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Exact form factors in integrable quantum field theories: the scaling $Z(N)$-Ising model

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Abstract

A general form factor formula for the scaling $Z(N)$-Ising model is constructed. Exact expressions of all matrix elements are obtained for several local operators. In addition, the commutation rules for order, disorder parameters and para-Fermi fields are derived. Because of the unusual statistics of the fields, the quantum field theory seems not to be related to any classical Lagrangian or field equation.

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1. Introduction

The 'form factor program' is part of the so-called 'bootstrap program' for integrable quantum field theories in $1+1$ dimensions. This program *classifies* integrable quantum field theoretic models and in addition it provides their explicit exact solutions in terms of all Wightman functions. This means, in particular, that we do not *quantize* a classical field theory. In fact the quantum field theory considered in this paper is not related (at least to our knowledge) to any classical Lagrangian or field equations of massive particles. The reason for this seems to be the

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unusual anyonic statistics of the fields, turning this form factor investigation even more fascinating. The bootstrap program consists of three main steps: First the S-matrix is calculated by means of general properties as unitarity and crossing, the Yang–Baxter equations and the additional assumption of 'maximal analyticity'. Second, matrix elements of local operators

$$^{\text{out}}\langle p'_m, \ldots, p'_1|\mathcal{O}(x)|p_1, \ldots, p_n\rangle^{\text{in}}$$

are calculated using the 2-particle S-matrix as an input. As a third step the Wightman functions can be obtained by inserting a complete set of intermediate states.

The generalized form factors [1] are defined by the vacuum—n-particle matrix elements

$$\langle 0|\mathcal{O}(x)|p_1, \ldots, p_n\rangle^{\text{in}}_{\alpha_1\ldots\alpha_n} = e^{-ix(p_1+\cdots+p_n)} F^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1, \ldots, \theta_n),$$

where the α_i denote the type (charge) and the θ_i are the rapidities of the particles ($p_i = M_i(\cosh\theta_i, \sinh\theta_i)$). This definition is meant for $\theta_1 > \cdots > \theta_n$, in the other sectors of the variables the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = F^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1, \ldots, \theta_n)$ is given by analytic continuation with respect to the θ_i. General matrix elements are obtained from $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ by crossing which means in particular the analytic continuation $\theta_i \to \theta_i \pm i\pi$. Using general LSZ assumptions and maximal analyticity in [2] the following properties of form factors have been derived[1]:

(o) The form factor function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ is meromorphic with respect to all variables $\theta_1, \ldots, \theta_n$.

(i) It satisfies Watson's equations

$$F^{\mathcal{O}}_{\ldots\alpha_i\alpha_j\ldots}(\ldots, \theta_i, \theta_j, \ldots) = F^{\mathcal{O}}_{\ldots\alpha_j\alpha_i\ldots}(\ldots, \theta_j, \theta_i, \ldots) S_{\alpha_i\alpha_j}(\theta_{ij}).$$

(ii) The crossing relation means for the connected part (see e.g. [4]) of the matrix element

$$\begin{aligned} {}_{\bar{\alpha}_1}\langle p_1|\mathcal{O}(0)|p_2, \ldots, p_n\rangle^{\text{in,conn.}}_{\alpha_2\ldots\alpha_n} &= \sigma^{\mathcal{O}}(\alpha_1) F^{\mathcal{O}}_{\alpha_1\alpha_2\ldots\alpha_n}(\theta_1 + i\pi, \theta_2, \ldots, \theta_n) \\ &= F^{\mathcal{O}}_{\alpha_2\ldots\alpha_n\alpha_1}(\theta_2, \ldots, \theta_n, \theta_1 - i\pi), \end{aligned}$$

where $\sigma^{\mathcal{O}}(\alpha)$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle α.

(iii) The function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has poles determined by one-particle states in each sub-channel. In particular, if α_1 is the anti-particle of α_2, it has the so-called annihilation pole at $\theta_{12} = i\pi$ which implies the recursion formula

$$\begin{aligned} \operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{\underline{\alpha}}(\theta_1, \ldots, \theta_n) &= 2i\mathbf{C}_{\alpha_1\alpha_2} F^{\mathcal{O}}_{\alpha_3\ldots}(\theta_3, \ldots, \theta_n) \\ &\quad \times \big(\mathbf{1} - \sigma^{\mathcal{O}}(\alpha_2) S_{\alpha_1\alpha_n}(\theta_{2n}) \ldots S_{\alpha_2\alpha_3}(\theta_{23})\big). \end{aligned}$$

(iv) Bound state form factors yield another recursion formula

$$\operatorname*{Res}_{\theta_{12}=i\eta} F^{\mathcal{O}}_{\alpha\beta\ldots}(\theta_1, \theta_2, \theta_3, \ldots) = \sqrt{2} F^{\mathcal{O}}_{\gamma\ldots}(\theta_{(12)}, \theta_3, \ldots) \Gamma^{\gamma}_{\alpha\beta}$$

if $i\eta$ is the position of the bound state pole. The bound state intertwiner $\Gamma^{\gamma}_{\alpha\beta}$ (see e.g. [4,5]) is defined by

$$i \operatorname*{Res}_{\theta=i\eta} S_{\alpha\beta}(\theta) = \Gamma^{\beta\alpha}_{\gamma} \Gamma^{\gamma}_{\alpha\beta}.$$

[1] The formulae have been proposed in [3] as a generalization of formulae in [1]. The formulae are written here for the case of no backward scattering, for the general case see [4].

(v) Since we are dealing with relativistic quantum field theories Lorentz covariance in the form

$$F_{\underline{\alpha}}^{\mathcal{O}}(\theta_1, \dots, \theta_n) = e^{s\mu} F_{\underline{\alpha}}^{\mathcal{O}}(\theta_1 + \mu, \dots, \theta_n + \mu)$$

holds if the local operator transforms as $\mathcal{O} \to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

Note that consistency of (ii), (iii) and (v) imply a relation of spin and statistics $\sigma^{\mathcal{O}}(\alpha) = e^{-2\pi i s}$ and also $\sigma^{\mathcal{O}}(\alpha) = 1/\sigma^{\mathcal{O}}(\bar{\alpha})$ where $\bar{\alpha}$ is the anti-particle of α, which has the same charge as $\mathcal{O}$. All solutions of the form factor equations (i)–(v) should provide the matrix elements of all fields in an integrable quantum filed theory with a given S-matrix.

Generalized form factors are of the form [1]

$$F_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) \prod_{1 \leqslant i < j \leqslant n} F(\theta_{ij}) \quad (\theta_{ij} = \theta_i - \theta_j), \tag{1}$$

where $F(\theta)$ is the 'minimal' form factor function. It is the solution of Watson's equation [6] and the crossing relation for $n = 2$

$$F(\theta) = F(-\theta)S(\theta), \qquad F(i\pi - \theta) = F(i\pi + \theta) \tag{2}$$

with no poles and zeros in the physical strip $0 < \operatorname{Im}\theta \leqslant \pi$ (and a simple zero at $\theta = 0$). In [4] a general integral representation for the K-function $K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ in terms of the *off-shell Bethe Ansatz* [7,8] has been presented, which transforms the complicated equations (i)–(v) for the form factors to simple ones for the *p-functions* (see [4] and (41) below)

$$K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta}) = \sum_m c_{nm} \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, h(\underline{\theta}, \underline{z}) p^{\mathcal{O}}(\underline{\theta}, \underline{z}) \Psi_{\underline{\alpha}}(\underline{\theta}, \underline{z}). \tag{3}$$

The symbols $\mathcal{C}_{\underline{\theta}}$ denote specific contours in the complex z_i-planes. The function $h(\underline{\theta}, \underline{z})$ is scalar and encodes only data from the scattering matrix. The function $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ on the other hand depends on the explicit nature of the local operator $\mathcal{O}$. We discuss these objects in more detail below. For the case of a diagonal S-matrix, as in this paper, the off-shell Bethe Ansatz vector $\Psi_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ is trivial. The K-function $K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ is an meromorphic function and has the 'physical poles' in $0 < \operatorname{Im}\theta_{ij} \leqslant \pi$ corresponding to the form factor properties (iii) and (iv). It turns out that for the examples we consider in this paper there is only one term in the sum of (3).

In this paper we will focus on the determination of the form factors of the scaling $Z(N)$-Ising quantum field theory in $1 + 1$ dimensions. An Euclidean field theory is obtained as the scaling limit of a classical statistical lattice model in 2-dimensions given by the partition function

$$Z = \sum_{\{\sigma\}} \exp\left(-\frac{1}{kT} \sum_{\langle ij \rangle} E(\sigma_i, \sigma_j)\right), \quad \sigma_i \in \{1, \omega, \dots, \omega^{N-1}\}, \ \omega = e^{2\pi i/N}$$

as a generalization of the Ising model. It was conjectured by Köberle and Swieca [9] that there exists a $Z(N)$-invariant interaction $E(\sigma_i, \sigma_j)$ such that the resulting massive quantum field theory is integrable. In particular for $N = 2$ the scaling $Z(2)$-Ising model is the well investigated model [10–13] which is equivalent to a massive free Dirac field theory. In this paper we investigate the general $Z(N)$-model. It has also been discussed as a deformation [14,15] of a conformal $Z(N)$ para-Fermi field theory [16]. The $Z(N)$-Ising model in the scaling limit possesses $N - 1$ types of particles: $\alpha = 1, \dots, N - 1$ of charge α, mass $M_\alpha = M \sin \pi \frac{\alpha}{N}$ and $\bar{\alpha} = N - \alpha$ is the antiparticle of α. The n-particle S-matrix factorizes in terms of two-particle ones since the model is integrable. The two-particle S-matrix for the $Z(N)$-Ising model has been proposed by Köberle and Swieca [9]. The scattering of two particles of type 1 is

$$S(\theta) = \frac{\sinh\frac{1}{2}(\theta + \frac{2\pi i}{N})}{\sinh\frac{1}{2}(\theta - \frac{2\pi i}{N})}. \tag{4}$$

This S-matrix is consistent with the picture that the bound state of $N-1$ particles of type 1 is the anti-particle of 1. This will be essential also for the construction of form factors below. We construct generalized form factors of an operator $\mathcal{O}(x)$ and n particles of type 1 and for simplicity we write $F_n^{\mathcal{O}}(\underline{\theta}) = F_{1\dots1}^{\mathcal{O}}(\underline{\theta})$. Note that all further matrix elements with different particle states of the field operator $\mathcal{O}(x)$ are obtained by the crossing formula (ii) and the bound state formula (iv). As an application of this form factor approach we compute the commutation relations of fields. In particular, we consider the fields $\psi_{Q\tilde{Q}}(x)$, $(Q, \tilde{Q} = 0, \dots, N-1)$ with charge Q and 'dual charge' $\tilde{Q}$. There are in particular the order parameters $\sigma_Q(x) = \psi_{Q0}(x)$, the disorder parameters $\mu_{\tilde{Q}}(x) = \psi_{0\tilde{Q}}(x)$ and the para-Fermi fields $\psi_Q(x) = \psi_{QQ}(x)$. We show that they satisfy the space like commutation rules:

$$\begin{aligned}
&\sigma_Q(x)\sigma_{Q'}(y) = \sigma_{Q'}(y)\sigma_Q(x),\\
&\mu_{\tilde{Q}}(x)\mu_{\tilde{Q}'}(y) = \mu_{\tilde{Q}'}(y)\mu_{\tilde{Q}}(x),\\
&\mu_{\tilde{Q}}(x)\sigma_Q(y) = \sigma_Q(y)\mu_{\tilde{Q}}(x)e^{\theta(x^1-y^1)2\pi i Q\tilde{Q}/N},\\
&\psi_Q(x)\psi_Q(y) = \psi_Q(y)\psi_Q(x)e^{\epsilon(x^1-y^1)2\pi i Q^2/N}.
\end{aligned} \tag{5}$$

It turns out that the nontrivial statistics factors $\sigma^{\mathcal{O}}(\alpha)$ in the form factor equations (ii) and (iii) lead to the nontrivial order–disorder and the anyonic statistics of the fields.

The form factor bootstrap program has been applied in [13] to the $Z(2)$-model. Form factors for the $Z(3)$-model were investigated by one of the present authors in [17]. There the form factors of the order parameter σ_1 were proposed for up to four particles. Kirilov and Smirnov [18] proposed all form factors of the $Z(3)$-model in terms of determinants. Some related work can be found in [19]. For general N form factors for charge-less states (n particles of type 1 and n particles of type $N-1$) were calculated in [20]. In the present paper we present for the scaling $Z(N)$-Ising model integral representations for all matrix elements of several field operators.

Recently, there has been a renewed interest in the form factors program in connection to condensed matter physics [21–23] and atomic physics [24]. In particular, applications to Mott insulators and carbon nanotubes as well as in the field of Bose–Einstein condensates of ultra-cold atomic gases have been discussed and in some instances correlation functions have been computed.

The paper is organized as follows: In Section 2 we construct the general form factor formula for the simplest $N = 2$ case, which corresponds to the well-known scaling $Z(2)$-Ising model, and show that the known results can be reproduced by our new general approach. In Section 3 we construct the general form factor formula for the $Z(3)$ case, which is more complex due to the presence of bound states, and discuss several explicit examples. We extend these results in Section 4, where the general form factors for $Z(N)$ are constructed and discussed in detail. Section 5 contains the derivation of the commutation rules of the fields and some applications of this formalism are presented. Some results of the present article have been published previously [25] without proofs.

2. $Z(2)$-form factors

To make our method more transparent and with the hope that our construction will help to calculate form factors for all primary and descendant fields, we start with the simplest case $N = 2$, which corresponds to the well-known Ising model in the scaling limit. This model, already investigated in [10,11,13,20], is equivalent to a massive free Dirac field theory. The model possesses one particle with mass M and the 2-particle S-matrix is $S = -1$. In [13] the form factor approach has been applied to this case with the result for the order parameter field $\sigma(x)$

$$F_n^\sigma(\underline{\theta}) = (2i)^{(n-1)/2} \prod_{1\leqslant i<j\leqslant n} \tanh \frac{1}{2}\theta_{ij} \tag{6}$$

for n odd. It is easy to see that this expression satisfies the form factor equations (i)–(iii) with statistics factor $\sigma^\sigma = 1$. For the $Z(2)$ case in this section we skip the proof that the functions obtained by our general formula satisfy the form factor equations (i)–(v), the reader may easily reduce the proofs for the $Z(3)$ and the general $Z(N)$ case of the following sections to this simple one. Rather, we present the results for several fields, in particular, we will show that our general formula reproduces the known results.

2.1. *The general formula for n-particle form factors*

We propose the n-particle form factors of an operator $\mathcal{O}(x)$ as given by (1) with the minimal form factor function

$$F(\theta) = \sinh \frac{1}{2}\theta \tag{7}$$

which is the minimal solution of Watson's equations and crossing (2) for $S = -1$. The K-function is given by our general formula (3) where the Bethe Ansatz vector is trivial (because there is no backward scattering) and the sum consists only of one term

$$K_n^{\mathcal{O}}(\underline{\theta}) = N_n I_{nm}\big(\underline{\theta}, p^{\mathcal{O}}\big). \tag{8}$$

The *fundamental building blocks* of form factors are

$$I_{nm}(\underline{\theta}, p^{\mathcal{O}}) = \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_1}{R} \cdots \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_m}{R} h(\underline{\theta}, \underline{z}) p^{\mathcal{O}}(\underline{\theta}, \underline{z}), \tag{9}$$

$$h(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n}\prod_{j=1}^{m} \phi(z_j - \theta_i) \prod_{1\leqslant i<j\leqslant m} \tau(z_i - z_j). \tag{10}$$

The h-function does not depend on the operator but only on the S-matrix of the model, whereas the p-function depends on the operator. Both are analytic functions of θ_i $(i = 1, \dots, n)$ and z_j $(j = 1, \dots, m)$ and are symmetric under $\theta_i \leftrightarrow \theta_j$ and $z_i \leftrightarrow z_j$. For all integration variables z_j the integration contours $\mathcal{C}_{\underline{\theta}} = \sum \mathcal{C}_{\theta_i}$ enclose clock wise oriented the points $z_j = \theta_i$ $(i = 1, \dots, n)$. This means we assume that $\phi(z)$ has a pole at $z = 0$ such that $R = \int_{\mathcal{C}_\theta} dz\, \phi(z - \theta)$. The functions $\phi(z)$ and $\tau(z)$ are given in terms of the minimal form factor function as

$$\phi(z) = \frac{1}{F(\theta)F(\theta + i\pi)} = \frac{-2i}{\sinh z}, \tag{11}$$

$$\tau(z) = \frac{1}{\phi(z)\phi(-z)} = \frac{1}{4}\sinh^2 z. \tag{12}$$

The following properties of the p-functions guarantee that the form factors satisfy (i)–(iii)

(i$'_2$) $p^{\mathcal{O}}_{nm}(\underline{\theta}, \underline{z})$ is symmetric under $\theta_i \leftrightarrow \theta_j$
(ii$'_2$) $\sigma^{\mathcal{O}} p^{\mathcal{O}}_{nm}(\theta_1 + 2\pi i, \theta_2, \ldots, \underline{z}) = p^{\mathcal{O}}_{nm}(\theta_1, \theta_2, \ldots, \underline{z})$
(iii$'_2$) if $\theta_{12} = i\pi$

$$p^{\mathcal{O}}_{nm}(\underline{\theta}, \underline{z})\big|_{z_1=\theta_1} = \sigma^{\mathcal{O}} p^{\mathcal{O}}_{nm}(\underline{\theta}, \underline{z})\big|_{z_1=\theta_2} = \sigma^{\mathcal{O}} p^{\mathcal{O}}_{n-2m-1}(\theta_3, \ldots, \theta_n, z_2, \ldots, z_m) + \tilde{p},$$

where $\sigma^{\mathcal{O}}$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle. The function $\tilde{p}$ must not contribute after integration, which means in particular that is does not depend on the z_i (in most cases $\tilde{p} = 0$). For convenience we have introduced the indices $_{nm}$ to denote the number of variables. For the recursion relation (iii) in addition the normalization coefficients have to satisfy

$$N_n = i N_{n-2}. \tag{13}$$

One may convince oneself that the form factor satisfies (i) and (ii). Not so trivial is the residue relation (iii), however, it follows from a simplified version of the proofs for the $Z(3)$ and the general $Z(N)$ case below.

2.2. *Examples of fields and their p-functions*

We present the following correspondence of operators and p-functions which are solutions of (i$'_2$)–(iii$'_2$). For the order parameter $\sigma(x)$, the disorder parameter $\mu(x)$, the Fermi field $\psi(x)$, and the higher conserved currents $J^{\mu}_L(x)$ ($L \in \mathbb{Z}$) we propose the following p-functions and statistics parameters

$$\begin{aligned}
&\sigma(x) \leftrightarrow p^{\sigma}_{nm}(\underline{\theta}, \underline{z}) = 1 && \text{for } n = 2m+1 \text{ with } \sigma^{\sigma} = 1,\\
&\mu(x) \leftrightarrow p^{\mu}_{nm}(\underline{\theta}, \underline{z}) = i^m e^{(\sum z_i - \frac{1}{2}\sum \theta_i)} && \text{for } n = 2m \text{ with } \sigma^{\mu} = -1,\\
&\psi^{\pm}(x) \leftrightarrow p^{\psi^{\pm}}_{nm}(\underline{\theta}, \underline{z}) = e^{\pm(\sum z_i - \frac{1}{2}\sum \theta_i)} && \text{for } n = 2m+1 \text{ with } \sigma^{\psi} = -1,\\
&J^{\pm}_L(x) \leftrightarrow p^{J^{\pm}_L}_{nm}(\underline{\theta}, \underline{z}) = \sum e^{\pm\theta_i} \sum e^{L z_i} && \text{for } n = 2m \text{ with } \sigma^{\mu} = 1.
\end{aligned} \tag{14}$$

Note that $\tilde{p} \neq 0$ in (iii$'_2$) only for $J^{\pm}_L$. The motivation of these choices will be more obvious when we investigate the commutation rules of fields in Section 5 and will follow from the properties of the form factors which we now discuss in more detail.

Explicit expressions of the form factors Now we have to check that the proposed p-functions really provide the well-known form factors for the order, disorder and Fermi fields. In order to get simple expressions for these form factors, we have to calculate the integral (9) with (10) for each p-function separately.

For the order parameter Only for odd numbers of particles the form factors are non-zero. We calculate

$$I_{nm}(\underline{\theta}, 1) = 2^m \prod_{1 \leqslant i < j \leqslant n} \frac{1}{\cosh \frac{1}{2}\theta_{ij}} \quad \text{for } n = 2m+1. \tag{15}$$

The proof of this formula can be found in Appendix B. The general formulae (1), (3), (7), and (13) with $N_n = i^{(n-1)/2}$ then imply for n odd

$$F_n^\sigma(\underline{\theta}) = 2^m \prod_{1\leqslant i<j\leqslant n} \frac{F(\theta_{ij})}{\cosh\frac{1}{2}\theta_{ij}} = (2i)^{(n-1)/2} \prod_{1\leqslant i<j\leqslant n} \tanh\frac{1}{2}\theta_{ij}, \tag{16}$$

which agrees with the known result (6). The proof that the integral $I_{nm}(\underline{\theta}, 1)$ vanishes for even n and $m > 0$ is simple: If $m(m-1) < n$ we may decompose for real θ_i the contours $\mathcal{C}_{\underline{\theta}} = \mathcal{C}_0 - \mathcal{C}_{0-i\pi}$ where $\mathrm{Re}\,\mathcal{C}_0$ goes from $-\infty$ to ∞ and $\mathrm{Im}(\theta_i + i\pi) < \mathrm{Im}\,\mathcal{C}_0 < \mathrm{Im}(\theta_i)$. The shift $z_i \to z_i - i\pi$ implies $h(\underline{\theta}, \underline{z}) \to (-1)^n h(\underline{\theta}, \underline{z})$ such that the integrals along $\mathcal{C}_0$ and $\mathcal{C}_{0-i\pi}$ cancel for even n.

For the disorder parameter Only for even numbers of particles the form factors are non-zero. We calculate with p^μ as given in (14)

$$I_{nm}\big(\underline{\theta}, p^\mu\big) = 2^m \prod_{1\leqslant i<j\leqslant n} \frac{1}{\cosh\frac{1}{2}\theta_{ij}} \quad \text{for } n = 2m.$$

The proof of this formula is analog to that in Appendix B, therefore with $N_n = i^{n/2}$ the form factors are for n even

$$F_n^\mu(\underline{\theta}) = (2i)^{n/2} \prod_{1\leqslant i<j\leqslant n} \tanh\frac{1}{2}\theta_{ij}. \tag{17}$$

Similar as above for the order parameter one can show that the integral $I_{nm}(\underline{\theta}, p^\mu)$ vanishes for odd n and $m > 0$. It is also interesting to investigate the asymptotic behavior of the form factors when one of the rapidities goes to infinity. From the integral representation it is easy to check that

$$F_n^\sigma(\underline{\theta}) \overset{\theta_1\to\infty}{\to} F_{n-1}^\mu(\underline{\theta}') \overset{\theta_2\to\infty}{\to} 2i F_{n-2}^\sigma(\underline{\theta}'').$$

Of course, this result follows easily from the explicit expressions (16) and (17). This asymptotic behavior is another motivation for our choice (14) of the p-function for $\sigma(x)$ and $\mu(x)$.

For the Fermi field Only for $n = 1$ the form factors are non-zero. We calculate with $p^{\psi^\pm}$ as given in (14)

$$I_{nm}\big(\underline{\theta}, p^{\psi^\pm}\big) = \delta_{n1} e^{\mp\frac{1}{2}\theta} \quad \text{for } n = 2m+1.$$

The proof that $I_{nm}(\underline{\theta}, p^{\psi^\pm}) = 0$ for $n = 2m+1$ odd and $m > 0$ is the same as for the disorder parameter. Therefore with the normalization $N_1 = \sqrt{M}$ we obtain

$$F_1^\psi(\theta) = \langle 0|\psi(0)|\theta\rangle = u(\theta) = \sqrt{M}\begin{pmatrix} e^{-\frac{1}{2}\theta} \\ e^{\frac{1}{2}\theta} \end{pmatrix}. \tag{18}$$

The property that all form factors of the Fermi field vanish except the vacuum one-particle matrix element reflects the fact that $\psi(x)$ is a free field, in particular for the Wightman functions one easily obtains

$$\langle 0|\psi(x_1)\cdots\psi(x_n)|0\rangle = \langle 0|\psi(x_1)\cdots\psi(x_n)|0\rangle_{\text{free}}.$$

For the infinite set of conserved higher currents Only for $n = 2$ the form factors are non-zero. We calculate with $p^{J_L^\pm}$ as given in (14)

$$I_{nm}\big(\underline{\theta}, p^{J_L^\pm}\big) = \delta_{n2}\big(e^{\pm\theta_1} + e^{\pm\theta_2}\big)2i\left(\frac{e^{L\theta_1}}{\sinh\theta_{12}} + \frac{e^{L\theta_2}}{\sinh\theta_{21}}\right) \quad \text{for } n = 2m.$$

The proof that $I_{nm}(\underline{\theta}, p^{J_L^\pm}) = 0$ for $n = 2m > 2$ is again similar as above. With the normalization $c_{21} = \pm iM$ the form factors are

$$\langle 0|J_L^\pm(0)|\theta_1,\ldots,\theta_n\rangle^{\text{in}} = \mp\delta_{n2}2M\big(e^{\pm\theta_1} + e^{\pm\theta_2}\big)\big(e^{L\theta_1} - e^{L\theta_2}\big)\frac{1}{\sinh\theta_{12}}$$

such that as in [26] the charges $Q_L = \int dx\, J_L^0(x)$ satisfy the eigenvalue equation

$$\left(Q_L - \sum_{i=1}^n e^{L\theta_i}\right)|\theta_1,\ldots,\theta_n\rangle^{\text{in}} = 0 \quad \text{for } L = \pm 1, \pm 3, \ldots.$$

Obviously we get the energy–momentum tensor from $J_{\pm 1}^\pm(x)$.

The property $F_n^\psi = 0$ for odd $n > 1$ and $F_n^J = 0$ for even $n > 2$ is related to the fact that in the recursion relation (iii) the factor $(1 - \sigma^{\mathcal{O}} \prod S)$ is zero in both cases.

3. $Z(3)$-form factors

Let us now consider $N = 3$, which corresponds to the scaling $Z(3)$-Ising model. In this case we have two particles, 1 and 2, and the bound state of two particles of type 1 is the particle 2, which in turn is the anti-particle of particle 1. Conversely, the bound state of two particles of type 2 is the particle of type 1, which in turn is again the anti-particle of 2. So, our construction should take into account this structure of the bound states. We construct the form factors for particles of type 1, the others can then be obtained by the form factor bound state formula (iv).

3.1. *The general formula for n-particle form factors*

In order to obtain a recursion relation where only form factors for particles of type 1 are involved, we have to apply the bound state relation (iv) to get the anti-particle and then the creation annihilation equation (iii) to obtain

$$\begin{aligned}\operatorname*{Res}_{\theta_{23}=i\eta}\operatorname*{Res}_{\theta_{12}=i\eta} F^{\mathcal{O}}_{1111\ldots1}(\theta_1,\ldots) &= \operatorname*{Res}_{\theta_{(12)3}=i\pi} F^{\mathcal{O}}_{211\ldots1}(\theta_{(12)},\theta_3,\ldots)\sqrt{2}\varGamma \\ &= 2i F^{\mathcal{O}}_{1\ldots1}(\theta_4,\ldots)\left(1 - \sigma_1^{\mathcal{O}}\prod_{i=4}^n S(\theta_{3i})\right)\sqrt{2}\varGamma,\end{aligned} \tag{19}$$

where $\theta_{(12)} = \frac{1}{2}(\theta_1 + \theta_2)$ is the bound state rapidity, $\eta = \frac{2}{3}\pi$ is the bound state fusion angle and $\varGamma = i|\operatorname{Res}_{\theta=i\eta} S_{11}(\theta)|^{1/2}$ is the bound state intertwiner (see [4,5]). In the following we use again the short notation $F^{\mathcal{O}}_{1\ldots1}(\theta_1,\ldots,\theta_n) = F_n^{\mathcal{O}}(\underline{\theta})$ and also write the statistics factor as $\sigma^{\mathcal{O}}(1) = \sigma_1^{\mathcal{O}}$. We write the form factors again in the form (1) where minimal form factor function

$$F(\theta) = c_3 \exp\int_0^\infty \frac{dt}{t}\frac{2\cosh\frac{1}{3}t\sinh\frac{2}{3}t}{\sinh^2 t}\left(1 - \cosh t\left(1 - \frac{\theta}{i\pi}\right)\right) \tag{20}$$

is the solution of Watson's equations (2) with the S-matrix (4) for $N=3$. The constant c_3 is given by (A.1) in Appendix A. Similar as above we make the Ansatz for the K-functions

$$K_n^{\mathcal{O}}(\underline{\theta}) = N_n I_{nmk}\big(\underline{\theta}, p^{\mathcal{O}}\big) \tag{21}$$

with the fundamental building blocks of form factors

$$I_{nmk}(\underline{\theta}, p) = \frac{1}{m!k!}\int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_1}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_m}{R}\int_{\mathcal{C}_{\underline{\theta}}} \frac{du_1}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}}} \frac{du_k}{R} h(\underline{\theta}, \underline{z}, \underline{u})\, p(\underline{\theta}, \underline{z}, \underline{u}), \tag{22}$$

$$\begin{aligned} h(\underline{\theta}, \underline{z}, \underline{u}) = {}& \prod_{i=1}^{n}\left(\prod_{j=1}^{m}\phi(z_j-\theta_i)\prod_{j=1}^{k}\phi(u_j-\theta_i)\right) \\ & \times \prod_{1\leqslant i<j\leqslant m}\tau(z_{ij})\prod_{1\leqslant i<j\leqslant k}\tau(u_{ij})\prod_{1\leqslant i\leqslant m}\prod_{1\leqslant j\leqslant k}\varkappa(z_i-u_j). \end{aligned} \tag{23}$$

Again the integration contours $\mathcal{C}_{\underline{\theta}} = \sum \mathcal{C}_{\theta_i}$ enclose the points θ_i such that $R = \int_{\mathcal{C}_\theta} dz\, \phi(z-\theta)$. Equations (iii) and (iv), in particular (19) lead to the relations

$$\prod_{k=0}^{1}\phi(\theta + ki\eta)\prod_{k=0}^{2}F(\theta+ki\eta) = 1, \tag{24}$$

$$\tau(z)\phi(z)\phi(-z) = 1, \qquad \varkappa(z)\phi(z) = 1, \tag{25}$$

as an extension of (11) and (12) for the $Z(2)$ case. The solution for ϕ is

$$\phi(z) = \frac{1}{\sinh\frac{1}{2}z\sinh\frac{1}{2}(z+i\eta)} \tag{26}$$

if the constant c_3 is fixed as in (A.1). The phi-function satisfies the 'Jost function' property

$$\frac{\phi(-\theta)}{\phi(\theta)} = S(\theta). \tag{27}$$

We will now show again that by the Ansatz (21)–(23) we have transformed the form factor equations (i)–(v), in particular equation (19) to simple equations for the p-functions.

Assumptions for the p-functions The function $p_{nmk}^{\mathcal{O}}(\underline{\theta}, \underline{z}, \underline{u})$ is analytic in all variables and satisfies:

(i$'_3$) $p_{nmk}^{\mathcal{O}}(\underline{\theta}, \underline{z}, \underline{u})$ is symmetric under $\theta_i \leftrightarrow \theta_j$,
(ii$'_3$) $\sigma_1^{\mathcal{O}} p_{nmk}^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, \ldots, \underline{z}, \underline{u}) = p_{nmk}^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \underline{z}, \underline{u})$,
(iii$'_3$) if $\theta_{12} = \theta_{23} = i\eta$

$$p_{nmk}^{\mathcal{O}}(\underline{\theta}, \underline{z}, \underline{u})\Big|_{\substack{z_1=\theta_1\\u_1=\theta_2}} = \sigma_1^{\mathcal{O}} p_{nmk}^{\mathcal{O}}(\underline{\theta}, \underline{z}, \underline{u})\Big|_{\substack{z_1=\theta_2\\u_1=\theta_3}} = \sigma_1^{\mathcal{O}} p_{n-3m-1k-1}^{\mathcal{O}}(\underline{\theta}', \underline{z}', \underline{u}') + \tilde{p}, \tag{28}$$

(iv$'_3$) $p_{nmk}^{\mathcal{O}}(\underline{\theta}+\mu, \underline{z}+\mu, \underline{u}+\mu) = e^{s\mu} p_{nmk}^{\mathcal{O}}(\underline{\theta}, \underline{z}, \underline{u})$,

where $\underline{\theta}' = (\theta_4, \ldots, \theta_n)$, $\underline{z}' = (z_2, \ldots, z_m)$ and $\underline{u}' = (u_2, \ldots, u_k)$. In (ii$'_3$) and (iii$'_3$) $\sigma_1^{\mathcal{O}}$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle of type 1 and in (v$'_3$) s is the spin of the operator $\mathcal{O}$. Again $\tilde{p}$ must not contribute after integration (in most cases $\tilde{p}=0$). Again one may convince oneself that the form factor satisfies (i) and (ii) if $h(\underline{\theta},\underline{z})$ is symmetric under $\theta_i \leftrightarrow \theta_j$ and periodic with respect to $\theta_i \to \theta_i + 2\pi i$. Not so trivial is the residue relation (iii) which is proved in the following lemma.

Lemma 1. *The form factors $F_n^{\mathcal{O}}(\underline{\theta})$ defined by* (1) *and* (20)–(23) *satisfies* (i)–(v)*, in particular* (19)*, if the p-functions satisfy* (i$'_3$)–(v$'_5$) *of* (28)*, the functions ϕ, τ and $\varkappa$ the relations* (24), (25) *and the normalization constants in* (21) *the recursion relation*

$$N_n\Big(\operatorname*{Res}_{\theta=i\eta}\phi(-\theta)\Big)^2\phi(i\eta)\ F^2(i\eta)F(2i\eta)=N_{n-3}2i\sqrt{2}\Gamma. \tag{29}$$

Proof. The form factor equations (i), (ii), and (v) follow obviously from the equations for the p-functions (i$'_3$), (ii$'_3$), and (v$'_3$), respectively. As already stated, we will prove properties (iii) and (iv) together in the form of (19). Taking the residues $\operatorname{Res}_{\theta_{23}=i\eta}\operatorname{Res}_{\theta_{12}=i\eta}$ there will be mk equal terms originating from pinchings for z_i and u_i. We pick them from z_1 and u_1 and rewrite the products that appear in the expression for I_{nmk} in a convenient form, such that the location of the poles turns out to be separated from the general expression. Then, the essential calculation to be performed is

$$\begin{aligned}\operatorname*{Res}_{\theta_{23}=i\eta}\operatorname*{Res}_{\theta_{12}=i\eta} I_{nmk}\big(\underline{\theta},p^{\mathcal{O}}\big)=\frac{mk}{m!k!}&\int_{\mathcal{C}_{\underline{\theta}'}}\frac{dz_2}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}'}}\frac{dz_m}{R}\int_{\mathcal{C}_{\underline{\theta}'}}\frac{du_2}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}'}}\frac{du_k}{R}\\&\times\prod_{i=4}^{n}\left(\prod_{j=2}^{m}\phi(z_j-\theta_i)\prod_{j=2}^{k}\phi(u_j-\theta_i)\right)\\&\times\prod_{2\leqslant i<j\leqslant m}\tau(z_{ij})\prod_{2\leqslant i<j\leqslant k}\tau(u_{ij})\prod_{i=2}^{m}\prod_{j=2}^{k}\varkappa(z_i-u_j)\\&\times\prod_{i=1}^{3}\left(\prod_{j=2}^{m}\phi(z_j-\theta_i)\prod_{j=2}^{k}\phi(u_j-\theta_i)\right)r\end{aligned}$$

with

$$\begin{aligned}r=\operatorname*{Res}_{\theta_{23}=i\eta}\operatorname*{Res}_{\theta_{12}=i\eta}&\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz_1}{R}\int_{\mathcal{C}_{\underline{\theta}}}\frac{du_1}{R}\prod_{i=1}^{n}\big(\phi(z_1-\theta_i)\phi(u_1-\theta_i)\big)\\&\times\prod_{j=2}^{m}\tau(z_{1j})\prod_{j=2}^{k}\tau(u_{1j})\varkappa(z_1-u_1)\prod_{j=2}^{k}\varkappa(z_1-u_j)\prod_{j=2}^{m}\varkappa(z_j-u_1)p_n^{\mathcal{O}}(\underline{\theta},\underline{z},\underline{u}).\end{aligned}$$

Replacing $\mathcal{C}_{\underline{\theta}}$ by $\mathcal{C}_{\underline{\theta}'}$ where $\underline{\theta}'=(\theta_4,\dots,\theta_n)$ we have used $\tau(0)=\tau(\pm i\eta)=\varkappa(0)=\varkappa(-i\eta)=0$ and the fact that the z_1-, u_1-integrations give non-vanishing results only for (z_1,u_1) at (θ_1,θ_2) and (θ_2,θ_3). This is because for $\theta_{12},\theta_{23}\to i\eta$ pinching appears at $z_1=\theta_2, u_1=\theta_3$ and $z_1=\theta_1$, $u_1=\theta_2$. Defining the function

$$f(z_1,u_1)=\prod_{j=2}^{m}\tau(z_{1j})\prod_{j=2}^{k}\tau(u_{1j})\varkappa(z_1-u_1)\prod_{j=2}^{k}\varkappa(z_1-u_j)\prod_{j=2}^{m}\varkappa(z_j-u_1)p_n^{\mathcal{O}}(\underline{\theta},\underline{z},\underline{u})$$

and using the property that $f(z,z) = f(z, z - i\eta) = 0$, we calculate

$$r = \operatorname*{Res}_{\theta_{23}=i\eta} \operatorname*{Res}_{\theta_{12}=i\eta} \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_1}{R} \int_{\mathcal{C}_{\underline{\theta}}} \frac{du_1}{R} f(z_1, u_1) \prod_{i=1}^{n} \big(\phi(z_1 - \theta_i)\phi(u_1 - \theta_i)\big)$$
$$= \Big(\operatorname*{Res}_{\theta=i\eta} \phi(-\theta)\Big)^2 \phi^2(i\eta) \prod_{i=4}^{n} \big(\phi(\theta_{2i})\phi(\theta_{3i})\big) \left(f(\theta_2, \theta_3) - f(\theta_1, \theta_2) \prod_{i=4}^{n} S(\theta_{3i}) \right).$$

We have used the symmetries of the phi-function $\phi(\theta) = \phi(\theta + 2\pi i) = \phi(-\theta - i\eta)$ the Jost property (27) and $\theta_{12}, \theta_{23} = i\eta = \frac{2}{3}i\pi$ which imply that

$$\operatorname*{Res}_{\theta_{12}=i\eta} \phi(\theta_{21}) \operatorname*{Res}_{\theta_{23}=i\eta} \phi(\theta_{32}) = \operatorname*{Res}_{\theta_{12}=i\eta} \phi(\theta_{13}) \operatorname*{Res}_{\theta_{23}=i\eta} \phi(\theta_{21}) = \Big(\operatorname*{Res}_{\theta=i\eta} \phi(-\theta)\Big)^2,$$
$$\phi(\theta_{23})\phi(\theta_{31}) = \phi(\theta_{12})\phi(\theta_{23}) = \phi^2(i\eta),$$
$$\frac{\phi(\theta_{1i})}{\phi(\theta_{3i})} = \frac{\phi(\theta_{3i} - i\eta)}{\phi(\theta_{3i})} = \frac{\phi(-\theta_{3i})}{\phi(\theta_{3i})} = S(\theta_{3i}).$$

With the help of the defining equations (25) for τ and $\varkappa$ which imply

$$\left(\prod_{i=1}^{3} \phi(z - \theta_i)\right)^{-1} = \tau(\theta_1 - z)\varkappa(z - \theta_2) = \tau(\theta_2 - z)\varkappa(\theta_1 - z)$$
$$= \tau(\theta_2 - z)\varkappa(z - \theta_3) = \tau(\theta_3 - z)\varkappa(\theta_1 - z),$$

we obtain the relations for $f(\theta_2, \theta_3)$ and $f(\theta_1, \theta_2)$

$$\prod_{i=1}^{3} \left(\prod_{j=2}^{m} \phi(z_j - \theta_i) \prod_{j=2}^{k} \phi(u_j - \theta_i) \right) f(\theta_1, \theta_2) = \varkappa(\theta_{12}) p_n^{\mathcal{O}}(\underline{\theta}, \theta_1, \underline{z}', \theta_2, \underline{u}'),$$
$$\prod_{i=1}^{3} \left(\prod_{j=2}^{m} \phi(z_j - \theta_i) \prod_{j=2}^{k} \phi(u_j - \theta_i) \right) f(\theta_2, \theta_3) = \varkappa(\theta_{23}) p_n^{\mathcal{O}}(\underline{\theta}, \theta_2, \underline{z}', \theta_3, \underline{u}').$$

Finally we obtain using the defining relation (24) for the phi-function

$$\operatorname*{Res}_{\theta_{23}=i\eta} \operatorname*{Res}_{\theta_{12}=i\eta} I_{nmk}\big(\underline{\theta}, p^{\mathcal{O}}\big) = \Big(\operatorname*{Res}_{\theta=i\eta} \phi(-\theta)\Big)^2 \phi^2(i\eta)\varkappa(i\eta) \prod_{i=1}^{3} \prod_{j=4}^{n} \big(F(\theta_{ij})\big)^{-1}$$
$$\times I_{n-3m-1k-1}\big(\underline{\theta}', p^{\mathcal{O}}\big) \left(1 - \sigma_1^{\mathcal{O}} \prod_{i=4}^{n} S(\theta_{3i})\right)$$

if the p-function satisfies (iii$'_3$). Therefore the form factor given by (1) and (20)–(23) satisfies (14) and (iii) if the normalization constants satisfy (29). □

3.2. *Examples of fields and p-functions*

We present solutions of the equations for the p-functions (i$'_3$)–(v$'_3$) of (28) and some explicit examples of the resulting form factors. We identify the fields by the properties of their matrix elements. In Section 5 we show that the field satisfy the desired commutation rules. This motivates to propose a correspondence of fields $\phi(x)$ and p-functions $p^{\phi}_{nmk}(\underline{\theta}, \underline{z}, \underline{u})$.

The order parameter $\sigma_Q(x)$ We look for a solution of (i$'_3$)–(v$'_3$) with

$$\begin{cases} \text{charge } Q = 1, 2, \\ \text{spin } s = 0, \\ \text{statistics } \sigma_1^{\sigma_Q} = 1. \end{cases}$$

Since the fields carry the charge Q the only non-vanishing form factors with n particles of type 1 are the ones with $n = Q \mod 3$. We propose the correspondence of the field and the p-function:

$$\sigma_Q \leftrightarrow p_{nmk}^{\sigma_Q} = 1 \quad \text{with } n = 3l + Q, \quad \begin{cases} m = l+1,\ k = l & \text{for } Q = 1, \\ m = l+1,\ k = l+1 & \text{for } Q = 2. \end{cases}$$

The normalization constants N_n follow from (29).

Examples for $Q = 1$ The form factors of the order parameter $\sigma_1(x)$ for one and four particles of type 1 are

$$F_1^{\sigma_1} = \langle 0|\sigma_1(0)|p\rangle_1 = N_1 I_{110} = 1,$$
$$F_{1111}^{\sigma_1}(\underline{\theta}) = \langle 0|\sigma_1(0)|p_1, p_2, p_3, p_4\rangle_{1111}^{\text{in}} = N_4 I_{421}(\underline{\theta}, 1) \prod_{1 \leqslant i < j \leqslant 4} F(\theta_{ij}),$$

where we calculate from our integral representation (22)

$$I_{421}(\underline{\theta}, 1) = \text{const} \times \left(\sum e^{-\theta_i} \sum e^{\theta_i} - 1 \right) \prod_{i<j} \frac{1}{\sinh \frac{1}{2}(\theta_{ij} - i\eta) \sinh \frac{1}{2}(\theta_{ij} + i\eta)}.$$

This result has already been obtained in [17] where also the form factor equation (iv) has been discussed, in particular (up to normalizations)

$$\operatorname*{Res}_{\theta_{34} = 2\pi i/3} \langle 0|\sigma_1(0)|p_1, p_2, p_3, p_4\rangle_{1111}^{\text{in}} = \text{const} \times \langle 0|\sigma_1(0)|p_1, p_2, p_3 + p_4\rangle_{112}^{\text{in}}$$

with

$$\langle 0|\sigma_1(0)|p_1, p_2, p_3\rangle_{112}^{\text{in}} = \frac{\text{const} \times F(\theta_{12})}{\sinh \frac{1}{2}(\theta_{12} - i\eta) \sinh \frac{1}{2}(\theta_{12} + i\eta)} \prod_{i=1}^{2} \frac{F_{12}^{\min}(\theta_{i3})}{\cosh \frac{1}{2}\theta_{i3}},$$

where $F_{12}^{\min}$ is the minimal form factor function for the S-matrix S_{12}. Further it has been found in [17] that

$$\operatorname*{Res}_{\theta_{12} = 2\pi i/3} \langle 0|\sigma_1(0)|p_1, p_2, p_3\rangle_{112}^{\text{in}} = \text{const} \times \langle 0|\sigma_1(0)|p_1 + p_2, p_3\rangle_{22}^{\text{in}}$$

with

$$\langle 0|\sigma_1(0)|p_1, p_2\rangle_{22}^{\text{in}} = \frac{\text{const} \times F(\theta_{12})}{\sinh \frac{1}{2}(\theta_{12} - i\eta) \sinh \frac{1}{2}(\theta_{12} + i\eta)} \tag{30}$$

and the form factor equation (iii) has been checked

$$\operatorname*{Res}_{\theta_{23} = i\pi} \langle 0|\sigma_1(0)|p_1, p_2, p_3\rangle_{112}^{\text{in}} = \text{const} \times \big(S(\theta_{12}) - 1 \big).$$

Example for $Q = 2$ The form factor of the order parameter $\sigma_2(x)$ for two particles of type 1 is

$$F_{11}^{\sigma_2}(\underline{\theta}) = \langle 0|\sigma_2(0)|p_1, p_2\rangle_{11}^{\text{in}} = N_2 I_{211}(\underline{\theta}, 1) F(\theta_{12}),$$

where we calculate

$$I_{211}(\underline{\theta}, 1) = \text{const} \times \frac{1}{\sinh\frac{1}{2}(\theta_{12} - i\eta)\sinh\frac{1}{2}(\theta_{12} + i\eta)},$$

which agrees with the result (30) of [17]. This is to be expected because of charge conjugation.

The disorder parameter $\mu_{\tilde{Q}}(x)$ We look for a solution of (i$'_3$)–(v$'_3$) with

$$\begin{cases} \text{charge } Q = 0, \\ \text{spin } s = 0, \\ \text{statistics } \sigma_1^{\mu_{\tilde{Q}}} = \omega^{\tilde{Q}}, \end{cases}$$

where $\omega = e^{i\eta}$, $\eta = 2/3$. We call the number $\tilde{Q} = 1, 2$ the 'dual charge' of the field $\mu_{\tilde{Q}}(x)$. Since the fields carry the charge $Q = 0$ the only non-vanishing form factors with n particles of type 1 are the ones with $n = 0 \mod 3$. We propose the correspondence of the field and the p-function $\tilde{Q}$

$$\mu_{\tilde{Q}} \leftrightarrow \begin{cases} p_{nmk}^{\mu_1} = \rho \exp\left(\sum_{i=1}^m z_i - \frac{1}{3}\sum_{i=1}^n \theta_i\right) \\ p_{nmk}^{\mu_2} = \rho \exp\left(\sum_{i=1}^m z_i + \sum_{i=1}^k u_i - \frac{2}{3}\sum_{i=1}^n \theta_i\right) \end{cases} \quad \text{with} \quad \begin{cases} n = 3m, \\ k = m, \end{cases}$$

where $\rho = \sqrt{\omega}^{\tilde{Q}(\tilde{Q}-N+2)m}$. Again the normalization constants N_n follow from (29).

Examples for $\tilde{Q} = 1, 2$ The form factors of the disorder parameter $\mu_{\tilde{Q}}(x)$ for 0 and 3 particles of type 1 are

$$F^{\mu_{\tilde{Q}}} = \langle 0|\mu_{\tilde{Q}}(0)|0\rangle = 1,$$
$$F_{111}^{\mu_{\tilde{Q}}}(\underline{\theta}) = \langle 0|\mu_{\tilde{Q}}(0)|p_1, p_2, p_3\rangle_{111}^{\text{in}} = N_3 I_{311}\left(\underline{\theta}, p^{\mu_{\tilde{Q}}}\right) \prod_{1\leqslant i<j\leqslant 3} F_{11}(\theta_{ij}).$$

We calculate from our integral representation (22)

$$I_{311}\left(\underline{\theta}, p^{\mu_{\tilde{Q}}}\right) = \text{const} \times e^{\mp\frac{1}{3}\sum_{i=1}^n \theta_i} \sum_{i=1}^3 e^{\pm\theta_i} \prod_{i<j} \frac{1}{\sinh\frac{1}{2}(\theta_{ij} - i\eta)\sinh\frac{1}{2}((\theta_{ij} + i\eta)},$$

where the upper sign is for $\tilde{Q} = 1$ and the lower one for $\tilde{Q} = 2$. Using the form factor bound state formula (iv) we obtain (up to a constant)

$$F_{12}^{\mu_{\tilde{Q}}}(\underline{\theta}) = e^{\mp\frac{1}{6}\theta_{12}} \frac{1}{\cosh\frac{1}{2}\theta_{12}} F_{12}^{\min}(\theta_{12}).$$

It is interesting to note that for $\operatorname{Re}\theta_1 \to \infty$ we have the relation of order and disorder parameter form factors (up to constants)

$$\lim_{\operatorname{Re}\theta_1\to\infty} \langle 0|\sigma_1(0)|p_1, p_2, p_3, p_4\rangle_{1111}^{\text{in}} = \langle 0|\mu_2(0)|p_2, p_3, p_4\rangle_{111}^{\text{in}},$$

which follows from the asymptotic behavior

$$F(\theta_{1i}) \to e^{\frac{2}{3}\theta_{1i}},$$
$$I_{421}(\underline{\theta}, 1) \to e^{\theta_1}\left(\sum_{i=2}^{4} e^{-\theta_i}\right)\prod_{j=2}^{4} e^{-\theta_{1j}} \prod_{1<i<j} \frac{1}{\sinh\frac{1}{2}(\theta_{ij}-a)\sinh\frac{1}{2}(\theta_{ij}+a)}.$$

The para-Fermi field $\psi_Q(x)$ We look for a solution of (i$'_3$)–(v$'_3$) with

$$\begin{cases} \text{charge } Q = Q, \\ \text{spin } s = Q(3-Q)/3, \\ \text{statistics } \sigma_1^{\psi_Q} = \omega^Q. \end{cases}$$

These fields have charge $Q = 1, 2$ and dual charge $\tilde{Q} = Q$. The only non-vanishing form factors with n particles of type 1 are the ones with $n = Q \mod 3$. We propose the correspondence of the field and the p-function:

$$\psi_Q \leftrightarrow \begin{cases} p_{nmk}^{\psi_1} = \rho \exp\left(\sum_{i=1}^{m} z_i - \frac{1}{3}\sum_{i=1}^{n}\theta_i\right) \\ p_{nmk}^{\psi_2} = \rho \exp\left(\sum_{i=1}^{m} z_i + \sum_{i=1}^{k} u_i - \frac{2}{3}\sum_{i=1}^{n}\theta_i\right) \end{cases} \quad \text{with} \quad \begin{cases} n = 3l + Q, \\ m = l + 1, \\ k = l + Q - 1, \end{cases}$$

where $\rho = \sqrt{\omega}^{\tilde{Q}(\tilde{Q}-1)l}$. Again the normalization constants N_n follow from (29).

Examples for $Q = 1, 2$ The form factors of the para-Fermi field $\psi_1(x)$ for 1 and 4 particles of type 1 are

$$F_1^{\psi_1}(\theta) = \langle 0|\psi_1(0)|p\rangle_1 = N_1 I_{110}\left(\theta, p^{\psi_1}\right) = e^{\frac{2}{3}\theta},$$
$$F_{1111}^{\psi_1}(\underline{\theta}) = \langle 0|\psi_1(0)|p_1, p_2, p_3, p_4\rangle_{1111}^{\text{in}} = N_4 I_{421}\left(\underline{\theta}, p^{\psi_1}\right) \prod_{1\leqslant i<j\leqslant 4} F(\theta_{ij})$$
$$= \text{const} \times e^{-\frac{2}{3}\sum_{i=1}^{4}\theta_i} \sum_{i<j} e^{\theta_i+\theta_j} \prod_{1\leqslant i<j\leqslant 4} \frac{F(\theta_{ij})}{\sinh\frac{1}{2}(\theta_{ij}-i\eta)\sinh\frac{1}{2}(\theta_{ij}+i\eta)}$$

and the one of the para-Fermi field $\psi_2(x)$ for 2 particles of type 1 is

$$F_{11}^{\psi_2}(\underline{\theta}) = \langle 0|\psi_2(0)|p_1, p_2\rangle_{11} = N_2 I_{211}\left(\underline{\theta}, p^{\psi_2}\right) F(\theta_{12})$$
$$= \text{const} \times e^{\frac{1}{3}(\theta_1+\theta_2)} \frac{F(\theta_{12})}{\sinh\frac{1}{2}(\theta_{12}-i\eta)\sinh\frac{1}{2}(\theta_{12}+i\eta)}.$$

All these examples agree with the results of [18].

The higher currents $J_L^{\pm}(x)$ We look for a solution of (i$'_3$)–(v$'_3$) with

$$\begin{cases} \text{charge } Q = 0, \\ \text{spin } s = L \pm 1, \\ \text{statistics } \sigma_1^{J_L^{\pm}} = 1. \end{cases}$$

Since the currents are $Z(3)$-charge-less the only non-vanishing form factors with n particles of type 1 are the ones with $n = 0 \mod 3$. We propose the correspondence of the currents and the

p-functions for $L \in \mathbb{Z}$

$$J_L^\pm \leftrightarrow p_{nmk}^{J_L^\pm} = \pm\left(\sum_{i=1}^n e^{\pm\theta_i}\right)\left(\sum_{i=1}^m e^{Lz_i} + \sum_{i=1}^m e^{Lu_i}\right) \quad \text{for} \quad \begin{cases} n = 3m, \\ k = m. \end{cases}$$

Note that for this case the function $\tilde{p}$ in (28) is non-vanishing, however, it does not contribute because $I_{nmm}(\underline{\theta}, 1) = 0$ for $n = 3m$. The proof of this fact is similar to the one given in Appendix B. The higher charges $Q_L = \int dx\, J_L^0(x)$ satisfy the eigenvalue equations

$$\left(Q_L - \sum_{i=1}^n e^{L\theta_i}\right)|p_1, \ldots, p_n\rangle^{\text{in}} = 0.$$

Obviously, from $J_{\pm 1}^\pm(x)$ we obtain the energy–momentum tensor.

Examples The form factors of the energy momentum tensor that is $J_L^\pm(x)$ for $L = \pm 1$ for 0 and 3 particles of type 1 are

$$\begin{aligned} F^{J_L^\pm} &= \langle 0|J_L^\pm(0)|0\rangle = 0, \\ F_{111}^{J_L^\pm}(\underline{\theta}) &= \langle 0|J_L^\pm(0)|p_1, p_2, p_3\rangle_{111}^{\text{in}} = c_{311} I_{311}\big(\underline{\theta}, p^{J_L^\pm}\big) \prod_{1\leqslant i<j\leqslant 3} F(\theta_{ij}) \\ &= \pm\text{const} \times \big(e^{\pm\theta_1} + e^{\pm\theta_2} + e^{\pm\theta_3}\big)\big(e^{L\theta_1} + e^{L\theta_2} + e^{L\theta_3}\big) \\ &\quad \times \prod_{1\leqslant i<j\leqslant 3} \frac{F(\theta_{ij})}{\sinh\frac{1}{2}(\theta_{ij} - i\eta)\sinh\frac{1}{2}(\theta_{ij} + i\eta)}. \end{aligned}$$

By (iv) we obtain the bound state form factor (up to a normalization) for $L = \pm 1$

$$F_{12}^{J_L^\pm}(\underline{\theta}) = \pm\text{const} \times e^{\frac{1}{2}(L\pm 1)(\theta_1+\theta_2)} F_{12}^{\min}(\theta_{12}).$$

Notice that this last expression agrees with the results of [20] when $N = 3$.

4. $Z(N)$-form factors

The scaling $Z(N)$-Ising model possesses particles of type $\alpha = 1, \ldots, N-1$ with $Z(N)$-charge $Q_\alpha = \alpha$ such that the anti-particle of α is $\bar{\alpha} = N - \alpha$. The bound state fusion rules are $(\alpha\beta) = \alpha + \beta \bmod N$, in particular the bound state of $N-1$ particles of type 1 is the anti-particle $\bar{1}$. Therefore applying $N-2$ times formula (iv) and once (iii) we obtain the recursion relations for form factors where only particles of type 1 are involved

$$\begin{aligned} &\operatorname*{Res}_{\theta_{N-1N}=i\eta} \ldots \operatorname*{Res}_{\theta_{12}=i\eta} F_n(\theta_1, \ldots, \theta_n) \\ &\quad = 2i F_{n-N}(\theta_{N+1}, \ldots, \theta_n)\left(1 - \sigma_1^{\mathcal{O}} \prod_{i=N+1}^n S(\theta_{Ni})\right) \prod_{\alpha=1}^{N-2} \sqrt{2}\,\Gamma_{1\alpha}^{1+\alpha}, \end{aligned} \tag{31}$$

where $\eta = \frac{2\pi i}{N}$ and the $\Gamma_{1\alpha}^{1+\alpha} = i\,|\operatorname{Res}_{\theta=i\eta_\alpha} S_{1\alpha}(\theta)|^{1/2}$ are the bound state intertwiners of the fusion $(1\alpha) = 1 + \alpha$. We will construct the form factors of particles of type 1, all the others are then obtained by the bound state formula (iv).

4.1. The general Z(N)-form factor formula

Following [1] we write the form factors again in the form (1)

$$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) \prod_{1\leqslant i<j\leqslant n} F(\theta_{ij}), \tag{32}$$

where minimal form factor function [17]

$$F(\theta) = c_N \exp \int_0^\infty \frac{dt}{t} \frac{2\cosh \frac{1}{N}t \sinh \frac{N-1}{N}t}{\sinh^2 t}\left(1-\cosh t\left(1-\frac{\theta}{i\pi}\right)\right) \tag{33}$$

is the solution of Watson's equations (2) with the S-matrix (4). The constant c_N is given by (A.1) in Appendix A. The K-function $K_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)$ is totally symmetric in the rapidities θ_i, $2\pi i$ periodic, containing the entire pole structure and determines the asymptotic behavior for large values of the rapidities. Similar as above we make the Ansatz for the K-functions

$$K_n^{\mathcal{O}}(\underline{\theta}) = N_n I_{n\underline{m}}\big(\underline{\theta}, p^{\mathcal{O}}\big) \tag{34}$$

with the fundamental building blocks of form factors

$$I_{n\underline{m}}(\underline{\theta}, p^{\mathcal{O}}) = \left(\prod_{k=1}^{N-1} \frac{1}{m_k!} \prod_{j=1}^{m_k} \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_{kj}}{R}\right) h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z}), \tag{35}$$

$$\begin{aligned} h(\underline{\theta},\underline{z}) = {} & \prod_{k=1}^{N-1}\left(\prod_{j=1}^{m_k}\prod_{i=1}^{n}\phi(z_{kj}-\theta_i) \prod_{1\leqslant i<j\leqslant m_k}\tau(z_{ki}-z_{kj})\right) \\ & \times \prod_{1\leqslant k<l\leqslant N-1} \prod_{i=1}^{m_k}\prod_{j=1}^{m_l} \varkappa(z_{ki}-z_{lj}), \end{aligned} \tag{36}$$

where $\underline{m} = (m_1,\dots,m_{N-1})$ and $\underline{z} = (z_{ki}), k = 1,\dots,N-1, i = 1,\dots,m_k$. Again the integration contours $\mathcal{C}_{\underline{\theta}} = \sum \mathcal{C}_{\theta_i}$ enclose the points θ_i such that $R = \int_{\mathcal{C}_\theta} dz\, \phi(z-\theta)$. Equations (iii) and (iv), in particular (31) lead to the relations

$$\prod_{k=0}^{N-2} \phi(z+ki\eta) \prod_{k=0}^{N-1} F(z+ki\eta) = 1, \tag{37}$$

$$\tau(-z)\phi(z+i\pi)\phi(z) = 1, \qquad \varkappa(z)\phi(z) = 1, \tag{38}$$

$$N_n\Big(\operatorname*{Res}_{\theta=i\eta}\phi(-\theta)\Big)^{N-1} \prod_{k=1}^{N-2}\phi^k(ki\eta) \prod_{k=1}^{N-1} F^{N-k}(ki\eta) = N_{n-N} 2i \prod_{k=1}^{N-2}\prod_{\alpha=1}^{N-2} \sqrt{2}\Gamma_{1\alpha}^{1+\alpha}, \tag{39}$$

where $\underline{m}-1 = (m_1-1,\dots,m_{N-1}-1)$. The solution of (37) for ϕ is again

$$\phi(z) = \frac{1}{\sinh \frac{1}{2}z \sinh \frac{1}{2}(z+i\eta)} \tag{40}$$

if the constant c_N is fixed as in (A.1). The phi-function satisfies again the 'Jost function' property $\phi(-\theta)/\phi(\theta) = S(\theta)$. The *p-function* $p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z})$ is analytic in all variables and satisfies:

(i′) $p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z})$ is symmetric under $\theta_i \leftrightarrow \theta_j$,
(ii′) $\sigma^{\mathcal{O}}_1 p^{\mathcal{O}}_{n\underline{m}}(\theta_1 + 2\pi i, \theta_2, \dots, \underline{z}) = p^{\mathcal{O}}_{n\underline{m}}(\theta_1, \theta_2, \dots, \underline{z})$,
(iii′) if $\theta_{kk+1} = i\eta = 2\pi i/N$ for $k = 1, \dots, N-1$

$$p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z})|_{z_{k1}=\theta_k} = \sigma^{\mathcal{O}}_1 p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta}, \underline{z})|_{z_{k1}=\theta_{k+1}} = \sigma^{\mathcal{O}}_1 p^{\mathcal{O}}_{n-N\underline{m}-1}(\underline{\theta}', \underline{z}') + \tilde{p}, \tag{41}$$

(v′) $p^{\mathcal{O}}_{nmk}(\underline{\theta} + \mu, \underline{z} + \mu) = e^{s\mu} p^{\mathcal{O}}_{nmk}(\underline{\theta}, \underline{z})$,

where $\underline{\theta}' = (\theta_{N+1}, \dots, \theta_n)$, $\underline{z}' = (z_{ki})$, $k = 1, \dots, N-1$, $i = 2, \dots, m_k$. In (ii′) and (iii′) $\sigma^{\mathcal{O}}_1$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle of type 1 and in (v′) s is the spin of the operator $\mathcal{O}$. Again $\tilde{p}$ must not contribute after integration (in most cases $\tilde{p} = 0$).

By means of the off-shell Bethe Ansatz (3) and (35) we have transformed the complicated form factor equations (i)–(v) to simple equations for the p-functions (i′)–(v′). Again one may convince oneself that the form factor satisfies (i) and (ii) if $h(\underline{\theta}, \underline{z})$ is symmetric under $\theta_i \leftrightarrow \theta_j$ and periodic with respect to $\theta_i \to \theta_i + 2\pi i$. Not so trivial is again the residue relation (iii) which is proved in the following lemma.

Lemma 2. *The form factors given by equations* (32)–(36) *satisfy the form factor equations* (i)–(v) *if the functions ϕ, τ, $\varkappa$ satisfy* (37) *and* (38), *the normalization constants satisfy* (39) *and the p-functions satisfy* (i′)–(v′) *of* (41).

The proof of this lemma follows the same strategy of the previous $Z(3)$ case. Here, however, the essential calculation is much more involved, due to the existence more types of particles. Details of this proof can be found in Appendix C.

4.2. Examples of fields and p-functions

We present solutions of the equations for the p-functions (i′)–(v′) of (41) and some explicit examples of the resulting form factors. We identify the fields by the properties of their matrix elements. In Section 5 we show that the fields satisfy the desired commutation rules. This motivates to propose a correspondence of fields $\phi(x)$ and p-functions $p^{\phi}(\underline{\theta}, \underline{z})$.

The fields $\psi_{Q,\tilde{Q}}(x)$ These fields have the charge $Q = 0, \dots, N-1$ and the dual charge $\tilde{Q} = 0, \dots, N-1$. We look for a solution of (i′)–(v′) with

$$\begin{cases} \text{charge } Q \mod N, \\ \text{spin } s^{\psi} = \min(Q, \tilde{Q}) - Q\tilde{Q}/N, \\ \text{statistics } \sigma^{\psi}_1 = \omega^{\tilde{Q}} \end{cases} \tag{42}$$

with $\omega = e^{i\eta} = e^{2\pi i/N}$. The phase factor σ^{ψ}_1 is the statistics factor of the field $\psi_{Q,\tilde{Q}}(x)$ with respect to the particle of type 1. Since the fields carry the charge Q the only non-vanishing form factors with n particles of type 1 are the ones with $n = Q \mod N$. We propose the correspondence of the field and the p-function:

$$\psi_{Q,\tilde{Q}} \leftrightarrow p_{n\underline{m}}^{Q\tilde{Q}} = \rho \exp\left(\sum_{k=1}^{\tilde{Q}} \sum_{j=1}^{m_k} z_{kj} - \frac{\tilde{Q}}{N} \sum_{i=1}^{n} \theta_i\right)$$
$$\text{with } n = 3l + Q,\ l = 0, 1, 2, \ldots \quad \text{and} \quad \begin{cases} m_k = l + 1 & \text{for } k \leqslant Q, \\ m_k = l & \text{for } Q < k, \end{cases} \tag{43}$$

where $\rho = \sqrt{\omega}^{\tilde{Q}(\tilde{Q}-N+2)n/N}$. One easily checks that this p-function satisfies the equations (i′)–(v′) and the requirements (42). The normalization constants N_n follow from (39). In particular we have for

$\tilde{Q} = 0$ the order parameters $\sigma_Q(x) = \psi_{Q0}(x)$,

$Q = 0$ the disorder parameter $\mu_{\tilde{Q}}(x) = \psi_{0\tilde{Q}}(x)$,

$Q = \tilde{Q}$ the para-Fermi fields $\psi_Q(x) = \psi_{QQ}(x)$.

They satisfy space like commutation rules (5), derived in the next section. The para-Fermi fields $\psi_Q(x)$ are the massive analogs of the para-Fermi fields in the conformal quantum field theory of [14,15]. One obtains a second set of fields $\tilde{\psi}_{Q,\tilde{Q}}(x)$ by changing the sign in the exponent of (43).

The higher currents $J_L^{\pm}(x)$ These fields are charge-less, have bosonic statistics and spin $L \pm 1$. The only non-vanishing form factors with n particles of type 1 are the ones with $n = 0 \mod N$. We propose the correspondence of the currents and the p-functions for $L \in \mathbb{Z}$

$$J_L^{\pm} \leftrightarrow p_{n\underline{m}}^{J_L^{\pm}} = \pm \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{k=1}^{N-1} \sum_{j=1}^{m} e^{L z_{kj}} \quad \text{for } n = 3m.$$

The higher charges $Q_L = \int dx\, J_L^0(x)$ satisfy again the eigenvalue equations

$$\left(Q_L - \sum_{i=1}^{n} e^{L\theta_i}\right) |p_1, \ldots, p_n\rangle^{\text{in}} = 0.$$

Obviously, from $J_{\pm 1}^{\pm}(x)$ we obtain the energy–momentum tensor.

Examples Up to normalization constants we calculate for the order parameters $\sigma_1(x)$ and $\sigma_2(x)$

$$\langle 0|\sigma_1(0)|\theta\rangle_1 = 1,$$
$$\langle 0|\sigma_2(0)|\theta_1, \theta_2\rangle_{11}^{\text{in}} = \frac{F(\theta_{12})}{\sinh\frac{1}{2}(\theta_{12} - 2\pi i/N)\sinh\frac{1}{2}(\theta_{12} + 2\pi i/N)}$$

and for the para-Fermi fields $\psi_Q(x)$ and $\psi_2(x)$

$$\langle 0|\psi_Q(0)|\theta\rangle_Q = e^{\frac{Q(N-Q)}{N}\theta},$$
$$\langle 0|\psi_2(0)|\theta_1, \theta_2\rangle_{11}^{\text{in}} = \frac{e^{(1-\frac{2}{N})(\theta_1+\theta_2)} F(\theta_{12})}{\sinh\frac{1}{2}(\theta_{12} - 2\pi i/N)\sinh\frac{1}{2}(\theta_{12} + 2\pi i/N)}, \tag{44}$$

where $|\theta\rangle_Q$ denotes a one-particle state of charge Q and $|\theta_1, \theta_2\rangle_{11}^{\text{in}}$ a state of two particles of charge 1.

5. Commutation rules

5.1. The general formula

Techniques similar to the ones used in this section have been applied for the simpler case of no bound states and bosonic statistics in [3,27]. A generalization for the case of bound states has been discussed in [28]. Here we generalize these techniques for the case of more general statistics and also discuss the contribution of poles related to the double poles of bound state S-matrices.[2] In order to discuss commutation rules of two fields $\phi(x)$ and $\psi(y)$ we have to use a general crossing formula for form factors which was derived in [4] (see also [3]). For quantum field theories with general statistics we introduce assumptions on the statistics factor of a field $\psi(x)$ and a particle α. It is easy to see that for consistency of (ii) and (iii) the condition $\sigma^\psi(\alpha)\sigma^\psi(\bar\alpha)=1$ has to hold if $\bar\alpha$ is the anti-particle of α. We assume that

$$\sigma^\psi(\alpha)=\sigma^\psi(Q_\alpha) \tag{45}$$

depends on the charge of the particle such that $\sigma^\psi(Q+Q')=\sigma^\psi(Q)\sigma^\psi(Q')$. A stronger assumption (which holds for the $Z(N)$-model) is the existence of a 'dual charge' $\tilde Q_\psi$ of the fields such that

$$\sigma^\psi(\alpha)=\omega^{\tilde Q_\psi Q_\alpha}, \tag{46}$$

where $|\omega|=1$.

In order to write the following long formulae we introduce a short notation: For a field $\mathcal{O}(x)$ and for ordered sets of rapidities $\theta_1>\cdots>\theta_n$ and $\theta'_1<\cdots<\theta'_m$ we write the general matrix element of $\mathcal{O}(0)$ as

$$\mathcal{O}_{\underline\alpha}^{\underline\beta}(\underline\theta'_\beta,\underline\theta_\alpha):={}^{\text{out}}\big\langle\beta_m(\theta'_m),\dots,\beta_1(\theta p'_1)\big|\mathcal{O}\big|\alpha_1(\theta_1),\dots,\alpha_n(\theta_n)\big\rangle^{\text{in}}, \tag{47}$$

where $\underline\theta_\alpha=(\theta_1,\dots,\theta_n)$ and $\underline\theta'_\beta=(\theta'_1,\dots,\theta'_m)$. The array of indices $\underline\alpha=(\alpha_1,\dots,\alpha_n)$ denote a set of particles ($\alpha_i\in\{$types of particles$\}$) and correspondingly for $\underline\beta$ (we also write $|\alpha|=n$, etc.). Similar as for form factors this matrix element is given for general order of the rapidities by the symmetry property (i) for both the in- and out-states which takes the general form:

$$\mathcal{O}_{\underline\alpha}^{\underline\beta}(\underline\theta'_\beta,\underline\theta_\alpha)=S_{\underline\delta}^{\underline\beta}(\underline\theta'_\delta)\mathcal{O}_{\underline\gamma}^{\underline\delta}(\underline\theta'_\delta,\underline\theta_\gamma)S_{\underline\alpha}^{\underline\gamma}(\underline\theta_\alpha)$$

if $\underline\theta_\gamma$ is a permutation of $\underline\theta_\alpha$ and $\underline\theta'_\delta$ a permutation of $\underline\theta'_\beta$. The matrix $S_{\underline\alpha}^{\underline\gamma}(\underline\theta_\alpha)$ is defined as the representation of the permutation $\pi(\underline\theta_\alpha)=\underline\theta_\gamma$ generated by the two-particle S-matrices $S_{\alpha_1\alpha_2}^{\gamma_2\gamma_1}(\theta_{12})$, for example $S_{\alpha_1\alpha_2\alpha_3}^{\gamma_3\gamma_1\gamma_2}(\theta_1,\theta_2,\theta_3)=S_{\alpha_1\lambda_3}^{\gamma_3\gamma_1}(\theta_{13})S_{\alpha_2\alpha_3}^{\lambda_3\gamma_2}(\theta_{23})$ (cf. [4]).

We consider an arbitrary matrix element of products of fields $\mathcal{O}=\phi(x)\psi(y)$ and $\mathcal{O}=\psi(y)\phi(x)$. Inserting a complete set of intermediate states $|\underline{\tilde\theta}_\gamma\rangle_{\underline\gamma}^{\text{in}}$ we obtain

$$\big(\phi(x)\psi(y)\big)_{\underline\alpha}^{\underline\beta}(\underline\theta'_\beta,\underline\theta_\alpha)=e^{iP'_\beta x-iP_\alpha y}\frac{1}{\gamma!}\int_{\underline{\tilde\theta}_\gamma}\phi_{\underline\gamma}^{\underline\beta}(\underline\theta'_\beta,\underline{\tilde\theta}_\gamma)\psi_{\underline\alpha}^{\underline\gamma}(\underline{\tilde\theta}_\gamma,\underline\theta_\alpha)e^{-i\tilde P_\gamma(x-y)}, \tag{48}$$

[2] For bound state form factors there are also higher order 'physical poles' (see e.g. [29–32]).

where $\phi = \phi(0)$, $\psi = \psi(0)$, $P_\alpha =$ the total momentum of the state $|\underline{\theta}_\alpha\rangle^{\text{in}}_{\underline{\alpha}}$ etc. and $\int_{\underline{\tilde\theta}_\gamma} = \prod_{k=1}^{|\gamma|}\int\frac{d\tilde\theta_k}{4\pi}$. Einstein summation convention over all sets $\underline{\gamma}$ is assumed. We also define $\gamma! = \prod_\alpha n_\alpha!$ where n_α is the number of particles of type α in $\underline{\gamma}$. We apply the general crossing formula (31) of [4] which is obtained by taking into account the disconnected terms in (ii) and iterating that formula. Strictly speaking, we apply the second version of the crossing formula to the matrix element of ϕ

$$\phi^{\underline{\beta}}_{\underline{\gamma}}(\underline{\theta}'_\beta, \underline{\tilde\theta}_\gamma) = \sum_{\substack{\underline{\theta}'_\rho\cup\underline{\theta}'_\tau=\underline{\theta}'_\beta \\ \underline{\tilde\theta}_\varsigma\cup\underline{\tilde\theta}_\varkappa=\underline{\tilde\theta}_\gamma}} S^{\underline{\beta}}_{\underline{\rho\tau}}(\underline{\theta}'_\rho, \underline{\theta}'_\tau)\phi_{\underline{\varsigma\bar\rho}}(\underline{\tilde\theta}_\varsigma, \underline{\theta}'_{\bar\rho} - i\pi_-)\mathbf{C}^{\underline{\rho\bar\rho}}\mathbf{1}^{\underline{\tau}}_{\underline{\sigma}}(\underline{\theta}'_\tau, \underline{\tilde\theta}_\varkappa)S^{\underline{\varsigma\sigma}}_{\underline{\gamma}}(\underline{\tilde\theta}_\gamma), \tag{49}$$

where $\underline{\bar\rho} = (\bar\rho_{|\rho|}, \dots, \bar\rho_1)$ with $\bar\rho =$ antiparticle of ρ and $\underline{\theta}'_{\bar\rho} - i\pi_-$ means that all rapidities are taken as $\theta' - i(\pi - \epsilon)$. The matrix $\mathbf{1}^{\underline{\tau}}_{\underline{\sigma}}(\underline{\theta}'_\tau, \underline{\tilde\theta}_\varkappa)$ is defined by (47) with $\mathcal{O} = \mathbf{1}$ the unit operator. The summation is over all decompositions of the sets of rapidities $\underline{\theta}'_\beta$ and $\underline{\tilde\theta}_\gamma$. To the matrix element of ψ we apply the first version of the crossing formula

$$\psi^{\underline{\gamma}}_{\underline{\alpha}}(\underline{\tilde\theta}_\gamma, \underline{\theta}_\alpha) = \sigma^{\psi}_{(\underline{\bar\gamma})}\sum_{\substack{\underline{\tilde\theta}_\nu\cup\underline{\tilde\theta}_\pi=\underline{\tilde\theta}_\gamma \\ \underline{\theta}_\mu\cup\underline{\theta}_\lambda=\underline{\theta}_\alpha}} S^{\underline{\gamma}}_{\underline{\nu\pi}}(\underline{\tilde\theta}_\nu, \underline{\tilde\theta}_\pi)\mathbf{1}^{\underline{\nu}}_{\underline{\mu}}(\underline{\tilde\theta}_\nu, \underline{\theta}_\mu)\mathbf{C}^{\underline{\pi\bar\pi}}\psi_{\underline{\bar\pi\lambda}}(\underline{\tilde\theta}_{\bar\pi} + i\pi_-, \underline{\theta}_\lambda)S^{\underline{\mu\lambda}}_{\underline{\gamma}}(\underline{\theta}_\gamma), \tag{50}$$

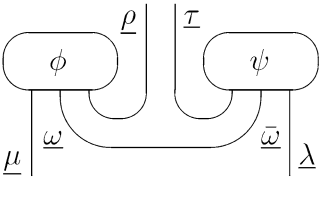
where we assume that the statistics factor $\sigma^{\psi}_{(\underline{\bar\gamma})}$ of the field ψ with respect to all particles in $\underline{\gamma}$ is the same for all $\underline{\gamma}$ which contribute to (48) (see below). Inserting (49) and (50) in (48) we use the product formula $S^{\underline{\varsigma\sigma}}_{\underline{\gamma}}(\underline{\tilde\theta}_\gamma)S^{\underline{\gamma}}_{\underline{\nu\pi}}(\underline{\tilde\theta}_\nu, \underline{\tilde\theta}_\pi) = S^{\underline{\varsigma\sigma}}_{\underline{\nu\pi}}(\underline{\tilde\theta}_\nu, \underline{\tilde\theta}_\pi)$. Let us first assume that the sets of rapidities in the initial state $\underline{\theta}_\alpha$ and the ones of the final state $\underline{\theta}'_\beta$ have no common elements which implies that also $\underline{\tilde\theta}_\nu \cap \underline{\tilde\theta}_\sigma = \emptyset$. Then we may use (ii) to get $S^{\underline{\varsigma\sigma}}_{\underline{\nu\pi}}(\underline{\tilde\theta}_\nu, \underline{\tilde\theta}_\pi) = 1$ and we can perform the $\underline{\tilde\theta}_\nu$- and $\underline{\tilde\theta}_\varkappa$-integrations. The remaining $\tilde\theta$-integration variables are $\underline{\tilde\theta}_\omega = \underline{\tilde\theta}_\varsigma \cap \underline{\tilde\theta}_\pi$, then we may write for the sets of particles $\underline{\varsigma} = \underline{\mu\omega}$, $\underline{\pi} = \underline{\omega\tau}$ and $\underline{\gamma} = \underline{\mu\omega\tau}$ and similar for the rapidities and momenta. Eq. (48) simplifies as

$$\begin{aligned}&\big(\phi(x)\psi(y)\big)^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}'_\beta, \underline{\theta}_\alpha) \\ &\quad = \sum_{\substack{\underline{\theta}'_\rho\cup\underline{\theta}'_\tau=\underline{\theta}'_\beta \\ \underline{\theta}_\mu\cup\underline{\theta}_\lambda=\underline{\theta}_\alpha}} \frac{\mu!\tau!}{\mu\omega\tau!} S^{\underline{\beta}}_{\underline{\rho\tau}}(\underline{\theta}'_\rho, \underline{\theta}'_\tau)\int_{\underline{\tilde\theta}_\omega} X^{\underline{\rho\tau}}_{\underline{\mu\lambda}} S^{\underline{\mu\lambda}}_{\underline{\alpha}}(\underline{\theta}_\alpha)e^{i(P'_\rho - P_\mu)x - i(P_\lambda - P'_\tau)y},\end{aligned} \tag{51}$$

where

$$\begin{aligned}X^{\underline{\rho\tau}}_{\underline{\mu\lambda}} &= \sigma^{\psi}_{(\underline{\bar\gamma})}\phi_{\underline{\mu\omega\bar\rho}}(\underline{\theta}_\mu, \underline{\tilde\theta}_\omega, \underline{\theta}'_{\bar\rho} - i\pi_-)\mathbf{C}^{\underline{\bar\rho\rho}}\mathbf{C}^{\underline{\tau\bar\tau}}\mathbf{C}^{\underline{\omega\bar\omega}} \\ &\quad \times \psi_{\underline{\bar\tau\bar\omega\lambda}}(\underline{\theta}'_{\bar\tau} + i\pi_-, \underline{\tilde\theta}_{\bar\omega} + i\pi_-, \underline{\theta}_\lambda)e^{-i\tilde P_\omega(x-y)}.\end{aligned} \tag{52}$$

The integrand $X^{\underline{\rho\tau}}_{\underline{\mu\lambda}}$ may be depicted as

$X^{\underline{\rho\tau}}_{\underline{\mu\lambda}} = \sigma^{\psi}_{(\underline{\bar\gamma})}$ 

Similarly, if we apply for the operator product $\psi(y)\phi(x)$ again the second crossing formula to the matrix element of ϕ and the first one the matrix element of ψ we obtain Eq. (51) where $X^{\rho\tau}_{\underline{\mu\lambda}}$ is replaced by

$$Y^{\rho\tau}_{\underline{\mu\lambda}} = \sigma^{\psi}_{(\underline{\bar\beta})}\phi_{\underline{\mu\omega\bar\rho}}(\underline{\theta}_\mu, \underline{\tilde\theta}_\omega - i\pi_-, \underline{\theta}'_{\bar\rho} - i\pi_-)\mathbf{C}^{\underline{\bar\rho\rho}}\mathbf{C}^{\underline{\tau\bar\tau}}\mathbf{C}^{\underline{\omega\bar\omega}}$$
$$\times\, \psi_{\underline{\bar\tau\bar\omega\lambda}}(\underline{\theta}'_{\bar\tau} + i\pi_-, \underline{\tilde\theta}_{\bar\omega}, \underline{\theta}_\lambda)e^{i\tilde P_\omega(x-y)}, \tag{53}$$

which means that only $\sigma^{\psi}_{(\underline{\bar\gamma})}$ is replaced by $\sigma^{\psi}_{(\underline{\bar\beta})}$, $\tilde P_\omega$ by $-\tilde P_\omega$ and the integration variables $\underline{\tilde\theta}_\omega$ by $\underline{\tilde\theta}_{\bar\omega} - i\pi_-$.

No bound states In this case there are no singularities in the physical strip and we may shift in the matrix element of $\psi(y)\phi(x)$ (51) with (53) for equal times and $x^1 < y^1$ the integration variables by $\tilde\theta_i \to \tilde\theta_i + i\pi_-$. Note that the factor $e^{i\tilde P_\omega(x-y)}$ decreases for $0 < \operatorname{Re}\tilde\theta_i < \pi$ if $x^1 < y^1$. Because $\tilde P_\omega \to -\tilde P_\omega$ we get the matrix element of $\phi(x)\psi(y)$ (51) with (52) up to the statistics factors. Therefore we conclude

$$\phi(x)\psi(y) = \psi(y)\phi(x)\sigma^{\psi\phi} \quad \text{for } x^1 < y^1, \tag{54}$$

where $\sigma^{\psi\phi} = \sigma^{\psi}_{(\underline{\bar\gamma})}/\sigma^{\psi}_{(\underline{\bar\beta})}$. Using the assumption (45) we have with $Q_{\underline{\bar\gamma}} = \sum_{\gamma\in\underline{\gamma}} Q_{\bar\gamma}$

$$\sigma^{\psi}_{(\underline{\bar\gamma})} = \prod_{\gamma\in\underline{\gamma}} \sigma^{\psi}(\bar\gamma) = \sigma^{\psi}(Q_{\underline{\bar\gamma}}) = \sigma^{\psi}(Q_{\underline{\bar\beta}} - Q_\phi),$$

which is the same for all $\underline{\gamma}$, as assumed above. The last equation follows from $Q_{\bar\gamma} = -Q_\gamma$ and charge conservation which means that the matrix elements $\phi^{\underline{\beta}}_{\underline{\gamma}}$ in (48) are non-vanishing if $Q_{\underline{\beta}} + Q_\phi = Q_{\underline{\gamma}}$. Therefore the statistics factor of the fields ψ with respect to ϕ is

$$\sigma^{\psi\phi} = \frac{\sigma^{\psi}(Q_{\underline{\bar\gamma}})}{\sigma^{\psi}(Q_{\underline{\bar\beta}})} = \sigma^{\psi}(-Q_\phi) = 1/\sigma^{\psi}(Q_\phi), \tag{55}$$

which is in general not symmetric under the exchange of ψ and ϕ. Finally, we obtain the space like commutation rules

$$\phi(x)\psi(y) = \psi(y)\phi(x)\begin{cases} 1/\sigma^{\psi}(Q_\phi) & \text{for } x^1 < y^1, \\ \sigma^{\phi}(Q_\psi) & \text{for } x^1 > y^1, \end{cases} \tag{56}$$

where the second relation is obtained from (54) by exchanging $\phi \leftrightarrow \psi$ and $x \leftrightarrow y$. The same result appears when there are bound states. This is proved in Appendix D where also the existence of double poles in bound state S-matrices is taken into account.

5.2. *Application to the $Z(N)$-model*

The statistics factors in this model are of the form (46) $\sigma^{\psi}(\alpha) = \omega^{\tilde Q_\psi Q_\alpha}$ where $\tilde Q_\psi$ is the dual charge of the field ψ and Q_α is the charge of the particle α, therefore $\sigma^{\psi\phi} = \omega^{-\tilde Q_\psi Q_\phi}$. The general equal time commutation rule (56) for fields $\psi_{Q\tilde Q}(x)$ defined by (43) in Section 4 reads as

$$\psi_{Q\tilde Q}(x)\psi_{R\tilde R}(y) = \psi_{R\tilde R}(y)\psi_{Q\tilde Q}(x)\begin{cases} \omega^{-\tilde R Q} = e^{-2\pi i\tilde R Q/N} & \text{for } x^1 < y^1, \\ \omega^{\tilde Q R} = e^{2\pi i\tilde Q R/N} & \text{for } x^1 > y^1. \end{cases} \tag{57}$$

Notice that in this model we have a more general anyonic statistics.

Examples

(1) The order parameters have bosonic commutation rules with respect to each other

$$\sigma_Q(x)\sigma_{Q'}(y) = \sigma_{Q'}(y)\sigma_Q(x).$$

(2) The disorder parameters have again bosonic commutation rules with respect to each other.
(3) For the order-disorder parameters we obtain the typical commutation rule

$$\mu_{\tilde{Q}}(x)\sigma_Q(y) = \sigma_Q(y)\mu_{\tilde{Q}}(x)\begin{cases} 1 & \text{for } x^1 < y^1, \\ \omega^{\tilde{Q}Q} = e^{2\pi i \tilde{Q}Q/N} & \text{for } x^1 > y^1. \end{cases}$$

(4) The para-Fermi fields have anyonic commutation rules

$$\psi_Q(x)\psi_R(y) = \psi_R(y)\psi_Q(x)e^{\epsilon(x^1-y^1)2\pi i QR/N}. \tag{58}$$

These results prove the commutation rules (5) in the introduction.

The 2-point Wightman function In order to compare these commutation rules with the explicit results of the previous section we calculate the 2-point Wightman function for the para-Fermi fields ψ_Q and ψ_{N-Q} (with spin $s = Q(N-Q)/N$) in 1-particle (charge Q) intermediate state approximation. Using the result (44) we obtain

$$\begin{aligned} \langle 0|\psi_Q(x)\psi_{N-Q}(0)|0\rangle &= \int \frac{d\theta}{4\pi} \langle 0|\psi_Q(x)|\theta\rangle_{Q\,Q}\langle\theta|\psi_{N-Q}(0)|0\rangle + \cdots \\ &= \frac{1}{2\pi}\left(\frac{x^- - i\epsilon}{x^+ - i\epsilon}\right)^{\nu/2} K_\nu\bigl(M\sqrt{i(x^+ - i\epsilon)}\sqrt{i(x^- - i\epsilon)}\bigr) + \cdots, \end{aligned}$$

where $\nu = 2Q(N-Q)/N$ and $x^\pm = t \mp x$. This agrees with the commutation rule (58), because for $t = 0$ and $x > 0$ using the symmetry $Q \leftrightarrow N - Q$, $x \to -x$ and translation invariance we obtain

$$\begin{aligned} \langle 0|\psi_Q(x)\psi_{N-Q}(0)|0\rangle &= \langle 0|\psi_{N-Q}(x)\psi_Q(0)|0\rangle \\ &= e^{i\pi\nu}\langle 0|\psi_{N-Q}(-x)\psi_Q(0)|0\rangle \\ &= e^{i\pi\nu}\langle 0|\psi_{N-Q}(0)\psi_Q(x)|0\rangle, \end{aligned}$$

where $((x - i\epsilon)/(-x - i\epsilon))^{\nu/2} = e^{i\pi\nu\epsilon(x)/2}$ has bee used. The asymptotic behavior is obtained from

$$2K_\nu(z) \to \begin{cases} \Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} + \Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu} & \text{for } z \to 0, \\ \sqrt{\frac{2\pi}{z}}e^{-z} & \text{for } z \to \infty, \end{cases}$$

for $\nu \neq 0$. Therefore the leading short distance behavior is up to constants

$$\begin{aligned} \langle 0|\psi_Q(x)\psi_{N-Q}(0)|0\rangle &\sim \bigl(x^+ - i\epsilon\bigr)^{-\nu}, \\ \langle 0|\tilde{\psi}_Q(x)\tilde{\psi}_{N-Q}(0)|0\rangle &\sim \bigl(x^- - i\epsilon\bigr)^{-\nu}, \end{aligned}$$

where the fields $\tilde{\psi}_Q(x)$ are obtained by changing the sign in the exponent of (43).

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Appendix A. Some useful formulae

In this appendix we provide some explicit formulae for the scattering matrices and the minimal form factors which we frequently employ in the explicit computations. The S-matrix of two 'fundamental' particles (i.e. of type 1) is [9]

$$S(\theta)=\frac{\sinh\frac{1}{2}(\theta+\frac{2\pi i}{N})}{\sinh\frac{1}{2}(\theta-\frac{2\pi i}{N})}=-\exp\int_0^\infty\frac{dt}{t}2\frac{\sinh t(1-2/N)}{\sinh t}\sinh tx,$$

where θ is the rapidity difference defined by

$$p_1p_2=M^2\cosh\theta.$$

A particle of type α $(0<\alpha<N)$ is a bound state $\alpha=(\alpha_1\cdots\alpha_l)$ of particles of type α_i where $\alpha=\alpha_1+\cdots+\alpha_l$, in particular $\alpha=(\underbrace{1\cdots1}_{\alpha})$ for all $\alpha_i=1$. For the scattering of the bound state α and β we have [17]

$$S_{\alpha\beta}(\theta)=\exp 2\int_0^\infty\frac{dx}{x}\frac{\cosh x(1-\frac{|\beta-\alpha|}{N})-\cosh x(1-\frac{\beta+\alpha}{N})}{\sinh x\tanh(x/N)}\sinh x\frac{\theta}{i\pi}.$$

The minimal form factor functions, which satisfies Watson's equations, are obtained from the S-matrix formulae [1] and are given as $(\beta>\alpha)$

$$F_{\alpha\beta}^{\min}(\theta)=\exp\int_0^\infty\frac{dt}{t}2\frac{\sinh t(1-\frac{\beta}{N})\sinh t(\frac{\alpha}{N})}{\sinh^2 t\tanh t/N}\left(1-\cosh t\left(1-\frac{\theta}{i\pi}\right)\right)$$

in particular[17]

$$F_{11}^{\min}(\theta)=\exp\int_0^\infty\frac{dt}{t}2\frac{\sinh t(1-\frac{1}{N})\cosh t\frac{1}{N}}{\sinh^2 t}\left(1-\cosh t\left(1-\frac{\theta}{i\pi}\right)\right)$$

$$=-i\sinh\frac{1}{2}\theta\exp\int_0^\infty\frac{dt}{t}\frac{\sinh t(1-\frac{2}{N})}{\sinh^2 t}\left(1-\cosh t\left(1-\frac{\theta}{i\pi}\right)\right)$$

$$= -i \sinh \frac{1}{2}\theta \prod_{k=0}^{\infty} \frac{\Gamma(k+1-\frac{1}{N}+\frac{x}{2})}{\Gamma(k+\frac{1}{N}+\frac{x}{2})} \frac{\Gamma(k+2-\frac{1}{N}-\frac{x}{2})}{\Gamma(k+1+\frac{1}{N}-\frac{x}{2})} \left(\frac{\Gamma(k+\frac{1}{2}+\frac{1}{N})}{\Gamma(k+\frac{3}{2}-\frac{1}{N})}\right)^2$$

and with $\bar{1} = (N-1)$

$$F_{1\bar{1}}^{\min}(\theta) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh t \frac{2}{N}}{\sinh^2 t} \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right)$$
$$= \prod_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2}+\frac{1}{N}+\frac{\theta}{2\pi i})}{\Gamma(k+\frac{1}{2}-\frac{1}{N}+\frac{\theta}{2\pi i})} \frac{\Gamma(k+\frac{3}{2}+\frac{1}{N}-\frac{\theta}{2\pi i})}{\Gamma(k+\frac{3}{2}-\frac{1}{N}-\frac{\theta}{2\pi i})} \left(\frac{\Gamma(k+1-\frac{1}{N})}{\Gamma(k+1+\frac{1}{N})}\right)^2.$$

There are simple relations between the minimal form factors which we essentially use in our construction which are up to constants

$$F_{11}^{\min}\left(\theta + \frac{i\pi}{N}\right) F_{11}^{\min}\left(\theta - \frac{i\pi}{N}\right) \propto \sinh \frac{1}{2}\left(\theta + \frac{i\pi}{N}\right) \sinh \frac{1}{2}\left(\theta - \frac{i\pi}{N}\right) F_{12}^{\min}(\theta)$$
$$\prod_{k=0}^{N-1} F_{11}^{\min}\left(\theta + \frac{k}{N} 2\pi i\right) \propto \prod_{k=0}^{N-2} \sinh \frac{1}{2}\left(\theta + \frac{k}{N} 2\pi i\right) \sinh \frac{1}{2}\left(\theta + \frac{k+1}{N} 2\pi i\right)$$
$$F_{11}^{\min}(\theta) F_{1\bar{1}}^{\min}(\theta + i\pi) \propto \sinh \frac{1}{2}\theta \sinh \frac{1}{2}(\theta + 2i\pi/N).$$

In Eqs. (20) and (33) we used the function $F(\theta) = c_N F_{11}^{\min}(\theta)$ with

$$c_N = e^{i\pi \frac{N-1}{N}} \exp\left(\int_0^\infty \frac{dt}{t \sinh t} \left(\left(1 - \frac{2}{N}\right) - \frac{\sinh t(1-\frac{2}{N})}{\sinh t}\right)\right), \tag{A.1}$$

such that the normalizations in (37) and (40) hold.

Appendix B. Integrals for the $Z(2)$-model

The claim (15) follows from the following lemma

Lemma 3. *For $n = 2m+1$ odd and $x_i = e^{\theta_i}$*

$$f_n(\underline{x}) := I_{nm}(\underline{\theta}, 1) - (2i)^{(n-1)/2} \prod_{1 \leqslant i < j \leqslant n} \frac{\tanh \frac{1}{2}\theta_{ij}}{F(\theta_{ij})} = 0.$$

Proof. Again as in the proof of Lemma 2 in [33] we apply induction and Liouville's theorem. One easily verifies $f_1(\underline{x}) = f_3(\underline{x}) = 0$. As induction assumptions we take $f_{n-2} = 0$. The functions $f_n(\underline{x})$ are a meromorphic functions in terms of the x_i with at most simple poles at $x_i = -x_j$ since pinchings appear for $z_k = \theta_i = \theta_j \pm i\pi$. The residues of the poles are proportional to f_{n-2} as follows from the recursion relations (iii) for both terms. Furthermore $f_n(\underline{x}) \to 0$ for $x_i \to \infty$. Therefore $f_n(\underline{x})$ vanishes identically by Liouville's theorem. □

Note that the integrations in the definition (9) of I_{nm} can easily be performed and with $\mathcal{N}=\{1,\dots,n\}$ and $|\mathcal{K}|=m$

$$I_{nm}(\underline{\theta},1)=\sum_{\mathcal{K}\subset\mathcal{N}}\prod_{k\in\mathcal{K}}\prod_{i\in\mathcal{N}\setminus\mathcal{K}}\frac{2i}{\sinh\theta_{ki}}.$$

Appendix C. Proof of the main lemma

In this appendix we prove the main Lemma 2 which provides the general $Z(N)$-form factor formula.

Proof. Similar as in the proof of Lemma 1 we calculate

$$\begin{aligned}
&\operatorname*{Res}_{\theta_{N-1N}=i\eta}\dots\operatorname*{Res}_{\theta_{12}=i\eta} I_{n\underline{m}}(\underline{\theta},p^{\mathcal{O}}_{n\underline{m}})\\
&\quad=\frac{m_1\cdots m_{N-1}}{m_1!\cdots m_{N-1}!}\left(\prod_{k=1}^{N-1}\prod_{j=2}^{m_k}\int_{\mathcal{C}_{\underline{\theta}'}}\frac{dz_{kj}}{R}\right)\\
&\qquad\times\prod_{k=1}^{N-1}\left(\prod_{i=N+1}^{n}\prod_{j=2}^{m_k}\phi(z_{kj}-\theta_i)\prod_{2\leqslant i<j\leqslant m_k}\tau(z_{ki}-z_{kj})\right)\\
&\qquad\times\prod_{1\leqslant k<l\leqslant N-1}\prod_{i=2}^{m_k}\prod_{j=2}^{m_l}\varkappa(z_{ki}-z_{lj})\left(\prod_{k=1}^{N-1}\prod_{i=1}^{N}\prod_{j=2}^{m}\phi(z_{kj}-\theta_i)\right)r
\end{aligned}$$

with

$$\begin{aligned}
r={}&\operatorname*{Res}_{\theta_{N-1N}=i\eta}\dots\operatorname*{Res}_{\theta_{12}=i\eta}\left(\prod_{k=1}^{N-1}\int_{\mathcal{C}_{\underline{\theta}}}dz_{k1}\right)\prod_{k=1}^{N-1}\left(\prod_{i=1}^{n}\phi(z_{k1}-\theta_i)\prod_{2\leqslant j\leqslant m}\tau(z_{k1}-z_{kj})\right)\\
&\times\prod_{1\leqslant k<l\leqslant N-1}\left(\varkappa(z_{k1}-z_{l1})\prod_{i=2}^{m_k}\varkappa(z_{ki}-z_{l1})\prod_{j=2}^{m_l}\varkappa(z_{k1}-z_{lj})\right)p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta},\underline{z}).
\end{aligned}$$

Replacing $\mathcal{C}_{\underline{\theta}}$ by $\mathcal{C}_{\underline{\theta}'}$ where $\underline{\theta}'=(\theta_{N+1},\dots,\theta_n)$ we have used $\tau(0)=\tau(\pm i\eta)=\varkappa(0)=\varkappa(-i\eta)=0$ and the fact that the z_{k1}-integrations give non-vanishing results only for $z_{k1}=\theta_k$ and θ_{k+1}, $k=1,\dots,N-1$. This is because for $\theta_{12},\dots,\theta_{N-1N}\to i\eta$ pinching appears at $(z_{11},\dots,z_{N-11})=(\theta_2,\dots,\theta_N)$ and $(\theta_1,\dots,\theta_{N-1})$. Defining the function

$$\begin{aligned}
f(z_{11},\dots,z_{N-11})={}&\prod_{k=1}^{N-1}\prod_{2\leqslant j\leqslant m_k}\tau(z_{k1}-z_{kj})\prod_{1\leqslant k<l\leqslant N-1}\varkappa(z_{k1}-z_{l1})\\
&\times\prod_{1\leqslant k<l\leqslant N-1}\left(\prod_{j=2}^{m_l}\varkappa(z_{k1}-z_{li})\prod_{i=2}^{m_k}\varkappa(z_{ki}-z_{l1})\right)p^{\mathcal{O}}_{n\underline{m}}(\underline{\theta},\underline{z})
\end{aligned}$$

one obtains by means of (38) after some lengthy but straightforward calculation

$$r = \operatorname*{Res}_{\theta_{N-1N}=i\eta} \dots \operatorname*{Res}_{\theta_{12}=i\eta} \left(\prod_{k=1}^{N-1} \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_{k1}}{R} \right) \prod_{k=1}^{N-1} \left(\prod_{i=1}^{n} \phi(z_{k1} - \theta_i) \right) f(z_{11}, \dots, z_{N-11})$$

$$= \left(\operatorname*{Res}_{\theta=i\eta} \phi(-\theta) \prod_{k=1}^{N-2} \phi(ki\eta) \right)^{N-1} \left(\prod_{k=1}^{N-1} \prod_{i=N+1}^{n} \phi(\theta_{k+1i}) \right)$$

$$\times \left(f(\theta_2, \dots, \theta_N) - \left(\prod_{i=N+1}^{n} \frac{\phi(\theta_{1i})}{\phi(\theta_{Ni})} \right) f(\theta_1, \dots, \theta_{N-1}) \right).$$

It has been used that $f(\dots, z, \dots, z \dots) = f(\dots, z, \dots, z - i\eta \dots) = 0$ because of $\varkappa(0) = \varkappa(-i\eta) = 0$. Using further the defining relation of ϕ in terms of F (37), the Jost property (27) of the ϕ-function and the properties (iii′) of (41) for the p-function we get

$$\operatorname*{Res}_{\theta_{N-1N}=i\eta} \dots \operatorname*{Res}_{\theta_{12}=i\eta} I_{n\underline{m}}\big(\underline{\theta}, p^{\mathcal{O}}_{n\underline{m}}\big)$$

$$= \big(\operatorname*{Res}_{\theta=i\eta} \phi(-\theta) \big)^{N-1} \prod_{k=1}^{N-2} \phi^k(ki\eta)$$

$$\times I_{n-N\underline{m}-1}\big(\underline{\theta}', p^{\mathcal{O}}_{n-N\underline{m}-1}\big) \left(\prod_{k=1}^{N} \prod_{i=N+1}^{n} F(\theta_{ki}) \right)^{-1} \left(1 - \sigma_1^{\mathcal{O}} \prod_{i=N+1}^{n} S(\theta_{Ni}) \right),$$

which together with the relation for the normalization constants (39) proves the claim. □

Appendix D. Proof of the commutation rules

In this appendix we prove that we find the same commutation rules for two fields $\phi(x)$ and $\psi(y)$ when there are bound states poles or even when the S-matrix has double poles.[3]

Bound states We now show that the same result (54) appears when there are bound states[4] which means that there are poles in the physical strip. Let $\gamma = (\alpha\beta)$ be a bound state of α and β with fusion angle $\eta^{\gamma}_{\alpha\beta}$ which means that at $\theta_{\alpha\beta} = i\eta^{\gamma}_{\alpha\beta}$ the S-matrix $S_{\alpha\beta}(\theta)$ has a pole. The momentum and the rapidity of the bound state are

$$p_\gamma = p_\alpha + p_\beta,$$
$$\theta_\gamma = \theta_\alpha - i\big(\pi - \eta^{\bar\beta}_{\bar\gamma\alpha}\big) = \theta_\beta + i\big(\pi - \eta^{\bar\alpha}_{\beta\bar\gamma}\big),$$

where $\eta^{\bar\beta}_{\bar\gamma\alpha}$ and $\eta^{\bar\alpha}_{\beta\bar\gamma}$ are the fusion angles of the bound states $\bar\beta = (\bar\gamma\alpha)$ and $\bar\alpha = (\beta\bar\gamma)$, respectively.

We start matrix element of $\psi(y)\phi(x)$ (given by (51) with (53)). First we consider the contribution in the sum over the intermediate states where $\alpha \in \underline{\bar\omega}$ and $\beta \in \underline{\lambda}$. All the particles which are not essential for this discussion will be suppressed. Then the function $\psi_{\alpha\beta}(\tilde\theta_\alpha, \theta_\beta)$ has a pole at $\tilde\theta_\alpha - \theta_\beta = i\eta^{\gamma}_{\alpha\beta}$ such that by shifting the integration $\tilde\theta_\alpha \to \tilde\theta_\alpha + i\pi_-$ there will be the additional

[3] These poles appear typically for bound state–bound state scattering. The case of higher order poles may be discussed similarly and will be published elsewhere.
[4] Here we follow the arguments of Quella [28].

contribution

$$\frac{i}{2}\operatorname*{Res}_{\tilde\theta_\alpha=\theta_\alpha}\phi_{\bar\alpha}(\tilde\theta_\alpha-i\pi_-)\mathbf{C}^{\bar\alpha\alpha}\psi_{\alpha\beta}(\tilde\theta_\alpha,\theta_\beta)e^{i\tilde P_\alpha(x-y)}e^{-iyP_\beta}$$
$$=\frac{i}{2}\phi_{\bar\alpha}(\theta_\alpha-i\pi_-)\mathbf{C}^{\bar\alpha\alpha}\psi_\gamma(\theta_\gamma)\sqrt{2}\Gamma^\gamma_{\alpha\beta}e^{ixP_\alpha-iyP_\gamma}$$

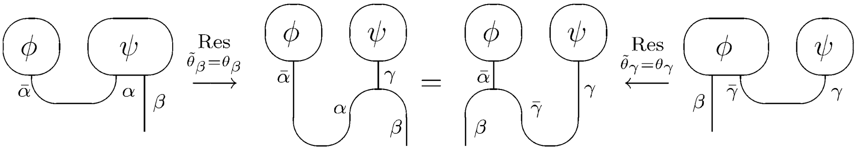
with $\theta_\alpha=\theta_\beta+i\eta^\gamma_{\alpha\beta}$, $\theta_\gamma=\theta_\beta+i(\pi-\eta^{\bar\alpha}_{\beta\gamma})$ and the fusion intertwiner $\Gamma^\gamma_{\alpha\beta}$ (see e.g. [33]). Next we consider the contribution to the sum over the intermediate states where $\gamma\in\underline{\bar\omega}$ and$\beta=\underline{\mu}$. Then the function $\phi_{\beta\bar\gamma}(\theta_\beta,\tilde\theta_\gamma-i\pi)$ has a pole at $\theta_\beta-\tilde\theta_\gamma+i\pi=i\eta^{\bar\alpha}_{\beta\bar\gamma}$ such that by shifting the integration $\tilde\theta_\gamma\to\tilde\theta_\gamma+i\pi_-$ there will be the additional contribution

$$\frac{i}{2}\operatorname*{Res}_{\tilde\theta_\gamma=\theta_\gamma}\phi_{\beta\bar\gamma}(\theta_\beta,\tilde\theta_\gamma-i\pi_-)\mathbf{C}^{\bar\gamma\gamma}\psi_\gamma(\tilde\theta_\gamma)e^{i\tilde P_\gamma(x-y)}e^{-ixP_\beta}$$
$$=-\frac{i}{2}\phi_{\bar\alpha}(\theta_\alpha-i\pi_-)\sqrt{2}\Gamma^{\bar\alpha}_{\beta\bar\gamma}\mathbf{C}^{\bar\gamma\gamma}\psi_\gamma(\theta_\gamma)e^{ixP_\alpha-iyP_\gamma}$$

with $\theta_\alpha,\theta_\gamma$ as above and the fusion intertwiner $\Gamma^{\bar\alpha}_{\beta\gamma}$. From the crossing relation of the fusion intertwiners

$$\mathbf{C}^{\bar\alpha\alpha}\Gamma^\gamma_{\alpha\beta}=\Gamma^{\bar\alpha}_{\beta\bar\gamma}\mathbf{C}^{\bar\gamma\gamma}$$

we conclude that these residue terms form bound state poles cancel. The steps of the argument may be depicted as



Double poles[5] Form factors have additional poles which are not related to bound states, they belong to higher poles of the S-matrix. First we consider the contribution to the sum over the intermediate states where the particles $\bar 1,1\in\underline{\bar\omega}$, $2\in\underline{\mu}$. Again we suppress all particles which are not essential for our discussion. Then the function $\phi_{2\bar 11}(\theta,\tilde\theta-i\pi,\tilde\theta'-i\pi)$ has a pole at $\tilde\theta=\theta_1=\theta+i\pi/N$ which correspond to the bound state $(2\bar 1)=1$ with the fusion angle $\eta^1_{2\bar 1}=\pi(1-1/N)$. We shift the integrations $\tilde\theta\to\tilde\theta+i\pi_-$ and $\tilde\theta'\to\tilde\theta'+i\pi_-$ such that during the shift $0<\operatorname{Im}(\tilde\theta-\tilde\theta')<\epsilon$. From the $\tilde\theta$-integration there will be the additional contribution

$$\frac{i}{2}\operatorname*{Res}_{\tilde\theta=\theta_1}\phi_{2\bar 11}(\theta,\tilde\theta-i\pi,\tilde\theta'-i\pi)\mathbf{C}^{\bar 111\bar 1}\psi_{1\bar 1}(\tilde\theta',\tilde\theta)e^{i\tilde P(x-y)}e^{-ixP}$$
$$=-\frac{i}{2}\phi_{11}(\theta-i\pi/N,\tilde\theta'-i\pi)\sqrt{2}\Gamma^{11}_2\mathbf{C}^{\bar 111\bar 1}\psi_{1\bar 1}(\tilde\theta',\theta_1)e^{i\tilde P(x-y)}e^{-ixP}.$$

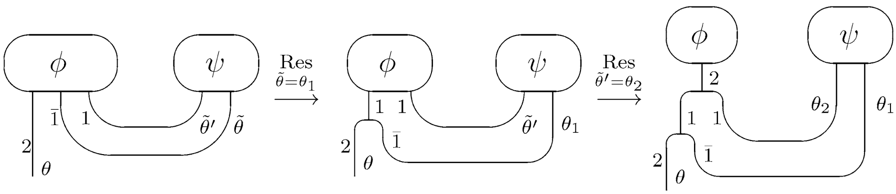
Further the function $\phi_{11}(\theta-i\pi/N,\tilde\theta'-i\pi)$ has a pole at $\tilde\theta'=\theta_2=\theta+i\pi(1-3/N)$ which correspond to the bound state $(11)=2$ with the fusion angle $\eta^2_{11}=\pi 2/N$. From the $\tilde\theta'$-integration

[5] This discussion is new.

there will be the additional contribution

$$\left(-\frac{i}{2}\right)^2 \sqrt{2} \operatorname*{Res}_{\tilde\theta'=\theta_2} \phi_{11}(\theta - i\pi/N, \tilde\theta' - i\pi)\sqrt{2}\Gamma_2^{11}\mathbf{C}^{\bar 1 1 1 \bar 1}\psi_{1\bar 1}(\tilde\theta', \theta_1)e^{i\tilde P(x-y)}e^{-ixP}$$
$$= -\frac{1}{2}\phi_2(\theta - i\pi 2/N)\Gamma_{11}^2\Gamma_2^{11}\mathbf{C}^{\bar 1 1 1 \bar 1}\psi_{1\bar 1}(\theta_2, \theta_1)e^{i(P_1+P_2)(x-y)}e^{-ixP}.$$

This procedure may be depicted as



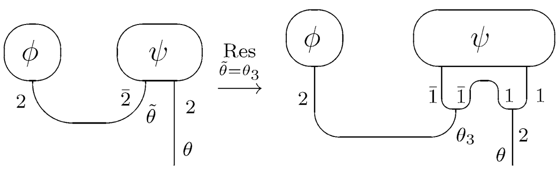
We show this the additional term is cancelled by a contribution to the sum over the intermediate states where $\bar 2 \in \underline{\omega}$, $2 \in \underline{\lambda}$. Then the function $\psi_{\bar 2 2}(\tilde\theta, \theta)$ has a pole at $\tilde\theta = \theta_3 = \theta + i\pi(1 - 2/N)$ which correspond to the double pole of the S-matrix

$$S_{\bar 2 2}(\theta) = \left(\frac{\sin\frac{\pi}{2}(\frac{\theta}{i\pi} + \frac{N-2}{N})}{\sin\frac{\pi}{2}(\frac{\theta}{i\pi} - \frac{N-2}{N})}\right)^2 \frac{\sin\frac{\pi}{2}(\frac{\theta}{i\pi} + \frac{N-4}{N})}{\sin\frac{\pi}{2}(\frac{\theta}{i\pi} - \frac{N-4}{N})}$$

at $\theta = i\pi(1 - 2/N)$. From the $\tilde\theta$-integration there will be the additional contribution

$$\frac{i}{2} \operatorname*{Res}_{\tilde\theta=\theta_3} \phi_2(\tilde\theta - i\pi)\mathbf{C}^{2\bar 2}\psi_{\bar 2 2}(\tilde\theta, \theta)e^{i\tilde P(x-y)}e^{-iyP}$$
$$= \frac{i}{2} i\phi_2(\theta - i\pi 2/N)\mathbf{C}^{2\bar 2}(-i)\left(\psi_{\bar 1 1}(\theta_2, \theta_1)\Gamma_{\bar 2}^{\bar 1 \bar 1}\mathbf{C}_{\bar 1 1}\Gamma_2^{11}\right)e^{iP_3(x-y)}e^{-iyP}.$$

This procedure may be depicted as



The crossing relation of the fusion intertwiners

$$\Gamma_{11}^2\Gamma_2^{11}\mathbf{C}^{1\bar 1} = \mathbf{C}^{2\bar 2}\Gamma_{\bar 2}^{\bar 1 \bar 1}\mathbf{C}_{\bar 1 1}\Gamma_2^{11}$$

implies that this contribution again cancelled the one above. It has been used that the form factor of bound states $\bar 2 2$ has a simple pole where the S-matrix $S_{\bar 2 2}$ has a double pole and the residue is

$$\operatorname*{Res}_{\tilde\theta=\theta_3} \psi_{\bar 2 2}(\tilde\theta, \theta) = -i\left(\psi_{\bar 1 1}(\theta_2, \theta_3)\Gamma_{\bar 2}^{\bar 1 \bar 1}\mathbf{C}_{\bar 1 1}\Gamma_2^{11}\right).$$

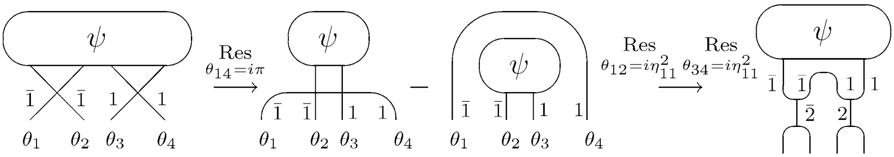
This may be calculated as follows. By the form factor equation (iv) we have

$$\operatorname*{Res}_{\theta_{12}=i\eta_{11}^2} \operatorname*{Res}_{\theta_{34}=i\eta_{11}^2} \psi_{\bar 1 \bar 1 1 1}(\theta_1, \theta_2, \theta_3, \theta_4) = 2\psi_{\bar 2 2}(\theta_{(12)}, \theta_{(34)})\Gamma_{\bar 1 \bar 1}^{\bar 2}\Gamma_{11}^2.$$

Therefore using the form factor equation (iii) and the definition of the fusion intertwiners $\operatorname{Res} iS = \Gamma\Gamma$ we obtain

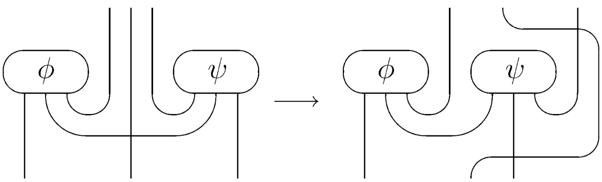
$$\begin{aligned}
&\operatorname*{Res}_{\theta_{(12)(34)}=i\pi(1-2/N)} \psi_{\bar{2}2}(\theta_{(12)},\theta_{(34)})\Gamma^{\bar{2}}_{\bar{1}\bar{1}}\Gamma^{2}_{11}\\
&\quad= \frac{1}{2}\operatorname*{Res}_{\theta_{14}=i\pi}\operatorname*{Res}_{\theta_{12}=i\eta^2_{11}}\operatorname*{Res}_{\theta_{34}=i\eta^2_{11}} \psi_{\bar{1}\bar{1}11}(\theta_2,\theta_1,\theta_4,\theta_3)S_{\bar{1}\bar{1}}(\theta_{12})S_{11}(\theta_{34})\\
&\quad= \frac{1}{2}\operatorname*{Res}_{\theta_{14}=i\pi}\operatorname*{Res}_{\theta_{12}=i\eta^2_{11}} 2i\mathbf{C}_{\bar{1}1}\big(\psi_{\bar{1}1}(\theta_2,\theta_3)S_{\bar{1}\bar{1}}(\theta_{12})S_{11}(\theta_{34}) - \psi_{\bar{1}1}(\theta_2,\theta_3)\big)\\
&\quad= i\mathbf{C}_{\bar{1}1}\big(\psi_{\bar{1}1}(\theta_2,\theta_3)(-i)\Gamma^{\bar{1}\bar{1}}_{\bar{2}}\Gamma^{\bar{2}}_{\bar{1}\bar{1}}(-i)\Gamma^{11}_{2}\Gamma^{2}_{11}\big),
\end{aligned}$$

which implies the residue formula used above. This procedure may be depicted as



Note that the last graph, as an on-shell graph, resembles (half of) the 'box' Feynman diagram which was used to investigate the double poles of bound state S-matrices (see e.g. [29,30]).

The general case Finally we consider the general case that the sets of rapidities in the initial state $\underline{\theta}_\alpha$ and the ones of the final state $\underline{\theta}'_\beta$ have also common elements. Then after inserting (49) and (50) in (48) there will be S-matrices $S^{\underline{\varsigma\sigma}}_{\underline{\nu\pi}}(\underline{\tilde{\theta}}_\nu, \underline{\tilde{\theta}}_\pi)$ which produce additional poles in the physical strip which would produce additional residue contributions while shifting the integration contours. However, we can remove these S-matrices by using again the crossing relation (ii) and move all the lines of common rapidities to the left or right as depicted as follows



Then we can apply the procedure as above.

References

[1] M. Karowski, P. Weisz, Nucl. Phys. B 139 (1978) 455.
[2] H.M. Babujian, A. Fring, M. Karowski, A. Zapletal, Nucl. Phys. B 538 (1999) 535.
[3] F. Smirnov, Advanced Series in Mathematical Physics, vol. 14, World Scientific, 1992.
[4] H. Babujian, M. Karowski, Nucl. Phys. B 620 (2002) 407.
[5] M. Karowski, Nucl. Phys. B 153 (1979) 244.
[6] K.M. Watson, Phys. Rev. 95 (1954) 228.
[7] H.M. Babujian, in: Gosen 1990, Proceedings, Theory of elementary particles, 12–23 (see high energy physics index 29 (1991) No. 12257).
[8] H. Babujian, J. Phys. A 26 (1993) 6981.
[9] R. Köberle, J.A. Swieca, Phys. Lett. B 86 (1979) 209.

[10] R.Z. Bariev, Phys. Lett. A 55 (1976) 456.
[11] B.M. McCoy, C.A. Tracy, T.T. Wu, Phys. Rev. Lett. 38 (1977) 793.
[12] M. Sato, T. Miwa, M. Jimbo, Proc. Jpn. Acad. 53 (1977) 6.
[13] B. Berg, M. Karowski, P. Weisz, Phys. Rev. D 19 (1979) 2477.
[14] A.B. Zamolodchikov, Int. J. Mod. Phys. A 3 (1988) 743.
[15] V.A. Fateev, Int. J. Mod. Phys. A 6 (1991) 2109.
[16] V.A. Fateev, A.B. Zamolodchikov, Sov. Phys. JETP 62 (1985) 215.
[17] M. Karowski, in: Lecture Notes in Physics, vol. 126, Springer, 1979, p. 344.
[18] A.N. Kirillov, F.A. Smirnov, ITF preprint 88-73P, Kiev, 1988.
[19] G. Delfino, J.L. Cardy, Nucl. Phys. B 519 (1998) 551.
[20] M. Jimbo, H. Konno, S. Odake, Y. Pugai, J. Shiraishi, J. Stat. Phys. 102 (2001) 883.
[21] F.H.L. Essler, R.M. Konik, in: M. Shifman, et al. (Eds.), From Fields to Strings, vol. 1, 2004, pp. 684–830.
[22] A.M. Tsvelik, Quantum Field Theory in Condensed Matter Physics, Cambridge Univ. Press, Cambridge, 1995, p. 332.
[23] A.O. Gogolin, A.A. Nersesyan, A.M. Tsvelik, Bosonization in Strongly Correlated Systems, Cambridge Univ. Press, Cambridge, 1999.
[24] J. Links, H. Zhou, R. McKenzie, M. Gould, J. Phys. A 36 (2003) R63.
[25] H. Babujian, M. Karowski, Phys. Lett. B 575 (2003) 144.
[26] H. Babujian, M. Karowski, J. Phys. A 35 (2002) 9081.
[27] M.Y. Lashkevich, LANDAU-94-TMP-4, 1994, unpublished.
[28] T. Quella, Diploma thesis FU-Berlin, 1999, unpublished.
[29] S.R. Coleman, H.J. Thun, Commun. Math. Phys. 61 (1978) 31.
[30] H.W. Braden, E. Corrigan, P.E. Dorey, R. Sasaki, Nucl. Phys. B 338 (1990) 689.
[31] G. Delfino, G. Mussardo, Nucl. Phys. B 455 (1995) 724.
[32] C. Acerbi, G. Mussardo, A. Valleriani, J. Phys. A 30 (1997) 2895.
[33] H. Babujian, M. Karowski, Phys. Lett. B 471 (1999) 53.