^{m-1-\beta},$$

whence for

$n \equiv 0 \pmod 4$, $\quad \alpha = \frac{1}{2}m + 1$, $\beta = \frac{1}{2}m$;

$\equiv 2$ $\quad \alpha = \frac{1}{2}(m+1)$, $\beta = \frac{1}{2}(m-1)$.

The following table summarises these results, and also gives the values of p in ( ):

n	$\Delta_n(\rho, k)$	$p \equiv \pmod 4$
$4s$	$(k-1)^{s+1}(k+1)^{s}(k-i)^{s}(k+i)^{s-1}$	$2s+1$
$4s+1$	$-(k-1)^{s+1}(k+1)^{s}(k-i)^{s}(k+i)^{s}$	$2s$
$4s+2$	$(k-1)^{s+1}(k+1)^{s+1}(k-i)^{s}(k+i)^{s}$	$2s+2$
$4s+3$	$-(k-1)^{s+1}(k+1)^{s+1}(k-i)^{s+1}(k+i)^{s}$	$2s+3$

§49. Taking q, r, s, t as in (63), the canonical form* of $S(\rho)$ is the following:

$$\xi'_{\alpha} = \xi_{\alpha}, \quad (\alpha = 1, 2, \cdots q),$$
$$\xi'_{\beta} = -\xi_{\beta}, \quad (\beta = q+1, \cdots q+r),$$
$$\xi'_{\gamma} = i\xi_{\gamma}; \quad (\gamma = q+r+1, \cdots q+r+s),$$
$$\xi'_{\delta} = -i\xi_{\delta}, \quad (\delta = q+r+s+1, \cdots n),$$

the ξ being a set, not in general unique, of linear invariants of $S(\rho)$. It remains to find expressions for the ξ in terms of the original coordinates x.

A separation can be made between the ξ with real and those with imaginary multipliers; the former appearing as the invariants of a new substitution in $(q+r)$ variables, and the latter in the same relation to another substitution

---

* Cf. Burnside, Theory of Groups, §194.

in $(\underline{s}+\underline{t})$ variables. The reduction shows ~~that~~ the $\underline{\xi}$ are in all cases real functions of the $\underline{x}$; thus the linear invariants, and therefore also the poles of $\underline{S}(\underline{\rho})$ are all real.

An invariant of $\underline{S}(\rho)$ with real multiplier (± 1) is an invariant of $\underline{S}^2(\underline{\rho})$ with multiplier $+1$, and therefore by (61) is unaltered by interchange throughout of $\underline{x}_\alpha, \underline{x}_{n-\alpha}$, $(\underline{\alpha} = 1, 2, \cdots \underline{n}-1)$. An invariant of $\underline{S}(\underline{\rho})$ with imaginary multiplier $(\pm \underline{i})$ is an invariant of $\underline{S}^2(\rho)$ with multiplier -1, and therefore simply changes sign on interchange of the $\underline{x}_\alpha, \underline{x}_{n-\alpha}$. Write

$$x_\alpha + x_{n-\alpha} = y_\alpha,$$
$$x_\alpha - x_{n-\alpha} = z_\alpha,$$

where $\underline{\alpha}$ takes all integral values from 1 to $\underline{m}$ or $\underline{m-1}$ according as $\underline{n} = 2\underline{m}+1$ or $2\underline{m}$. In the latter case write $\underline{x}_m = \underline{y}_m$, and in both cases $\underline{x}_n = \underline{y}_{m+1}$. Then the invariants with real multipliers are linear functions of the $\underline{y}$, those with imaginary multipliers of the $\underline{z}$.

Putting

$$\left.\begin{aligned} \rho^\alpha + \rho^{-\alpha} &\equiv 2\cos(2\pi\alpha/n) = \sigma_\alpha \\ -i(\rho^\alpha - \rho^{-\alpha}) &\equiv 2\sin(2\pi\alpha/n) = \tau_\alpha \end{aligned}\right\}, \quad \text{---} \; (65)$$

and taking *

(i) $\underline{n} = 2\underline{m}+1$, we have from (60)

$$\left.\begin{aligned} \sqrt{n}\,.\,y'_\alpha &= \sum_{\beta=1}^{m} \sigma_{\alpha\beta} y_\beta + 2y_{m+1}, \quad (\alpha = 1, 2, \cdots m) \\ \sqrt{n}\,.\,y'_{m+1} &= \sum_{\beta=1}^{m} y_\beta + y_{m+1} \end{aligned}\right\}, \quad \text{---} \; (66)$$

and

$$\sqrt{n}\,.\,z'_\alpha = \sum_{\beta=1}^{m} i\tau_{\alpha\beta} z_\beta, \quad (\alpha = 1, 2, \cdots m). \quad \text{--} \; (67)$$

* The suffix $\underline{\alpha\beta}$ in this paragraph represents of course the product of α, β.

The former is a substitution of order 2 in the ($\underline{m}+1$) variables $\underline{y}_1, \underline{y}_2, \cdots \underline{y}_{m+1}$, of which the invariants, when expressed in terms of the $\underline{x}$, are those of (60) with real coefficients and have real coefficients throughout. The invariants of (~~67~~ 64) are likewise identical with those of (60) which have imaginary ~~coefficients~~ multipliers; and since, if we omit the $\underline{i}$ on the right-hand side of (67), the only adjustment required is to alter the multipliers from $\pm\underline{i}$ to ± 1 respectively, the invariants themselves remaining unchanged, it is clear ~~that~~ they are real functions of the $\underline{z}$, and hence also of the $\underline{x}$.

(ii) Similarly for $\underline{n} = 2\underline{m}$. Equations (66) are replaced by

$$\left.\begin{aligned}
\sqrt{n}\,.\,y'_\alpha &= \sum_{\beta=1}^{m-1} \sigma_{\alpha\beta}\, y_\beta + (-)^\alpha 2y_m + 2y_{m+1}, \quad (\underline{\alpha} = 1, 2, \cdots \underline{m}-1) \\
\sqrt{n}\,.\,y'_m &= \sum_{\beta=1}^{m-1} (-)^\beta y_\beta + (-)^m y_m + y_{m+1} \\
\sqrt{n}\,.\,y'_{m+1} &= \sum_{\beta=1}^{m-1} y_\beta + y_m + y_{m+1}
\end{aligned}\right\} \quad (68)$$

and (67) by

$$\sqrt{n}\,.\,z'_\alpha = \sum_{\beta=1}^{m-1} i\,\tau_{\alpha\beta}\, z_\beta, \quad (\alpha = 1, 2, \cdots m-1). \quad \text{---} \quad (69)$$

On comparing the forms assumed by the characteristic equation in (66) - (69) with those of the table of §48, various identities are obtained, of which the following may be noted:

With $\underline{n} = 2\underline{m}+1 = 4\underline{s}+1$, the $\underline{\sigma}, \underline{\tau}$ being as in (65),

$$\begin{vmatrix}
\sigma_1 - \kappa, & \sigma_2, & \cdots & \sigma_m, & 2 \\
\sigma_2, & \sigma_4 - \kappa, & \cdots & \sigma_{2m}, & 2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\sigma_m, & \sigma_{2m}, & \cdots & \sigma_{m^2} - \kappa, & 2 \\
1, & 1, & \cdots & 1, & 1-\kappa
\end{vmatrix} = -(\kappa - \sqrt{n})(\kappa^2 - n)^s,$$

From Proc. Edin. Math. Soc. Vol XXXI (1912-13).

TABLE OF LINEAR INVARIANTS OF $S(\rho)$.

n	mult $+1$	mult $+i$
2	$\Sigma x + \sqrt{2} \,.\, x_2$	none
3	$\Sigma x + \sqrt{3} \,.\, x_3$	$x_1 - x_2$
4	$\Sigma x + \sqrt{4} \,.\, x_4$ $x_2 + x_4$	$x_1 - x_3$
5	$\Sigma x + \sqrt{5} \,.\, x_5$ $(x_1 + x_4) - (x_2 + x_3)$	$(\tau_1 + \sqrt{5})(x_1 - x_4) + \tau_2(x_2 - x_3)$
6	$\Sigma x + \sqrt{6} \,.\, x_6$ $(2 + \sqrt{6})(x_1 + x_5) - \sqrt{6}(x_2 + x_4) - 4x_3$	$(\tau_1 + \sqrt{6})(x_1 - x_5) + \tau_2(x_2 - x_4)$ (here $\tau_1 = \tau_2 = + \sqrt{3}$)
7	$\Sigma x + \sqrt{7} \,.\, x_7$ $(\sigma_1 - \sigma_2 + \sqrt{7})(x_1 + x_6) + (\sigma_2 - \sigma_3 - \sqrt{7})(x_2 + x_5) + (\sigma_3 - \sigma_1)(x_3 + x_4)$	$\tau_2(x_1 - x_6) - (\tau_1 - \sqrt{7})(x_2 - x_5)$ $(x_1 - x_6) + (x_2 - x_5) - (x_3 - x_4)$
8	$\Sigma x + \sqrt{8} \,.\, x_8$ $4(x_1 + x_7) - \sqrt{8} \,.\, x_4 + \sqrt{8} \,.\, x_8$ $(x_1 + x_7) - (x_3 + x_5)$	$4(x_1 - x_7) + \sqrt{8}(x_2 - x_6)$ $(x_1 - x_7) + (x_3 - x_5)$

$$\begin{vmatrix} \tau_1 - K, & \tau_2, & --- & \tau_m \\ \tau_2, & \tau_4 - K, & --- & \tau_{2m} \\ - & - & - & - \\ \tau_m, & \tau_{2m}, & --- & \tau_{m^2} - K \end{vmatrix} = (K^2+n)^s.$$

The invariants ξ obtained by the help of (66) – (69) as far as $n = 8$ are tabulated below. Only those with multipliers $+1, +i$ are written down; those with multipliers $-1, -i$ are deduced therefrom by changing the sign of $\sqrt{n}$ wherever it occurs explicitly. In the cases of two or more invariants with the same multiplier, those given are not unique, and any linear functions of them would serve equally well. The form

$$\sum_{\alpha=1}^{n} x_\alpha + \sqrt{n}.\, x_n$$

which appears first in each set, is at once seen to be invariant from (60). It is to be remembered that the σ, τ depend on the value of n.

Since the determinant of (60) is symmetrical, the pole corresponding to any invariant

$$\sum_{\alpha=1}^{n} p_\alpha x_\alpha$$

can be immediately deduced: it is in fact the point with coordinates $(p_1, p_2, -- p_n)$. The extension to the case of equal multipliers is easily made.

Relations come to light in the process of finding the forms, which are exemplified in the case of $n = 4$. For this case the substitution (66) will have as an invariant with multiplier $+1$ the function

$$\sum_{\alpha=1}^{4} p_\alpha y_\alpha$$

$$-(\sqrt{\gamma}-\sigma_1)p_1 + \sigma_2 p_2 + \sigma_3 p_3 + p_4 = 0,$$
$$\sigma_2 p_1 - (\sqrt{\gamma}-\sigma_3)p_2 + \sigma_1 p_3 + p_4 = 0,$$
$$\sigma_3 p_1 + \sigma_1 p_2 - (\sqrt{\gamma}-\sigma_2)p_3 + p_4 = 0,$$
$$2p_1 + 2p_2 + 2p_3 - (\sqrt{\gamma}-1)p_4 = 0.$$

Since there are two invariants of this kind, only two of these four equations can be independent; in other words, each first minor of the determinant of the coefficients must vanish. This can be verified in virtue of the relations

$$\sigma_1 + \sigma_2 + \sigma_3 = -1,$$
$$\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2 = -2;$$

and the ~~statement~~ remains true when the sign of the radical is changed.

Similarly the conditions ~~that~~

$$\sum_{\alpha=1}^{3} p_\alpha z_\alpha$$

be an invariant of (6γ) for this case, with multiplier $\pm i$, are

$$-(\sqrt{\gamma}-\tau_1)p_1 + \tau_2 p_2 + \tau_3 p_3 = 0,$$
$$\tau_2 p_1 - (\sqrt{\gamma}+\tau_3)p_2 + \tau_1 p_3 = 0,$$
$$\tau_3 p_1 - \tau_1 p_2 - (\sqrt{\gamma}-\tau_2)p_3 = 0,$$

only one of which can be independent. This is evident on ~~remembering~~ that

$$\tau_1 + \tau_2 - \tau_3 = \sqrt{\gamma},$$
$$\tau_2\tau_3 + \tau_3\tau_1 - \tau_1\tau_2 = 0,$$

but is no longer true when the sign of the radical in the three conditions is changed.