USING MULTIPLICATION AND DIVISION CONTEXTS TO BUILD PLACE-VALUE UNDERSTANDING

Jenny Young-Loveridge, Brenda Bicknell, University of Waikato, New Zealand

Abstract

The paper describes a study with five-year-old children to explore how multiplication and division problems helped them to develop early place-value understanding. Two teachers taught a series of focussed lessons over two four-week periods. The children solved problems using familiar materials grouped in twos, fives, and tens. By the end of the instructional period, virtually all children knew that two fives make ten; the majority could work with tens. Half of them could add tens and ones, fewer partitioned tens, and few could work with multi-unit processes. We propose a 5-level framework that describes developmental progressions in children's awareness of groups of five and ten as building blocks for place-value understanding.

Key words: division, multiplication, place-value understanding, primary/elementary

Introduction

Whole Number Arithmetic (WNA) continues to have a prominent place in most school mathematics curricula. A key aspect of WNA is the numeration system, where each digit in a multi-digit number has a different value according to its position within the numeral. Understanding place value requires students to be part-whole thinkers so they can partition numbers into different-sized units. Typically, mathematics in the early years of school focuses on counting, and this tends to be within the context of addition and subtraction. Place value is usually introduced as part of addition and subtraction with multi-digit numbers, before children have experienced meaningful multiplication and division. It is not until children have been at school for more than two years that multiplication and division become the focus of mathematics instruction.

Increasingly mathematics education researchers recognise that place value is inherently multiplicative (Askew, 2013; Bakker and van den Heuvel-Panhuizen, 2014; Nunes et al., 2009). Ross's (2002) work identified four key major properties of place value, including: positional, base-ten, multiplicative, and additive. It has been suggested that experiences with multiplication and division may be important in helping children develop a deep and connected understanding of place value (e.g., Askew, 2013).

Because place-value understanding is inherently multiplicative, it is far more complex than additive thinking (Clark and Kamii, 1996; Vergnaud; 1994). In contrast to additive thinking, where quantities of the same kind are manipulated (one variable), multiplicative thinking involves working with two variables (number of groups and number of items per group), and these are in a fixed ratio to each other, in a many-to-one relationship (Nunes et al., 2009). For example, a problem about four monkeys each with five bananas involves a 5:1 ratio between a monkey and its bananas. This many-to-one ratio must be strictly maintained to work out that four monkeys would have 20 bananas altogether. A division problem such as the number of boxes needed for 30 cupcakes if each box holds five cupcakes, requires the decomposition of 30 into groups of five (quotitive division). According to Vergnaud (1994, p. 47), 'multiplication and division are only the most visible part of an enormous conceptual iceberg' (the multiplicative conceptual field), that includes fractions, ratios, proportions, and measurement – all concepts involving proportionality.

Evidence clearly shows that quite young children are able to solve multiplication and division problems, although their strategies may differ from those of older children and adults (e.g., Bakker and van den Heuvel-Panhuizen, 2014; Blote, Lieffering and Ouwehand, 2006; Squire and Bryant, 2003). It makes sense for teachers to capitalise on that prior knowledge in the mathematics classroom.

More recently it has been argued that the development of number sense has an important spatial dimension (e.g., Papic, Mulligan and Mitchelmore, 2011; Thomas et al, 2002; van Nes and de Lange, 2007). A spatial structure is about the relationship between elements of a pattern, which has regularity in terms of number or space, including shape, spacing, or alignment.

Research on children's awareness of mathematical pattern and structure (AMPS) has shown the importance of students developing an awareness of structural relationships in mathematics (e.g., Mulligan, 2011). Low level of AMPS is associated with poor visual and working memory. Mulligan found that students with low AMPS tended to "rely on superficial unitary counting by ones" (p. 36), and did not develop efficient and flexible strategies for solving problems. AMPS also impacts on the development of measurement concepts and proportional reasoning. Mulligan's work on promoting awareness of pattern and structure is consistent with other research on the importance of helping children develop knowledge of place-value structure (Cobb, 2000; Fuson, Smith and Cicero, 1997; Thomas, Mulligan and Goldin, 2002). Many of the tasks used to assess AMPS involve the presentation of structured groups of objects for which shape, spacing, and alignment are important aspects of the structure. Children are asked reproduce displayed patterns by drawing them on paper (Mulligan, Mitchelmore and Stephanou, 2015). Mulligan's (2011) work on students' awareness of mathematical pattern and structure show the importance of constructing and representing composite units (multiples) and unit iteration (unit of repeat).

Recent research on so-called "groupitizing" has shown that grouped arrays can be quantified more quickly than ungrouped arrays because children can capitalise on the grouping structure to quantify objects in a display (Starkey and McCandliss, 2014). The advantage of structure becomes increasingly marked with grade level. A growing awareness of number composition in terms of part-whole relationships among the quantities accounts for the improvements in performance with age. This is consistent with research showing that intervention focused on enhancing children's awareness of pattern and structure leads to improvements in mathematics achievement (Mulligan, 2010, 2011).

A key feature of place-value development is the shift from a unitary (by ones) way of thinking about numbers to a multi-unit conception (e.g., tens & ones). The recent work on pattern and structure includes familiarity and use of structured groups of ten (ten-frames consisting of two rows of five) in the assessment of AMPS (Mulligan et al., 2015). Children with high AMPS construct multi-digit quantities quickly using structured material (ten-frames).

Research comparing the place-value understanding of children whose languages vary in the transparency of their decade-based structure for the “teen” numbers has found that children with the most transparent language structure (e.g., Korean, Japanese) have better place-value understanding than those with irregularity (Miura et al., 1993). Most of this research has focused on children from Confucian-heritage countries such as Japan and Korea. However, there are other less well-known languages that also have transparent decade structure, such as the Māori language used by some indigenous New Zealanders.

Overemphasis on counting in the context of addition and subtraction has detracted from an important idea of the composite unit or the notion of multiplicative or additive thinking (Behr et al., 1994; Lamon, 1996; Sophian, 2007). Although many teachers encourage children to skip count by twos, fives, and tens, links are not always made between these number-word sequences and the groups they represent in meaningful multiplicative contexts. A foundational idea underpinning all of mathematics learning is the concept of the unit, and this is the focus of much research on topics such as proportional reasoning and measurement (Mulligan and Mitchelmore, 2013). According to Behr et al. (1994, p. 123), a hidden assumption underpinning primary mathematics is that “all quantities are represented in terms of units of one”. Thus the idea of equal groups or composite units leading to multiplicative thinking is not linked to that learning.

The New Zealand Number Framework is embedded in the primary mathematics curriculum and this is linked to the expectations outlined in the Mathematics Standards (Ministry of Education, 2008, 2009). Expectations for the first two years of school are specified in terms of increasingly sophisticated counting strategies to join collections together. After three years, it is expected that children use so-called “part-whole strategies that utilise number properties.

The Study

This exploratory study was set in an urban school (medium SES) in New Zealand. The participants were 35 five-year-olds (21 girls & 14 boys) in two Year 1 classes. The average age of the students was 5.4 years at the start of the study (range 5.0 to 5.8 years). Children came from a diverse range of ethnic backgrounds, with about one third Māori (the indigenous people of New Zealand), one quarter European, one quarter Asian, and the remainder including African and Pasifika (Pacific Islands people). Children were assessed initially using an individual diagnostic task-based interview. The interview was completed again after the second 4-week teaching block (six months later). Tasks included: word problems involving addition, subtraction, multiplication, and division, subitizing, known facts, counting sequences, and place value.

Two series of 12 focused lessons were taught in May and August. Children were introduced to groups of two, using familiar contexts such as pairs of socks and shoes. Multiplication was introduced using simple word problems, such as:

> Three children each get 2 socks from the bag. How many socks do they have altogether?

Once children were familiar with groups of two, fives were introduced using contexts such as gloves (five fingers). Tens were introduced using egg cartons that held exactly ten eggs. For example:

> There are 20 eggs. Each carton holds 10 eggs. How many full cartons are there?

Later problems included numbers that were not multiples of ten, resulting in 'leftover' ones (i.e., the remainder).

> There are 23 chocolates. Each tray holds 10 chocolates. How many full trays are there?

Lessons began with the whole class solving a problem together. The teacher recorded children's problem-solving processes (e.g., drawings and number sentences) in a "modelling book" (a blank scrapbook). Following whole-class discussion, children completed a problem in their individual project books. These problems used the same context and language as the class problem, with a range of numbers to cater for varying abilities.

The Framework

Tasks related to place-value understanding and groups of ten were selected for analysis. Individual profiles were constructed by putting tasks in descending order of difficulty and grouping tasks according to similarity. Students' totals were then ordered to reveal a hierarchical pattern of acquisition (See Appendix – Tab. 1). The easiest task was knowledge of ten as two groups of five (quinary), while the hardest was working with multi-unit processes such as division by ten with remainder.

Results

Initially, only about half the students knew that ten is two groups of five (see Fig. 1). Some were starting to quantify two or three groups of ten and combine tens and ones. Only three students could halve 20, and this was the extent to which they could partition tens within whole decades. Final assessment after intervention showed marked improvement on the selected tasks.

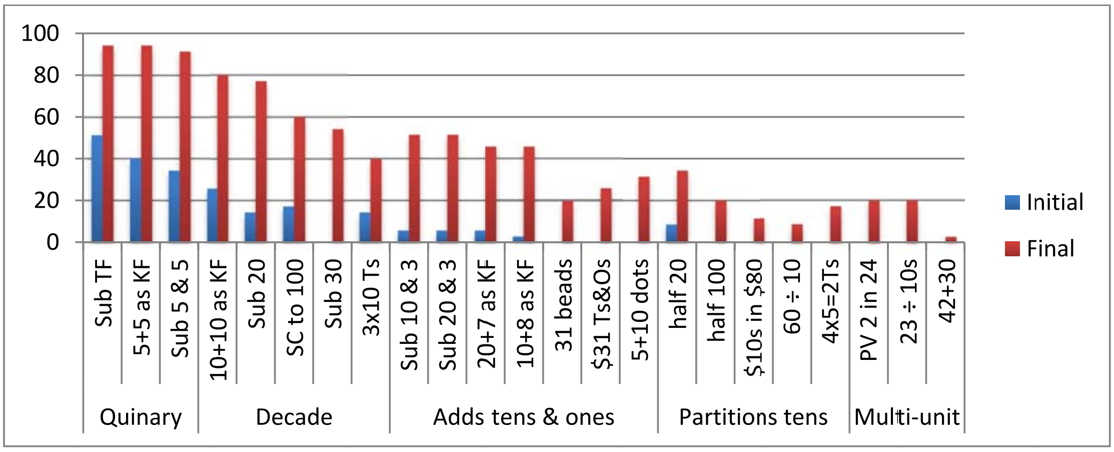

Fig. 1: Percentages of children succeeding on each task: Initial and Final

After the lessons, nearly all students knew that ten is two fives (Level 1). Successful performance on decades ranged from 40-80% (Level 2), 20-50% could add tens and ones (Level 3), and 9-37% could partition tens (Level 4). Only seven students identified tens within multi-digit numbers beyond the 'teens' (Level 5). The two students who could show 2 in 24 and divide 23 into 10s were the only ones who could explain the meaning of 2 in 24 as "two tens".

Discussion

Literature on the development of place value focuses on groups of ten without acknowledging the role of structure within ten, whereas the quinary structure (e.g., 2 rows of 5 dots) has been emphasised by some writers (e.g., Mulligan, 2010, 2011). This study supports Mulligan's (2011) assertion that the quinary structure is foundational for developing base-ten understanding. By the end of the study, all students were successful on at least one quinary task, but for some this was all they could manage (see Tab. 1). This finding is consistent with Mulligan's (2011) point about the importance of explicit features of structural development, including unitising, congruence, and collinearity. Awareness of mathematical pattern and structure (AMPS) is necessary for learning mathematical concepts. Moreover AMPS is multiplicative structure based on grouping and spatial visualisation groupitizing Starkey and McCandliss, 2014). Low levels of AMPS are explained by poor visualisation skills and visual memory, but intervention using the Pattern and Structure Mathematics Awareness Programme (PASMAP) can address this (Mulligan, 2011).

The analysis of performance taking ethnicity into account showed that Asian students performed better than the other two groups (see Tab. 1). This finding is consistent with research showing more advanced place-value understanding for children from Confucian-heritage cultures (e.g., Miura et al, 1993). Māori students did not perform as well as either of the other groups. Although the counting words used in the Māori language have a transparent decade structure, only children who are taught through the medium of Māori develop the fluency to speak and think in the Māori language. In reality, many teachers and students learn Māori as a second language, rather than being truly bilingual. The majority of Māori children are educated in mainstream (English) classrooms and experience only limited Māori language at school. This could explain why the Māori children in our study did not perform as well as the other two groups.

Although some curriculum documents suggest that basic facts should initially be restricted to small sums, we found that the children were more successful with 5 + 5 and 10 + 10 than with tasks where the sum was five or smaller. This may be a result of the salience of five fingers, and the early emphasis on numbers to 20. In the final assessment, some children visualised two groups of five bananas as a group of ten, then added this to the other two groups, finally adding the two tens.

The study showed that five-year-olds can work with multiplication and division problems using familiar contexts (e.g., fingers in gloves, eggs in a carton) and materials to work with fives and tens. This contrasts with Thompson's (2000, p. 291) claim that place value "is too sophisticated for many young children to grasp". It also challenges the many curricula that introduce place value before multiplication and division. The study has some important implications for teachers who could support the place-value understanding of their students by providing meaningful multiplication and division. Further exploratory studies are needed to focus on refining the framework, and explore other ways to support place-value ideas.

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Appendix

Tasks	Initial Total n=35	Final Total n=35	Initial Ma n=13	Initial As n=9	Initial Oth n=13	Final Ma n=13	Final As n=9	Final Oth n=13
1. Quinary (10=2 fives)								
Subitizes 1 full ten-frame	51	94	31	56	69	85	100	100
5 + 5 as a Known Fact	40	94	31	56	38	85	100	100
Subitizes 2 dice patterns of 5 dots	34	91	31	44	31	85	100	92
2. Decade (groups of ten)								
10 + 10 as a Known Fact	26	80	23	33	23	54	100	92
Subitizes 2 full ten-frames	14	77	0	22	23	69	89	77
Counts by 10s to 100 verbally	17	60	0	33	23	54	78	54
Subitizes 3 full ten-frames	0	54	0	0	0	54	78	38
3 rows of 10 by 10s or Known Fact	14	40	0	0	0	38	67	15
3. Adds tens & ones								
Subitizes 1 ten-frame & 3 single dots	6	51	0	0	15	31	78	54
Subitizes 2 ten-frames & 3 single dots	6	51	0	0	15	46	78	38
20 + 7 as a Known Fact	6	46	0	0	15	23	89	38
10 + 8 as a Known Fact	3	46	0	0	8	23	100	31
Show 31 beads by 10s & 1s	0	26	0	0	0	0	56	31
Get $31 by $10 notes & $1 coins	0	26	0	0	0	0	44	23
Dot strips 5 + 10 as Known Fact	0	31	0	0	0	8	78	23
4. Partitions into tens								
half 20	9	34	0	11	15	23	56	31
half 100	0	20	0	0	0	15	33	15
$10 notes for $80	0	11	0	0	0	8	22	8
60 sticks in 10s	0	9	0	0	0	8	11	8
4 groups of 5 = 10+10 or 2Ts	0	17	0	0	0	23	11	15
5. Multiple units								
PV for 2 in 24	0	20	0	0	0	23	22	15
23 eggs ÷ 10s	0	20	0	0	0	8	44	15
42 + 30 sheep	0	3	0	0	0	0	0	8

Tab. 1: Percentages of students who were successful on tasks at each Framework level for progressions in place-value development