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Obliquely propagating electromagnetic waves in magnetized kappa plasmas

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Velocity distribution functions (VDFs) that exhibit a power-law dependence on the high-energy tail have been the subject of intense research by the plasma physics community. Such functions, known as kappa or superthermal distributions, have been found to provide a better fitting to the VDFs measured by spacecraft in the solar wind. One of the problems that is being addressed on this new light is the temperature anisotropy of solar wind protons and electrons. In the literature, the general treatment for waves excited by (bi-)Maxwellian plasmas is well-established. However, for kappa distributions, the wave characteristics have been studied mostly for the limiting cases of purely parallel or perpendicular propagation, relative to the ambient magnetic field. Contributions to the general case of obliquely propagating electromagnetic waves have been scarcely reported so far. The absence of a general treatment prevents a complete analysis of the wave-particle interaction in kappa plasmas, since some instabilities can operate simultaneously both in the parallel and oblique directions. In a recent work, Gaelzer and Ziebell [J. Geophys. Res. **119**, 9334 (2014)] obtained expressions for the dielectric tensor and dispersion relations for the low-frequency, quasi-perpendicular dispersive Alfvén waves resulting from a kappa VDF. In the present work, the formalism is generalized for the general case of electrostatic and/or electromagnetic waves propagating in a kappa plasma in any frequency range and for arbitrary angles. An isotropic distribution is considered, but the methods used here can be easily applied to more general anisotropic distributions such as the bi-kappa or product-bi-kappa. [http://dx.doi.org/10.1063/1.4941260]

I. INTRODUCTION

A significant effort has been made in recent years on the study of the properties of plasmas composed by particles described by the so-called superthermal or kappa velocity distribution functions (κVDFs). These distributions distinguish themselves from the usual Maxwell-Bolzmann VDF by the presence of a tail (the high-velocity portion of the VDF) that decays with velocity according to a power law, instead of the Gaussian profile characteristic of the Maxwellian distribution.

Various space plasma environments, such as planetary magnetospheres, the solar corona, or the solar wind, are composed by particles whose observed VDFs are better fitted by a kappa or by combinations of kappa distributions, instead of any possible combination of Maxwellians.[1–4] The morphological distinction between a Maxwellian VDF and a kappa VDF is not a mere mathematical or observational curiosity. As a consequence of this difference, the physical processes that occur inside these environments are strongly influenced by the particular profile of the distribution and can significantly differ from the behavior one would expect from a quasi-thermal plasma.

Evidence of the importance of the particular morphology of superthermal distributions has been appearing in the literature during the last decade. Just to cite some examples, kappa distributions were employed by Viñas *et al.*[5] to provide a better description of the plasma resonances observed in Earth's magnetosphere. Recent studies concerning the quasi-thermal emission of magnetized plasmas resulting from the single-particle fluctuations have revealed distinctive differences whether the VDF is (bi-) Maxwellian or (bi-) kappa.[6–11] Simultaneously, several other studies have been conducted concerning the wave-particle resonance in Maxwellian or kappa plasmas, both in the low- and in the high-frequency regions of parallel-propagating waves.[12–22] The last couple of research subjects are relevant to the problem of the observed temperature anisotropy of both electronic and protonic populations of the solar wind. In laboratory and tokamak plasmas, kappa distributions have also been used to address discrepancies between experiments and theory when Maxwellian VDFs are employed.[23] Important problems, such as cyclotron heating and wave resonance with runaway (or superthermal) electrons, have been considered by this approach.[24,25] Finally, it has been also observed that in a dusty plasma, the excess of superthermal plasma particles not only affects the wave-resonance characteristics (dispersion relations and damping/growth rates) but also alters the resulting electrical charge of the dust particles as well.[26–31]

Some theories have been proposed to address the origin of κVDFs from a fundamental set of postulates. The most accepted explanation nowadays is based on the principle of nonadditive entropy proposed by Tsallis.[32] According to Tsallis's postulate, many-particle physical systems that evolve subjected to long-scale correlations and nonlinear effects can reach a quasi-stationary (or truly stationary) state in which the probability distribution functions of the physical

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quantities (such as the VDF) are not given by a Boltzmann distribution, but rather by probability functions that arise from the maximization of the nonadditive q-entropy proposed by Tsallis. A detailed account about the appearance of the κVDF as a consequence of the q-entropic principle and the resulting implications for space (or laboratory) plasma physics was given by Livadiotis and McComas.[33] Some of the alternative or related theories proposed in recent years are briefly mentioned in Ref. 34.

During the last generation, the yearly number of papers published on the subject of kappa distributions has been growing by a measurable exponential law.[33] In spite of the evident interest of the plasma community (mostly by space physicists) on kappa distributions, the overwhelming majority of papers concerned with the propagation and absorption/amplification of electrostatic/electromagnetic waves in superthermal magnetized plasmas has been restricted to the particular case of parallel propagation (relative to the ambient magnetic field).[35]

The more general case of obliquely propagating waves has been tackled by relatively few works so far. A basic formalism was proposed by Summers *et al.*[36] In their work, however, the results were mostly obtained by numerical integrations. A similar approach was later adopted by Basu[37] and Liu *et al.*[38] Important as these works are, the evaluation of the dielectric tensor components via numerical quadratures hampers the derivation of analytical expressions for the dispersion relations and damping/growth rates. Moreover, the analytical results obtained by the above authors were derived from power series expansions that do not account for all the possible mathematical properties of the dielectric tensor, as will be demonstrated below.

Recently, Gaelzer and Ziebell[34] proposed for the first time a mathematical formulation for dispersive Alfvén waves (DAWs), which are low-frequency, quasi-perpendicular waves propagating, in this case, in an isotropic kappa plasma. The proposed formulation obtains analytical expressions for the dielectric tensor components in terms of closed-form special functions that describe both the wave-particle resonance and the Larmor radius effects due to a superthermal VDF on the characteristics of the dispersive Alfvén waves. The formalism that was introduced in Ref. 34 rendered the problem of wave propagation more tractable, and new expressions for the dispersion relations of DAWs were in consequence obtained.

More recently, Astfalk *et al.*[39] reported the numerical implementation of the method proposed in Ref. 36 to study wave propagation at arbitrary angles in anisotropic superthermal plasmas described by a bi-kappa VDF.

Even more recently, Sugiyama *et al.*[40] implemented a dispersion equation solver for electromagnetic ion-cyclotron waves propagating at arbitrary angles in a kappa-Maxwellian plasma. From the purely mathematical point of view, the treatment of kappa-Maxwellian VDFs is simpler than isotropic kappa distributions, since in the former the dependence of the distribution function on the parallel component of the particle velocity (given by a κVDF) factors from the dependence on the perpendicular component, which is Maxwellian. Such simplification is not possible with an isotropic κVDF.

In this work, the initial formulation proposed by Gaelzer and Ziebell[34] is generalized for any number of particle species, wave frequency range, and propagation angle. A thorough and comprehensive analysis is conducted on the mathematical aspects and properties of the dielectric tensor components for an isotropic kappa distribution function, which are described by analytical, closed-form expressions. The said components are given by special functions for a kappa plasma that are either generalizations of already known expressions or are completely new definitions. As a demonstration on the feasibility of the proposed formalism, the dispersion equation is solved for high-frequency waves propagating in various angles and for several values of the κ parameter.

This paper is organized as follows. In Section II, the velocity distribution function and the dispersion equation for waves in a kappa plasma are considered. The dielectric tensor components are written in terms of (thermal) Stix parameters. Section III contains the bulk of the formalism. In there, the special functions that appear in the dielectric tensor for a kappa plasma are discussed in detail. Section IV describes a simple implementation of the formalism presented in Secs. II and III. Finally, Section V contains the conclusions. Additional and supporting materials are provided in Appendixes A–C.

II. THEORETICAL FORMULATION

A. The velocity distribution function

Calling $f_a(\boldsymbol{v})$ the VDF for the plasma species a, we will adopt in this work the isotropic kappa VDF form already introduced in Ref. 34

$$f_{\kappa,a}(\boldsymbol{v}) = \frac{1}{\pi^{3/2} w_a^3} \frac{\kappa_a^{-3/2}\Gamma(\sigma_a)}{\Gamma(\sigma_a - 3/2)} \left(1 + \frac{v^2}{\kappa_a w_a^2}\right)^{-\sigma_a}, \qquad (1)$$

which is valid when $\sigma_a > 3/2$. In (1), $\boldsymbol{v}$ is the particle's velocity, $\sigma_a = \kappa_a + \alpha_a$, where κ_a is the kappa index for the a-th species and α_a is a free real parameter, $w_a = w_a(\kappa_a)$ is another parameter with the same physical dimension and meaning as the particle's thermal speed and which depends on the κ parameter. Finally, $\Gamma(z)$ is the gamma function.

For all practical purposes, it is assumed in this work the κVDF form first proposed by Summers and Thorne,[41] which can be obtained from (1) by setting $\alpha_a = 1$ and $w_a^2 = (1 - 3/2\kappa_a)v_{Ta}^2$, where $v_{Ta}^2 = 2T_a/m_a$ is the thermal speed of species a with mass m_a and temperature T_a (in energy dimension). For any application of the general formalism presented in this work, which assumes an isotropic kappa distribution for a given species, the general form given in Eq. (1) reduces to the particular form introduced by Summers and Thorne (hereafter called the ST91 model) with the choices just given. Nevertheless, the index α_a will be kept throughout this work because it can be useful in order to extend the formalism for anisotropic κVDFs, such as the bi-kappa[42] or product-bi-kappa[42,43] models. The specific value of the α-parameter may then depend on the dimensionality of the distribution and on the correlation between the different degrees of freedom of the plasma particles.[43]

The distribution (1) contains the expected limiting form of the Maxwell-Boltzmann distribution, $f_{M,a}(\boldsymbol{v}) = \pi^{-3/2} v_{Ta}^{-3} \exp(-v^2/v_{Ta}^2)$, when $\kappa_a \to \infty$. This property is easily verified by using the exponential limit

$$\lim_{\kappa\to\infty}\left(1+\frac{y^2}{\kappa}\right)^{-\kappa} = e^{-y^2} \tag{2}$$

and Stirling's formula.[44,68] This process of limit evaluation will be dubbed here as the *Maxwellian limit*.

The ST91 form has the additional property that the *kinetic temperature* of the a-th species, defined from the second moment of the distribution by $T_{K,a} = \frac{1}{3} m_a \langle v^2 \rangle = \frac{1}{3} m_a \int d^3v\, v^2 f_{\kappa,a}(\boldsymbol{v})$, equals the thermodynamic measure of temperature, i.e., $T_{K,a} = T_a$.[33,43] This is an important property, since a single macroscopic parameter, namely, T_a, can be used to measure the velocity spread of plasma species, independent on the particular value of the κ parameter ($1/2 < \kappa < \infty$). Moreover, the ST91 form is also the most probable velocity distribution function for a kappa plasma, as obtained from Tsallis's entropic principle.[33,43] This is another important property, since it establishes a theoretical background with nonequilibrium statistical mechanics. These are the main reasons why the ST91 form is by far the most frequently employed to describe kappa plasmas.

B. The dispersion equation

Starting from the well-known expression for the dielectric tensor of a homogeneous magnetized plasma[45]

$$\varepsilon_{ij}(\boldsymbol{k},\omega) = \delta_{ij} + \sum_a \frac{\omega_{pa}^2}{\omega^2}\left[\sum_{n\to-\infty}^{\infty}\int d^3v \frac{v_\perp (\Xi_{na})_i (\Xi_{na}^*)_j \mathcal{L} f_a}{\omega - n\Omega_a - k_\parallel v_\parallel} + \delta_{iz}\delta_{jz}\int d^3v \frac{v_\parallel}{v_\perp} L f_a\right], \tag{3}$$

where $\{i,j\} = \{x, y, z\}$ identifies the Cartesian (in the E^3 space) components of ε_{ij}, with $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ being the basis vectors of E^3, $\boldsymbol{\Xi}_{na} = n\rho_a^{-1} J_n(\rho_a)\hat{\boldsymbol{x}} - iJ_n'(\rho_a)\hat{\boldsymbol{y}} + (v_\parallel/v_\perp) J_n(\rho_a)\hat{\boldsymbol{z}}$, where $J_n(z)$ is the Bessel function of the first kind,[46] $\rho_a = k_\perp v_\perp/\Omega_a$, $L f_a = v_\perp \partial f_a/\partial v_\parallel - v_\parallel \partial f_a/\partial v_\perp$, and $\mathcal{L} f_a = \omega \partial f_a/\partial v_\perp + k_\parallel L f_a$. Also, $\omega_{pa}^2 = 4\pi n_a q_a^2/m_a$ and $\Omega_a = q_a B_0/m_a c$ are the plasma and cyclotron frequencies of the a-th species, respectively, ω and $\boldsymbol{k} = k_\perp \hat{\boldsymbol{x}} + k_\parallel \hat{\boldsymbol{z}}$ are the wave frequency and wavenumber, $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$ ($B_0 > 0$) is the ambient magnetic induction field, and the symbols $\parallel(\perp)$ denote the usual parallel (perpendicular) components of vectors/tensors, respective to $\boldsymbol{B}_0$.

The wave equation in Fourier space can be written as $\Lambda_{ij}(\boldsymbol{k},\omega) E_j(\boldsymbol{k},\omega) = 0$, where the Einstein convention of implicit sum over repeated indices is adopted, $\Lambda_{ij}(\boldsymbol{k},\omega) = N_i N_j - N^2\delta_{ij} + \varepsilon_{ij}(\boldsymbol{k},\omega)$ is the dispersion tensor, and where $\boldsymbol{N} = \boldsymbol{k}c/\omega$ is the refractive index. Finally, the dispersion relations are the solutions of the dispersion equation

$$\Lambda(\boldsymbol{k},\omega) = \det(\Lambda_{ij}) = 0. \tag{4}$$

There are several known approximations and different expressions for the dispersion equation, depending on physical parameters and propagation characteristics such as wave frequency range, propagation angle, and plasma species. Since this work develops a general formulation for wave propagation in kappa plasmas, valid for any such characteristics, a general form for the dispersion equation will be employed which, albeit possibly not the more adequate for a particular situation, was nevertheless able to provide initial explicit results from the formalism.

With this objective in mind, it was found more convenient to change the reference frame from the Cartesian to a *rotated frame*, in which the usual limiting expressions for parallel or perpendicular propagation angles are readily identified. In the rotated frame, the dielectric tensor components are given by[45]

$$\begin{aligned}
&\varepsilon_{++} = \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy}) - i\varepsilon_{xy} \quad \varepsilon_{+-} = \frac{1}{2}(\varepsilon_{xx}-\varepsilon_{yy}) \\
&\varepsilon_{--} = \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy}) + i\varepsilon_{xy} \quad \varepsilon_{+\parallel} = \frac{1}{\sqrt{2}}(\varepsilon_{xz}+i\varepsilon_{yz}) \\
&\varepsilon_{\parallel\parallel} = \varepsilon_{zz} \quad \varepsilon_{-\parallel} = \frac{1}{\sqrt{2}}(\varepsilon_{xz}-i\varepsilon_{yz}),
\end{aligned} \tag{5}$$

in terms of the Cartesian components.

Finally, the rotated components can be expressed in terms of the *thermal Stix parameters* $\hat{L}$, $\hat{R}$, $\hat{P}$, $\hat{\tau}$, $\hat{\mu}$, and $\hat{\nu}$ as[45]

$$\begin{aligned}
&\varepsilon_{++} = \hat{L} - N_-^2(\hat{\tau}-1) \quad \varepsilon_{--} = \hat{R} - N_+^2(\hat{\tau}-1) \\
&\varepsilon_{\parallel\parallel} = \hat{P} \quad \varepsilon_{+-} = N_+ N_-(\hat{\tau}-1) \\
&\varepsilon_{+\parallel} = N_+ N_\parallel(\hat{\mu}-1) \quad \varepsilon_{-\parallel} = N_- N_\parallel(\hat{\nu}-1),
\end{aligned}$$

where the first three parameters are, respectively, the usual $\hat{L}$eft, $\hat{R}$ight, and $\hat{P}$lasma Stix parameters (with thermal corrections), which still exist in the cold plasma limit, whereas the last three are only present when there are thermal (kinetic) effects.[47]

Inserting now the kappa distribution function (1) into Eqs. (3) and (5), one obtains, after a fair amount of algebra, the following compact expressions for the (kappa) Stix parameters:

$$\begin{pmatrix}\hat{L}_\kappa \\ \hat{R}_\kappa\end{pmatrix} = 1 + \sum_a \frac{\omega_{pa}^2}{\omega^2}\xi_{0a}\sum_{n\to-\infty}^{\infty} n\left(\frac{n}{\mu_a} \pm \frac{\partial}{\partial\mu_a}\right)\mathcal{Z}_{n,\kappa_a}^{(\alpha_a,2)}, \tag{6a}$$

$$\hat{P}_\kappa = 1 - \sum_a \frac{\omega_{pa}^2}{\omega^2}\xi_{0a}\sum_{n\to-\infty}^{\infty}\xi_{na}\frac{\partial}{\partial\xi_{na}}\mathcal{Z}_{n,\kappa_a}^{(\alpha_a,1)}, \tag{6b}$$

$$\hat{\tau}_\kappa = 1 + \sum_a \frac{\omega_{pa}^2}{\Omega_a^2}\frac{w_a^2}{c^2}\xi_{0a}\sum_{n\to-\infty}^{\infty}\mathcal{Y}_{n,\kappa_a}^{(\alpha_a,2)}, \tag{6c}$$

$$\begin{pmatrix}\hat{\mu}_\kappa \\ \hat{\nu}_\kappa\end{pmatrix} = 1 - \frac{1}{2}\sum_a \frac{\omega_{pa}^2}{\omega\Omega_a}\frac{w_a^2}{c^2}\xi_{0a}^2 \times \sum_{n\to-\infty}^{\infty}\frac{\partial}{\partial\xi_{na}}\left(\frac{n}{\mu_a} \pm \frac{\partial}{\partial\mu_a}\right)\mathcal{Z}_{n,\kappa_a}^{(\alpha_a,1)}, \tag{6d}$$

where $\mu_a = \nu_a^2/2$, $\nu_a = k_\perp w_a/\Omega_a$, $\xi_{na} = (\omega - n\Omega_a)/k_\parallel w_a$, and where $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ are special functions that will be defined *a posteriori* in Section III C. In the same section, it will be shown that in the Maxwellian limit the functions $\mathcal{Z}$ and $\mathcal{Y}$ reduce to

$$\lim_{\kappa\to\infty} \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \mathscr{H}_n(\mu)Z(\xi)$$
$$\lim_{\kappa\to\infty} \mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \mathscr{H}'_n(\mu)Z(\xi), \tag{7}$$

where $Z(\zeta)$ is the usual plasma dispersion (or Fried and Conte) function, given below by definition (10), and

$$\mathscr{H}_n(z) = e^{-z}I_n(z) \tag{8}$$

is called here the (Maxwellian) *plasma gyroradius function*, with $I_n(z)$ being the modified Bessel function.[46] The expressions for the Stix parameters given in (6) in the Maxwellian limit reduce to the well-known expressions that can be found, e.g., in Ref. 45.

It is worth mentioning here that with the definition of the special functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}$, the Stix parameters for a kappa plasma can be expressed in a form as compact as the correspondent Maxwellian expressions. It will be also shown in Section III C that these functions can always be evaluated from analytical, closed-form expressions that do not depend on any residual numerical integration. Therefore, with the formulation developed in this work, the evaluation of (6a)–(6d) can be accomplished in a computational timeframe comparable to the usual Maxwellian limit, with no increased overhead due to lengthy numerical quadratures. Additionally, the analytical closed-form expressions obtained here simplify the determination of mathematical properties of the special functions that arise from the kappa distribution (1), which in turn allows the derivation of adequate approximations to the dielectric tensor and dispersion relations, relevant to a particular wave propagation regime.

Finally, using the Stix parameters given in Eqs. (6a)–(6d), the dispersion equation (4) can be cast in the following form:[45]

$$H_X(\boldsymbol{k},\omega)H_O(\boldsymbol{k},\omega) + \frac{1}{2}N_\perp^2 N_\parallel^2 K(\boldsymbol{k},\omega) = 0, \tag{9a}$$

where

$$H_X(\boldsymbol{k},\omega) = \left(\hat{L} - N_\parallel^2\right)\left(\hat{R} - N_\parallel^2\right) - N_\perp^2\left[\frac{1}{2}\left(\hat{L}+\hat{R}\right) - N_\parallel^2\right]\hat{\tau}, \tag{9b}$$

$$H_O(\boldsymbol{k},\omega) = \hat{P} - N_\perp^2, \tag{9c}$$

$$K(\boldsymbol{k},\omega) = \frac{1}{2}N_\perp^2(\hat{\nu}+\hat{\mu})^2\hat{\tau} - \left(\hat{L} - N_\parallel^2\right)\hat{\nu}^2 - \left(\hat{R} - N_\parallel^2\right)\hat{\mu}^2. \tag{9d}$$

Equation (9a) is convenient because it reduces to simple forms for limiting propagation angles. For parallel propagation ($k_\perp = 0$), it factors into the equations for the left- and right-handed circularly polarized modes and the longitudinal (or plasma) mode. On the other hand, for perpendicular propagation ($k_\parallel = 0$), Eq. (9a) factors into the equations for the extraordinary and ordinary modes.

For the sake of completeness, and should the necessity arise, the corresponding expressions for the Cartesian components of the dielectric tensor for a kappa plasma are given in Appendix C.

In Sec. III, the definitions and mathematical properties of the various special functions that appear in the treatment of a kappa plasma are discussed.

III. THE SPECIAL FUNCTIONS FOR A KAPPA PLASMA

The Stix parameters for a kappa plasma, shown in Eqs. (6a)–(6d), are given in terms of the two-variable special functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$. Before the proper definitions for these functions can be made, it is necessary to give prior definitions and discuss the properties of two related one-variable functions, namely, the *superthermal plasma dispersion function* and the *superthermal plasma gyroradius function*. The most important properties will be shown in this section, with additional properties given in Appendix A.

A. The superthermal plasma dispersion function

This special function is the equivalent, for a kappa plasma, to the well-known plasma dispersion (Fried and Conte) function, defined by[45,48]

$$Z(\xi) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{y-\xi}, \quad (\text{for } \xi_i > 0), \tag{10}$$

where ξ_i is the imaginary part of ξ.

The definition of the superthermal plasma dispersion function (κPDF) follows from the generalized expression adopted in this work for the distribution function, Eq. (1), and is given by

$$Z_\kappa^{(\alpha,\beta)}(\xi) = \frac{1}{\pi^{1/2}\kappa^{\beta+1/2}}\frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)} \times \int_{-\infty}^{\infty} ds\,\frac{\left(1+s^2/\kappa\right)^{-(\lambda-1)}}{s-\xi}, \quad \begin{pmatrix}\xi_i > 0\\ \lambda > 1\end{pmatrix}. \tag{11}$$

The function $Z_\kappa^{(\alpha,\beta)}(\xi)$ was first defined by Gaelzer and Ziebell.[34] The parameter α is the same as it appears in (1), whereas β is another real parameter. Moreover, $\lambda = \sigma + \beta$ ($\sigma = \kappa + \alpha$). It is also noteworthy that the definition (11) is valid for $\xi_i > 0$, as is the definition of the Fried and Conte function, but the integrand of $Z_\kappa^{(\alpha,\beta)}(\xi)$ has additional branch points at $s = \pm i\sqrt{\kappa}$, when λ is noninteger.

Again, for an isotropic κVDF, one can simply adopt the ST91 form, set $\alpha = 1$ and erase all reference to the parameter α. However, the new parameter β should be kept, because its value is related to the wave polarization. For instance, using the ST91 form and setting $\beta = 1$, Eq. (11) reduces to the original function $Z_\kappa^*(\xi)$ employed by Summers and Thorne[41] and Mace and Hellberg[49] for a kappa (Lorentzian) plasma. It is interesting to mention that the function $Z_\kappa^*(\xi)$ appears in the dispersion equation for longitudinal (Langmuir, ion-sound) waves propagating in a kappa plasma along the ambient magnetic field.

As another example, setting $\beta = 0$ in (11), one obtains exactly the function $Z_{\kappa M}(\xi)$ defined in Ref. 50 and related to parallel-propagating electromagnetic waves in a kappa-Maxwellian plasma, or the function $Z_\kappa^0(g)$ defined in Ref. 12, also related to circularly polarized waves propagating in a kappa plasma.

Hence, with the definition for the function $Z_\kappa^{(\alpha,\beta)}(\xi)$ given by Eq. (11), one can describe the propagation of different waves in a magnetized kappa plasma with a single expression, in which case the parameter β will be related to the wave polarization and other characteristics, as will be shown below.

Several mathematical properties of $Z_\kappa^{(\alpha,\beta)}(\xi)$ will be derived in this section and in Appendix A. Some of the results shown here are generalizations of properties previously obtained in various works found in the literature.[41,42,49–54]

Particular values: Direct integration of (11) provides special values of $Z_\kappa^{(\alpha,\beta)}(\xi)$ at special points.

Value at $\xi = 0$: At this point,

$$Z_\kappa^{(\alpha,\beta)}(0) = \frac{1}{\pi^{1/2}\kappa^{1/2+\beta}}\frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)} \times \int_{-\infty}^{\infty} ds\, s^{-1}\left(1+s^2/\kappa\right)^{-(\lambda-1)},$$

where the integration must be done following Landau prescription. This means that it is possible to employ Plemelj formula to evaluate the integral, which results equal to $i\pi$. Therefore,

$$Z_\kappa^{(\alpha,\beta)}(0) = \frac{i\sqrt{\pi}\,\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}, \quad (\lambda > 1). \tag{12}$$

At the Maxwellian limit, one obtains $Z_\kappa^{(\alpha,\beta)}(0) \xrightarrow{\kappa\to\infty} i\sqrt{\pi} = Z(0)$ as expected.

Values at $\xi = \pm i\sqrt{\kappa}$: At these points, it can be shown that for $\lambda > 1$,

$$\int_{-\infty}^{\infty} ds \frac{\left(1+s^2/\kappa\right)^{1-\lambda}}{s \mp i\sqrt{\kappa}} = \pm\frac{2i}{\sqrt{\kappa}}\int_0^\infty ds\left(1+\frac{s^2}{\kappa}\right)^{-\lambda}.$$

Since the remaining integral is a special case of Euler's Beta integral,[44] then

$$Z_\kappa^{(\alpha,\beta)}\left(\pm i\sqrt{\kappa}\right) = \pm i\frac{\kappa^{-\beta-1/2}\Gamma(\lambda-1/2)}{(\lambda-1)\Gamma(\sigma-3/2)}.$$

Representations for $Z_\kappa^{(\alpha,\beta)}(\xi)$: The κPDF has an already well-known representation in terms of the Gauss hypergeometric function ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)$, defined in Eq. (B4). This representation was first derived in Ref. 49 via an elegant application of the residue theorem. An alternative and equivalent derivation was already used in Refs. 26 and 34 and will be employed again here.

First, by means of the variable transformation $t^{-1} = 1 + s^2/\kappa$, the integral in (11) becomes

$$Z_\kappa^{(\alpha,\beta)}(\xi) = \frac{\xi}{\pi^{1/2}\kappa^{\beta+1}}\frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)} \times \int_0^1 t^{\lambda-3/2}(1-t)^{-1/2}\left[1-\left(1+\frac{\xi^2}{\kappa}\right)t\right]^{-1} dt.$$

Identifying with the integral representation (B5), one can write

$$Z_\kappa^{(\alpha,\beta)}(\xi) = \frac{\kappa^{-\beta-1}\Gamma(\lambda-1/2)}{(\lambda-1)\Gamma(\sigma-3/2)}\xi \times {}_2F_1\left(\begin{matrix}1,\lambda-\frac{1}{2}\\\lambda\end{matrix};1+\frac{\xi^2}{\kappa}\right), \quad \left(\lambda > \frac{1}{2}\right). \tag{13}$$

Although a valid representation of $Z_\kappa^{(\alpha,\beta)}(\xi)$, its principal branch is restricted to the sector $0 < \arg\xi \leq \pi$ (i.e., to $\xi_i > 0$). Hence, this is not the adequate representation when $\xi_i \leq 0$, in other words, it does not obey the Landau prescription. Nevertheless, if one employs the analytic continuation formula (B6), a valid representation is obtained, which is shown in Eq. (A1).

For plasma physics applications, one is interested in a mathematical representation of $Z_\kappa^{(\alpha,\beta)}(\xi)$ that satisfies the Landau prescription, i.e., is continuous at the limit $\xi_i \to 0$. Moreover, it is also required that the sought representation lends itself to the numerical evaluation of $Z_\kappa^{(\alpha,\beta)}(\xi)$, as carried out by computer programming languages such as `Fortran`, `C/C++`, or `python`, and/or by computer algebra software. Such representations are called in this work *computable representations* of the κPDF. Expression (13) does not satisfy this requisite, but it can be used within convenient transformations for the Gauss function, thereby rendering other representations which are indeed computable.

One such representation is obtained by inserting (13) into the quadratic transformation (B7d), resulting

$$Z_\kappa^{(\alpha,\beta)}(\xi) = i\frac{\kappa^{-\beta-1/2}\Gamma(\lambda-1/2)}{(\lambda-1)\Gamma(\sigma-3/2)} \times {}_2F_1\left[\begin{matrix}1,2(\lambda-1)\\\lambda\end{matrix};\frac{1}{2}\left(1+\frac{i\xi}{\kappa^{1/2}}\right)\right]. \tag{14}$$

Setting $\alpha = \beta = 1$, representation (14) of $Z_\kappa^{(1,1)}(\xi)$ is exactly the original result obtained in Ref. 49. On the other hand, with $\alpha = 1$ and $\beta = 0$, there results again the function $Z_{\kappa M}(\xi)$ obtained in Ref. 50. Hence, result (14) generalizes these well-known representations.

It is important to mention at this point that although the form (14) is indeed continuous across the real line of ξ, the branch cut has not disappeared. It has just moved to the line $-\sqrt{\kappa} \geq \xi_i > -\infty$. Therefore, it would be necessary to evaluate the analytical continuation of (14), should the branch line ever been crossed during the dynamical evolution of the waves in the plasma. The same caveat applies to all computable representations of $Z_\kappa^{(\alpha,\beta)}(\xi)$ found in this work.

Expression (14) is well-suited for computational purposes, however, other forms are more convenient to derive further analytical expression for $Z_\kappa^{(\alpha,\beta)}(\xi)$. A representation that directly renders a series expansion for the κPDF is obtained by inserting now (13) into (B7b), resulting in

$$Z_\kappa^{(\alpha,\beta)}(\xi) = -\frac{2\Gamma(\lambda-1/2)\xi}{\kappa^{\beta+1}\Gamma(\sigma-3/2)}\,{}_2F_1\left(\begin{matrix}1,\lambda-1/2\\3/2\end{matrix};-\frac{\xi^2}{\kappa}\right) + \frac{i\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-1)}, \quad (15)$$

also valid for $\lambda > 1/2$ and where identity (B9) was also used. This form immediately renders the power series expansion, given in (A2).

The representation (15) is also important because its Maxwellian limit reduces to a known representation of the Fried and Conte function. This fact can be verified by applying the limit $\kappa \to \infty$ to (15), employing both the exponential limit (2) and the Stirling formula, obtaining thus

$$Z_\kappa^{(\alpha,\beta)}(\xi) \xrightarrow{\kappa\to\infty} -2\xi\,{}_1F_1\left(\begin{matrix}1\\3/2\end{matrix};-\xi^2\right) + i\pi^{1/2}e^{-\xi^2}.$$

The function ${}_1F_1\left(\begin{matrix}1\\3/2\end{matrix};-\xi^2\right)$, according to (B2), is a particular case of the Kummer confluent hypergeometric function, and this result is another known representation for the Fried and Conte function.[55]

The last representation for $Z_\kappa^{(\alpha,\beta)}(\xi)$ to be shown in this section is obtained from (15) with the use of (B7c), resulting in

$$Z_\kappa^{(\alpha,\beta)}(\xi) = -\frac{\Gamma(\lambda-3/2)\xi^{-1}}{\kappa^{\beta}\Gamma(\sigma-3/2)}\,{}_2F_1\left(\begin{matrix}1,1/2\\5/2-\lambda\end{matrix};-\frac{\kappa}{\xi^2}\right) + [i-\tan(\lambda\pi)]\frac{\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{1-\lambda}, \quad (16)$$

which is now valid for $\lambda > 3/2$ and $\lambda \neq m + 3/2$, where $m = 1, 2, \ldots$. This form is well-suited to obtain an asymptotic approximation for $Z_\kappa^{(\alpha,\beta)}(\xi)$, shown in Appendix A. However, now there are two additional restrictions. First, when λ is semi-integer, each term has a singularity. Fortunately, they cancel out and in Appendix A it is also shown the expression for (16) when $\lambda = m + 3/2$. The other restriction appears due to the fact that now the branch line is stretched along the whole imaginary axis $-\infty < \xi_i < \infty$.

Recurrence relations: Some important recurrence relations for $Z_\kappa^{(\alpha,\beta)}(\xi)$ can be obtained. Performing integrations by parts in (11), one obtains

$$\left(1+\frac{\xi^2}{\kappa}\right)Z_\kappa^{(\alpha+1,\beta)}(\xi) = \frac{\lambda-1}{\sigma-3/2}Z_\kappa^{(\alpha,\beta)}(\xi) - \frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+1}\Gamma(\sigma-1/2)}\xi, \quad (17a)$$

$$\left(1+\frac{\xi^2}{\kappa}\right)Z_\kappa^{(\alpha,\beta+1)}(\xi) = \frac{\lambda-1}{\kappa}Z_\kappa^{(\alpha,\beta)}(\xi) - \frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+2}\Gamma(\sigma-3/2)}\xi. \quad (17b)$$

Derivatives: Deriving (11) with respect to ξ and then integrating by parts, one obtains the first derivative

$$Z_\kappa^{(\alpha,\beta)\prime}(\xi) = -2\left[\frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+1}\Gamma(\sigma-3/2)} + \xi Z_\kappa^{(\alpha,\beta+1)}(\xi)\right]. \quad (18a)$$

The Maxwellian limit of (18a) is the well-known formula[48] $Z'(\xi) = -2[1+\xi Z(\xi)]$.

Applying now the operator $d^n/d\xi^n$ on (11) and integrating by parts, one arrives at a first recurrence relation for the derivatives

$$Z_\kappa^{(\alpha,\beta)(n+1)}(\xi) = -2\left[\xi Z_\kappa^{(\alpha,\beta+1)(n)}(\xi) + nZ_\kappa^{(\alpha,\beta+1)(n-1)}(\xi)\right] = -2\frac{d^n}{d\xi^n}\left[\xi Z_\kappa^{(\alpha,\beta+1)}(\xi)\right], \quad (n\geqslant 1). \quad (18b)$$

The Maxwellian limit of (18b) is also a known result.[56]

Expressions (18a) and (18b) are useful, but they relate Z_κ functions with different values of β. In order to obtain the same-β expressions for the derivatives, one must first return to (18a) and insert (17b) to obtain

$$\left(1+\frac{\xi^2}{\kappa}\right)Z_\kappa^{(\alpha,\beta)\prime}(\xi) = -2\left[\frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+1}\Gamma(\sigma-3/2)} + \frac{\lambda-1}{\kappa}\xi Z_\kappa^{(\alpha,\beta)}(\xi)\right]. \quad (18c)$$

Deriving now both sides $n \geq 1$ times with respect to ξ, using Leibniz formula for the derivative,[57] and reorganizing, there results

$$\left(1+\frac{\xi^2}{\kappa}\right)Z_\kappa^{(\alpha,\beta)\prime\prime}(\xi) = -2\left[\frac{\lambda-1}{\kappa}Z_\kappa^{(\alpha,\beta)}(\xi) + \xi Z_\kappa^{(\alpha,\beta)\prime}(\xi)\right], \quad (18d)$$

$$\left(1+\frac{\xi^2}{\kappa}\right)Z_\kappa^{(\alpha,\beta)(n+1)}(\xi) = -2\left[\frac{\lambda+n-1}{\kappa}\xi Z_\kappa^{(\alpha,\beta)(n)}(\xi) + n\left(\frac{\lambda}{\kappa}+\frac{n-3}{2\kappa}\right)Z_\kappa^{(\alpha,\beta)(n-1)}(\xi)\right], \quad (n\geqslant 2). \quad (18e)$$

A general representation for the n-th derivative of $Z_\kappa^{(\alpha,\beta)}(\xi)$ is also obtainable. Applying the operator $d^n/d\xi^n$ on the form (14) and using the formula (B8), the following expression if found:

$$\frac{Z_\kappa^{(\alpha,\beta)(n)}(\xi)}{i^{n+1}n!} = \frac{\Gamma(\lambda+n/2-1)\Gamma(\lambda+n/2-1/2)}{\kappa^{\beta+(n+1)/2}\Gamma(\sigma-3/2)\Gamma(\lambda+n)} \times {}_2F_1\left[\begin{matrix}n+1, 2(\lambda-1)+n\\\lambda+n\end{matrix};\frac{1}{2}\left(1+\frac{i\xi}{\kappa^{1/2}}\right)\right]. \quad (19)$$

Another representation is given in (A3).

B. The superthermal plasma gyroradius function

In the same gist that the plasma dispersion function quantifies the (linear) wave-particle interactions in a finite-temperature plasma, leading to substantial effects on wave dispersion and absorption, the *plasma gyroradius function* (PGF), so called in this work, affects wave propagation at arbitrary angles when the particle's cyclotron (or Larmor) radius is finite.

In a thermal, Maxwellian plasma, the gyroradius function has a well-known and simple form, given in Eq. (8). In a kappa plasma, on the other hand, despite the large number of papers already present in the literature, the corresponding κPGF was not categorized and systematically studied until Ref. 34 proposed a first definition and derived some basic properties. In this section, a more systematic study of the κPGF is made, thereby complementing the initial formulation given in Ref. 34.

Definition and basic properties: The kappa (or superthermal) PGF is defined by the integral

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = 2\int_0^\infty dx\, \frac{xJ_n^2(yx)}{(1+x^2/\kappa)^{\lambda-1}}, \quad (y^2 = 2z), \tag{20}$$

where the parameters α, β, and λ are the same as for the κPDF, whereas $n = \ldots, -2, -1, 0, 1, 2, \ldots$ is the harmonic (of the gyrofrequency) number. For plasma physics applications, the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ describes the effects of finite particle gyroradius, and thus the argument will be $z = \mu_a$, where $\mu_a = k_\perp^2\rho_a^2$, with ρ_a being the said Larmor radius.

The definition (20) is slightly different from the definition given in Ref. 34, but the significance and importance are the same.

At the Maxwellian limit, if one employs identity (2), the remaining integral can be found in any table of mathematical formulae, and one readily obtains (8). Here, this limit will be demonstrated by the general representation of the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, which will be given below.

The first property to be derived is the value at the origin ($z = 0$). At that point, direct integration shows that[46]

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(0) = \frac{\kappa}{\lambda-2}\delta_{n,0}, \quad (\lambda > 2), \tag{21}$$

where the definition of the beta function[44] was also employed, and where $\delta_{n,m}$ is the Kronecker delta.

Since $J_{-n}(z) = (-)^n J_n(z)$ for integer n, another straightforward property is $\mathcal{H}_{-n,\kappa}^{(\alpha,\beta)}(z) = \mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, which is shared by the function $\mathscr{H}_n(z)$ as well.

Representations for $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$: When encumbered with the task of finding a computable representation for $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, one could argue, quite naturally, that since the function depends on the particle gyroradius, and since in many practical applications this quantity can be considered small, a "first-order" approximation for $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ could be obtained by first expanding $J_n^2(z)$ in a power series, and then evaluating the resulting integrals for the first few terms. This is an usual procedure to obtain a small-Larmor-radius approximation for the function $\mathscr{H}_n(z)$.

Applying this method blindly to the κPGF, however, one would invariably obtain a result that either imposes an artificially high constraint on the value of the κ parameter, or is plainly incorrect, from both the mathematical and physical points of view. The reason for this unfortunate outcome is simple. Considering an arbitrary value for the harmonic number n, the lowest-order approximation for the Bessel function is[46]

$$J_n^2(z) \simeq \frac{1}{(|n|!)^2}\left(\frac{z}{2}\right)^{2|n|}.$$

Inserting this approximation into (20), one would obtain, to the lowest order,

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) \simeq \frac{\kappa\Gamma(\lambda-|n|-2)}{2^{|n|}|n|!\Gamma(\lambda-1)}(\kappa z)^{|n|},$$

which only exists for $\lambda > |n| + 2$. Higher-order terms can be included from the series for $J_n^2(z)$, with the outcome that the term of order $k > 0$ will be proportional to $\Gamma(\lambda - |n| - k - 2)$, thereby imposing the even stricter constraint $\lambda > |n| + k + 2$.

Consequently, it would appear as if for a plasma where Larmor radius effects are important up to order $K \geq 0$ and thermal effects demand the inclusion of up to $N \geq 0$ harmonics, the particles in such a plasma could only have a VDF with $\lambda > N + K + 2$, where the lowest possible value is $\lambda > 2$, the constraint already imposed in Eq. (21). Moreover, the minimum value for the parameter κ, resulting from this constraint, would be linked with the harmonic number n. This is an undesirable and altogether unobserved restriction to the allowable values for the κ index, which has been measured to be as low as $\kappa \simeq 2$, both in the solar wind and in the magnetosphere.[1,3,4]

On the other hand, recalling that the asymptotic behavior of the Bessel function for large argument is[46]

$$J_n^2(z) \sim \frac{2}{\pi z}\cos^2\left(z - \frac{1}{2}|n|\pi - \frac{1}{4}\pi\right) \leq \frac{2}{\pi z},$$

the ensuing asymptotic behavior of the integration in (20) is

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) \sim \frac{\Gamma(\lambda-3/2)}{\Gamma(\lambda-1)}\sqrt{\frac{2\kappa}{\pi z}},$$

showing the constraint $\lambda > 3/2$, independent on the harmonic number and near the observed values of κ. Hence, the exact value of the integral in (20) must always exist for any harmonic number n, and the correct constraint imposed over κ must be independent of n and close to the small values observed in the interplanetary environment.

Additionally, the assumption that the kappa gyroradius function can be somehow expressed as a simple power series, as is the case with the function $\mathscr{H}_n(z)$, is simply incorrect. As it will be shown below, for certain values of the λ (or κ) parameter, the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ displays a logarithmic behavior, not present in the Maxwellian PGF. It will also be shown that even when $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ is represented by a power series, the lowest-order terms contain a contribution proportional to a noninteger power. These terms are completely overlooked by the simplified approach described above. As a consequence, even if one could argue that the κ parameter is small enough to justify a power-series approximation from the outset, the result would be incorrect due to the lacking terms.

Therefore, it is the contention in this work, as it was in Ref. 34, that approximate expressions for the kappa gyroradius function can only be sought after an exact, closed-form representation is found. It is that representation that will provide the mathematically (and physically) correct approximation for the function.

Starting again from definition (20), the gyroradius function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ will now be represented in terms of the Meijer G-function discussed in Appendix B 2. First, inserting representation (B15c) into (20), one can write

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\kappa^{\lambda-1}}{\sqrt{\pi}}\int_0^\infty dw\, \frac{G_{1,3}^{1,1}\left[2zw\Big|_{n,-n,0}^{1/2}\right]}{(\kappa+w)^{\lambda-1}},$$

after a simple transformation of variable. Identifying this integral in formula (B12) and using identity (B11b), there results

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)}G_{1,3}^{2,1}\left[2\kappa z\Big|_{\lambda-2,n,-n}^{1/2}\right]. \tag{22}$$

Formula (22) is the exact, closed-form representation of the kappa plasma gyroradius function. According to the definition of the G-function, the only restriction imposed in (22) is $\lambda \neq 3/2,\ 1/2, -1/2, \ldots$. Hence, the overall constraint imposed over λ is still determined in (21), namely, $\lambda > 2$. For the ST91 form, this implies that $\kappa > 0$ or 1, depending solely on the parameter β and quite within the measured values for κ in the solar wind VDFs.

Using formula (22) and the definition (B10), one obtains another proof of the Maxwellian limit. Accordingly,

$$\begin{aligned}\lim_{\kappa\to\infty}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) &= \frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_L ds\frac{\Gamma(n-s)\Gamma(1/2+s)}{\Gamma(1+n+s)}\\ &\quad\times\lim_{\kappa\to\infty}\left[\frac{\kappa\Gamma(\lambda-2-s)}{\Gamma(\lambda-1)}(2\kappa z)^s\right]\\ &= \frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_L\frac{\Gamma(n-s)\Gamma(1/2+s)}{\Gamma(1+n+s)}(2z)^s ds\\ &= \frac{1}{\sqrt{\pi}}G_{1,2}^{1,1}\left[2z\Big|_{n,-n}^{1/2}\right].\end{aligned}$$

Identifying with (B15d) and (8), one concludes that indeed $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) \overset{\kappa\to\infty}{\longrightarrow} \mathscr{H}_n(z)$.

Looking now for computable representations, the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ is divided in two cases, depending on the nature of parameter λ: integer or noninteger. The second case is handled first. If $\lambda-2-n$ is noninteger, then formula (B14) is valid and one obtains

$$\begin{aligned}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)}&\left[\frac{\Gamma(n+2-\lambda)\Gamma(\lambda-3/2)}{\Gamma(\lambda-1+n)}\right.\\ &\times(2\kappa z)^{\lambda-2}{}_1F_2\left(\begin{matrix}\lambda-3/2\\ \lambda-1-n,\lambda-1+n\end{matrix};2\kappa z\right)\\ &+\frac{\Gamma(\lambda-n-2)\Gamma(n+1/2)}{\Gamma(2n+1)}(2\kappa z)^n\\ &\left.\times{}_1F_2\left(\begin{matrix}n+1/2\\ n+3-\lambda,2n+1\end{matrix};2\kappa z\right)\right],\end{aligned} \tag{23}$$

where the ${}_1F_2\left(\begin{matrix}a\\ b,c\end{matrix};z\right)$ function is defined in (B3).

Two important observations about (23) can be made. First, one can clearly observe that if λ is integer, either term will always contain a singularity. Fortunately, they cancel out and the result can be written in terms of known functions. This case will be treated below. The other observation is that when λ is not integer, if one writes (23) explicitly as a power series, as shown in Eq. (A4), each term in the expansion will be proportional to a noninteger power of z.

The second case occurs when λ is integer, i.e., $\lambda = 2+k$ $(k=1,2,\ldots)$. In order to show that now the κPGF is also represented by known functions, the limit $\lambda=2$ will be temporarily considered in (22), in which case formula (B15e) shows that

$$\begin{aligned}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) &\overset{(\lambda=2)}{=}\frac{\kappa}{\sqrt{\pi}}G_{1,3}^{2,1}\left[2\kappa z\Big|\begin{matrix}1/2\\ 0,n,-n\end{matrix}\right]\\ &= 2\kappa K_n\left(\sqrt{2\kappa z}\right)I_n\left(\sqrt{2\kappa z}\right),\end{aligned}$$

where $K_n(z)$ is the second modified Bessel function.[46] This result shows clearly that in this case the function has a logarithmic singularity since, for $z\approx 0$,[46]

$$K_n(z) \simeq \frac{1}{2}\Gamma(n)\left(\frac{z}{2}\right)^{-n} + (-)^n\ln\left(\frac{z}{2}\right)I_n(z).$$

Now, in order to obtain the expression for physical values of $\lambda = 3, 4, \ldots$, one must take into account the formula (B13), the Leibniz differentiation formula,[57] and the identities[58]

$$\begin{aligned}\frac{\partial^n}{\partial z^n}\left[z^{\pm\nu/2}I_\nu(a\sqrt{z})\right] &= \left(\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}I_{\nu\mp n}(a\sqrt{z})\\ \frac{\partial^n}{\partial z^n}\left[z^{\pm\nu/2}K_\nu(a\sqrt{z})\right] &= \left(-\frac{a}{2}\right)^n z^{(\pm\nu-n)/2}K_{\nu\mp n}(a\sqrt{z}).\end{aligned}$$

After some simple algebra, one obtains

$$\begin{aligned}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) &= \frac{2\kappa}{\Gamma(\lambda-1)}\left(\frac{\kappa z}{2}\right)^{\lambda/2-1}\sum_{s=0}^{\lambda-2}(-)^s\binom{\lambda-2}{s}\\ &\quad\times K_{n-(\lambda-2)+s}\left(\sqrt{2\kappa z}\right)I_{n+s}\left(\sqrt{2\kappa z}\right),\end{aligned} \tag{24}$$

which is valid for $\lambda = 2+k$ $(k=1,2,\ldots)$.

With identities (23) and (24), all possible values for the parameters and the argument are covered and adequate approximations for $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ can be obtained. As an example, in Ref. 34 dispersion relations for dispersive Alfvén waves propagating in a kappa plasma were derived from suitable approximations to the κPGF. From the computational point of view, representations (23) and (24) also come in handy. The Meijer G-function is supported by some computer algebra software and also by the `mpmath` library.[59] However, for computationally intensive applications, codes written in `Fortran` or `C/C++` are better suited.

In order to evaluate the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, a special code written in `Modern Fortran`[60] was developed. A complete description of the code structure will not be given here. Suffice it to say, that for the case λ integer, given in (24), library routines that evaluate the modified Bessel functions were employed, whereas for noninteger λ, although the series (23) formally converges for any $0 < z < \infty$, rounding errors corrupt the accuracy of the result for $2\kappa z \gtrsim 10$ and other strategies are needed. The code that was written evaluates $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ in roughly the same time-frame taken by any library routine that evaluates a transcendental function, with an accuracy of the order of the machine precision.

Figure 1 shows some plots of the function $\mathcal{H}_{n,\kappa}^{(0,0)}(z)$, evaluated with the code that was developed. The Maxwellian form $\mathscr{H}_n(z)$ is clearly reached for $\kappa \gg 1$ and the departure from the Maxwellian increases as κ diminishes, as expected. The plots also show that the greatest relative departure occurs for small z ($z \lesssim 1$), implying that the effect of the κVDF is more pronounced on small-gyroradius particles.

A situation where rounding errors could be important in the developed code might occur when λ is in the vicinity of an integer, when the code still evaluates formula (23), but near the poles of the gamma functions. However, for the test cases considered, the code demonstrated a robust performance, as illustrated in Fig. 2, which shows plots of $\mathcal{H}_{n,\kappa}^{(0,0)}(1/2)$ varying κ, for some values of n. The graph shows that the code provides smooth results for all $5/2 \le \kappa \le 50$.

Derivative and recurrence relations: As the last mathematical properties for the κPGF, their recurrence relations will be deducted now.

Starting from the representation (22), if one writes down the explicit Mellin-Barnes integral from (B10) and evaluates the derivative of the argument, one obtains

$$\frac{d}{dy}G_{1,3}^{2,1}\left[y\middle|\begin{matrix}1/2\\ \lambda-2,n,-n\end{matrix}\right] = \frac{y^{-1}}{2\pi i}\int_L \frac{\Gamma(\lambda-2-s)\Gamma(n-s)\Gamma(1/2+s)}{\Gamma(n+1+s)} s y^s ds.$$

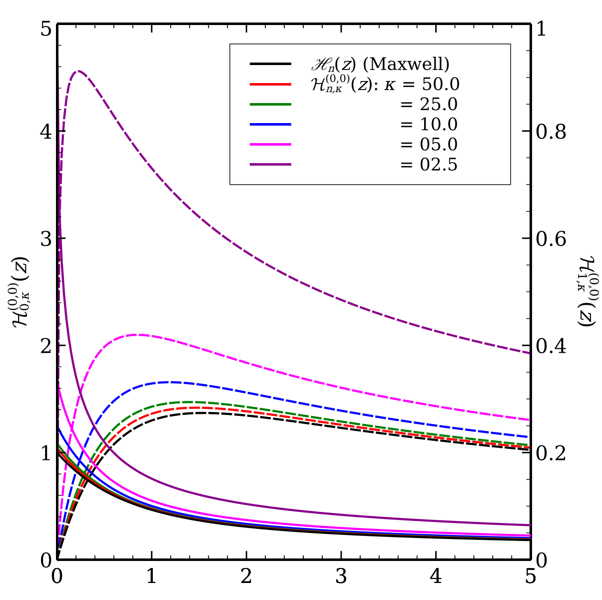

FIG. 1. Plots of function $\mathcal{H}_{n,\kappa}^{(0,0)}(z)$ for $n=0$ (full lines) and $n=1$ (dashed lines) and several values of the κ parameter. The limiting form $\mathscr{H}_n(z)$ for $\kappa \to \infty$ is also included.

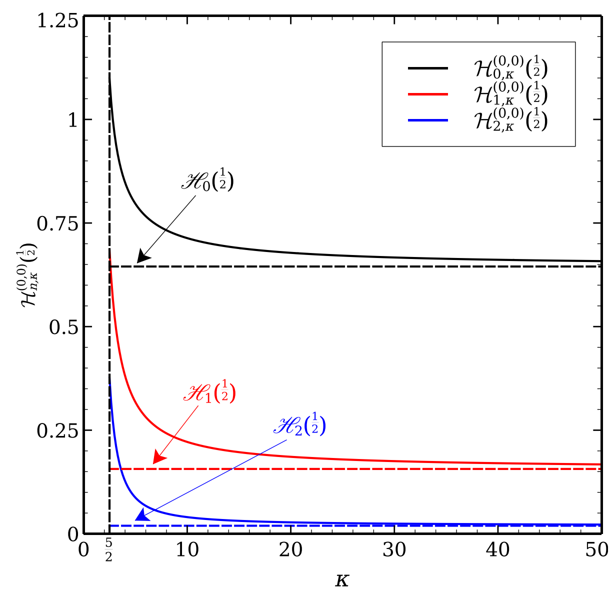

FIG. 2. Plots of $\mathcal{H}_{n,\kappa}^{(0,0)}(1/2)$ as a function of κ for $n = 0, 1, 2$. The limit value $\mathscr{H}_n(1/2)$ is also shown.

Since $\Gamma(n-s) = (n-s)^{-1}\Gamma(n+1-s)$, adding and subtracting n in the integrand above and then changing the integration variable as $s \to s+1$, there results

$$\begin{aligned}\frac{d}{dy}G_{1,3}^{2,1}&\left[y\middle|\begin{matrix}1/2\\ \lambda-2,n,-n\end{matrix}\right]\\ &= ny^{-1}G_{1,3}^{2,1}\left[y\middle|\begin{matrix}1/2\\ \lambda-2,n,-n\end{matrix}\right]\\ &-\frac{1}{2\pi i}\int_L \frac{\Gamma(\lambda-3-s)\Gamma(n-s)\Gamma(3/2+s)}{\Gamma(n+2+s)} y^s ds.\end{aligned}$$

Finally, writing $1 = (n+1+s)-(n-s)$, reorganizing the terms, and identifying the result back to (22), one obtains

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)\prime}(z) = \frac{n}{z}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) - \frac{\kappa}{\lambda-2}\left[\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)}(z) - \mathcal{H}_{n+1,\kappa}^{(\alpha,\beta-1)}(z)\right], \tag{25a}$$

which has the constraint $\lambda > 3$ imposed.

Going back now to the first expression for the derivative, employing the same identity for the gamma function, but now cancelling the denominator, one obtains in a similar way

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)\prime}(z) = -\frac{n}{z}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) - \frac{\kappa}{\lambda-2}\left[\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)}(z) - \mathcal{H}_{n-1,\kappa}^{(\alpha,\beta-1)}(z)\right], \tag{25b}$$

which has the same condition $\lambda > 3$.

Finally, by adding and subtracting (25a) and (25b), one obtains the additional relations

$$\begin{aligned}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)\prime}(z) &= -\frac{\kappa}{\lambda-2}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)}(z)\\ &+\frac{1}{2}\frac{\kappa}{\lambda-2}\left[\mathcal{H}_{n-1,\kappa}^{(\alpha,\beta-1)}(z) + \mathcal{H}_{n+1,\kappa}^{(\alpha,\beta-1)}(z)\right],\end{aligned} \tag{25c}$$

and

$$2\frac{n}{z}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\kappa}{\lambda-2}\left[\mathcal{H}_{n-1,\kappa}^{(\alpha,\beta-1)}(z) - \mathcal{H}_{n+1,\kappa}^{(\alpha,\beta-1)}(z)\right]. \quad (25d)$$

In the Maxwellian limit, the recurrence relations (25a)–(25d) will reduce to the corresponding expressions for $\mathscr{H}_n(z)$ and $\mathscr{H}'_n(z)$ that can be obtained from the recurrence relations of the modified Bessel function.[46]

Further properties of the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ are given in Appendix A.

C. The two-variable kappa plasma functions

After discussing at length about the functions $Z_\kappa^{(\alpha,\beta)}(\xi)$ and $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, one is finally ready to return to the two-variable functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ that appear in the Stix parameters (6a)–(6d) for a kappa plasma.

In order to define these functions, one has to first go back to Eq. (3), introduce the κVDF (1) into the integrals, evaluate the derivatives, identify the rotated components via Eq. (5), and then finally identify the Stix parameters in (6). By an adequate change of integration variables and some algebra, one can verify that the functions in question can be defined as

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = 2\int_0^\infty dx\, \frac{xJ_n^2(\nu x)}{(1+x^2/\kappa)^{\lambda-1}}\, Z_\kappa^{(\alpha,\beta)}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right), \quad (26a)$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \frac{2}{\mu}\int_0^\infty dx\, \frac{x^3 J_{n-1}(\nu x)J_{n+1}(\nu x)}{(1+x^2/\kappa)^{\lambda-1}} \times Z_\kappa^{(\alpha,\beta)}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right), \quad (26b)$$

where $\nu^2 = 2\mu$.

Applying the limit $\kappa\to\infty$, one obtains the expressions

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) \to 2Z_\kappa^{(\alpha,\beta)}(\xi)\int_0^\infty dx\, xe^{-x^2}J_n^2(\nu x)$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) \to \frac{2}{\mu}Z_\kappa^{(\alpha,\beta)}(\xi)\int_0^\infty dx\, x^3 e^{-x^2}J_{n-1}(\nu x)J_{n+1}(\nu x),$$

that can be found in any table of integrals. The final result will be Eqs. (7).

As it was already mentioned, functions similar to $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$, also defined in terms of a remaining integral, were already considered in Ref. 36 and have recently been numerically implemented in Ref. 39. However, the advantages of having exact, analytically closed-form expressions for these functions are plenty. First of all, they simplify the derivation of adequate approximations for dispersion relations, damping or growth-rate coefficients, among other quantities related to wave propagation. Moreover, analytical and computable representations are usually advantageous for numerical applications, both in terms of computing time and accuracy. As an example, the availability of analytical expressions for the kappa gyroradius function allowed the authors of Ref. 34 to obtain physically correct expressions for the dispersion relations of dispersive Alfvén waves.

As it will be shown presently, the two-variable functions defined above have contributions from both one-variable functions $Z_\kappa^{(\alpha,\beta)}(\xi)$ and $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$. It was shown here that the former can always be written in terms of the Gauss function; hence, it is hypergeometric (i.e., given by a power series) in nature. However, the second one is not in general of the same nature. Therefore, the functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ are not representable in general by any two-variable hypergeometric function such as the Appell or Horn series.[61]

The analytical, closed-form representations sought for functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ can be derived in the following manner. Considering first the function $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}$ in (26a), if the quantity ξ is a point inside the domain of the principal branch of $Z_\kappa^{(\alpha,\beta)}(\zeta)$, then, as the integration is carried out, the argument $\zeta(x) = \xi/\sqrt{1+x^2/\kappa}$ follows a curve in the complex plane attaching the point $\zeta(0)=\xi$ with the origin. Since it is always possible to find a region R where the function $Z_\kappa^{(\alpha,\beta)}(\zeta)$ is analytic and that contains the whole integration path, Taylor's theorem assures that it is possible to expand the function in a power series around any interior point of R. Therefore, for any $0\le x<\infty$, the function $Z_\kappa^{(\alpha,\beta)}(\zeta(x))$ can be expanded in a power series around $\zeta=\xi$, as

$$Z_\kappa^{(\alpha,\beta)}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right) = \sum_{k=0}^\infty\sum_{\ell=0}^k(-)^\ell\binom{k}{\ell}\frac{(-\xi)^k}{k!} \times Z_\kappa^{(\alpha,\beta)(k)}(\xi)\left(1+\frac{x^2}{\kappa}\right)^{-\ell/2}. \quad (27)$$

Inserting (27) into (26a), one can immediately identify the $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}$ function from the definition (20) and obtain

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \sum_{k=0}^\infty\sum_{\ell=0}^k(-)^\ell\binom{k}{\ell}\frac{(-\xi)^k}{k!} \times Z_\kappa^{(\alpha,\beta)(k)}(\xi)\mathcal{H}_{n,\kappa}^{(\alpha,\beta+\ell/2)}(\mu). \quad (28a)$$

Following the same procedure for $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ in (26b), one obtains

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \sum_{k=0}^\infty\sum_{\ell=0}^k\binom{k}{\ell}\frac{(-)^\ell\kappa}{\lambda+\ell/2-2}\frac{(-\xi)^k}{k!} \times Z_\kappa^{(\alpha,\beta)(k)}(\xi)\mathcal{H}_{n,\kappa}^{(\alpha,\beta+\ell/2-1)\prime}(\mu), \quad (28b)$$

where the identity (A5) was also employed.

Although expressions (28a) and (28b) look formidable, they allow a significant speed-up for numerical evaluations of functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$. A short test was performed, comparing the evaluation of $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ both via numerical quadrature applied to formula (26a) and by truncating the k-series in (28a) to a value $k_{\max}\ge 0$.

Call $\mathcal{Z}_{\rm sum}(\mu)$ the result obtained by truncating the series (28a) up to $k=k_{\max}$ and $\mathcal{Z}_{\rm int}(\mu)$ the result of (26a), evaluated via an adaptive quadrature routine, with requested relative accuracy of $\epsilon=10^{-5}$. This will be considered the "correct" value, against which the truncated series result can be compared. Figure 3 shows plots of the relative difference $\Delta = |1 - \mathcal{Z}_{\rm sum}/\mathcal{Z}_{\rm int}|$, both for the real and imaginary parts of the functions. The results were obtained by keeping fixed $n=0$, $\alpha=\beta=0$, $\kappa=2.5$ and $\xi=1+i$, and varying μ.

Figure 4 shows the same results, but now for the harmonic number $n=1$. The results show that indeed the relative difference decreases with $k_{\rm max}$. Interestingly, the approximation with $k_{\rm max}=0$ was indeed better than with $k_{\rm max}=1$ in both figures, but from $k_{\rm max}=2$ the quantity Δ steadily decreases, for both the real and imaginary parts, by roughly one order of magnitude for each successive value of $k_{\rm max}$.

In both figures, for $k_{\rm max}=4$ one gets roughly $\Delta_r \simeq 10^{-3}$ and $\Delta_i \simeq 10^{-4}$. If one were to plot both $\mathcal{Z}_{\rm int}(\mu)$ and $\mathcal{Z}_{\rm sum}(\mu)$ for $k_{\rm max}=4$ in the same slide, the graph of the latter would be almost indistinguishable from the former; however, calling $T_{\rm sum}$ and $T_{\rm int}$ the average computing time per point for $\mathcal{Z}_{\rm sum}$ and $\mathcal{Z}_{\rm int}$, respectively, for $n=1$ and $k_{\rm max}=4$ in Fig. 4, the ratio of time per point resulted $T_{\rm sum}\ /T_{\rm int} \simeq 10^{-2}$. In (28a), the total number of $Z_\kappa^{(\alpha,\beta)(k)}(\xi)$ evaluations is $k_{\rm max}+1$, whereas the total number of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta+\ell/2)}(\mu)$ evaluations is $\frac{1}{2}(k_{\rm max}+1)(k_{\rm max}+2)$ (15, for $k_{\rm max}=4$). Nevertheless, the evaluation of $\mathcal{Z}_{\rm sum}(\mu)$ was in average almost 100 times faster than the evaluation $\mathcal{Z}_{\rm int}(\mu)$. The speed-up can be further increased if one employs the recurrence relations for $Z_\kappa^{(\alpha,\beta)(k)}(\xi)$ and $\mathcal{H}_{n,\kappa}^{(\alpha,\beta+\ell/2)}(\mu)$, derived in Secs. III A and III B, which were not used in the tests. However, in this case, a stability analysis of the recurrence relations must be first carried out.

Therefore, although Eqs. (26a) and (26b) are simpler to implement, for computer-intensive applications, one should employ the closed-forms shown in (28a) and (28b). As was also argued, these last representations also allow for the correct derivation of approximations, valid for some particular range of particle species, frequencies, propagation angles, and wave polarization.

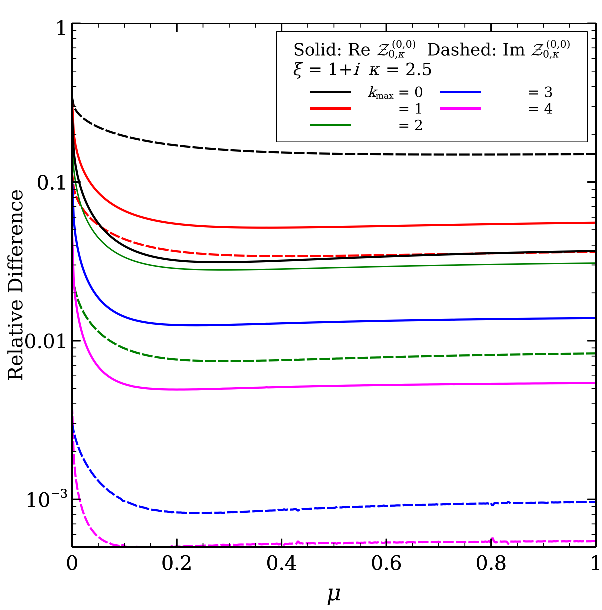

FIG. 3. Plots of the real (solid lines) and imaginary (dashed) parts of the relative difference Δ between functions $\mathcal{Z}_{\rm sum}$ and $\mathcal{Z}_{\rm int}$ for $n=0$, $\alpha=\beta=0$, $\kappa=2.5$, $\xi=1+i$ and different values of $k_{\rm max}$.

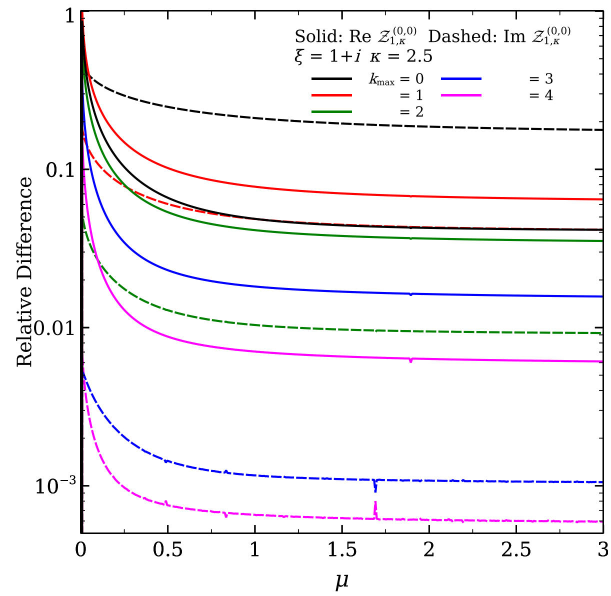

FIG. 4. Same as Fig. 3, but for $n=1$.

IV. NUMERICAL APPLICATIONS

As a simple application of the formalism developed in Secs. II and III, the dispersion equation (9) was solved for some particular cases.

Since the intention was to solely provide a simple demonstration of the formalism, it was assumed an electron-proton plasma, both formally described by the same distribution function (1), and both in the ST91 form. However, the dispersion equation was solved only for high-frequency waves propagating at arbitrary angles with $\boldsymbol{B}_0$. Hence, the protons only serve to provide a stationary background by taking $m_p \to \infty$ (m_p: proton mass). The Stix parameters in Eqs. (6) were evaluated with $a=e$ only and by truncating the harmonic number series to $-n_{\rm max} \le n \le n_{\rm max}$. In all solutions presented below, $n_{\rm max}=1$.

The kappa plasma functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and their derivatives in Eqs. (6a)–(6d) were evaluated in (28a) and (28b) on the lowest possible order, i.e., $k_{\rm max}=0$. In this case, the exact expressions (28a) and (28b) are replaced by the approximations

$$\tilde{\mathcal{Z}}_{n,\kappa}^{(1,\beta)}(\mu,\xi) = \mathcal{H}_{n,\kappa}^{(1,\beta)}(\mu) Z_\kappa^{(1,\beta)}(\xi)$$
$$\tilde{\mathcal{Y}}_{n,\kappa}^{(1,\beta)}(\mu,\xi) = \frac{\kappa}{\lambda-2} \mathcal{H}_{n,\kappa}^{(1,\beta-1)\prime}(\mu) Z_\kappa^{(1,\beta)}(\xi),$$

in which case their formal structure is the same as the Maxwellian limits (7). One must point out here that although $\tilde{\mathcal{Z}}_{n,\kappa}^{(1,\beta)}(\mu,\xi)$ and $\tilde{\mathcal{Y}}_{n,\kappa}^{(1,\beta)}(\mu,\xi)$ are approximations when the distribution function is the isotropic κVDF given in (1), or when one employs a bi-kappa model, if one employs instead another anisotropic distribution model, such as the product-bi-kappa or kappa-Maxwellian VDFs, which describe statistical distributions of particles with uncorrelated velocity directions, the same structure will correspond to the *exact* expressions for the functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$, since the expansion (27) will not be needed in these cases.

Figure 5 shows the numerical solutions of the dispersion equation (9), i.e., the dispersion relations, for high-frequency waves propagating at three oblique propagation angles, relative to $\boldsymbol{B}_0$: $\theta = 10°$, 45°, and 80°. For each angle, the dispersion equation is solved for different values of the κ parameter, including the Maxwellian limit. The other physical parameters adopted in Fig. 5 are: $\omega_{pe}^2/\Omega_e^2 = 0.5$, corresponding to a low-density plasma, and $v_{Te}^2/c^2 = 10^{-4}$. Both these quantities determine the electron beta parameter

$$\beta_e = \frac{\omega_{pe}^2}{\Omega_e^2}\frac{v_{Te}^2}{c^2} = \frac{n_e T_e}{B_0^2/8\pi} = 5 \times 10^{-5},$$

which measures the ratio of the thermal to magnetic field energy densities. Hence, the dispersion relations in Fig. 5 are typical to a low-beta plasma.

Since for oblique propagation the polarizations of the eigenmodes are elliptic, in this work, we adopted a simplistic nomenclature that combines the names for both exactly parallel or perpendicular directions. In the slide for $\theta = 10°$, for instance, the blue curves are labelled "*RX*" because in the parallel direction case ($\theta = 0°$), these would be the modes with right-hand circular polarization (*R*). On the other hand, at the perpendicular direction ($\theta = 90°$), they would correspond to the fast extraordinary mode (*X*). Hence, this elliptic mode is termed the *RX*, or "right-fast extraordinary" mode. The "*PO*" mode is the "plasma-ordinary" mode, since for $\theta = 0°$ this mode would be the longitudinal (plasma, *P*) mode, whereas for $\theta = 90°$ it would be the ordinary (*O*) mode. The mode "*LZ*" or "left-slow extraordinary" would be the left-hand circularly polarized mode (*L*) for parallel or the slow extraordinary (*Z*) for perpendicular directions. Finally, the "*W*" or "whistler" mode is the lower-frequency branch of the *R* mode for $\theta = 0°$, which disappears as $\theta \to 90°$, since only the electron inertia is included in the dispersion equation.

In each panel of Fig. 5, the cut-off frequencies ω_{RX} and ω_{LZ} corresponding to the *R* and *L* modes cut-offs for a Maxwellian plasma,[45] which are independent of the propagation angle θ, are also identified.

Since in a low-beta plasma the thermal effects are less important, the effect of the long tail of the κVDF is not very pronounced. Notwithstanding, the dispersion relations noticeably depart from the Maxwellian case as κ decreases, with the $\kappa = 20$ case practically identical to the Maxwellian limit. One also notices that the departure is roughly constant in wavenumber for almost all modes, although the *W* mode is the least affected by the shape of the VDF.

A quite different scenario appears for a high-density, high-temperature and high-beta plasma. Figures 6–8 show the solutions of (9) for $\omega_{pe}^2/\Omega_e^2 = 50$, $v_{Te}^2/c^2 = 10^{-2}$ and,

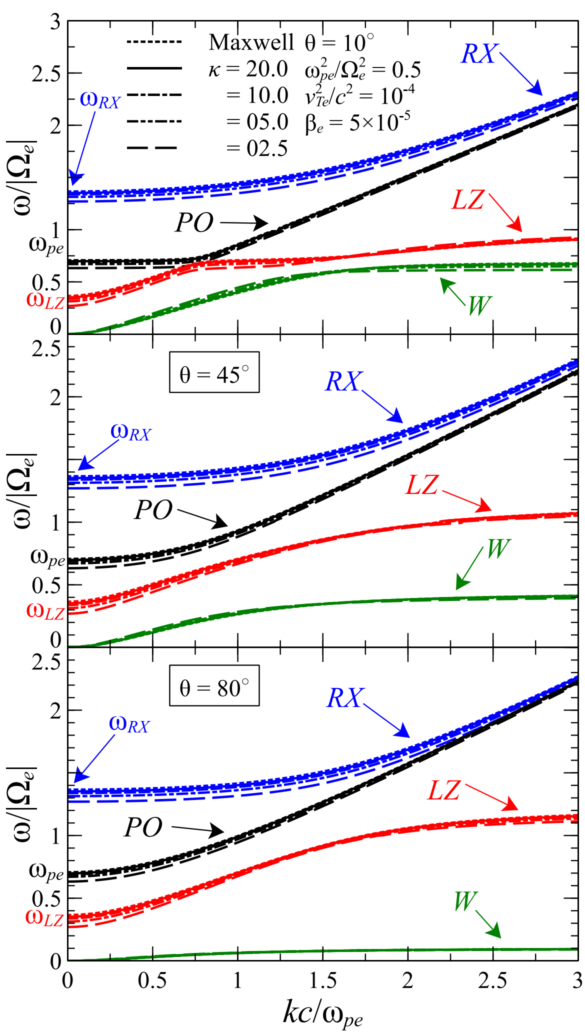

FIG. 5. Plots of high-frequency dispersion relations in a low-beta, kappa plasma, as the numerical solutions of the dispersion equation (9), for propagation angles $\theta = 10°$, 45°, and 80°. For each angle, the dispersion equation is solved for several values of the κ parameter, including the limit $\kappa \to \infty$ (Maxwell). The modes with elliptic polarization are named: *RX* (right-fast extraordinary), *PO* (plasma-ordinary), *LZ* (left-slow extraordinary), and *W* (whistler).

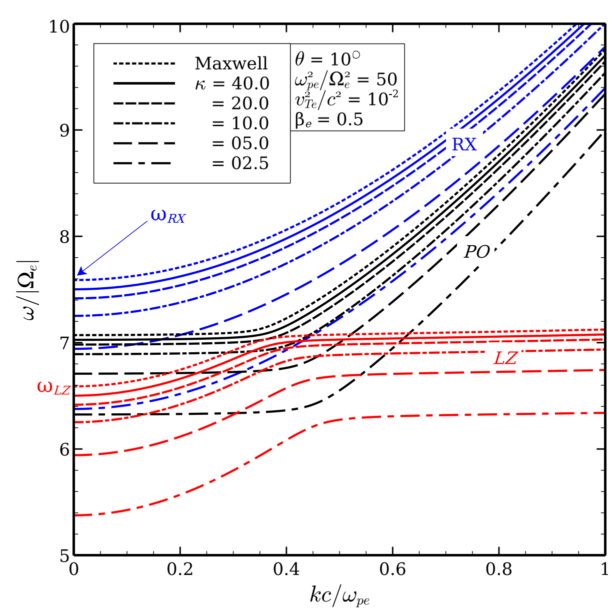

FIG. 6. Plots of high-frequency dispersion relations in a high-beta, kappa plasma, as the numerical solutions of the dispersion equation (9), for propagation angle $\theta = 10°$. The dispersion equation is solved for several values of the κ parameter, including the limit $\kappa \to \infty$ (Maxwell). The modes with elliptic polarization are named: *RX* (right-fast extraordinary), *PO* (plasma-ordinary), and *LZ* (left-slow extraordinary).

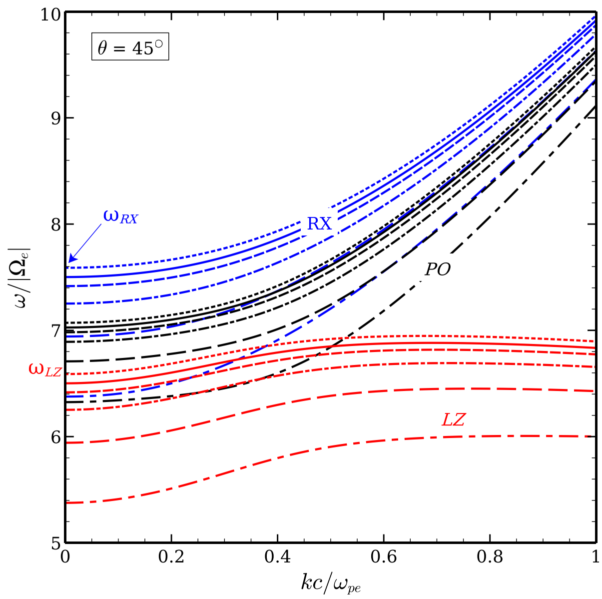

FIG. 7. Same as Fig. 6, but with propagation angle $\theta = 45°$.

consequently, $\beta_e = 0.5$. In this case, the effect of the superthermal particles on wave dispersion is considerably more important than it is in a low-beta plasma. Now the W mode is not shown because it occupies a region well below the displayed range of frequencies. In all propagation angles, for a sufficiently small κ the dispersion relation for a given mode and at the same wavenumber can assume a range of frequencies previously taken by another mode for a larger κ. Therefore, it is expected that the effect of the superthermal tails on the VDFs will be significant for moderate to high-beta plasmas in all propagation angles.

Further studies of oblique waves propagating in superthermal plasmas will be conducted in future papers, including the effects of anisotropies in temperature.

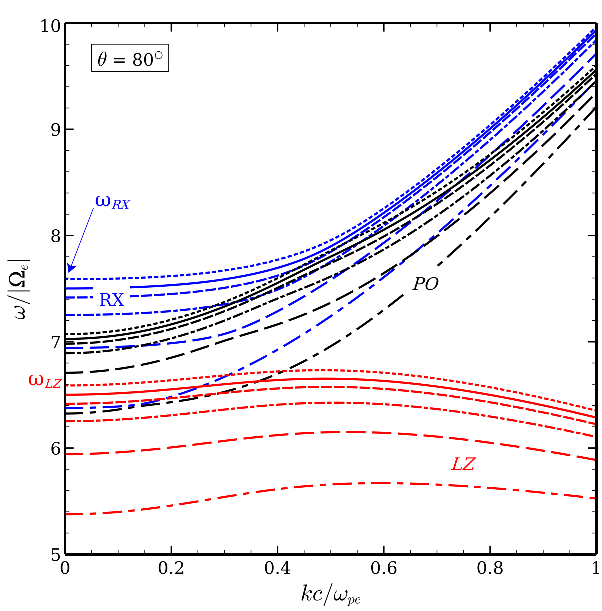

FIG. 8. Same as Figs. 6 and 7, but with propagation angle $\theta = 80°$.

V. CONCLUSIONS

In this work, a general treatment for the problem of wave propagation in a magnetized superthermal plasma was proposed. The formulation is valid for any number of particle species, wave frequency, and propagation angle.

The dielectric tensor components are written in terms of thermal Stix parameters, which in turn are written in terms of special functions that appear when the velocity distribution functions are isotropic kappa distributions. The mathematical properties of the special functions are discussed in detail and several useful identities and formulae are derived.

As a demonstration of the usefulness of the formulation proposed here, the dispersion relations for high-frequency waves propagating at several angles relative to the ambient magnetic field are obtained as the numerical solutions of the dispersion equation for a kappa plasma.

It is expected that this formulation will be very useful for the study of wave propagation and amplification/damping in superthermal plasmas such as the solar wind or planetary magnetospheres.

ACKNOWLEDGMENTS

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APPENDIX A: ADDITIONAL PROPERTIES OF THE KAPPA PLASMA FUNCTIONS

1. Properties of $Z_\kappa^{(\alpha,\beta)}(\xi)$

Other representations: Employing formula (B6) and the function (B9), result (13) can be analytically continued across the line $\xi_i = 0$, resulting in the representation

$$Z_\kappa^{(\alpha,\beta)}(\xi) = Z_{\kappa,NC}^{(\alpha,\beta)}(\xi) + \gamma \frac{2i\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)} \left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-1)}, \tag{A1}$$

where $Z_{\kappa,NC}^{(\alpha,\beta)}(\xi)$ is the non-continued representation (13), and

$$\gamma = \begin{cases} 0, & 0 < \arg\xi \le \pi \\ 1, & \pi < \arg\xi \leqslant 2\pi \ \ (\text{or } -\pi < \arg\xi \leqslant 0). \end{cases}$$

Notice that with (A1), the branch cut moved to the line $-\sqrt{\kappa} \ge \xi_i > -\infty$, as is the case of all other computable representations of $Z_\kappa^{(\alpha,\beta)}(\xi)$.

It is easy to show that $Z_\kappa^{(\alpha,\beta)}(\xi)$ always reduces to a polynomial when λ is integer. First, starting from (15) and inserting into (B7a), there results

$$Z_\kappa^{(\alpha,\beta)}(\xi) = \left[\frac{i\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)} - \frac{2\Gamma(\lambda-1/2)\xi}{\kappa^{\beta+1}\Gamma(\sigma-3/2)} \times {}_2F_1\left(\begin{matrix}2-\lambda, 1/2\\ 3/2\end{matrix}; -\frac{\xi^2}{\kappa}\right)\right]\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-1)}.$$

According to (B7g), if $\lambda = 2 + m$ ($m = 0, 1, 2, \ldots$), the Gauss function reduces to a polynomial of degree $\lambda - 2$. This result is also interesting because its Maxwellian limit is another well-known representation of the Fried and Conte function,[48] $Z(\xi) = i\sqrt{\pi}e^{-\xi^2}\mathrm{erfc}(-i\xi)$, where $\mathrm{erfc}(z)$ is the complementary error function.[62]

Series representations: Several expansions were obtained for the κPDF.

Power series: Inserting definition (B4) into the form (15), one readily obtains

$$Z_\kappa^{(\alpha,\beta)}(\xi) = -\frac{\pi^{1/2}\kappa^{-\beta-1}}{\Gamma(\sigma-3/2)}\xi\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k-1/2)}{\Gamma(k+3/2)}\left(-\frac{\xi^2}{\kappa}\right)^k + i\frac{\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-1)}, \tag{A2}$$

which converges within the radius $|\xi^2| < \kappa$. The Maxwellian limit of (A2) reduces, via convolution, to the well-known series for $Z(\xi)$.[48]

Asymptotic series: Inserting definition (B4) into the form (16), one readily obtains

$$Z_\kappa^{(\alpha,\beta)}(\xi) = -\frac{\Gamma(\lambda-3/2)}{\kappa^\beta\Gamma(\sigma-3/2)}\frac{1}{\xi}\sum_{k=0}^{\infty}\frac{(1/2)_k}{(5/2-\lambda)_k}\left(-\frac{\kappa}{\xi^2}\right)^k + \frac{\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}[i-\tan(\lambda\pi)]\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-1)}.$$

When $\lambda = 3/2 + m$ ($m = 1, 2, \ldots$), the result above is not valid, since singularities appear in both terms. In this case, one must return to (15) and employ the special case (B7f), in which case there results

$$Z_\kappa^{(\alpha,\beta)}(\xi) = -\frac{\pi^{-1/2}\kappa^{-\beta}}{\Gamma(\kappa+\alpha-3/2)}\frac{1}{\xi}\left\{\sum_{k=0}^{m-1}\Gamma\left(k+\frac{1}{2}\right)\times(m-k-1)!\left(\frac{\kappa}{\xi^2}\right)^k + \left(\frac{\kappa}{\xi^2}\right)^m\sum_{k=0}^{\infty}\frac{\Gamma(k+m+1/2)}{k!}\times\left[\ln\left(\frac{\xi^2}{\kappa}\right)+t_{m,k}\right]\left(-\frac{\kappa}{\xi^2}\right)^k\right\} + i\pi^{1/2}\frac{\Gamma(m+1/2)}{\kappa^{\beta+1/2}\Gamma(\kappa+\alpha-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{-(m+1/2)},$$

where

$$t_{m,k} = \psi(k+1) - \psi\left(k+m+\frac{1}{2}\right).$$

This result shows that there is a logarithmic singularity for the case $\lambda = m + 3/2$ when $|\xi^2/\kappa| \to \infty$.

Derivatives: Starting from (19) and applying in sequence transformations (B7e), (B7a), and (B7b), one obtains

$$\frac{Z_\kappa^{(\alpha,\beta)(n)}(\xi)}{-i^n\sqrt{\pi}n!} = \frac{\kappa^{-(\beta+1/2+n/2)}}{\Gamma(\sigma-3/2)}\left[\frac{\Gamma\left(\lambda-\frac{1}{2}+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}\frac{2\xi}{\sqrt{\kappa}}\times{}_2F_1\left(\begin{matrix}\lambda-\frac{1}{2}+\frac{n}{2}, 1+\frac{n}{2}\\ 3/2\end{matrix}; -\frac{\xi^2}{\kappa}\right) - i\frac{\Gamma\left(\lambda-1+\frac{n}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}\times{}_2F_1\left(\begin{matrix}\lambda-1+\frac{n}{2}, \frac{1}{2}+\frac{n}{2}\\ 1/2\end{matrix}; -\frac{\xi^2}{\kappa}\right)\right], \tag{A3}$$

which immediately renders the power series expansion for $Z_\kappa^{(\alpha,\beta)(n)}(\xi)$.

Taking the limit $x \to \infty$ in (27), and using the value (12), the resulting expression,

$$\sum_{k=0}^{\infty}\frac{(-\xi)^k}{k!}Z_\kappa^{(\alpha,\beta)(k)}(\xi) = i\frac{\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)},$$

is a sum rule for the derivatives of the κPDF. Once again, the Maxwellian limit returns a known result[56]

$$\sum_{k=0}^{\infty}\frac{(-\xi)^k}{k!}Z^{(k)}(\xi) = i\sqrt{\pi}.$$

2. Properties of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$

An explicit power series expansion when λ is not integer can be written from (23) and (B3) as

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\kappa(2\kappa z)^n}{\sqrt{\pi}\Gamma(\lambda-1)}\sum_{k=0}^{\infty}H_{n,k}^{(\lambda)}(z)\frac{(2\kappa z)^k}{k!}, \tag{A4}$$

where

$$H_{n,k}^{(\lambda)}(z) = \frac{\Gamma(\lambda-n-2)\Gamma(n+1/2+k)}{\Gamma(2n+1+k)(n+3-\lambda)_k} + \frac{\Gamma(n+2-\lambda)\Gamma(\lambda-3/2+k)}{\Gamma(\lambda-1+n+k)(\lambda-1-n)_k}(2\kappa z)^{\lambda-n-2}.$$

Notice that each term in this series is proportional to a noninteger power of z.

Another identity related to the derivative of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ will be obtained now, regarding the integral in the definition (26b). Starting from (B15b) and employing formula (B13), one obtains

$$\int_0^\infty dx\,\frac{x^3J_{n-1}(\nu x)J_{n+1}(\nu x)}{(1+x^2/\kappa)^{\lambda-1}} = \frac{\mu}{\sqrt{\pi}}\frac{d}{d\mu}\int_0^\infty dx\,\frac{x^3G_{1,3}^{1,1}\left[2\mu x^2\middle|\begin{matrix}1/2\\ n,-n,-1\end{matrix}\right]}{(1+x^2/\kappa)^{\lambda-1}},$$

recalling that $\nu^2 = 2\mu$. Then, using formula (B12) and identifying the result with the definition (20), one concludes that

$$\int_0^\infty dx\, \frac{x^3 J_{n-1}(\nu x) J_{n+1}(\nu x)}{(1+x^2/\kappa)^{\lambda-1}} = \frac{1}{2}\frac{\kappa\mu}{\lambda-2}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)\prime}(\mu). \quad \text{(A5)}$$

APPENDIX B: HYPERGEOMETRIC FUNCTIONS EMPLOYED IN KAPPA PLASMAS

There are two classes of special functions that are employed in this and other theoretical works concerning superthermal plasmas, namely, the generalized hypergeometric series and the Meijer G functions. Some of their properties will be presented here.

1. The generalized hypergeometric series

The general expression for the hypergeometric series is

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}\frac{z^k}{k!}, \quad \text{(B1)}$$

where p, q are natural numbers, the sets $\{a_p\}$, $\{b_q\}$ and the argument z are in general complex, and $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ is the Pochhammer symbol. Unless explicitly stated, all properties presented here can be found in Ref. 63.

Except when any of the inferior parameters $b_1,\ldots, b_q$ is a nonpositive integer, the hypergeometric series ${}_pF_q(\cdots;z)$ belongs to the class $\mathbb{C}^{p+q+1}$ within its convergence radius, which divides it in three classes: (i) $p \le q$, (ii) $p = q+1$, and (iii) $p \ge q+2$. In this work, we employ functions of the first two classes. Series of class 3 are not convergent except at the origin. The Meijer G-function, discussed in Appendix B 2, lends an analytical representation for these functions.

a. Class 1 series: The Kummer and ${}_1F_2$ hypergeometric functions

The Kummer confluent hypergeometric function: The Kummer function is defined from (B1) as

$$ {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k}\frac{z^k}{k!}. \quad \text{(B2)}$$

The ${}_1F_2\left(\begin{matrix} a \\ b, c \end{matrix}; z\right)$ *hypergeometric function*: From (B1),

$$ {}_1F_2\left(\begin{matrix} a \\ b, c \end{matrix}; z\right) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k (c)_k}\frac{z^k}{k!}. \quad \text{(B3)}$$

Notice that both these series, for ${}_1F_1$ and ${}_1F_2$, converge for any $|z| < \infty$. Thus, the ${}_1F_2(\cdots;z)$ function belongs to the class $\mathbb{C}^4$, is meromorphic on the b, c planes except at the nonpositive integer points and entire on the a, z planes.

b. Class 2 series: The Gauss function

The Gauss hypergeometric series is defined from (B1) as[63,64]

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!}. \quad \text{(B4)}$$

This series, and all other functions belonging to the same class, is convergent within the unit circle $|z| < 1$ and conditionally convergent along it. Some properties of the Gauss function are presented below.

Convergence at $|z| = 1$ and analyticity: Over the unit circle, the series (B4) converges: (i) absolutely if $\Re(c-a-b) > 0$, (ii) conditionally if $-1 < \Re(c-a-b) \leqslant 0$, and (iii) diverges if $\Re(c-a-b) \le -1$. For $|z| > 1$ the function must be analytically continued.

The Gauss function has a branch point at $z = 1$, with the branch line running over $1 \le z < \infty$. The principal branch is defined as the region $0 < \arg(z-1) \leqslant 2\pi$.

When $a = -m$ $(m = 0, 1, 2, \ldots)$ and $c \neq 0, -1, -2, \ldots$, the function ${}_2F_1(\begin{matrix} -m, b \\ c \end{matrix}; z)$ reduces to a polynomial of degree m. Obviously, ${}_2F_1(\begin{matrix} a, b \\ c \end{matrix}; z) = {}_2F_1(\begin{matrix} b, a \\ c \end{matrix}; z)$.

Integral representation: The Gauss function can be expressed, when $\Re c > \Re b > 0$, by the integral

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt. \quad \text{(B5)}$$

Analytic continuation: Writing $z = x + iy$, the difference of the values of ${}_2F_1$ across the branch line $x > 1$ is

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x+i0\right) - {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x-i0\right) \\ &= \frac{2\pi i \Gamma(c)(x-1)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \\ &\times {}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-x\right). \end{aligned} \quad \text{(B6)}$$

Linear and quadratic transformations: The analytical continuation of the series (B4) can also be accomplished employing several known transformations of ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$, either linear or nonlinear. Some of these transformations are given below

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right) \quad \text{(B7a)}$$

$$\begin{aligned} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ &\times {}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-z\right) \left(|\arg(1-z)| < \pi\right) \end{aligned} \quad \text{(B7b)}$$

$$\begin{aligned} &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}; \frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}; \frac{1}{z}\right) \\ &\left(|\arg(-z)| < \pi\right) \end{aligned} \quad \text{(B7c)}$$

$$
{}_2F_1\left(\begin{matrix}a,b\\a+b-1/2\end{matrix};z\right) = -(1-z)^{-1/2}{}_2F_1\left(\begin{matrix}2a-1,2b-1\\a+b-1/2\end{matrix};\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right) \tag{B7d}
$$

$$
{}_2F_1\left(\begin{matrix}a,b\\\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\end{matrix};z\right) = {}_2F_1\left(\begin{matrix}\frac{1}{2}a,\frac{1}{2}b\\\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\end{matrix};4z-4z^2\right). \tag{B7e}
$$

Some special cases of the transformations above are also relevant for this work. If $b-a=m$ ($m=0, 1, 2,\ldots$), then

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}a,a+m\\c\end{matrix};z\right)
&= \frac{\Gamma(c)}{\Gamma(a+m)}(-z)^{-a}\sum_{k=0}^{m-1}\frac{(a)_k(m-k-1)!}{k!\Gamma(c-a-k)}z^{-k}\\
&\quad+\frac{\Gamma(c)}{\Gamma(a)}(-z)^{-a}\sum_{k=0}^{\infty}\frac{(-)^k(a+m)_k z^{-k-m}}{k!(k+m)!\Gamma(c-a-k-m)}\\
&\quad\times\left[\ln(-z)+\psi(k+1)+\psi(k+m+1)\right.\\
&\quad\left.-\psi(a+k+m)-\psi(c-a-k-m)\right],\\
&\quad\left(\begin{matrix}|z|>1\\|\arg(-z)|<\pi\end{matrix}\right),
\end{aligned} \tag{B7f}
$$

where $\psi(z)$ is the digamma function.[44]

On the other hand, if $a=-m$ ($m=0, 1, 2, \ldots$) and neither b nor c are nonpositive integers, the Gauss function reduces to the polynomial

$$
F\left(\begin{matrix}-m,b\\c\end{matrix};z\right) = \sum_{n=0}^{m}\frac{(-m)_n(b)_n}{(c)_n}\frac{z^n}{n!} = \sum_{n=0}^{m}\binom{m}{n}\frac{(b)_n}{(c)_n}(-z)^n. \tag{B7g}
$$

Derivatives: The general formula for the n-th derivative of the Gauss function is

$$
\frac{d^n}{dz^n}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) = \frac{(a)_n(b)_n}{(c)_n}{}_2F_1\left(\begin{matrix}a+n,b+n\\c+n\end{matrix};z\right). \tag{B8}
$$

Other functions in the same class: We have also employed the function

$$
{}_1F_0\left(\begin{matrix}a\\-\end{matrix};z\right) \equiv {}_2F_1\left(\begin{matrix}a,c\\c\end{matrix};z\right) = (1-z)^{-a}. \tag{B9}
$$

2. The Meijer *G*-function

The Meijer G-function is that function whose Mellin transform[65] can be expressed as a ratio of certain products of gamma functions. Consequently, its definition is given by the Mellin-Barnes contour integral

$$
G^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_L \Phi\left(\begin{matrix}(a_p)\\(b_q)\end{matrix};s\right)z^s ds, \tag{B10}
$$

where

$$
\Phi\left(\begin{matrix}(a_p)\\(b_q)\end{matrix};s\right) = \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}.
$$

In (B10), p, $q=0, 1, 2,\ldots$, $0\le m\le q$ and $0\le n\le p$. If $m+1>q$ or $n+1>p$, the product is replaced by one. The notation is such that $(a_p)\doteq a_1, a_2, \ldots, a_p$ and $(b_q)\doteq b_1, b_2, \ldots, b_q$. It is assumed that the a_j's and b_j's are such that no pole of $\Gamma(b_j-s)$ $(j=1,\ldots,m)$ coincides with any pole of $\Gamma(1-a_k+s)$ $(k=1,\ldots,n)$, i.e., $a_k-b_j\neq 1, 2,\ldots$. It is also assumed that $z\neq 0$, since the origin is a branch point.

The integration contour L in (B10) corresponds to that of the inverse Mellin transform, but deformed in such a way that the poles of $\Gamma(b_j-s)$ ($j=1,\ldots, m$) lie to the right of the contour, whereas the poles of $\Gamma(1-a_j+s)$ ($j=1,\ldots, n$) lie to the left of the same path. A detailed account on all possible integration paths can be found in Refs. 63, 66, and 67. All properties of the G-function shown here are likewise found in these sources.

The G-function has remarkable properties. For instance, it contains all generalized hypergeometric functions, but it also represents functions that cannot be expanded in a power series anywhere, such as functions with logarithmic singularities. The definition (B10) forms a class that is closed under reflections of the argument ($z\to -z$ and $z\to z^{-1}$), multiplication by powers of z, differentiation, Laplace transform, and integration. The last property, in particular, means that the integration of one or a product of two G-functions is a G-function. Computer algebra software takes advantage of this property in order to analytically evaluate integrals by representing the integrand as G-function(s).

Elementary properties: The following identities can be easily obtained from the definition (B10):

$$
\begin{aligned}
G^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] &= G^{n,m}_{q,p}\left[z^{-1}\middle|\begin{matrix}1-(b_q)\\1-(a_p)\end{matrix}\right]\\
z^{\sigma}G^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] &= G^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_p)+\sigma\\(b_q)+\sigma\end{matrix}\right],
\end{aligned} \tag{B11a}
$$

$$
\begin{aligned}
G^{m,n}_{p,q}&\left[z\middle|\begin{matrix}\alpha, a_2, \ldots a_n, a_{n+1}, \ldots, a_p\\b_1, \ldots, b_m, b_{m+1}, \ldots, b_{q-1}, \alpha\end{matrix}\right]\\
&= G^{m,n-1}_{p-1,q-1}\left[z\middle|\begin{matrix}a_2, \ldots, a_p\\b_1, \ldots, b_{q-1}\end{matrix}\right],
\end{aligned} \tag{B11b}
$$

where for (B11b), $n, p, q\ge 1$.

Integrals containing the G-function: Amongst the myriad integration formulae available, this work makes use of

$$\int_0^\infty y^{\alpha-1}(y+\beta)^{-\sigma} G^{m,n}_{p,q}\left[zy\middle| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right]dy = \frac{\beta^{\alpha-\sigma}}{\Gamma(\sigma)} G^{m+1,n+1}_{p+1,q+1}\left[\beta z\middle| \begin{matrix}1-\alpha,(a_p)\\ \sigma-\alpha,(b_q)\end{matrix}\right]. \quad \text{(B12)}$$

Derivatives: The following formulas are employed:

$$\frac{d^k}{dz^k}\left\{z^{-b_1} G^{m,n}_{p,q}\left[z\middle| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right]\right\} = (-)^k z^{-b_1-k} G^{m,n}_{p,q}\left[z\middle| \begin{matrix}(a_p)\\ b_1+k,b_2,\ldots,b_q\end{matrix}\right], \ (m\geqslant 1) \quad \text{(B13a)}$$

$$z^k \frac{d^k}{dz^k} G^{m,n}_{p,q}\left[z\middle| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right] = G^{m,n+1}_{p+1,q+1}\left[z\middle| \begin{matrix}0,(a_p)\\(b_q),k\end{matrix}\right]. \quad \text{(B13b)}$$

Representations of special functions: If no two of the b_h ($h=1,., m$) parameters differ by an integer, all poles of $\Phi(\cdots;s)$ in (B10) are simple and then the G-function can be expressed as a combination of the hypergeometric series (B1) as

$$G^{m,n}_{p,q}\left[z\middle| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right] = \sum_{h=1}^{m} \frac{\prod_{j=1}^{m}\Gamma(b_j-b_h)^* \prod_{j=1}^{n}\Gamma(1+b_h-a_j)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_h)} \times z^{b_h}\, {}_pF_{q-1}\left(\begin{matrix}1+b_h-(a_p)\\ 1+b_h-(b_q)^*\end{matrix};(-)^{p-m-n}z\right), \quad \text{(B14)}$$

which is valid for $p<q$ or $p=q$ and $|z|<1$. The notation $\Gamma(b_j-b_h)^*$ means that this term is absent when $h=j$.

If any pair of b_h parameters differ by an integer, then expression (B14) is no longer valid and the singularities in different terms have to be cancelled out by a limiting process. In this case, the final result will contain a logarithmic singularity and the G-function will then represent a function that is not simply expandable in a power series. A detailed account of this lengthy process is given in Ref. 66. Although this case is relevant in this work, the obtained results can always be expressed by known functions.

A short list of function representations is given below

$${}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix};z\right) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} G^{1,p}_{p,q+1}\left[-z\middle| \begin{matrix}1-(a_p)\\0,1-(b_q)\end{matrix}\right], \quad \text{(B15a)}$$

$$J_\mu(\sqrt{z})J_\nu(\sqrt{z}) = \frac{1}{\sqrt{\pi}} G^{1,2}_{2,4}\left[z\middle| \begin{matrix}0,1/2\\ \frac{\mu+\nu}{2},-\frac{\mu+\nu}{2},\frac{\mu-\nu}{2},-\frac{\mu-\nu}{2}\end{matrix}\right], \quad \text{(B15b)}$$

$$J^2_\nu(\sqrt{z}) = \frac{1}{\sqrt{\pi}} G^{1,1}_{1,3}\left[z\middle| \begin{matrix}1/2\\ \nu,-\nu,0\end{matrix}\right], \quad \text{(B15c)}$$

$$e^{-z/2} I_\nu\left(\frac{z}{2}\right) = \frac{1}{\sqrt{\pi}} G^{1,1}_{1,2}\left[z\middle| \begin{matrix}1/2\\ \nu,-\nu\end{matrix}\right], \quad \text{(B15d)}$$

$$I_\nu(\sqrt{z})K_\nu(\sqrt{z}) = \frac{1}{2\sqrt{\pi}} G^{2,1}_{1,3}\left[z\middle| \begin{matrix}1/2\\ 0,\nu,-\nu\end{matrix}\right]. \quad \text{(B15e)}$$

In particular, representation (B15a) gives meaning to a hypergeometric function ${}_pF_q(\cdots;z)$ when $p>q+1$.

APPENDIX C: CARTESIAN COMPONENTS OF THE DIELECTRIC TENSOR

The components of the dielectric tensor in Cartesian coordinates and in terms of the Stix parameters are[45]

$$\varepsilon_{xx} = \hat{S} \quad \varepsilon_{\binom{xy}{yx}} = \mp i\hat{D},$$
$$\varepsilon_{\binom{xz}{zx}} = N_\perp N_\parallel \eta \quad \varepsilon_{yy} = \hat{S} - N_\perp^2 \hat{\tau} + N_\perp^2,$$
$$\varepsilon_{\binom{yz}{zy}} = \pm i N_\perp N_\parallel \zeta \quad \varepsilon_{zz} = \hat{P},$$

where $\hat{S} = \frac{1}{2}(\hat{R}+\hat{L})$ and $\hat{D} = \frac{1}{2}(\hat{R}-\hat{L})$ are, respectively, the $\hat{S}$um and $\hat{D}$ifference (thermal) Stix parameters, and $\eta = \frac{1}{2}(\hat{\mu}+\hat{\nu}) - 1$ and $\zeta = \frac{1}{2}(\hat{\nu}-\hat{\mu})$ are kinetic parameters.

Then, using the expressions (6) for the kappa Stix parameters, one obtains

$$\hat{S}_\kappa = 1 + \sum_a \frac{\omega_{pa}^2}{\omega^2} \sum_{n\to-\infty}^{\infty} \frac{n^2}{\mu_a} \xi_{0a} \mathcal{Z}^{(\alpha_a,2)}_{n,\kappa_a},$$
$$\hat{D}_\kappa = -\sum_a \frac{\omega_{pa}^2}{\omega^2} \sum_{n\to-\infty}^{\infty} n\xi_{0a} \frac{\partial \mathcal{Z}^{(\alpha_a,2)}_{n,\kappa_a}}{\partial \mu_a},$$
$$\eta_\kappa = -\frac{1}{2}\sum_a \frac{\omega_{pa}^2}{\omega\Omega_a}\frac{w_a^2}{c^2} \sum_{n\to-\infty}^{\infty} \frac{n}{\mu_a} \xi_{0a}^2 \frac{\partial \mathcal{Z}^{(\alpha_a,1)}_{n,\kappa_a}}{\partial \xi_{na}},$$
$$\zeta_\kappa = \frac{1}{2}\sum_a \frac{\omega_{pa}^2}{\omega\Omega_a}\frac{w_a^2}{c^2} \sum_{n\to-\infty}^{\infty} \xi_{0a}^2 \frac{\partial^2 \mathcal{Z}^{(\alpha_a,1)}_{n,\kappa_a}}{\partial \xi_{na}\partial \mu_a}.$$

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