Universidade Federal do Rio Grande do Sul
Instituto de Matemática
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UNIQUE ERGODICITY OF INTERVAL EXCHANGE MAPS

Luiz Fernando Carvalho da Rocha

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LUIZ FERNANDO C. DA ROCHA
Instituto de Matemática,
Universidade Federal do Rio Grande do Sul
91500 Porto Alegre RS – Brazil

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Abstract

Necessary and sufficient conditions are given in order that an interval exchange map satisfying Keane's infinite and distinct orbit condition be uniquely ergodic. This is done through the development of a theory for interval exchange maps that parallels the classical theory of continued fractions.

1 Introduction

In this paper we give necessary and sufficient conditions for an interval exchange map satisying Keane's infinite and distinct orbit condition, i.d.o.c., to be uniquely ergodic. To this end we develop a theory for interval exchange maps paralelling the classical theory of continued fractions which might be of independent interest.

An *interval exchange map* $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ of the half open interval [0,1) is defined by a permutation π of the set $\{1, ..., m\}$ and a column vector $\alpha = (\alpha_1, ..., \alpha_m)^t$ in the simplex:

$$\mathcal{S}_m = \{\alpha \in \mathbf{R}^m \mid \sum_{i=1}^{m} \alpha_i = 1 \text{ and } \alpha_i > 0 \text{ for } i = 1, ..., m\}$$

as follows:

Decompose [0,1) in sucessive half-open intervals $\mathbf{I}_1, ..., \mathbf{I}_m$ of lengths respectively $\alpha_1, ..., \alpha_m$. $\mathbf{T}$ is a translation in each of this intervals and permutes them in such a way that $\mathbf{T}(\mathbf{I}_i) = \pi(i)$-th permuted interval; i=1,...,m. Thus $\mathbf{T}$ is continuous but for the extremes of the intervals $\mathbf{I}_i$, where we assume it is only right continuous.

Interval exchange maps were first defined and studied by Keane [2].

From now on we will fix the permutation π and identify the interval exchange map $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ with $\alpha \in \mathcal{S}_m = \mathcal{S}_m(\pi)$.

$\mathbf{T}$ satisfies the *infinite distinct orbit condition, i.d.o.c.* if the $\mathbf{T}$–orbit of the $\mathbf{T}$-discontinuities are infinite and distinct. Keane [2] has shown that except for maps lying in a denumerable set of hyperplanes, all interval exchange maps satisfy i.d.o.c. and that i.d.o. interval exchange maps are *minimal* meaning by this that the positive orbit of every point is dense. It is clear that an interval exchange map $\mathbf{T}$ preserves Lebesgue measure on [0,1); if this is the only Borel probability preserved by $\mathbf{T}$ we say $\mathbf{T}$ is *uniquely ergodic*.

It is obvious that unique ergodicity implies minimality but the converse does not hold, Keynes and Newton [4], however it is true that for $\mathbf{T}$ in a set of full Lebesgue measure in $\mathcal{S}_m$ $\mathbf{T}$ is uniquely ergodic, Mazur [6] and Veech [8]. Boshernitzan has found sufficient conditions for a minimal interval exchange map to be uniquely ergodic, [1].

The main idea behind our characterization of the uniquely ergodic i.d.o. $\mathbf{T}$'s center around an analogy with the theory of continuous fractions and the quality of rational aproximation to irrational numbers, the part of rational numbers being played by the *primitive* interval exchange maps $\mathbf{T}$ i.e. the ones such that every orbit is periodic and the $\mathbf{T}$-orbit of 0 hits every $\mathbf{T}$-discontinuity, Keane and Rauzy [3]. These maps have a half open-periodic interval whose orbit sweeps out the entire [0,1) interval.

To get all primitive interval exchange maps $\mathbf{T}$, we fix $n > 0$, its period, decompose n as a sum of m non-negative integers $\alpha_1, ..., \alpha_m$ and define the permutation Π on the set $\{0, ..., n-1\}$ using π as we did in the definition of the interval exchange map $\mathbf{T}(\pi, \alpha)$. If Π is a cycle then $\mathbf{T} = (\alpha_1, ..., \alpha_m)/n$ is a primitive interval exchange map. In this case all entries of α are rational numbers $\alpha_i = p_i/q_i$ with $p_i, q_i > 0$ and $(p_i, q_i) = 1$ for i=1,...,m.

The theory of partial fractions can be developed throught the concept of Farey series and mediants as indicated in Khinchin [5, pages 13–15] and that is the point of view more akin to the one we take here.

We say $\mathbf{S}$ is a *proper approximant* to $\mathbf{T}$ of order $n \geq 0$ if $\mathbf{S}$ is primitive and the record of the visits that the orbit of 0 makes to the intervals $\mathbf{I}_i$ up to the time n is the same for $\mathbf{T}$ and $\mathbf{S}$. More precisely:

$$\mathbf{T}^k(0) \in \mathbf{I}_i(\mathbf{T}) \Longleftrightarrow \mathbf{S}^k(0) \in \mathbf{I}_i(\mathbf{S}) \tag{1}$$

for any $i \in \{1, \ldots, m\}$ and $k \in \{0, \ldots, n\}$.

We call the set of $\mathbf{S}$'s satisfying condition (1) above the *Farey cell of order n* around $\mathbf{T}$, $\mathcal{F}_n(\mathbf{T})$. $\mathcal{F}_n(\mathbf{T})$ is a convex polyhedron whose vertices in $\mathcal{S}_m$ are primitive interval exchange maps but, in general, at most one of these vertices is in $\mathcal{F}_n(\mathbf{T})$ and are thus proper approximants to $\mathbf{T}$. We call the vertices of $\mathcal{F}_n(\mathbf{T})$ the *improper approximants* or, more simply, *approximants* of order n to $\mathbf{T}$. Using these concepts we prove that an i.d.o. $\mathbf{T}$ is uniquely ergodic iff its n-th order approximants converge to $\mathbf{T}$ as $n \to \infty$ in other words $\mathcal{F}_n(\mathbf{T}) \to \mathbf{T}$ as $n \to \infty$.

We will show that there is a finite set of bounded polyhedra, the *abstract Farey cells* of type π:

$$\{\mathcal{C}_\gamma\}_{\gamma \in \mathcal{A}}\ ,\ \mathcal{C}_\gamma \subseteq \mathbf{R}^{2(m-1)}$$

such that for every i.d.o. $\mathbf{T}$, $\mathcal{F}_n(\mathbf{T})$ is projectively isomorphic to some $\mathcal{C}_\gamma, \gamma \in \mathcal{A}$, if n is great enough.

By a *projective isomorphism* we mean a bijection $\widetilde{\mathcal{L}}$ which can be expressed as $\widetilde{\mathcal{L}}(x) = \mathcal{L}(x)/||\mathcal{L}(x)||$, where $\mathcal{L}: \mathbf{R}^N \longrightarrow \mathbf{R}^M$ is linear, $x \in \mathbf{R}^N$ and $||y|| = \sum_{i=1}^M |y_i|$ for $y \in \mathbf{R}^M$. In this case we say $\widetilde{\mathcal{L}}$ is the projective map induced by $\mathcal{L}$. We call the vector $\tilde{y} = y/||y||$ the *normalized* of the vector $y \in \mathbf{R}^M$ thus projective maps are normalized linear maps.

To explain how we get the isomorphism between a Farey cell and its abstract model we need more definitions.

We say $n \geq 0$ or, more properly, $\mathbf{T}^n(0)$ is a *critical iterate* of $\mathbf{T}$ if from $\mathbf{T}^n(0)$ we can see a discontinuity of $\mathbf{T}$ without the interference of previous $\mathbf{T}$-iterates of 0 or, more precisely, if there is a discontinuity of $\mathbf{T}$, $D_i(\mathbf{T})$ for $i \in \{1, \ldots, m\}$ such that one of the intervals $[\mathbf{T}^n(0), D_i(\mathbf{T}))$ or $[D_i(\mathbf{T}), \mathbf{T}^n(0)]$ only intercepts $\{\mathbf{T}^k(0)\}_{k=0}^n$ at $\mathbf{T}^n(0)$.

If there is only one discontinuity satisfying this condition we say i is the *type of the critical iterate* and call the iterate *left* or *right critical iterate* accordingly to its position with respect to the discontinuity. If the condition above holds true for $p > n$ that is, one of the intervals $[\mathbf{T}^n(0), D_i(\mathbf{T}))$ or $[D_i(\mathbf{T}), \mathbf{T}^n(0)]$ only intercepts $\{\mathbf{T}^k(0)\}_{k=0}^p$ at $\mathbf{T}^n(0)$, we say n *remains critical up to the order* p.

Finally define the *distribution vector* of $\mathbf{T}$ between the iterates n and p, $n < p$, as the column vector whose i–th entry is the number of times $\mathbf{T}^k(0)$ hits $\mathbf{I}_i$ as k runs from n to $p-1$.

Using these concepts we can characterize the uniquely ergodic i.d.o.'s $\mathbf{T} = \mathbf{T}(\pi, \alpha)$'s as the ones which have the sequence of normalized distribution vectors between two consecutive critical iterates converging to α.

We say a Farey cell $\mathcal{F}_s$ is *small* if the set $\{\mathbf{T}^k(0)\}_{k=1}^s$, for $\mathbf{T} \in \mathcal{F}_s$, has at least a point in each one of the intervals $\mathbf{I}_i(\mathbf{T})$ and $\mathbf{T}(\mathbf{I}_i(\mathbf{T}))$, $i = 1, \ldots, m$. Now, given a small Farey cell $\mathcal{F}_s$ and $n > s$, the next critical iterate of the interval exchange map in the interior of $\mathcal{F}_s$, we are able to define the projective isomorphism between $\mathcal{F}_s$ and its abstract model $\mathcal{C}_\gamma$.

This isomorphism is induced by the $m \times 2(m-1)$ matrix (λ^n, ρ^n), the *distribution matrix* of $\mathcal{F}_s$, whose first $m-1$ columns λ_j^n are the distribution vectors of $\mathbf{T}$ between the critical left iterated of type j that remains critical up to the order n and the next remaining left critical iterate; $j = 1, ..., m-1$, and whose last $m-1$ columns ρ_j^n; $j = 1, ..., m-1$, are defined similarly using the right critical iterates that remain up to the order n, where $\mathbf{T} \in interior(\mathcal{F}_s)$.

On $\mathcal{C} = \sum_{\gamma \in \mathcal{A}} \mathcal{C}_\gamma$ where $\sum$ denotes disjoint union, we define a map $\mathcal{G}$, the *Gauss map*, which dynamically generates the approximants. $\mathcal{G}$ is a 2-1 onto map and the two branches of $\mathcal{G}^{-1}$ defined on an abstract Farey cell are projective isomorphisms onto their images which, in their turn, abstractly represent the "next" Farey cell.

To see how $\mathcal{G}$ generates the approximants to an i.d.o. $\mathbf{T}$, take the first small Farey cell around $\mathbf{T}$, $\mathcal{F}_{n_0}$. This Farey cell is isomorphic as described above to an abstract Farey cell $\mathcal{C}_{\gamma_0}$, $\gamma_0 \in \mathcal{A}$. We call $\mathcal{C}_{\gamma_0}$ the *integral cell* around $\mathbf{T}$, $\gamma_0 \in \mathcal{A}$ the *integral type* of $\mathbf{T}$ and $\mathbf{P}_0^{-1}$, the isomorphism $\mathcal{C}_{\gamma_0} \longrightarrow \mathcal{F}_{n_0}$, the *integral part* of $\mathbf{T}$.

Using this definitions we can write:

$$\alpha = \mathbf{P}_0^{-1}(r_0)$$

for some uniquely defined $r_0 \in \mathcal{C}_{\gamma_0}$. We call r_0 the *fractional part* of α. Now, using $\mathcal{G}$, we can write:

$$\alpha = \mathbf{P}_0^{-1}(r_0) = \mathbf{P}_0^{-1}(\mathcal{G}_{\gamma_1}^{-1}(\mathcal{G}(r_0))) = \mathbf{P}_0^{-1}(\mathcal{G}_{\gamma_1}^{-1}(\mathcal{G}_{\gamma_2}^{-1}(\mathcal{G}^2(r_0)))) =$$

$$= ... = \mathbf{P}_0^{-1}(\mathcal{G}_{\gamma_1}^{-1}(...\mathcal{G}_{\gamma_n}^{-1}(\mathcal{G}^n(r_0))...))$$

where γ_i is defined by $\mathcal{G}^i(r_0) \in \mathcal{C}_{\gamma_i}$ for $i = 1, ..., n$ and $\mathcal{G}_{\gamma_i}^{-1}$ is the branch of $\mathcal{G}^{-1}$ which takes $\mathcal{C}_{\gamma_i}$ into $\mathcal{C}_{\gamma_{i-1}}$.

We call the expansion:

$$\alpha = \mathbf{P}_0^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ ...\mathcal{G}_{\gamma_n}^{-1} \circ ...$$

the *generalized continued fraction expansion* of α associated to π.

Given α we get this expansion by following the $\mathcal{G}$-orbit of the fractional part of $\mathbf{T}$. Using the expansion we get the n-th order approximants by truncating the expansion at level n, substituting the remainder $\mathcal{G}^n(r_0)$ for the vertices of $\mathcal{C}_{\gamma_n}$ and carrying the indicated operations. Thus we can decide if an i.d.o. $\mathbf{T}$ is uniquely ergodic by looking at its generalized continued fraction expansion. In the opposite direction, under mild conditions, each sequence in the space of the subshift of finite type in the simbols $\mathcal{A}$ with a transition from γ_i to γ_j allowed iff $\mathcal{G}(\mathcal{C}_{\gamma_i}) \bigcap \mathcal{C}_{\gamma_j} \neq \emptyset$, gives the expansion of at least one i.d.o. uniquely ergodic $\mathbf{T}$. If we fix the integral part, this $\mathbf{T}$ is unique.

To finish this picture we give a description of the matrices (λ^n, ρ^n) which occur as integral parts of maps $\mathbf{T}$ with a specified integral type $\gamma \in \mathcal{A}$.

The techniques used in this paper are elementary and the only difficulty that might came in the way of an interested reader is the notation employed. Since the naming of things is unavoidable if we are bound to speak about them, it seems convenient to work now, though in a sketchy way, the case $m = 2$, in order to keep in mind a simple example of what goes on.

If $m = 2$ then π is the transpositon $(2, 1)$ and compactifying $[0, 1)$ to the circle

$$\mathbf{S}^1 = \{e^{2\pi i\theta} \mid \theta \in [0, 1)\} \subseteq \mathbf{C}$$

we see that $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ is conjugated to the rotation

$$\rho = \rho(\mathbf{T}): z \in \mathbf{S}^1 \mapsto e^{2\pi i\alpha_2} z \in \mathbf{S}^1$$

where $\alpha = (\alpha_1, \alpha_2)^t$.

Identifying rotations with exchange maps of two intervals, we see $\mathbf{T}$ satisfies i.d.o.c. iff the rotation is irrational and $\mathbf{T}$ is primitive iff the rotation is rational.

To get the n-th order approximants to $\mathbf{S} \in \mathcal{S}_2$ we start with $\mathbf{T} = \mathbf{S}$ and move $\mathbf{T}$ in the interval $\mathcal{S}_2$ to the left and then to the right watching for the first $\mathbf{T}$ such that the distribution of the piece of $\mathbf{T}$-orbit $\{\mathbf{T}^k(0)\}_{k=0}^n$ on the intervals $\mathbf{I}_i$, $i = 1, \ldots, m$ changes. This change is only possible if one iterate $\mathbf{T}^k(0)$, $k \in \{0, 1, \ldots, n\}$ crosses the discontinuity α_1 of $\mathbf{T}$ and it is clear that the first iterate that crosses the discontinuity must be one of the two critical iterates that remain up to the order n. Thus, without loss of generality, we can take n a critical iterate of $\mathbf{T}$. Take l and r in $\{0, \ldots, n-1\}$ given by:

$$\mathbf{T}^l(0) = \max\{\mathbf{T}^k(0) < \alpha_1 \mid k = 0, \ldots, n-1\}$$

and

$$\mathbf{T}^r(0) = \min\{\mathbf{T}^k(0) > \alpha_1 \mid k = 0, \ldots, n-1\}$$

and define the intervals $L = [\mathbf{T}^l(0), D_1(\mathbf{T}))$ and $R = [D_1(\mathbf{T}), \mathbf{T}^r(0)]$. If we stack the intervals $\mathbf{I} = [\mathbf{T}^i(0), \mathbf{T}^j(0)]$, $i \neq j$ and $i, j \in \{0, 1, \ldots, n\}$, $int(\mathbf{I}) \bigcap \{\mathbf{T}^k(0)\}_{k=1}^n \neq \emptyset$, by putting $\mathbf{I}_2$ on top of $\mathbf{I}_1$ iff $\mathbf{T}(\mathbf{I}_1) = \mathbf{I}_2$ we see these intervals fall into two stacks that partition the set $\{\mathbf{I}\}$. These stacks have in their tops a pair of contiguous intervals with $\mathbf{T}^n(0)$ as their common vertex. The union of these tops is $L \bigcup R$ and, as for their bottons, we have: $\mathbf{T}^2(L)$ is the botton of the right stack and $\mathbf{T}(R)$ of the left stack. Note that all intervals of each stack have the same length.

We will use these stacks to parametrize the Farey cell $\mathcal{F}_s$, where $s = \max\{l, r\}$, since it is clear that once we have a pair of stacks, specified by the lengths of the intervals L and R and their heights, specified by the number of slices $\mathbf{I}$ in each stack, we can reassemble $[0,1)$ by putting one slice next to the other and recover $\mathbf{T} \in \mathcal{F}_s$ by moving up in each stack. On the other hand, once we have these two stacks we can easily construct the next pair accordingly to $R \geq L$ or $L > R$, where we are denoting the lengths of the intervals L and R also by L and R, respectively.

In fact, if $R \geq L$ the next pair will have a left stack of width $R - L$ and the same height as the initial left stack and a right stack with the same width L as the initial right stack but height the sum of the previous two heights. If $L > R$ the next pair will have a right stack of width $L - R$ and the same height as the initial right stack and a left stack with the same width R as the initial left stack but height the sum of the previous two heights.

Normalizing the lengths of these intervals by the requirement $L + R = 1$ and using this equality to eliminate R we get the Gauss map $\mathcal{G}\colon [0,1) \longrightarrow [0,1)$, which is the map that takes a pair of stacks to the next pair, defined for $x = L$ by:

$$\mathcal{G}(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \leq x < \frac{1}{2}; \\ 2 - \frac{1}{x}, & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

2 Farey Cells

Fix an integer $m \geq 2$ and a permutation π of the set $\{1, 2, \ldots, m\}$. For each m rows columm matrix $\alpha = (\alpha_1, \ldots, \alpha_m)^t \in \mathcal{S}_m$, where

$$\mathcal{S}_m = \{\alpha \in \mathbf{R}^m \mid \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i > 0 \text{ for } i = 1, \ldots, m\}$$

π induces a bijection $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ of the unit half-open interval $[0,1)$ called the *interval exchange map* induced by π using α as follows:

Starting at zero, partition $[0,1)$ into m half-open intervals of lengths $\alpha_1, \ldots, \alpha_m$, respectively. Next permute these intervals using π, that is, in the first place put the $\pi^{-1}(1)$-th interval, in the second place put the $\pi^{-1}(2)$-th interval and so on and so forth .... Finally put the permuted intervals back, one next to the other, in order to reassemble $[0,1)$. $\mathbf{T}$ is the map that takes each inicial interval onto its permuted through a translation.

More precisely, given $\alpha \in \mathcal{S}_m$, we define $D_i = D_i(\alpha)$; $i = 0, \ldots, m$ as $D_0 = 0$ and $D_i = \sum_{k=1}^{i} \alpha_k$ and the intervals $\mathbf{I}_i = \mathbf{I}_i(\alpha); i = 1, \ldots, m$, by

$\mathbf{I}_i = [D_{i-1}, D_i)$. Thus $\{\mathbf{I}_i\}_{i=1}^m$ is a partition of $[0,1)$ and $length(\mathbf{I}_i) = \alpha_i$. Now, using π, define $\alpha_i^\pi; i = 1, \ldots, m$ as $\alpha_i^\pi = \alpha_{\pi^{-1}(i)}$ and, as before, the corresponding $D_i^\pi(\alpha) = D_i(\alpha^\pi); i = 0, \ldots, m$ and $\mathbf{I}_i^\pi(\alpha) = \mathbf{I}_i(\alpha^\pi); i = 1, \ldots, m$. $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ is given by :

$$\mathbf{T}(x) = x - D_{i-1} + D_{\pi(i)-1}^\pi \text{ for } x \in \mathbf{I}_i \text{ and } i = 1, \ldots, m$$

As we will fix π from now on, we are going to identify $\mathbf{T}$ with α via the map $\alpha \mapsto \mathbf{T}(\pi, \alpha)$. Thus $\mathcal{S}_m = \mathcal{S}_m(\pi)$, the space of interval exchange maps (induced by π), has a topology and an affine structure induced from $\mathbf{R}^m$ and we are allowed to make convex linear combinations of interval exchange maps (induced by π) and still get interval exchange maps (induced by π).

As $\mathbf{T}$ takes $\mathbf{I}_i$ isometric and increasingly onto $\mathbf{I}_{\pi(i)}^\pi$; $i = 1, \ldots, m$, $\mathbf{T}$ is continuous, but for points in $\{D_i\}$, where it is right continuous.

To ensure that $\mathbf{T}$ is discontinuous at the set $\{D_i\}$ and we have actually m intervals permuted we suppose from now on that π is *discontinuous*, meaning by this that:

$$\pi(1) + 1 \neq \pi(1 + 1); i = 1, \ldots, m - 1$$

Another restriction we are going to make on π is that π be *irreducible* which means:

$$\pi\{1, 2, \ldots, i\} = \{1, \ldots, i\} \text{ and } 1 \leq i \leq m \Longrightarrow i = m$$

This is not a serious restriction since clearly the dynamics of an interval exchange map induced by a non-irreducible permutation can be decomposed and analysed in terms of maps induced by irreducible ones.

If $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ is an interval exchange map, $\mathbf{T}^{-1}$ is a map of the same kind. $\mathbf{T}^{-1}$ is induced by π^{-1} using $\alpha^\pi \in \mathcal{S}_m$. It is also easy to see that $\mathbf{T}^n$, $n \neq 0$, is an interval exchange map and the extremes of the permuted intervals lie in the $\mathbf{T}$-orbit of the $\mathbf{T}$-discontinuities. Using this fact and noting that a bijection $\mathbf{T}: [0,1) \longrightarrow [0,1)$ is an interval exchange map if and only if its graph is made of a finite number of half-open intervals parallel to the graph of the identity, we see that if an interval exchange map $\mathbf{T}$ has a periodic point p of period n then p lies in maximal half-open interval of periodic points with the same period n and the extremes of this interval are in the $\mathbf{T}$-orbit of the $\mathbf{T}$-discontinuities. Thus if $\mathbf{T}$ has at least one positive dense orbit, say the positive orbit of 0, then $\mathbf{T}$ is free of periodic orbits. As a matter of fact, much more is true: in this case $\mathbf{T}$ is *minimal* which means that every positive orbit is dense, Keane [2]. On the other hand a sufficient condition for $\mathbf{T}$ to be minimal is that $\mathbf{T}$ satisfies Keane's *infinite and distinct orbit condition*, i.d.o.c., already defined in the introduction as the condition that the $\mathbf{T}$–orbit of the $\mathbf{T}$-discontinuities be infinite and distinct.

Finally, a sufficient condition for i.d.o.c. is that α be *irrational* which means that;

$$\sum_{i=1}^{m} n_i\alpha_i = n \text{ for } n, n_i \in \mathbf{Z} \Longrightarrow n_i = n,\ \forall i$$

Following Veech [7] we define the skew-symetric matrix $\mathbf{L} = \mathbf{L}^{\pi}$ as $\mathbf{L} = \mathbf{E} - \Pi^t\mathbf{E}\Pi$ where $\mathbf{E}$ is a $m \times m$ matrix with zeros on and bellow the main diagonal and ones above and Π is the matrix of the permutation π given by: $\Pi_{ij} = \delta_{i\pi(j)}$ where $\delta_{kl}; i, j = 1, \ldots, m$ are the entries of the $m \times m$ identity matrix. $\mathbf{L}$ is defined so that:

$$\mathbf{T}(x) = x + e_i\mathbf{L}\alpha\ ;\ x \in\ \mathbf{I}_i \text{ and } i = 1, \ldots, m.$$

holds true, where $e_i = i$-th row of the $m \times m$ identity matrix.

Defining the m columns row matrices $T^k; k = 0, 1, 2, \ldots$ by $T^0 = 0$ and $T_i^k = \#\{j \mid \mathbf{T}^j(0) \in \mathbf{I}_i;\ 0 \leq j < k\}$; $i = 1, \ldots, m$ and $k = 1, 2, \ldots$ we see that, for $k \geq 0$ and $i = 1, \ldots, m$, we have:

$$T^{k+1} - T^k = e_i \Longleftrightarrow \mathbf{T}^k(0) \in \mathbf{I}_i$$

Lemma 2.1 $\mathbf{T}^k(0) = T^k\mathbf{L}\alpha;\ k = 0, 1, \ldots$

Proof: Induction on $k \geq 0$. For $k = 0$ the lemma is clear. Suppose $\mathbf{T}^{k-1}(0) = T^{k-1}\mathbf{L}\alpha$ for $k \geq 1$. Let $i \in \{1, \ldots, m\}$ such that $\mathbf{T}^{k-1}(0) \in \mathbf{I}_i$. Using we have

$$\mathbf{T}^k(0) = \mathbf{T}(\mathbf{T}^{k-1}(0)) = \mathbf{T}^{k-1}(0) + e_iL\alpha =$$

$$T^{k-1}L\alpha + e_iL\alpha = (T^{k-1} + e_i)L\alpha = T^kL\alpha$$

which proves the lemma.

Let $\mathbf{F} = \mathbf{E} + \mathbf{Id}$ where $\mathbf{Id}$ is the $m \times m$ identity matrix. Using $\mathbf{E}$ and $\mathbf{F}$ we can write:

$$\mathbf{T}^k(0) \in \mathbf{I}_i \Longleftrightarrow e_i\mathbf{E}^t\alpha \leq T^k\mathbf{L}\alpha < e_i\mathbf{F}^t\alpha$$

for $k \geq 0$ and $i = 1, \ldots, m$. Thus:

$$(T^{k+1} - T^k)\mathbf{E}^t\alpha \leq T^k\mathbf{L}\alpha < (T^{k+1} - T^k)\mathbf{F}^t\alpha$$

for $k \geq 0$ and $\mathbf{T} = \mathbf{T}(\pi, \alpha)$.

Interval exchange maps are closely related to measured foliations on surfaces. To every interval exchange map $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ we are going to associate a Riemann surface $\mathcal{R}$ and a quadratic differential $w = w(\mathbf{T})$ on $\mathcal{R}$ whose vertical foliation,$\mathcal{V} = \mathcal{V}(\mathbf{T})$, when conveniently oriented, induces $\mathbf{T}$ as a first return map on the essentially unique non-singular horizontal leave. To get $\mathcal{R}$ start with the rectangle $[0,1]^2$ in the complex plane $z = x + iy \in \mathbf{C}$ and decompose $[0,1] \times \{0\}$ in the intervals $closure(\mathbf{I}_j^\pi)$ and $[0,1] \times \{1\}$ in the intervals $closure(\mathbf{I}_i)$;$i, j = 1, \ldots, m$. We get $\mathcal{R}$ from $[0,1]^2$ identifying through a translation the interval $closure(\mathbf{I}_i)$ with the interval $closure(\mathbf{I}_{\pi(i)}^\pi)$ for $i = 1, \ldots, m$ and the interval $\{0\} \times [0,1]$ with the interval $\{1\} \times [0,1]$. It is clear that $\mathcal{R}$ has a conformal structure induced from $\mathbf{C}$ and that dz goes down to a holomorphic differential whose square we denote by w. The vertical (horizontal) straight line segments of $[0,1]^2$ go down to form the vertical (resp.horizontal) leaves of w and the set of points $D_i \equiv D_{\pi(i)}^\pi$ contains the set of zeros of w and therefore the set of singularities of its vertical and horizontal foliations.

Using $\frac{\partial}{\partial y}$ $(\frac{\partial}{\partial x})$ to orient the vertical (resp. horizontal) foliation of w we look at the first return map induced by $\mathcal{V}$ on the horizontal leaf at height $y = 1/2$. This map is well defined except at the meeting points of this leaf with the separatrices of the singularities. Extending this map by demanding right continuity at these points we get the interval exchange map $\mathbf{T}$. Using $\mathcal{R}$ we see that $x_1 = \mathbf{T}(x_0)$ iff the union of segments of $[0,1]^2$, $(x_0 + \frac{1}{2}i, x_0 + i]$ and $(x_1, x_1 + i]$, go down in $\mathcal{R}$ to make a connected subset of a union of vertical leaves and singularities and the set $\{0, \mathbf{T}(0), \ldots, \mathbf{T}^n(0)\}$ is represented by a union of $n+1$ vertical segments that go down in $\mathcal{R}$ to make up a closed connected graph which is a union of singularities and segments of vertical leaves with total length n, starting at $\frac{1}{2}i$ and ending $\mathbf{T}^n(0) + \frac{1}{2}i$.

For each $n \geq 0$ define the equivalence relation $\overset{n}{\sim}$ on $\mathcal{S}_m$ as follows:

$$\mathbf{T} \overset{n}{\sim} \mathbf{S} \text{ iff } \mathbf{T}^k(0) \in \mathbf{I}_i(\mathbf{T}) \Leftrightarrow \mathbf{S}^k(0) \in \mathbf{I}_i(\mathbf{S});\ k = 1, \ldots, n \text{ and } i = 1, \ldots, m$$

The *Farey cell of order* $n \geq 0$ around an interval exchange map $\mathbf{T} = \mathbf{T}(\pi, \alpha)$, $\mathcal{F}_n = \mathcal{F}_n(\mathbf{T})$, is defined as the equivalence class of $\mathbf{T}$ under $\overset{n}{\sim}$. Thus:

$$\mathcal{F}_n = \{\mathbf{S}(\beta) \mid \mathbf{T}^k(0) \in \mathbf{I}_i(\mathbf{T}) \Leftrightarrow \mathbf{S}^k(0) \in \mathbf{I}_i(\mathbf{S});\ k = 1, \ldots, n \text{ and } i = 1, \ldots, m\}$$

Using the remark following the preceeding lemma have:

$$\mathbf{S}(\beta) \in \mathcal{F}_n(\mathbf{T}) \Leftrightarrow (T^{k+1} - T^k)\mathbf{E}^t\beta \leq T^k\mathbf{L}\beta < (T^{k+1} - T^k)\mathbf{F}^t\beta$$

for $k = 0, \ldots, n$ and conclude that $\mathcal{F}_n$ is a convex polyhedron of $\mathcal{S}_m$.

Thus, to get all the Farey cells of order $n \geq 1$ without talking about interval exchange maps we have to find all the sequences $\xi_1, \ldots, \xi_{n+1} \in \{e_1, \ldots, e_m\}$, $\xi_1 = e_1$, such that the set of diophantine inequalities :

$$\xi_{k+1}\mathbf{E}^t\alpha \leq (\sum_{i=1}^{k} \xi_i)\mathbf{L}\alpha < \xi_{k+1}\mathbf{F}^t\alpha$$

$k = 1, \ldots, n$, have a solution in the cone of positive $\alpha = (\alpha_1, \ldots, \alpha_m)^t$. If this is the case, the set of positive solutions of these inequalities in $\mathcal{S}_m$ is $\mathcal{F}_n(\mathbf{T})$, where $\mathbf{T} = \mathbf{T}(\alpha)$ and α is any solution of the above system in $\mathcal{S}_m$. If this is the case $T^k = \sum_{l=1}^{k} \xi_i$, $k = 1, \ldots, n+1$.

Two distinct Farey cells $\mathcal{F}_{n_1}$ and $\mathcal{F}_{n_2}$ are either disjoint or one, say $\mathcal{F}_{n_1}$, is contained in the other, $\mathcal{F}_{n_2}$, and in this case we have $n_1 > n_2$.

Proposition 2.1 *The interior of a Farey cell $\mathcal{F}_n$ in $\mathcal{S}_m$ is non-empty.*

Proof: Take $\mathbf{T} = \mathbf{T}(\alpha) \in \mathcal{S}_m$. If $\alpha \notin interior(\mathcal{F}_n(\mathbf{T}))$ we have $\mathbf{T}^k(0) = D_i(\alpha)$ for some $k \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, m\}$. We will show that for $\epsilon > 0$ small enough

$$\tilde{\alpha} = (\frac{\alpha_1 - \epsilon}{1 - \epsilon}, \frac{\alpha_2}{1 - \epsilon}, \ldots, \frac{\alpha_m}{1 - \epsilon}) \in interior(\mathcal{F}_n)$$

To see the truth of this assertion we will make use of the Riemann surface $\mathcal{R}(\mathbf{T})$, the quadratic differential $w = w(\mathbf{T})$ and the vertical foliation $\mathcal{V}(\mathbf{T})$ induced by w. Denote by F the union of segments of leaves and singularities of $\mathcal{V}$ representing the $\mathbf{T}$-positive orbit of 0 up to the n-th iterate and take a point $x + \epsilon + i \in [0,1]^2$ where $x, \epsilon > 0$ are so small $x + \epsilon < D_1$ and there is no point of $\{0, \mathbf{T}(0), \ldots, \mathbf{T}^n(0)\}$ in the interval $(0, x+\epsilon)$. Consider the polygonal line through the sequence of points $x + \epsilon + i$; $x + \epsilon + \frac{2}{3}i$; $D^{\pi}_{\pi(1)-1} + x + \epsilon + \frac{2}{3}i$ and $D^{\pi}_{\pi(1)-1} + x + \epsilon$ and the polygonal line through the sequence of points we get by dropping the summand $\epsilon > 0$ in the above sequence. These two polygonal lines cut the square $[0,1]^2$ in four components, two of them are rectangles of width $\epsilon > 0$; one, R', above the line $y = \frac{2}{3}$ the other, R'', bellow. If we cut these rectangles from the square, push to the left the remaining right connected component and make the obvious identifications in the rectangle thus obtained we get a Riemann surface $\widetilde{\widetilde{R}}$ which is $\widetilde{R}$ with the horizontal lengths scaled by a factor of $1 - \epsilon$. Now, the effect of this cutting and gluing on F is to displace by ϵ to the right each crossing of this set in the interval $[x + \epsilon + \frac{2}{3}i, D^{\pi}_{\pi(1)-1} + x + \epsilon + \frac{2}{3}i]$, which includes at least the crossing corresponding to $\mathbf{T}(0)$. Thus if we choose $\epsilon > 0$ small enough we can prevent $\widetilde{\widetilde{F}}$ in $\widetilde{\widetilde{R}}$ from hitting any $D_i = D_{\pi(i)}$ as we increase the widths of the rectangles R' and R'' from 0 to ϵ. Since a horizontal scaling of $1 - \epsilon$ does not destroy these relationships we see that if $\epsilon > 0$ is small enough $\widetilde{\alpha} \in \mathcal{F}_n$ and, on account of this right displacement of F starting at $\mathbf{T}(0)$, we have in fact $\widetilde{\alpha} \in interior(\mathcal{F}_n)$, proving the proposition.

It is easy to see that the map:

$$k \in \{0, \ldots, n\} \mapsto \mathbf{T}^k(0) \in [0, 1)$$

is injective, for every $\mathbf{T} \in interior(\mathcal{F}_n)$ and, using the proposition, we see that the map:

$$\mathbf{T} \in \mathcal{F}_n \mapsto \mathbf{T}^k(0) \in [0, 1)$$

$k = 1, \ldots, n + 1$ is the restriction of a non-trivial linear functional.

Proposition 2.2 *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be in the interior of a Farey cell* $\mathcal{F}_n$ *then:*

$$\mathbf{S}^k(0) < \mathbf{S}^l(0) \iff \mathbf{T}^k(0) < \mathbf{T}^l(0) \text{ for } k \text{ and } l \in \{0, \ldots, n\}$$

Proof: To get a contradiction, suppose there are k and l for which our hypothesis does not hold. Take k and l such that $k + l$ is minimum with this property. Without loss of generality we can suppose $\mathbf{S}^k(0) < \mathbf{S}^l(0)$ and $\mathbf{T}^k(0) \geq \mathbf{T}^l(0)$. But then $\mathbf{T}^k(0) > \mathbf{T}^l(0)$ otherwise 0 is $\mathbf{T}$-periodic and we would have $\mathbf{T}^j(0) = D_{\pi^{-1}(1)-1}(\alpha)$ for some $j \in \{1, \ldots, n\}$ contradicting the fact that $\mathbf{T} \in interior(\mathcal{F}_n)$. We can also suppose that $\mathbf{S}^{k-1}(0) < \mathbf{S}^{l-1}(0)$ and we have by the minimal property of k and l, $\mathbf{T}^{k-1}(0) < \mathbf{T}^{l-1}(0)$. By hypothesis $\mathbf{S}^{k-1}(0) \in \mathbf{I}_i(\beta)$, $\mathbf{T}^{k-1}(0) \in \mathbf{I}_i(\alpha)$, and $\mathbf{S}^{l-1}(0) \in \mathbf{I}_j(\beta)$, $\mathbf{T}^{l-1}(0) \in \mathbf{I}_j(\alpha)$ for i , $j \in \{1, \ldots, m\}$. As $\mathbf{S}^{k-1}(0) < \mathbf{S}^{l-1}(0)$ we see that $i \leq j$, but $i = j$ is an absurd since, using the fact that $\mathbf{T}$ is order preserving in $\mathbf{I}_i(\alpha)$, we would have $\mathbf{T}^k(0) < \mathbf{T}^l(0)$. Thus $i < j$. Now $\mathbf{S}^k(0) \in \mathbf{I}^{\pi}_{\pi(i)}(\beta)$ and $\mathbf{S}^l(0) \in \mathbf{I}^{\pi}_{\pi(j)}(\beta)$ and since $\mathbf{S}^k(0) < \mathbf{S}^l(0)$ we see that $\mathbf{I}^{\pi}_{\pi(i)}(\beta)$ is bellow $\mathbf{I}^{\pi}_{\pi(j)}(\beta)$; but $\mathbf{T}$ and $\mathbf{S}$

are induced by the same permutation π and therefore the same relation holds between the intervals $\mathbf{I}^{\pi}_{\pi(i)}(\alpha)$ and $\mathbf{I}^{\pi}_{\pi(j)}(\alpha)$ but then $\mathbf{T}^k(0) < \mathbf{T}^l(0)$, again a contradiction which proves the proposition.

This proposition shows that if we fix a Farey cell $\mathcal{F}_n$, the order induced on the set $\{0, \ldots, n\}$ by the natural order on $[0,1)$ via the injection $k \mapsto \mathbf{T}^k(0)$ is independent of $\mathbf{T} \in interior(\mathcal{F}_n)$ and that the bijective correspondence:

$$\mathbf{T}^k(0) \longleftrightarrow \mathbf{S}^k(0) \text{ and } D_i(\mathbf{T}) \longleftrightarrow D_i(\mathbf{S})$$

for $k \in \{0, \ldots, n\}$, $i \in \{0, \ldots, m\}$ and $\mathbf{T}, \mathbf{S} \in interior(\mathcal{F}_n)$ between the sets $\{\mathbf{T}^k(0)\}_{k=0}^n \bigcup \{D_i(\mathbf{T})\}_{i=0}^m$ and $\{\mathbf{S}^k(0)\}_{k=0}^n \bigcup \{D_i(\mathbf{T})\}_{i=0}^m$ is order preserving.

We say $n \geq 0$ or, more properly, $\mathbf{T}^n(0)$ is a *critical iterate* of $\mathbf{T}$ if there is a discontinuity of $\mathbf{T}$, $D_i(\mathbf{T})$, $i \in \{1, \ldots, m\}$, such that one of the intervals $[\mathbf{T}^n(0), D_i(\mathbf{T}))$ or $[D_i(\mathbf{T}), \mathbf{T}^n(0)]$ only intercepts $\{\mathbf{T}^k(0)\}_{k=0}^n$ at $\mathbf{T}^n(0)$.

If there is only one discontinuity satisfying this condition we say i is the *type of the critical iterate* and call the iterate *left* or *right critical iterate* accordingly to its position with respect to the discontinuity. If the condition above holds true for $p > n$ that is, one of the intervals $[\mathbf{T}^n(0), D_i(\mathbf{T}))$ or $[D_i(\mathbf{T}), \mathbf{T}^n(0)]$ only intercepts $\{\mathbf{T}^k(0)\}_{k=0}^p$ at $\mathbf{T}^n(0)$, we say n *remains critical up to the order p*.

The reason why we need the notion of critical iterates can be seen in the next proposition, but before we state and prove this proposition we will show that an interval exchange map has plenty of critical iterates.

Lemma 2.2 *An interval exchange map $\mathbf{T} \in \mathcal{S}_m$ has arbitrarily large critical iterates.*

Proof: If 0 is $\mathbf{T}$-periodic the result is clear. To get a contradiction suppose that there is $s \geq 0$ such that n is not critical for $n > s$. Take $i \in \{1, \ldots, m\}$ such that $\mathbf{T}^{s+1}(0) \in \mathbf{I}_i$. If $i < m$ using that $\mathbf{T}^{s+1}(0)$ is not a critical iterate, there is $0 \leq k \leq s$ such that $\mathbf{T}^{s+1}(0) < \mathbf{T}^k(0) < D_i(\mathbf{T})$. We can suppose:

$$\mathbf{T}^k(0) = \min\{\mathbf{T}^l(0) \mid \mathbf{T}^l(0) > \mathbf{T}^{s+1}(0) \text{ and } k \in \{0, \ldots, s\}\}$$

Consider the interval $I = [\mathbf{T}^{s+1}(0), \mathbf{T}^k(0)]$. $\mathbf{T}^n(I)$ is an interval for every $n \geq 0$ otherwise we would have a first $n \geq 0$ such that $\mathbf{T}^n(I) \bigcap \{D_j(\mathbf{T})\}_{j=1}^{m-1} \neq \emptyset$. Let $j \in \{1, \ldots, m\}$ be such that $D_j(\mathbf{T}) \in \mathbf{T}^n(I)$. Since $\mathbf{T}^{s+n+1}(0)$ is not a critical iterate we can get $0 \leq p \leq s+n$ such that $\mathbf{T}^p(0) \in interior(\mathbf{T}^n(I))$.

Decrease n if necessary in such a way that n is the first integer such that there is $0 \leq p \leq s+n$ with $\mathbf{T}^p(0) \in interior(\mathbf{T}^n(I))$. But then $\mathbf{T}^{p-1}(0) \in interior(\mathbf{T}^{n-1}(I))$ which is only possible if $p = 0$ and this is an absurd since then we would have $0 = \mathbf{T}^p(0) \in interior(\mathbf{T}^n(I))$. Thus we must have $\mathbf{T}^n(I)$ an interval for every n, but that is an absurd also since the sequence of intervals $I, \mathbf{T}^l(I), \mathbf{T}^{2l}(I), \ldots, l = s-k+1$ is an infinite sequence of contiguous non-trivial intervals in $[0,1)$. If $i = m$ we can use the discontinuity $D_{m-1}(\mathbf{T})$ to get $\mathbf{T}^{s+1}(0) > \mathbf{T}^k(0) > D_{m-1}(\mathbf{T})$ and repeat the above argument to get a contradiction. This concludes the proof of the lemma.

If $\mathcal{F}_n$ is a Farey cell then $c = c(\mathcal{F}_n) =$ the greatest $\mathbf{T}$-critical iterate not greater than n is independent of $\mathbf{T} \in interior(\mathcal{F}_n)$ since this integer only depends on the order induced on the set $\{\mathbf{T}^k(0)\}_{k=0}^n \bigcup \{D_i(\mathbf{T})\}_{i=0}^m$ by $[0,1)$.

Proposition 2.3 *$\mathcal{F}_n = \mathcal{F}_n(\mathbf{T}) = \mathcal{F}_c(\mathbf{T})$, for $c = c(\mathcal{F}_n)$ and every $\mathbf{T} \in interior(\mathcal{F}_n)$.*

Proof: It is clear that $\mathcal{F}_n(\mathbf{T}) \subseteq \mathcal{F}_c(\mathbf{T})$. We will prove that $\mathcal{F}_n(\mathbf{T}) \supseteq \mathcal{F}_c(\mathbf{T})$ for every $\mathbf{T} \in interior(\mathcal{F}_n(\mathbf{T}))$ by induction on n. For $n = 0$ the statement is trivial. Suppose, to get a contradiction, that the statement is true for orders $< n$ but $\mathcal{F}_n \neq \mathcal{F}_c(\mathbf{T})$ for some $\mathbf{T} \in interior(\mathcal{F}_n)$. Since $\mathcal{F}_c(\mathbf{T}) = \mathcal{F}_{n-1}(\mathbf{T})$ we have $\mathcal{F}_n(\mathbf{T}) \subseteq \mathcal{F}_{n-1}(\mathbf{T})$ but $\mathcal{F}_n(\mathbf{T}) \neq \mathcal{F}_{n-1}(\mathbf{T})$. Take $\mathbf{V} \in \mathcal{F}_{n-1}(\mathbf{T})$ but $\mathbf{V} \notin \mathcal{F}_n(\mathbf{T})$ and consider the closed interval $[\mathbf{T}, \mathbf{V}] \subseteq \mathcal{F}_{n-1}(\mathbf{T})$ oriented from $\mathbf{T}$ to $\mathbf{V}$. Let $\mathbf{S}$ be the supremum of the points $\mathbf{U} \in [\mathbf{T}, \mathbf{V}]$ such that $\mathbf{U} \in \mathcal{F}_n$ and take $i \in \{1, \ldots, m\}$ such that $\mathbf{T}^n(0) \in \mathbf{I}$. Since $\mathbf{V} \notin \mathcal{F}_n$ but $\mathbf{V} \in \mathcal{F}_{n-1}(\mathbf{T})$ we have $\mathbf{V}^n(0) < D_{i-1}(\mathbf{V})$ or $D_i(\mathbf{V}) \leq \mathbf{V}^n(0)$. In the first case, since n is not critical for $\mathbf{T}$ we can find $0 \leq l < n$ such that $D_{i-1}(\mathbf{T}) < \mathbf{T}^l(0) < \mathbf{T}^n(0)$ and considering the linear maps $\mathbf{U} \mapsto \mathbf{U}^l(0) - D_{i-1}(\mathbf{U})$ and $\mathbf{U} \mapsto \mathbf{U}^n(0) - D_{i-1}(\mathbf{U})$ on $[\mathbf{T}, \mathbf{V}]$, we have:

$$0 < \mathbf{U}^l(0) - D_{i-1}(\mathbf{U}) < \mathbf{U}^n(0) - D_{i-1}(\mathbf{U})$$

for $\mathbf{U} \in [\mathbf{T}, \mathbf{S})$ and $\mathbf{U}^n(0) - D_{i-1}(\mathbf{U}) < 0$ for $\mathbf{U} = \mathbf{V}$ which is only possible if $\mathbf{S}^l(0) = D_{i-1}(\mathbf{S})$ and therefore $\mathbf{U}^l(0) < D_{i-1}(\mathbf{U})$ for $\mathbf{U}$ very close and after $\mathbf{S}$, a contradiction with $\mathbf{S}^l(0) \in \mathbf{I}_i(\mathbf{U})$ since $\mathbf{S} \in \mathcal{F}_{n-1}(\mathbf{T})$. In the second case, since n is not critical for $\mathbf{T}$ we can find $0 \leq k < n$ such that $\mathbf{T}^n(0) < \mathbf{T}^k(0) < D_i(\mathbf{T})$ and considering the linear maps $\mathbf{U} \mapsto \mathbf{U}^k(0) - D_i(\mathbf{U})$ and $\mathbf{U} \mapsto \mathbf{U}^n(0) - D_i(\mathbf{U})$ on $[\mathbf{T}, \mathbf{V}]$, we have:

$$\mathbf{U}^n(0) - D_i(\mathbf{U}) < \mathbf{U}^k(0) - D_i(\mathbf{U}) < 0$$

for $\mathbf{U} \in [\mathbf{T}, \mathbf{S})$ and $0 \leq \mathbf{U}^n(0) - D_i(\mathbf{U})$ for $\mathbf{U} = \mathbf{V}$ which is only possible if $\mathbf{S}^k(0) = D_i(\mathbf{S})$, a contradiction with $\mathbf{S}^k(0) \in \mathbf{I}_i(\mathbf{U})$ since $\mathbf{S} \in \mathcal{F}_{n-1}(\mathbf{T})$.

Now, fix a Farey cell $\mathcal{F}_n$, take $\mathbf{T} \in interior(\mathcal{F}_n)$ and define $l_1 = l_1(\mathbf{T}), \ldots, l_{m-1} = l_{m-1}(\mathbf{T})$ and $r_1 = r_1(\mathbf{T}), ..., r_{m-1} = r_{m-1}(\mathbf{T})$ by:

$$\mathbf{T}^{l_j}(0) = \max\{\mathbf{T}^q(0) < D_j(\alpha) \mid 0 \leq q \leq n\}$$

$$\mathbf{T}^{r_k}(0) = \min\{\mathbf{T}^q(0) > D_k(\alpha) \mid 0 \leq q \leq n\}$$

In other words, l_j and r_k are, respectively, the critical left and right iterates that remain up to the order $n; j, k = 1, ..., m-1$. It is clear that these definitions are independent of $\mathbf{T} \in interior(\mathcal{F}_n)$ and we can write, in fact, $l_j = l_j(\mathcal{F}_n)$ and $r_k = r_k(\mathcal{F}_n)$.

We saw above that a Farey cell $\mathcal{F}_n$ is defined by a set of inequalities on $\mathbf{T} \in \mathcal{S}_m$:

$$D_{i_k-1}(\mathbf{T}) \leq \mathbf{T}^k(0) < D_{i_k}(\mathbf{T});\ k = 1, \ldots, n$$

where the i_k's are chosen in $\{1, \ldots, m\}$.

We can rewrite these inequalities more conveniently as:

$$0 \leq \mathbf{T}^k(0) - D_{i_k-1}(\mathbf{T}) \text{ and } \mathbf{T}^k(0) - D_{i_k}(\mathbf{T}) < 0 \text{ for } k = 1, \ldots, n \qquad (2)$$

The same kind of ideas that lead to the proof of our last result can be used to prove the next proposition.

Proposition 2.4 *The set of inequalities defining $\mathcal{F}_n$ is equivalent to the subset:*

$$0 \leq \mathbf{T}^{r_i}(0) - D_i(\mathbf{T}) \text{ and } \mathbf{T}^{l_i}(0) - D_i(\mathbf{T}) \ < 0 \text{ for } i = 1, \ldots, m-1 \qquad (3)$$

Proof: Start by removing one by one the inequalities of 2) that are not in 3) and are redundant. Reasoning by absurd, suppose that after the completion of this process there remains an inequality not in 3):

$$\mathbf{T}^k(0) - D_i(\mathbf{T}) < 0 \text{ for } k \neq l_i \qquad (4)$$

or

$$0 \leq \mathbf{T}^k(0) - D_i(\mathbf{T}) \text{ for } k \neq r_i \qquad (5)$$

This means there is a point $\mathbf{S} \in \mathcal{S}_m$ not verifying (4) or (5) but verifying all other remaining inequalities, which includes (3). Take $\mathbf{U} \in interior(\mathcal{F}_n)$.

Let $\mathbf{V}$ be the point of intersection of the interval $[\mathbf{U}, \mathbf{S}]$ with the hyperplane $\mathbf{T}^k(0) = D_i(\mathbf{T})$. We have $[\mathbf{U}, \mathbf{V}) \subseteq interior(\mathcal{F}_n)$. If $\mathbf{S}$ does not verify we have $0 \leq \mathbf{S}^k(0) - D_i(\mathbf{S})$. Using that $k \neq l_i$ we have:

$$\mathbf{T}^k(0) - D_i(\mathbf{T}) < \mathbf{T}^{l_i}(0) - D_i(\mathbf{T}) < 0 \text{ for } \mathbf{T} \in [\mathbf{U}, \mathbf{V})$$

and therefore, observing the linear maps $\mathbf{T} \mapsto \mathbf{T}^{l_i}(0) - D_i(\mathbf{T})$ and $\mathbf{T} \mapsto \mathbf{T}^k(0) - D_i(\mathbf{T})$ on the interval $[\mathbf{U}, \mathbf{S}]$, we get $0 \leq \mathbf{S}^{l_i}(0) - D_i(\mathbf{S})$ a contradiction since $\mathbf{S}$ satisfies all inequalities (3). If $\mathbf{S}$ does not verify (5) we have $\mathbf{S}^k(0) - D_i(\mathbf{S}) < 0$. Using that $k \neq r_i$ we have:

$$0 < \mathbf{T}^{r_i}(0) - D_i(\mathbf{T}) < \mathbf{T}^k(0) - D_i(\mathbf{T}) \text{ for } \mathbf{T} \in [\mathbf{U}, \mathbf{V})$$

and therefore, observing the linear maps $\mathbf{T} \mapsto \mathbf{T}^{r_i}(0) - D_i(\mathbf{T})$ and $\mathbf{T} \mapsto \mathbf{T}^k(0) - D_i(\mathbf{T})$ on the interval $[\mathbf{U}, \mathbf{S}]$, we get $\mathbf{S}^{r_i}(0) - D_i(\mathbf{S}) < 0$ again a contradiction since $\mathbf{S}$ satisfies all inequalities (3). The proposition is proved.

We finish this section recalling a definition from the introduction. A Farey cell $\mathcal{F}_n$ is called *small* iff each interval $\mathbf{I}_i(\mathbf{T})$ ($\mathbf{I}_i^\pi(\mathbf{T})$), $i = 1, \ldots, m$ has at least one point of the set $\{\mathbf{T}^k(0)\}_{k=1}^n$ for some (and therefore all) $\mathbf{T} \in interior(\mathcal{F}_n)$.

It is clear that given an i.d.o. $\mathbf{T}$ the Farey cell around $\mathbf{T}$, $\mathcal{F}_n(\mathbf{T})$, is small if n is great enough. In the next section we give a description of the small Farey cells.

3 Stacks

In this section we give a combinatorial description of the small Farey cells and show that they belong to a finite set of projective types.

Given π a permutation of $\{1, ..., m\}$ irreducible and discontinuous, define:

$$f = f(\pi) \colon \{0, ..., m-1\} \to \{1, ...m\}$$

by:

$$f(j) = \begin{cases} \pi^{-1}(1) - 1, & \text{if } j = 0; \\ m, & \text{if } j = \pi^{-1}(m); \\ \pi^{-1}(\pi(j) + 1) - 1, & \text{otherwise.} \end{cases}$$

if $\pi(m)+1=\pi(1)$ and

$$f(j)=\begin{cases}\pi^{-1}(1)-1, & \text{if } j=0;\\ m, & \text{if } j=\pi^{-1}(\pi(1)-1);\\ \pi^{-1}(\pi(m)+1)-1, & \text{if } j=\pi^{-1}(m);\\ \pi^{-1}(\pi(j)+1)-1, & \text{in the remaining cases.}\end{cases}$$

if $\pi(m)+1\neq\pi(1)$.

It is easy to see that f is bijective.

Now, using f define the set $\mathcal{A}=\mathcal{A}(\pi)$ of pairs $\gamma=(g,G)$ where:

$$g\colon\{0,...,m-1\}\rightarrow\{1,...,m-1\}$$

and

$$G\colon\{1,...,m\}\rightarrow\{1,...,m-1\}$$

satisfy:

1.
$$g=G\circ f \tag{6}$$

2.
$$\{g(0),g^2(0),...,g^{m-1}(0)\}=\{1,2,...m-1\}=$$
$$\{G(m),G^{\ 2}(m),...,G^{\ m-1}(m)\} \text{ and } f(g^{m-1}(0))\neq G^{\ m-1}(m) \tag{7}$$

3. C_γ, the convex subset of $\mathbf{R}^{2(m-1)}=\{0\}\times\mathbf{R}^{m-1}\times\mathbf{R}^{m-1}\times\{0\}\subseteq\mathbf{R}^{2m}$ given by the column matrices $(L_0,L_1,...,L_{m-1},R_1,R_2,...,R_m)^t$ satisfying:

   (a)
   $$L_i+R_i=\sum_{j\in g^{-1}(i)}L_j+R_{f(j)};\ i=1,...,m-1 \tag{8}$$

   (b)
   $$L_i>0 \text{ and } R_i\geq 0\ ;\ i=1,...,m-1$$

   (c)
   $$\sum_{i=1}^{m-1}(L_i+R_i)=1 \tag{9}$$

has dimension $m-1$.

We call the convex set C_γ the *abstract Farey cell of type* γ.

It follows from (7) that g and G are onto and there is precisely one $i_0 \in \{1, ..., m-1\}$ such that $\#g^{-1}(i_0) = \#G^{-1}(i_0) = 2$. We say that i_0 is the *type* of γ or, by abuse of language, the type of g (or G).

Note that we can also write (8) as:

$$L_i + R_i = \sum_{k \in G^{-1}(i)} L_{f^{-1}(k)} + R_k \ ; \ i = 1, ..., m-1$$

or, more simetrically :

$$L_i + R_i = L_{g^{-1}(i)} + R_{G^{-1}(i)} \ ; \ i = 1, ..., m-1 \text{ and } i \neq i_0$$

$$L_{i_0} + R_{i_0} = L_{g^{-1}(i_0)} + R_{G^{-1}(i_0)} + L_{g^{m-1}(0)} + R_{G^{m-1}(m)}$$

Where $g^{-1}(G^{-1})$ is the unique right inverse of g (resp. G) which misses $g^{m-1}(0)$ (resp. $G^{m-1}(m)$) in its image.

We are going to prove that, given a small Farey cell $\mathcal{F}_n$, there is $\gamma \in \mathcal{A}$ such that $\mathcal{F}_n(\mathbf{T})$ is projectively isomorphic to some $\mathcal{C}_\gamma$, which implies the 'asymptotic' finiteness of projective types of Farey cells around any i.d.o.. Recall from the introduction that by a *projective isomorphism* we mean a bijection $\tilde{\mathcal{L}}$ which can be expressed as $\tilde{\mathcal{L}}(x) = \mathcal{L}(x)/||\mathcal{L}(x)||$, where $\mathcal{L}$ is linear, $x \in \mathbf{R}^m$ and $||x|| = \sum_{i=1}^m |x_i|$; in this case we say $\tilde{\mathcal{L}}$ is the projective map induced by $\mathcal{L}$.

Now, fix $\mathbf{T}$ an interval exchange map and $n \geq 0$ such that:

$$\{\mathbf{T}^k(0)\}_{k=0}^n \bigcap \{D_i(\mathbf{T})\}_{i=1}^{m-1} = \emptyset \tag{10}$$

Then we have that $0, \mathbf{T}(0), \mathbf{T}^2(0), \ldots, \mathbf{T}^n(0)$ are all distinct and

$$\{\mathbf{T}^k(0)\}_{k=0}^n \bigcap \{D_i^\pi(\mathbf{T})\}_{i=1}^{m-1} = \{\mathbf{T}(0)\} = \{D^\pi_{\pi(1)-1}(\mathbf{T})\}.$$

Using $\mathbf{T}$ and n we define the set $\mathcal{I} = \mathcal{I}(\mathbf{T}, n) = \{\mathbf{I}\}$ given by:

1. $\mathbf{I}$ is a non-degenerate closed interval with extremes in $\{\mathbf{T}^k(0)\}_{k=0}^n$.

2. the only points of $\mathbf{I}$ in $\{\mathbf{T}^k(0)\}_{k=0}^n$ are its extremes.

Observe that if $D_i(\mathbf{T}) \in \mathbf{I}$, $i \in \{1, \ldots, m-1\}$, then $D_i(\mathbf{T}) \in interior(\mathbf{I})$ and if $D_i^{\pi}(\mathbf{T}) \in \mathbf{I}$, $i \in \{1, \ldots, m-1\}$ then $D_i^{\pi}(\mathbf{T}) \in interior(\mathbf{I})$ unless $D_i^{\pi}(\mathbf{T}) = \mathbf{T}(0)$.

On $\mathcal{I}$ define the relation $\preceq = \preceq (\mathbf{T}, n)$:

$\mathbf{I}_1 \preceq \mathbf{I}_2$ iff $\mathbf{I}_1 = \mathbf{I}_2$ or there are $q \geq 1$ and $\mathbf{I}^0, \mathbf{I}^1, \ldots, \mathbf{I}^q \in \{\mathbf{I}\}$ such that $\mathbf{I}^0 = \mathbf{I}_1, \mathbf{I}^q = \mathbf{I}_2$ and $\mathbf{T}(\mathbf{I}^{r-1}) = \mathbf{I}^r$ for $r = 1, \ldots, q$. It is easy to see that $\preceq$ is an order relation.

We are interested on subsets $\mathcal{P} \subseteq \{\mathbf{I}\}$ which are totally ordered by the relation $\preceq$ and maximal (with respect to set inclusion) with this property. These sets we call *stacks*. Stacks are disjoint and given a stack $\mathcal{P}$ we denote by $t(\mathcal{P})$, the *top* of $\mathcal{P}$, its last element and by $b(\mathcal{P})$, the *botton* of $\mathcal{P}$, its first element. Denoting by $[\mathbf{I})$ the half-open interval we get from $\mathbf{I}$ by dropping its last extreme we can write:

$$[0, 1) = \sum_{\mathcal{P}} \sum_{\mathbf{I} \in \mathcal{P}} [\mathbf{I}) + [M, 1) \tag{11}$$

where, $\sum$ and $+$ denote disjoint union, and $M = \max\{\mathbf{T}^k(0) \mid k = 0, \ldots, n\}$.

Lemma 3.1 *Let $\mathbf{T}$ be an interval exchange map and $n \geq 0$ such that holds and take $\mathcal{I} = \mathcal{I}(\mathbf{T}, n)$ and $\preceq = \preceq (\mathbf{T}, n)$ as defined above. Let $\mathbf{I}$ be the interval $[\mathbf{T}^k(0), \mathbf{T}^l(0)]$, $0 \leq l, k \leq n$.*

1. *$\mathbf{I} = t(\mathcal{P})$ for some stack $\mathcal{P}$ iff either $\mathbf{I}$ contains a discontinuity of $\mathbf{T}$ or $k = n$ or $l = n$.*

2. *$\mathbf{I} = b(\mathcal{P})$ for some stack $\mathcal{P}$ iff $k = 0$ or $l = 1$ or $\mathbf{T}^{n+1}(0) \in interior(\mathbf{I})$ or $\mathbf{I}$ has a discontinuity D_i^{π} of $\mathbf{T}^{-1}$ for some $i = 1, \ldots, m-1$, $i \neq \pi(1) - 1$.*

Proof: 1) To get a contradiction suppose $\mathbf{I} = t(\mathcal{P})$ but $\mathbf{I} \bigcap \{D_i(\mathbf{T})\}_{i=0}^{m-1} = \emptyset$ and $k, l < n$. Then $\mathbf{T}(\mathbf{I}) = [\mathbf{T}^{k+1}(0), \mathbf{T}^{l+1}(0)]$ and since $\mathbf{I} = t(\mathcal{P})$ we must have $\mathbf{T}^p(0) \in interior(\mathbf{T}(\mathbf{I}))$ for some $1 \leq p \leq n$ but then $\mathbf{T}^{p-1}(0) \in interior(\mathbf{I})$ which contradicts $\mathbf{I} \in \mathcal{I}$.

Conversely, take $\mathcal{P}$ the stack containing $\mathbf{I}$. If $D_i(\mathbf{T}) \in \mathbf{I}$ for $i \in \{1, \ldots, m-1\}$ then $D_i \in interior(\mathbf{I})$ and $\mathbf{T}(\mathbf{I})$ is not an interval and therefore $\mathbf{T}(\mathbf{I}) \notin \mathcal{I}$, from which we get $\mathbf{I} = t(\mathcal{P})$. We can suppose then that $\mathbf{I} \bigcap \{D_i\}_{i=1}^{m-1} = \emptyset$ which implies $\mathbf{T}(\mathbf{I}) = [\mathbf{T}^{k+1}(0), \mathbf{T}^{l+1}(0)]$. By hypothesis we have $k = n$ or $l = n$. If $\mathbf{T}(\mathbf{I}) \in \mathcal{I}$ we have $\mathbf{T}^{n+1}(0) = \mathbf{T}^p(0)$ for some $0 \leq p \leq n$ which means that $\mathbf{T}^{n-p+1}(0) = 0$ or $\mathbf{T}^{n-p}(0) = D_{\pi^{-1}(1)-1}$ for $0 \le n-p \le n$, a contradiction with (10). Thus $\mathbf{T}(\mathbf{I}) \notin \mathcal{I}$ and again we get $\mathbf{I} = t(\mathcal{P})$.

2) Suppose we have $\mathbf{I} = b(\mathcal{P})$, but $\mathbf{I} \bigcap \{D_i^{\pi}(\mathbf{T})\}_{i=0 \ i \neq \pi(1)-1}^{m-1} = \emptyset$, $k > 0$ and $l > 1$. Then $\mathbf{T}^{-1}(\mathbf{I}) = [\mathbf{T}^{k-1}(0), \mathbf{T}^{l-1}(0)]$ even if $D_{\pi(1)-1}^{\pi}(\mathbf{T}) = \mathbf{T}(0) \in \mathbf{I}$ because in this case this discontinuity is the left extreme of the interval $\mathbf{I}$. But then we must have $\mathbf{T}^{-1}(\mathbf{I}) \notin \mathcal{I}$ since $\mathbf{I} = b(\mathcal{P})$ and this means there is $\mathbf{T}^{p}(0) \in interior(\mathbf{T}^{-1}(\mathbf{I}))$ for $0 \le p \le n$ and then $\mathbf{T}^{p+1}(0) \in interior(\mathbf{I})$ which is possible only if $p = n$.

Now, going in the opposite direction, take $\mathcal{P}$ the stack that contains $\mathbf{I}$. If $D_i^{\pi}(\mathbf{T}) \in \mathbf{I}$, $i \in \{1, \ldots, m-1\}$ and $i \neq \pi(1)-1$ then $D_i^{\pi}(\mathbf{T}) \neq \mathbf{T}^k(0)$ for if $D_i^{\pi}(\mathbf{T}) = \mathbf{T}^k(0)$ we have $k > 0$ and $D_{\pi^{-1}(i+1)-1}(\mathbf{T}) = \mathbf{T}^{k-1}(0)$ from which we get $\pi^{-1}(i+1) - 1 = 0$ or $i = \pi(1) - 1$. Thus $D_i^{\pi}(\mathbf{T}) \neq \mathbf{T}^k(0)$ and therefore $\mathbf{T}^{-1}(\mathbf{I}) \notin \mathcal{I}$ since $\mathbf{T}^{-1}(\mathbf{I})$ is not an interval and we have $\mathbf{I} = b(\mathcal{P})$. We can assume from now on that $\mathbf{I}$ does not contain discontinuities of $\mathbf{T}^{-1}$ other then $D_{\pi(1)-1}^{\pi}(\mathbf{T}) = \mathbf{T}(0)$. If $l = 1$ then again $\mathbf{T}^{-1}(\mathbf{I})$ is not an interval and therefore $\mathbf{I} = b(\mathcal{P})$ thus we can assume also that $l > 1$, but then we have $\mathbf{T}^{-1}(\mathbf{I}) = [\mathbf{T}^{k-1}(0), \mathbf{T}^{l-1}(0)]$. If $k = 0$ then $\mathbf{T}^{-1}(\mathbf{I}) \notin \mathcal{I}$ and thus $\mathbf{I} = b(\mathcal{P})$. If $k > 0$ we have by hypothesis $\mathbf{T}^{n+1}(0) \in interior(\mathbf{I})$ which implies $\mathbf{T}^{n}(0) \in interior(\mathbf{T}^{-1}(\mathbf{I}))$ and again $\mathbf{T}^{-1}(\mathbf{I}) \notin \mathcal{I}$ and $\mathbf{I} = b(\mathcal{P})$ thus proving the lemma.

For the next three lemmas take $\mathcal{F}_s$ a small Farey cell with $s \ge 0$ critical iterate and fix $\tilde{\mathbf{T}} \in interior(\mathcal{F}_s)$. For n the first $\tilde{\mathbf{T}}$-critical iterate after s it follows that any $\mathbf{T} \in interior(\mathcal{F}_n(\tilde{\mathbf{T}}))$ satisfies (10) above. Fix $\mathbf{T} \in interior(\mathcal{F}_n(\tilde{\mathbf{T}}))$. From the lemma and definitions above it follows that $\mathcal{I} = \mathcal{I}(\mathbf{T}, n)$ has m stacks which we will index as $\mathcal{P}_0, \ldots, \mathcal{P}_{m-1}$ in such a way that $D_i^{\pi}(\mathbf{T}) \in b(\mathcal{P}_i)$ holds for $i = 0, \ldots, m-1$. Since $\mathcal{F}_s$ is small, $n > s$ has a type i_0 and the tops of the stacks are given by $[\mathbf{T}^{l_i}(0), \mathbf{T}^{r_i}(0)]$; $i = 1, \ldots, m-1$, $i \neq i_0$, $[\mathbf{T}^{l_{i_0}}(0), \mathbf{T}^{n}(0)]$ and $[\mathbf{T}^{n}(0), \mathbf{T}^{r_{i_0}}(0)]$ where $l_i = l_i(\mathcal{F}_s)$, $r_i = r_i(\mathcal{F}_s)$. We have then:

$$\mathbf{T}^{l_{i_0}}(0) < \mathbf{T}^{n}(0) < \mathbf{T}^{r_{i_0}}(0)$$

The natural order of the set $\{l_j\}_{j=1}^{m-1}$ induces, via the map $j \mapsto l_j$, an order on the set $\{1, \ldots, m-1\}$. To this ordered set add 0 as a first element and define $g = g(\mathbf{T})\colon \{0, 1, \ldots, m-1\} \to \{1, \ldots, m-1\}$ taking each point to its sucessor and the last one to i_0, the type of n. Define $G = G(\mathbf{T})\colon \{1, 2, \ldots, m\} \to \{1, \ldots, m-1\}$ analogously using the r's instead of the l's and adding m as a first element.

Let $L_1^\sharp = L_1^\sharp(\mathbf{T}), \ldots, L_{m-1}^\sharp = L_{m-1}^\sharp(\mathbf{T})$ and $R_1^\sharp = R_1^\sharp(\mathbf{T}), \ldots, R_{m-1}^\sharp = R_{m-1}^\sharp(\mathbf{T})$ be the left and right intervals defined by:

$$L_i^\sharp = [\mathbf{T}^{l_i}(0), D_i(\alpha)) \tag{12}$$

$$R_i^\sharp = [D_i(\alpha), \mathbf{T}^{r_i}(0)] \tag{13}$$

for $i = 1, \ldots, m-1$.

Lemma 3.2 *Let $i \in \{1, \ldots, m-1\}$ then:*

1.

$$\mathbf{T}(R_i^\sharp) = [D^\pi_{\pi(i+1)-1}(\alpha), \mathbf{T}^{r_i+1}(0)] \subseteq b(\mathcal{P}_{\pi(i+1)-1})$$

2.

$$\mathbf{T}(L_i^\sharp) = [\mathbf{T}^{l_i+1}(0), D^\pi_{\pi(i)}(\alpha))$$

and either $\pi(i) \neq m$ and $\mathbf{T}(L_i^\sharp) \subseteq b(\mathcal{P}_{\pi(i)})$ or $\pi(i) = m$ and $\mathbf{T}(L_i^\sharp) = [\mathbf{T}^{l_i+2}(0), D^\pi_{\pi(m)}(\alpha)) \subseteq b(\mathcal{P}_{\pi(m)})$ with $\mathbf{T}^{l_i+1}(0) = \max\{\mathbf{T}^p(0) \mid p = 0, \ldots, n\}$.

Proof: 1)Take $R_i^\sharp = [D_i(\alpha), \mathbf{T}^{r_i}(0)]$. $R_i^\sharp \subseteq \mathbf{I}_{i+1}(\alpha)$ by the definition of r_i and since each interval $\mathbf{I}_i$ has at least a point of $\{\mathbf{T}^k(0)\}_{k=1}^n$. Thus $\mathbf{T}(R_i^\sharp) = [D^\pi_{\pi(i+1)-1}(\alpha), \mathbf{T}^{r_i+1}(0)]$ and $\mathbf{T}(R_i^\sharp) \bigcap \{\mathbf{T}^k(0)\}_{k=1}^n = \{\mathbf{T}^{r_i+1}(0)\}$. If $0 \in \mathbf{T}(R_i^\sharp)$ then $D^\pi_{\pi(i+1)-1}(\alpha) = 0$ and we have $i = \pi^{-1}(1) - 1$ from which we get $\mathbf{T}(R_i^\sharp) = b(\mathcal{P}_0)$. If $0 \notin \mathbf{T}(R_i^\sharp)$ then $0 < D^\pi_{\pi(i+1)-1}(\alpha)$ or $\pi(i+1) - 1 \geq 1$ and there is k, $0 \leq k \leq n$, such that $\mathbf{T}^k(0) < D^\pi_{\pi(i+1)-1}(\alpha)$. Taking the greatest k with this property we see that $\mathbf{T}(R_i^\sharp) \subseteq [\mathbf{T}^k(0), \mathbf{T}^{r_i+1}(0)] \in \{\mathbf{I}\}$ and, by the preceeding lemma, this last interval is the botton of $\mathcal{P}_{\pi(i+1)-1}$. This completes the proof of 1).

2)Let $L_i = [\mathbf{T}^{l_i}(0), D_i(\alpha))$ be a left interval. As above, it is clear that $\mathbf{T}(L_i) = [\mathbf{T}^{l_i+1}(0), D^\pi_{\pi(i)}(\alpha))$ and this set has no points in $\{\mathbf{T}^k(0)\}_{k=1}^n$ besides $\mathbf{T}^{l_i+1}(0)$. If $\pi(i) \neq m$ then $\pi(i) < m$ or $D^\pi_{\pi(i)}(\alpha) < 1$ and it follows that there is a least one $\mathbf{T}^k(0)$, $0 \leq k \leq n$, such that $\mathbf{T}^k(0) > D^\pi_{\pi(i)}(\alpha)$. We have then $\mathbf{T}(L_i^\sharp) \subseteq [\mathbf{T}^{l_i+1}(0), \mathbf{T}^k(0)] =: \mathbf{I} \in \{\mathbf{I}\}$ and $\mathbf{I} = b(\mathcal{P}_{\pi(i)})$. If $\pi(i) = m$ then $\mathbf{T}(L_i^\sharp) = [\mathbf{T}^{l_i+1}(0), 1) \subseteq \mathbf{I}_m(\alpha)$ and $\mathbf{T}^{l_i+1}(0) = \max\{\mathbf{T}^k(0) \mid k = 0, \ldots, n\}$. Applying $\mathbf{T}$ again to $\mathbf{T}(L_i^\sharp)$ we have $\mathbf{T}^2(L_i^\sharp) = [\mathbf{T}^{l_i+2}(0), D_{\pi(m)})$ and $l_i + 2 \leq n$ otherwise $l_i + 1 = n$, but this lead us to conclude that $\mathbf{T}^n(0)$ is a critical iterate of $\mathbf{T}$ and $\mathbf{T}^n(0) = \max\{\mathbf{T}^k(0) \mid k = 0, \ldots, n\}$ which contradicts the fact that $\mathcal{F}_s$ is small. $\mathbf{T}^2(L_i^\sharp)$ doesn't contain points of $\{\mathbf{T}^k(0)\}_{k=0}^n$ besides $\mathbf{T}^{l+2}(0)$ and $\pi(m) < m$ or, which is the same, $D_{\pi(m)}(\alpha) < 1$ and arguing as we did before we get $k \in \{0, \ldots, n\}$ such that $\mathbf{T}^2(L_i^\sharp) \subseteq [\mathbf{T}^{l_i+2}(0), \mathbf{T}^k(0)] = b(\mathcal{P}_{\pi(m)})$ thus proving the lemma.

Now define the intervals $L_j^\flat$ and $R_k^\flat$ for $j = 0, \ldots, m-1$ and $k = 1, \ldots, m$ by:

$$L_j^\flat = \begin{cases} \mathbf{T}(L_j^\sharp), & \text{if } 0 < j \leq m-1 \text{ and } \pi(j) \neq m; \\ \mathbf{T}^2(L_j^\sharp), & \text{if } 0 < j \leq m-1 \text{ and } \pi(j) = m; \\ \emptyset, & \text{if } j = 0. \end{cases} \tag{14}$$

$$R_k^\flat = \begin{cases} \mathbf{T}(R_k^\sharp), & \text{if } 1 \leq k < m; \\ \{\mathbf{T}(0)\}, & \text{if } k = m. \end{cases} \tag{15}$$

with these definitions we have the following description of the bottons of the stacks:

Lemma 3.3 *1.*

$$b(\mathcal{P}_0) = [0, \mathbf{T}^{r_{\pi^{-1}(1)-1}+1}(0)] = L_0^\flat + R_{\pi^{-1}(1)-1}^\flat$$

2.

$$b(\mathcal{P}_{\pi(1)-1}) = \begin{cases} L_{\pi^{-1}(m)}^\flat + R_m^\flat = & \\ [\mathbf{T}^{l_{\pi^{-1}(m)}+2}(0), \mathbf{T}(0)], & \text{if } \pi(m) = \pi(1)-1; \\ L_{\pi^{-1}(\pi(1)-1)}^\flat + R_m^\flat = & \\ [\mathbf{T}^{l_{\pi^{-1}(\pi(1)-1)}+1}(0), \mathbf{T}(0)], & \text{if } \pi(m) \neq \pi(1)-1. \end{cases}$$

3.

$$b(\mathcal{P}_j) = \begin{cases} L_{\pi^{-1}(m)}^\flat + R_{\pi^{-1}(j+1)-1}^\flat = & \\ [\mathbf{T}^{l_{\pi^{-1}(m)}+2}(0), \mathbf{T}^{r_{\pi(j+1)-1}+1}(0)], & \text{if } j = \pi(m); \\ L_{\pi^{-1}(j)}^\flat + R_{\pi^{-1}(j+1)-1}^\flat = & \\ [\mathbf{T}^{l_{\pi^{-1}(j)}+1}(0), \mathbf{T}^{r_{\pi^{-1}(j+1)-1}+1}(0)], & \text{if } j \neq \pi(m). \end{cases}$$

if $j \in \{1, \ldots, m-1\}$ *and* $j \neq \pi(1)-1$.

Proof: 1)We have:

$$L_0^\flat + R_{\pi^{-1}(1)-1}^\flat = R_{\pi^{-1}(1)-1}^\flat = \mathbf{T}(R_{\pi^{-1}(1)-1}^\sharp) =$$

$$[D_0^\pi(\alpha), \mathbf{T}^{r_{\pi^{-1}(1)-1}+1}(0)] \subseteq b(\mathcal{P}_0)$$

which implies $L_0^\flat + R_{\pi^{-1}(1)-1}^\flat = b(\mathcal{P}_0)$.

2)If $\pi(m) = \pi(1) - 1$ we have:

$$L_{\pi^{-1}(m)}^\flat + R_m^\flat = \mathbf{T}^2(L_{\pi^{-1}(m)}^\sharp) + \{\mathbf{T}(0)\} =$$

$$[\mathbf{T}^{l_{\pi^{-1}(m)}+2}(0), D_{\pi(m)}^\pi(\alpha)) + \{D_{\pi(1)-1}^\pi(\alpha)\} =$$

$$[\mathbf{T}^{l_{\pi^{-1}(m)}+2}(0), D_{\pi(1)-1}^\pi(\alpha)] \subseteq b(\mathcal{P}_{\pi(1)-1})$$

and thus, as before, the equality must hold. If $\pi(m) \neq \pi(1) - 1$;

$$L_{\pi^{-1}(\pi(1)-1)}^\flat + R_m^\flat = \mathbf{T}(L_{\pi^{-1}(\pi(1)-1)}^\sharp) + \{\mathbf{T}(0)\} =$$

$$[\mathbf{T}^{l_{\pi^{-1}(\pi(1)-1)}+1}(0), D_{\pi(1)-1}^\pi(\alpha)) + \{\mathbf{T}(0)\} = [\mathbf{T}^{l_{\pi^{-1}(\pi(1)-1)}+1}(0), \mathbf{T}(0)]$$

which is contained, and therefore is equal to, $b(\mathcal{P}_{\pi(1)-1})$.

3)Take $j \in \{1, \ldots, m-1\}$ and distinct from $\pi(1) - 1$. If $j = \pi(m)$ we have:

$$L_{\pi^{-1}(m)}^\flat + R_{\pi^{-1}(j+1)-1}^\flat = [\mathbf{T}^{l_{\pi^{-1}(m)}+2}(0), D_{\pi(m)}^\pi(\alpha))+$$
$$[D_{\pi(m)}^\pi(\alpha), \mathbf{T}^{r_{\pi^{-1}(j+1)-1}+1}(0)] = b(\mathcal{P}_j)$$

If, on the other hand, $j \neq \pi(m)$,

$$L_{\pi^{-1}(j)}^\flat + R_{\pi^{-1}(j+1)-1}^\flat = [\mathbf{T}^{l_{\pi^{-1}(j)}+1}(0), D_j^\pi(\alpha))+$$

$$[D_j^\pi(\alpha), \mathbf{T}^{r_{\pi^{-1}(j+1)-1}+1}(0)] = b(\mathcal{P}_j)$$

and the lemma is proved.

Lemma 3.4 *Take $i \in \{1, \ldots, m-1\}$:*

1. *If $i \neq i_0$ the type of $\mathbf{T}^n(0)$ and $\mathcal{P}$ is the stack such that $t(\mathcal{P}) = L_i^\sharp + R_i^\sharp$ we have:*

$$b(\mathcal{P}) = L_{g^{-1}(i)}^\flat + R_{f(g^{-1}(i))}^\flat = L_{f^{-1}(G^{-1}(i))}^\flat + R_{G^{-1}(i)}^\flat$$

2. *If $i = i_0$ and $t(\mathcal{P}_1) + t(\mathcal{P}_2) = L^\sharp_{i_0} + R^\sharp_{i_0}$, where the stacks $\mathcal{P}_1$ and $\mathcal{P}_2$ are indexed in such a way that $t(\mathcal{P}_1)$ is to the left of $t(\mathcal{P}_2)$, we have:*

$$b(\mathcal{P}_1) = L^\flat_{j_0} + R^\flat_{G^{m-1}(m)} \text{ and } b(\mathcal{P}_2) = L^\flat_{g^{m-1}(0)} + R^\flat_{k_0}$$

where k_0 and j_0 are given by:

$$g^{-1}(i_0) = \{j_0, g^{m-1}(0)\} \text{ and } G^{-1}(i_0) = \{k_0, G^{\ m-1}(m)\}$$

Proof: Take $\mathcal{P}$ any stack. By the preceeding lemma we have $b(\mathcal{P}) = L^\flat_j + R^\flat_{f(j)}$ for some $j \in \{0, \ldots, m-1\}$.

1) Suppose $t(\mathcal{P}) = L^\sharp_i + R^\sharp_i$ for $i \neq i_0$. If we look at the $\mathbf{T}$-iterates of the left extremum of $L^\flat_j$ = left extremum of $b(\mathcal{P})$ and pay attention to the definitions of the l_i's the first description of $b(\mathcal{P})$ follows. The second description of $b(\mathcal{P})$, the one using G, follows if we look at the $\mathbf{T}$-iterates of the right extremum of $b(\mathcal{P})$.

2)We just have to recall that $\mathbf{T}^{l_{i_0}}(0) < \mathbf{T}^n(0) < \mathbf{T}^{r_{i_0}}(0)$ and $\mathbf{T}^{l_{i_0}}(0) < D_{i_0}(\alpha) < \mathbf{T}^{r_{i_0}}(0)$ from which we get $t(\mathcal{P}_1) = [\mathbf{T}^{l_{i_0}}(0), \mathbf{T}^n(0)]$ and

$t(\mathcal{P}_2) = [\mathbf{T}^n(0), \mathbf{T}^{r_{i_0}}(0)]$. The lemma follows as in 1) by the definition of g(or G) since $l_{g^{m-1}(0)}$ (resp. $r_{G^{\ m-1}(0)}$) is the greatest l_i (resp. r_i).

These last results show that given a small Farey cell $\mathcal{F}_s$ with $s \geq 0$ critical then n, the next $\mathbf{T}$-critical iterate after s, and i_0 the type of $\mathbf{T}^n(0)$ are independent of $\mathbf{T} \in interior(\mathcal{F}_s)$ and therefore $\mathcal{F}_s$ is divided into exactly two Farey cells of order n, these two cells being defined accordingly to $\mathbf{T}^n(0) \in R^\sharp_{i_0}$ or $\mathbf{T}^n(0) \in L^\sharp_{i_0}$ and separated by the hyperplane $\mathbf{T}^n(0) = D_{i_0}(\mathbf{T})$. In fact we can give a description of n and i_0 that depend only on the order of the points $\{\mathbf{T}^k(0)\}_{i=0}^s \bigcup \{D_i(\mathbf{T})\}_{i=0}^m$ in $[0,1)$ which, as we know, is independent of $\mathbf{T} \in interior(\mathcal{F}_s)$.

To see this, start by noting that $s = \max_{1 \leq i \leq m-1}\{l_i, r_i\}$ and since $\mathcal{F}_s$ is small s has a type $j_0 \in \{1, \ldots, m-1\}$. Suppose $s = l_{j_0}$ which means $\mathbf{T}^s(0)$ is the left extremum of $L^\sharp_{j_0}$. Then $\mathbf{T}^{s+1}(0)$ (or $\mathbf{T}^{s+2}(0)$) is the left extremum of $L^\flat_{j_0} \subseteq b(\mathcal{P}_{k_0})$ for $k_0 = \pi(j_0)$ (or $k_0 = \pi^2(j_0)$). Take $\mathbf{T}^t(0)$, $t = 1$ or $t \in \{r_i + 1\}_{i=1}^{m-1}$, the other extreme of $b(\mathcal{P}_{k_0})$ and $\mathbf{T}^u(0)$, $u \in \{r_i\}_{i=1}^{m-1}$ the point of $t(\mathcal{P}_{k_0})$ lying above $\mathbf{T}^t(0)$: i_0 is given by $u = r_{i_0}$ and $n = s + u - t$.

The case $s = r_{j_0}$ is similar, the key observation being again that the intervals $L^\sharp$, $R^\sharp$, $L^\flat$ and $R^\flat$ depend only on the order induced on the set $\{\mathbf{T}^k(0)\}_{k=0}^s$ by $[0,1)$.

A consequence of these observations is that $g = g(\mathbf{T})$ and $G = G(\mathbf{T})$ are also independent of $\mathbf{T} \in interior(\mathcal{F}_s)$.

Now that we have completed the description of the stacks of an interval exchange map $\mathbf{T}$ it will be useful to interpret these objects geometrically on $\mathcal{R}(\mathbf{T})$, the Riemann surface associated to $\mathbf{T}$. To this end it is more convenient to think of $\mathbf{T}$ as the first return map induced by $\mathcal{V}(\mathbf{T})$ on the union of leaves and singularities given by $y = 0$. If we do this it is clear that to each stack $\mathcal{P}$ is associated a rectangle $rec(\mathcal{P}) \subseteq \mathcal{R}(\mathbf{T})$ with vertical sides on the graph associated to the piece of orbit $0, \mathbf{T}(0), \ldots, \mathbf{T}^n(0)$, tops on the union of leaves and singularities $y = 1$, bottons on $y = 0$ and having the intervals $\mathbf{I} \in \mathcal{P}$ as horizontal equally spaced by 1 slices. To the rectangles $rec(\mathcal{P}(\mathbf{T}))$ we add the rectangle $[\max_{0 \leq k \leq n}\{\mathbf{T}^k(0)\}, 1] \times [0, 1]$ and get a decomposition of $\mathcal{R}(\mathbf{T})$ as a union of rectangles with disjoint interiors; the interior of these rectangles are embbeded disks in $\mathcal{R}(\mathbf{T})$.

Recall the definition of the distribution matrix of a Farey cell $\mathcal{F}_n$ from the introduction. This is the $m \times 2(m-1)$ matrix (λ^n, ρ^n) whose first $m-1$ columns λ^n_j ; $j = 1, \ldots, m-1$ are given by:
λ^n_{ij} =number of times $\mathbf{T}^k(L^\sharp_j)$ intercepts $\mathbf{I}_i$ as k runs from 0 up to the time just before $\mathbf{T}^k(L^\sharp_j)$ hits the next $t(\mathcal{P})$.
and whose last $m-1$ columns ρ^n_j ; $j = 1, \ldots, m-1$ are given by:
ρ^n_{ij} =number of times $\mathbf{T}^k(R^\sharp_j)$ intercepts $\mathbf{I}_i$ as k runs from 0 up to the time just before $\mathbf{T}^k(R^\sharp_j)$ hits the next $t(\mathcal{P})$.

It is easy to see that these definitions are independent of $\mathbf{T} \in interior(\mathcal{F}_s)$.

We are ready to show that every small Farey cell has the projective type of an abstract Farey cell. To this end take $\mathcal{F}_s$ a small Farey cell with $s \geq 0$ critical and take $n > s$ next critical iterate of the elements of $interior(\mathcal{F}_s)$; take $l_i = l_i(\mathcal{F}_s)$, $r_i = r_i(\mathcal{F}_s)$, $i = 1, \ldots, m-1$, $g = g(\mathcal{F}_s)$ and $G = G(\mathcal{F}_s)$ and define the linear map:

$$\alpha \mapsto (L_0(\alpha), L_1(\alpha), \ldots, L_{m-1}(\alpha), R_1(\alpha), \ldots, R_{m-1}(\alpha), R_m(\alpha))^t$$

where $L_i(\alpha) = D_i(\alpha) - \mathbf{T}^{l_i}(0)$ and $R_i(\alpha) = \mathbf{T}^{r_i}(0) - D_i(\alpha)$ for $i = 1, \ldots, m-1$, $L_0 = R_m = 0$ and $\mathbf{T} = \mathbf{T}(\pi, \alpha)$.

Theorem 3.1 *$\gamma = (g, G) \in \mathcal{A}$ and $\mathbf{P}$ the projective map induced by the linear map $\alpha \mapsto \begin{pmatrix} L \\ R \end{pmatrix}$ is a projective isomorphism of $\mathcal{F}_s$ onto $\mathcal{C}_\gamma$.*

Proof: We start by showing that $g = G \circ f$. Take $j \in \{0, \ldots, m-1\}$. If $g(j) \neq i_0$ then $j = g^{-1}(g(j))$ and from 1) in the preceeding lemma we have $L_j^b + R_{f(j)}^b = L_{f^{-1}(G^{-1}(g(j)))}^b + R_{G^{-1}(g(j))}^b$ from which we get $f(j) = G^{-1}(g(j))$ or $G(f(j)) = g(j)$. If $g(j) = i_0$ we have, using the notation of the preceeding lemma, $j = j_0$ or $j = g^{m-1}(0)$ and from 2) of the same lemma we have $b(\mathcal{P}_1) = L_{j_0}^b + R_{G^{m-1}(m)}^b$, and then $f(j_0) = G^{\,m-1}(m)$ which means that $G(f(j_0)) = G^{\,m}(m) = i_0 = g(j_0)$. Starting with $b(\mathcal{P}_2) = L_{g^{m-1}(0)}^b + R_{k_0}^b$ we have $f(g^{m-1}(0)) = k_0$ and again $G(f(g^{m-1}(0))) = G(k_0) = i_0$. These last results also show that $f(g^{m-1}(0)) = k_0 \neq G^{\,m-1}(m)$. From the definitions of g and G it is clear that (7) of the definition of γ holds.

Take $\widetilde{(L, R)}^t = \mathbf{P}(\alpha)$ where $(L, R)^t = (L_0, L_1, \ldots, L_{m-1}, R_1, R_2, \ldots, R_m)^t$. It is clear that $L_i > 0$ and $R_i \geq 0$ and as $length(t(\mathcal{P})) = length(b(\mathcal{P}))$, (8) follows from the preceeding lemma so that all that remains to be proved is $dimension(\mathcal{C}_\gamma) = m-1$. Now, this dimension is $\leq m-1$ for the equations in (8) are dependent since they add to the trivial equation $0 = 0$ and, using (9) we have $\mathcal{C}_\gamma$ defined by at most $m-1$ linearly independent equations. To prove that dimension $\geq m-1$, and finish the proof of the proposition all we have to do is show that $\mathbf{P}$ is a projective injection onto $\mathcal{F}_n$. Start by noting that L and R are linear in α and thus $\mathbf{P}$ is projective. In fact, we have:

$$L_i(\alpha) = D_i(\alpha) - \mathbf{T}^{l_i}(0) = \sum_{k=1}^{i} \alpha_k - T^{l_i}\mathbf{L}\alpha \tag{16}$$

$$R_i(\alpha) = \mathbf{T}^{r_i}(0) - D_i(\alpha) = T^{r_i}\mathbf{L}\alpha - \sum_{k=1}^{i} \alpha_k \tag{17}$$

for $i = 1, \ldots, m-1$.
Now, using (11) we can write:

$$\alpha = (\lambda^n, \rho^n)\begin{pmatrix} L \\ R \end{pmatrix} \tag{18}$$

Using this equality we can get a left inverse for $\mathbf{P}$ given by the projective map induced by $\begin{pmatrix} L \\ R \end{pmatrix} \mapsto \alpha$ where $\alpha = (\lambda^n, \rho^n)\begin{pmatrix} L \\ R \end{pmatrix}$. Since clearly $\mathbf{P}(\mathcal{F}_s) \subseteq \mathcal{C}_\gamma$, this inverse shows that $\mathbf{P}$ is injective and that the dimension of $\mathcal{C}_\gamma$ is $m-1$. Thus all we need to finish the proof of the theorem is to show that $\mathbf{P}$ is onto. Take $(L, R)^t \in \mathcal{C}_\gamma$ and construct a set of closed disjoint intervals J distributed into m disjoint subsets $\{\mathcal{P}_k\}_{k=1}^m$ satisfying:

1. Every interval of $\mathcal{P}_k$ has the same length $L_{f^{-1}(k)} + R_k$. $k = 1, \ldots, m$.

2. The cardinality of $\mathcal{P}_k$ is given by:

$$\#(\mathcal{P}_k) = \begin{cases} r_{G(k)} - r_k, & \text{if } k \neq G^{m-1}(m) \text{ and } k \neq m; \\ r_{G(m)}, & \text{if } k = m; \\ n - r_{G^{m-1}(m)}, & \text{if } k = G^{m-1}(m). \end{cases}$$

To this collection of intervals add a half-open interval $\tilde{J}$ of length $L_{\pi^{-1}(m)}$; call the set of intervals thus obtained $\mathcal{J}$. We linearly order each set $\mathcal{P}_k$, name these sets abstract stacks and carry the top, botton terminology to this abstract context.

Now, index the right extremes of the intervals in $\mathcal{J}$ distinct from $\tilde{J}$, $\{re_u\}_{u=1}^{n}$ and the left extremes of the intervals in $\mathcal{J}$, $\{le_u\}_{u=0}^{n}$ in such a way that:

1. The maps $re_u \mapsto u$ and $le_u \mapsto u$ are order preserving whithin each stack.

2. The index of the right extreme of $t(\mathcal{P}_k)$ is

$$\begin{cases} r_{G(k)}, & \text{if } k \neq G^{m-1}(m); \\ n, & \text{if } k = G^{m-1}(m). \end{cases}$$

   for $k = 1, \ldots, m$.

3. The index of the left extreme of $t(\mathcal{P}_k)$ is

$$\begin{cases} l_{G(k)}, & \text{if } k \neq f(g^{m-1}(0)); \\ n, & \text{if } k = f(g^{m-1}(0)). \end{cases}$$

   for $k = 1, \ldots, m$.

It is clear that there is only one way to index the extremes of the intervals satisfying the three above requirements.

Now glue the left and right extremes of the intervals in $\mathcal{J}$ identifying the extremes with the same index $le^u \equiv re^u$, $u = 1, \ldots, n$. After this identification we get an interval which, after a proper normalization, we can assume to be $[0, 1)$ and a sequence of $n+1$ distinct points $t^u := le^u = re^u$, $u = 1, \ldots, n$ and $t^0 := 0$ in $[0, 1)$. Using these points define:

1. The points $d_i \in [t^{l_i}, t^{r_i}]$, $i = 1, \ldots, m-1$ such that $d_i = t^{l_i} + \widetilde{L}_i$ where $\widetilde{L}_i = L_i / \parallel A \parallel$ and $A = (\lambda^n, \rho^n) \begin{pmatrix} L \\ R \end{pmatrix}$.

2. The intervals $L_i^\sharp$ and $R_i^\sharp$, $i = 1, \ldots, m-1$ defined as in (12) and (13) using t^k and d_i instead of $\mathbf{T}^k(0)$ and $D_i(\mathbf{T})$, respectively.

3. The intervals $L_k^\flat$ and $R_k^\flat$, $k = 1, \ldots, m-1$ such that $b(\mathcal{P}_k) = L^\flat_{f^{-1}(k)} + R_k^\flat$ where $L^\flat_{f^{-1}(k)}$ is half-open of length $L_{f^{-1}(k)}$ and $R_k^\flat$ is closed of length R_k.

The definition of τ such that $\mathbf{P}(\tau) = (L, R)^t$ and finishes the proof of the theorem should be obvious by now:
τ translates each point in an interval of a stack one interval up in the same stack; points in the the top intervals are mapped in such a way that (14) and (15) hold true for τ in place of $\mathbf{T}$ and, finally, $\tau(\widetilde{J}) = L^\flat_{\pi^{-1}(m)}$.

We call $\mathbf{P}$ defined above the *canonical isomorphism* of $\mathcal{F}_s$ onto $\mathcal{C}_\gamma$.

The next lemma describe the vertices of an abstract Farey cell and will help us in the understanding the approximants to an interval exchange map $\mathbf{T}$.

Lemma 3.5 *$\widetilde{X}^0 = (\widetilde{L}_0^0, \ldots, \widetilde{L}_{m-1}^0, \widetilde{R}_1^0, \ldots, \widetilde{R}_m^0)$ is a vertex of $\mathcal{C}_\gamma$, $\gamma = (g, G)$ iff $\widetilde{X}^0 = X^0 / \parallel X^0 \parallel$ where $X^0 = (L^0, R^0)$ satisfies :*

1. *(L^0, R^0) is a non-trivial solution of the system (8).*

2. *L_i^0 , $R_i^0 \in \{0, 1\}$; $i = 1, \ldots, m$*

3. *If we define the support of L^0, $supp(L^0)$ as $supp(L^0) = \{i \mid L^0 \neq 0\}$ and, analogously, $supp(R^0)$, then these supports must be disjoint and the function c defined on $supp(L^0) \bigcup supp(R^0)$ by $c \mid_{supp(L^0)} = g \mid_{supp(L^0)}$ and $c \mid_{supp(R^0)} = G \mid_{supp(R^0)}$ must be a cycle on the union of the supports $supp(L^0) \bigcup supp(R^0)$.*

Proof: Take (L^0, R^0) satisfying the three conditions above and let $X^p = (L^p, R^p)$; $p = 1, 2$, L_i^p and $R_i^p \geq 0$, $i = 1, \ldots, m$, non-trivially satisfying the equations (8) and such that X is the mid-point of the interval $[X^1, X^2]$. We are going to prove that X^p, $p = 1, 2$ are multiples of X^0 therefore concluding that X^0 is a vertex of $\mathcal{C}_\gamma$. It is clear that $supp(L^p) \subseteq supp(L^0)$ and

$supp(R^p) \subseteq supp(R^0)$, $p = 1,2$ since we are dealing with non-negative quantities. Thus, to prove the sufficiency of the conditions it is enough to show that $X_i^p = X_{c(i)}^p$; $i \in supp(L^0) \bigcup supp(R^0)$ and $p = 1,2$. The $c(i)$-th equation of (8) is:

$$L_{c(i)} + R_{c(i)} = L_{g^{-1}(c(i))} + R_{G^{-1}(c(i))}$$

or

$$L_{c(i)} + R_{c(i)} = L_{g^{-1}(c(i))} + L_{g^{m-1}(0)} + R_{G^{-1}(c(i))} + R_{G^{m-1}(m)}$$

In any case if we substitute (L^0, R^0) in this equation we see, by the disjointness of the supports of L^0 and R^0, that the left hand side of has exactly one non-zero summand; $L_{c(i)}$ if $c(i) \in supp(L^0)$ or $R_{c(i)}$ if $c(i) \in supp(R^0)$. Now, from 2), it is clear that the same situation must hold on the right hand of this expression. In other words exactly one summand of the right side must be non-zero; $L_{g^{-1}(c(i))}$ or $L_{g^{m-1}(0)}$ if $c(i) = g(i)$ or $R_{G^{-1}(c(i))}$ or $R_{G^{m-1}(m)}$ if $c(i) = G(i)$ and in any case these equations say that $X_{c(i)}^0 = X_i^0$. Using our observation above on the supports of X^1 and X^2 we see that the same relations must hold between the entries of X^1 and X^2 or $X_{c(i)}^p = X_i^p$, $p = 1,2$ thus proving the sufficiency of the conditions.

To show that the conditions are necessary let, $\widetilde{X^0} = (\tilde{L}^0, \tilde{R}^0)$ a vertex of $\mathcal{C}_\gamma$. $\widetilde{X^0}$ is a non-trivial solution of (8) and $\tilde{L}_i^0, \tilde{R}_i^0 \geq 0$. It is clear that if $i \in supp(L) \bigcup supp(R)$ then $g(i)$ or $G(i) \in supp(L) \bigcup supp(R)$ and thus, reasoning by induction, we can construct a cycle c with domain $D \subseteq supp(L) \bigcup supp(R)$ such that $c(i) = g(i)$ or $c(i) = G(i)$ for $i \in D$. Let $X^1 = (L^1, R^1)$ be given by $X_i^1 = 1$ if $i \in D$ and $X_i^1 = 0$ otherwise. $X_i^1 \geq 0$ and, it is easy to see, X^1 satisfies the system (8) and therefore the same holds for $tX^1 + \widetilde{X^0}$, where $t \in (-\epsilon, \epsilon)$ and $\epsilon > 0$ is small enough. But, by hypothesis, $\widetilde{X^0}$ is an extreme point of $\mathcal{C}_\gamma$ and this forces $\widetilde{X^0}$ be a multiple of X^1 proving the lemma since then $D = supp(L) \bigcup supp(R)$ and thus all entries of $\widetilde{X^0}$ must be equal.

Recall from the introduction that we call the vertices of $\mathcal{F}_n(\mathbf{T})$ the approximants to $\mathbf{T}$.

Theorem 3.2 *Let $\mathcal{F}_n$ be a Farey cell with n critical, then the vertices of $\mathcal{F}_n$ in $\mathcal{S}_m$ are primitive interval exchange maps.*

Proof: Take $\mathbf{S}$ vertex of $\mathcal{F}_n$ in $\mathcal{S}_m$. By the preceeding lemma $\mathbf{P}(\mathbf{S}) = \widetilde{X^0}$ is a vertex of $\mathcal{C}_\gamma$ where $\gamma \in \mathcal{A}$ is the abstract type of $\mathcal{F}_n$ and $\mathbf{P}$ is the canonical isomorphism $\mathcal{F}_n \longrightarrow \mathcal{C}_\gamma$. This means $\widetilde{X^0} = X^0 / \parallel X^0 \parallel$ where X^0 satisfies 1), 2) and 3) of the preceeding lemma. Take $\mathbf{T} \in interior(\mathcal{F}_n)$ and $\mathcal{R}(\mathbf{U})$, the Riemann surface associated to $\mathbf{U} \in [\mathbf{T}, \mathbf{S})$. $\mathcal{R}(\mathbf{U})$ is decomposed into rectangles associated to the stacks of $\mathbf{U}$. As we move $\mathbf{U}$ from $\mathbf{T}$ to $\mathbf{S}$ the rectangles whose basis are not in $supp(L^0) \bigcup supp(R^0)$ will colapse into graphs but since c , which maps the botton of a rectangle to its top, is a cycle on $supp(L^0) \bigcup supp(R^0)$, we see that $\mathbf{S}$ is primitive and this proves the theorem.

4 Unique Ergodicity

As before we fix π an irreducible and discontinuous permutation of the set $\{1, \ldots, m\}$, $m \geq 2$, and identify $\alpha \in \mathcal{S}_m$ with the interval exchange map $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ induced by π using α.

We want to consider now the Borel probabilities on $[0,1)$ that are invariant by $\mathbf{T}$. Since $\mathbf{T}$ is a translation on each interval $\mathbf{I}_i(\alpha)$ it is clear that the Lebesgue measure on $[0,1)$ is $\mathbf{T}$-invariant. If μ is a $\mathbf{T}$-invariant Borel probability (i.e. $\mu(\mathbf{T}^{-1}(B)) = \mu(B)$, for every Borel subset $B \subseteq [0,1))$, consider $\phi = \phi_\mu : [0,1) \to [0,1)$ its probability distribution given by $\phi(x) = \mu([0,x)), x \in [0,1)$. ϕ is non-decreasing, left-continuous and, in fact, if $\mathbf{T}$ is minimal, an homeomorphism of the interval $[0,1)$. Indeed, if ϕ has a jump this is due to a point $p \in [0,1)$ which has positive measure, an atom of μ. But $\mu(\{p\}) = \mu(\{\mathbf{T}^{-n}(p)\}), n \geq 0$ thus p must be $\mathbf{T}$-periodic since μ is finite and we have a contradiction with the minimality of $\mathbf{T}$. This proves that ϕ is continuous. If, on the other hand, ϕ is not injective, ϕ is constant in an open interval which has then μ-measure zero. Let $\{X\}$ be the set of open intervals that are maximal with the property of having zero μ-measure. Since

$$\mathbf{T}^{-1}(X) \in \{X\} \quad \text{if} \quad closure_{\mathbf{R}}(X) \bigcap \{D_i^\pi(\alpha)\}_{i=1}^m = \emptyset$$

and there are only a finite number of intervals X not satisfying this last property (as a matter of fact, at most $2m$ of them), we see that the set $\{X\}$ is finite. In fact, $\mathbf{T}$ has no periodic points, preserves the Lebesgue measure and it cannot take an arbitrary ammount of $\mathbf{T}$-iteration to hit the $\mathbf{T}$-discontinuities. Taking in account that $\mathbf{T}$ has dense orbits we see that the complement of $\bigcup X$ must be a finite set of points which is a contradiction since, as ϕ continuous, this implies $\mu = 0$. Thus ϕ is an increasing homeomorphism.

Let $\beta_i = \beta_i(\mu); i = 1, \ldots, m$ be given by $\beta_i = \mu(\mathbf{I}_i) = \phi(D_i) - \phi(D_{i-1})$. It is clear that $\beta \in \mathcal{S}_m$. Thus $\mathbf{S} = \mathbf{S}(\mu) = \mathbf{S}(\pi, \beta)$ is a well defined interval exchange map. An easy computation shows that ϕ conjugates $\mathbf{T}$ and $\mathbf{S}$ or, more precisely, $\mathbf{S} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{S}$, Veech [7].

The above considerations show that, if $\mathbf{T}$ is minimal, we have a map:

$$\mathcal{P}(\mathbf{T}) \longrightarrow \mathcal{C}(\mathbf{T}) \subset \mathcal{S}_m \ ; \quad \mu \mapsto \mathbf{S}(\pi, \beta(\mu))$$

where $\mathcal{P}(\mathbf{T})$ is the set of $\mathbf{T}$-invariant Borel probabilities on $[0, 1)$ and $\mathcal{C}(\mathbf{T})$ is the conjugacy class of $\mathbf{T}$ in the space of $\mathcal{S}_m$. We will only consider conjugacies by increasing homeomorphisms but refer to them simply as conjugacies.

Lemma 4.1 (Veech) *Let $\mathbf{T}$ be a minimal interval exchange map, then the map $\mu \mapsto \mathbf{S}$ defined above is an affine bijection of $\mathcal{P}(\mathbf{T})$ onto $\mathcal{C}(\mathbf{T})$.*

As a corollary of the lemma we see that $\mathbf{T}$ is uniquely ergodic iff its conjugacy class in $\mathcal{S}_m$ is trivial.

In what follows we relate the conjugacy class of $\mathbf{T}$, $\mathcal{C}(\mathbf{T})$, to the Farey sequence of cells around $\mathbf{T}$, $\mathcal{F}_n(\mathbf{T})$.

If $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ and $\mathbf{S} = \mathbf{S}(\pi, \beta)$ are two interval exchange maps and ϕ conjugates $\mathbf{S}$ and $\mathbf{T}$, $\phi \circ \mathbf{T} = \mathbf{S} \circ \phi$, then ϕ takes $\mathbf{T}$-discontinuities onto $\mathbf{S}$-discontinuities and as $\phi(0) = 0$, we see that :

$$\mathbf{T}^k(0) \in \mathbf{I}_i(\alpha) \Longleftrightarrow \mathbf{S}^k(0) \in \mathbf{I}_i(\beta)$$

for $i = 1, \ldots, m$ and $k \geq 0$.

Thus $\mathcal{C}(\mathbf{T}) \subseteq \bigcap_{n=0}^{\infty} interior(\mathcal{F}_n(\mathbf{T}))$.

Theorem 4.1 *If $\mathbf{T}$ satisfies Keane's i.d.o.c. then*

$$\mathcal{C}(\mathbf{T}) = \bigcap_{n=0}^{\infty} interior(\mathcal{F}_n(\mathbf{T})).$$

We just saw that $\mathcal{C}(\mathbf{T}) \subseteq \bigcap_{n=0}^{\infty} interior(\mathcal{F}_n(\mathbf{T}))$. To show the other inclusion and finish the proof the theorem we need some preliminary results.

Lemma 4.2 *Let* $\mathbf{T}$ *and* $\mathbf{S}$ *be interval exchange maps induced by* π *and satisfying:*

a)The positive $\mathbf{T}$*-orbit of zero,* $\{\mathbf{T}^n(0)\}_{n=0}^{\infty}$*, is dense in* $[0,1)$.

b)For k *and* $l \geq 0$ *we have:* $\mathbf{T}^k(0) < \mathbf{T}^l(0)$ *iff* $\mathbf{S}^k(0) < \mathbf{S}^l(0)$.

Then there exits $\phi: [0,1) \rightarrow [0,1)$ *increasing and right continuous such that* $\phi \circ \mathbf{T} = \mathbf{S} \circ \phi$.

Proof: Given $x \in [0,1)$ define $\phi(x)$ as follows: choose $n_k \geq 0$, increasing such that $\mathbf{T}^{n_k}(0) \downarrow x$ (i.e. $\mathbf{T}^{n_k}(0) > \mathbf{T}^{n_{k+1}}(0) > 0$ and $\mathbf{T}^{n_k}(0) \rightarrow x$ as $k \rightarrow \infty$). By b) we have $\mathbf{S}^{n_k}(0) > \mathbf{S}^{n_{k+1}}(0)$ so that $\mathbf{S}^{n_k}(0)$ is decreasing and therefore converges to some $y \in [0,1)$; take $\phi(x) := y$.

To prove that ϕ is well defined take $\mathbf{T}^{m_k}(0) \downarrow x$ and $\mathbf{S}^{m_k}(0) \downarrow z$ and, to get a contradiction, suppose that $y < z$. We have:

$$\mathbf{S}^{n_k}(0) < \mathbf{S}^{n_{k_0}}(0) < z < \mathbf{S}^{m_l}(0)$$

for some k_0 big enough and any l and $k > k_0$. Again by b) we have $\mathbf{T}^{n_k}(0) < \mathbf{T}^{n_{k_0}}(0) < \mathbf{T}^{m_l}(0)$. If we make $k \rightarrow \infty$ we have $x < \mathbf{T}^{n_{k_0}}(0) < \mathbf{T}^{m_l}(0)$ and now making $l \rightarrow \infty$ we get $x < x$, an absurd.

To see that ϕ is increasing and right-continuous is equally easy. Now, take $x \in [0,1)$ and $\mathbf{T}^{n_k}(0) \downarrow x$. Since $\mathbf{T}$ is right-continuous we have $\mathbf{T}^{n_k+1}(0) \downarrow \mathbf{T}(x)$, and, by definition, $\mathbf{S}^{n_k+1}(0) \downarrow \phi(\mathbf{T}(x))$. But then:

$$\phi(\mathbf{T}(x)) = \lim_{k\rightarrow\infty} \mathbf{S}^{n_k+1}(0) = \mathbf{S}(\lim_{k\rightarrow\infty} \mathbf{S}^{n_k}(0)) = \mathbf{S}(\phi(x))$$

by the right-continuity of $\mathbf{S}$. This proves the lemma.

Using that ϕ is increasing and right -continuous we can write $[0,1) - Image(\phi) = \sum K$ where $\sum$, as before, denotes disjoint union, and $\{K\}$ is the family of intervals $K = [\lim_{x\uparrow x_0} \phi(x), \phi(x_0))$; x_0 a discontinuity of ϕ, $K = [\sup \phi, 1)$ or $K = [0, \phi(0))$. From $\phi \circ \mathbf{T} = \mathbf{S} \circ \phi$ we conclude that $\mathbf{S}(Image(\phi)) = Image(\phi)$ and thus that $\mathbf{S}(\sum K) = \sum K$.

Lemma 4.3 *Let* $\mathbf{T}$ *,* $\mathbf{S}$ *,* ϕ *be as in the preceeding lemma and let* $\{K\}$ *be as above:*

a)If $closure_{\mathbf{R}}(K) \bigcap \{D_i(\beta)\}_{i=1}^{m-1} = \emptyset$ *then:*

$$\mathbf{S}(K) \in \{K\} \text{ and } \mathbf{S}(K) = [\mathbf{S}(\inf K), \mathbf{S}(\sup K))$$

b)If $closure_{\mathbf{R}}(K) \bigcap \{D_i^{\pi}(\beta)\}_{i=1}^{m-1} = \emptyset$ *then:*

$$\mathbf{S}^{-1}(K) \in \{K\} \text{ and } \mathbf{S}^{-1}(K) = [\mathbf{S}^{-1}(\inf K), \mathbf{S}^{-1}(\sup K))$$

Proof: The proof of a) is clear since by hypothesis the closure of K is contained in an open interval where $\mathbf{S}$ is continuous; the same idea holds for b).

Lemma 4.4 *Let* $\mathbf{T}$, $\mathbf{S}$, ϕ *and* $\{K\}$ *be as in the preceeding lemma. If* $\mathbf{T}$ *satisfies i.d.o.c. then* $\{K\}$ *is finite.*

Proof: Fix an interval K and suppose $closure_{\mathbf{R}}(\mathbf{S}^n(K)) \bigcap \{D_i(\beta)\}_{i=1}^{m-1} = \emptyset$, for every $n \geq 0$. By the preceding lemma we we have that $\mathbf{S}^n(K) \in \{K\}$ for every $n \geq 0$ and since the family $\{K\}$ is disjoint and $\mathbf{S}$ preserves the the Lebesgue measure there are $0 \leq n_1 < n_2$ such that $\mathbf{S}^{n_1}(K) = \mathbf{S}^{n_2}(K)$ or $\mathbf{S}^{n_2-n_1}(K) = K$ which implies that K is made of $\mathbf{S}$-periodic points and therefore is contained in a maximal interval M of $\mathbf{S}$-periodic points. If $\inf M < \inf K$ there is a x such that $\phi(x) \in M$ which implies that x is $\mathbf{T}$-periodic a contradiction with the fact that $\mathbf{T}$ is i.d.o.. If $\inf M = \inf K$ we get again a contradiction since the extremes of M under iteration by $\mathbf{S}$ hit the set $\{D_i(\beta)\}_{i=1}^{m-1}$. All these contradictions prove that we must have a first $n \geq 0$ such that $closure_{\mathbf{R}}(\mathbf{S}^n(K)) \bigcap \{D_i(\beta)\}_{i=0}^{m-1} \neq \emptyset$. Now the set of K's with a $D_i(\beta)$ in its real closure is finite and it can not take an arbitrary amount of iteration to hit $\{D_i(\beta)\}_{i=1}^{m-1}$ since $\mathbf{S}$ preserves the Lebesgue measure and this proves the lemma.

Using the fact that $\{K\}$ is finite and that the half-open intervals are a semi-algebra it follows that we can write $Image(\phi) = [0,1) - \sum K$ as a finite disjoint and non-contiguous set of half-open intervals $\{L\}$. It is clear also that a result analogous to Lemma 4.3 holds for the family $\{L\}$ instead of the family $\{K\}$.

Lemma 4.5 *Let* $\mathbf{T}$ *,* $\mathbf{S}$ *,* ϕ *and* $\{K\}$ *be as in the preceding lemma and let* $\{L\}$ *be as above. Suppose that the image of* ϕ *meets every interval* $\mathbf{I}_i(\beta)$ *and* $\mathbf{I}_i^{\tau}(\beta)$*;* $i = 1, \ldots, m$*, then:*

$$\{\phi(D_i(\alpha)\}_{i=1}^{m-1} = \{D_i(\beta) \mid D_i(\beta) \in Image(\phi);\ i = 1, \ldots, m-1\}+$$

$$\{\phi(x_0) \mid x_0 \text{ is a discontinuity of } \phi \ \&\ \{D_i(\beta)\}_{i=1}^{m-1} \bigcap [\lim_{x \uparrow x_0} \phi(x), \phi(x_0)) \neq \emptyset\}$$

Proof: Let M be the half-open interval obtained translating the the intervals L and laying then one next to the other starting from zero and without changing their order. Let $\psi: M \to Image(\psi)$ be the translation by parts that put the L's back to their original position, let $\widetilde{\mathbf{S}} = \psi^{-1} \circ \mathbf{S} \circ \psi$ be the map induced by $\mathbf{S}$ on M and $h = \psi^{-1} \circ \phi$.

But for its domain, which can be different from $[0,1)$, $\widetilde{\mathbf{S}}$ is an interval exchange map and h is an increasing homeomorphism that conjugates $\widetilde{\mathbf{S}}$ to $\mathbf{T}$. Thus $\widetilde{\mathbf{S}}$ is discontinuous at the points $\{h(D_i(\alpha))\}_{i=1}^{m-1}$ from which we have: $\psi(discontinuities\ of\ \widetilde{\mathbf{S}}) = \{\phi(D_i(\alpha))\}_{i=1}^{m-1}$ so that to prove the lemma we have to show that:

$$\psi(discontinuities\ of\ \widetilde{\mathbf{S}}) = \{D_i(\beta) \mid D_i(\beta) \in Image(\phi);\ i = 1, \dots, m-1\}$$

$$+\{\phi(x_0) \mid x_0 \text{ is a discontinuity of } \phi \ \&\ \{D_i(\beta)\}_{i=1}^{i=m-1} \bigcap [\lim_{x \uparrow x_0} \phi(x), \phi(x_0)) \neq \emptyset\}$$

and, considering that these sets have $m-1$ elements since no interval K can contain an interval $\mathbf{I}_i(\beta)$, we only need to prove the inclusion of the right hand side of the equality in the left hand. To do this take $B_i = D_i(\beta) \in Image(\phi)$ and let L be such that $B_i \in L$. If $B_i > \inf L$ it is clear that $\psi^{-1}(B_i)$ is a discontinuity of $\widetilde{\mathbf{S}}$ since there is $\mathbf{I}_j^\pi(\beta) \subseteq [\lim_{x \uparrow B_i} \mathbf{S}(x), \mathbf{S}(B_i))$ and $Image(\phi) \bigcap \mathbf{I}_j^\pi(\beta) \neq \emptyset$ which implies $\lim_{x \uparrow B_i} \widetilde{\mathbf{S}}(x) < \widetilde{\mathbf{S}}(B_i)$.

If, on the other hand, $B_i = \inf L$, take the interval $\mathbf{I}_i(\beta) = [D_{i-1}(\beta), B_i)$. By hypothesis this interval also meets $Image(\phi)$ from which it follows that there is an interval $L_1 \subseteq Image(\phi)$ immediately before L and again we see that $\psi^{-1}(B_i)$ is a discontinuity of $\widetilde{\mathbf{S}}$ since $\lim_{x \uparrow B_i} \widetilde{\mathbf{S}}(x) < \widetilde{\mathbf{S}}(B_i)$. Now, let x_0 be a discontinuity of ϕ such that the interval $N = [\lim_{x \uparrow x_0} \phi(x), \phi(x_0))$ contains the (necessarily) unique $\mathbf{S}$-discontinuity, $B_i = D_i(\beta)$. We have to prove that $\widetilde{\mathbf{S}}$ is discontinuous at $\psi^{-1}(\phi(x_0))$. As $Image(\phi)$ intercepts $\mathbf{I}_i(\beta)$ and $\mathbf{I}_{i+1}(\beta)$ there are intervals L_1 and L_2 that meet these intervals and, respectively, precede and follow N. These two intervals will came together when we make up M and $\widetilde{\mathbf{S}}$ will be discontinuous at $\psi^{-1}(\phi(x_0))$ since $\phi(x_0) = \inf L_2$ and, as before, there is an $\mathbf{I}(\beta)$-interval between $\lim_{x \uparrow B_i} \widetilde{\mathbf{S}}(x)$ and $\widetilde{\mathbf{S}}(B_i)$, thus proving the lemma.

Lemma 4.6 *Let* $\mathbf{T}$, $\mathbf{S}$, ϕ , $\{K\}$ *and* $\{L\}$ *be as in the preceeding lemma then* ϕ *is continuous.*

Proof: Suppose, to get a contradiction, that x_0 is a discontinuity of ϕ. Take $K = [\lim_{x \uparrow x_0} \phi(x), \phi(x_0))$. Reasoning as in lemma we can get a first $n \geq 0$ such that $closure_{\mathbf{R}}(\mathbf{S}^n(K)) \bigcap \{D_i(\beta)\}_{i=0}^{m-1} \neq \emptyset$ and for this n we still have

$\mathbf{S}^n(K) = [\mathbf{S}^n(\inf K), \mathbf{S}^n(\sup K)) \in \{K\}$. If $\mathbf{S}^n(K) \bigcap \{D_i(\beta)\}_{i=0}^{m} \neq \emptyset$ then by the previous lemma there is $i_0 \in \{1, \ldots, m-1\}$ such that $\mathbf{S}^n(\sup K) = \phi(D_{i_0})$ for $D_{i_0} = D_{i_0}(\alpha)$. But $\mathbf{S}^n(\sup K) = \mathbf{S}^n(\phi(x_0)) = \phi(\mathbf{T}^n(x_0))$ from which we get $\mathbf{T}^n(x_0) = D_{i_0}$. If $\mathbf{S}^n(K) \bigcap \{D_i(\beta)\}_{i=0}^{m-1} = \emptyset$ we have $\sup(\mathbf{S}^n(K)) \in \{D_i(\beta)\}_{i=0}^{m-1}$, but $\sup(\mathbf{S}^n(K)) = \mathbf{S}^n(\sup(K)) = \mathbf{S}^n(\phi(x_0)) = \phi(\mathbf{T}^n(x_0))$ and we conclude, again by the previous lemma, that $\mathbf{T}^n(x_0) = D_{i_0}(\alpha)$ for some $i_0 \in \{1, \ldots, m-1\}$.

Now start moving K backwards using $\mathbf{S}$. There must have a first $l \leq 0$ such that $closure_{\mathbf{R}}(\mathbf{S}^l(K)) \bigcap \{D_i^\pi(\beta)\}_{i=0}^{m-1} \neq \emptyset$ and, as before, $\mathbf{S}^l(K) = [\mathbf{S}^l(\inf K), \mathbf{S}^l(\sup K)) \in \{K\}$. We have again two possibilities, either

$$\mathbf{S}^l(K) \bigcap \{D_i^\pi(\beta)\}_{i=1}^{m-1} \neq \emptyset$$

and, in this case, by the previous lemma, we have $\mathbf{S}^{l-1}(\sup(K)) = \phi(D_{j_0}(\alpha))$ (or $\mathbf{S}^{l-2}(\sup(K)) = \phi(D_{j_0}(\alpha))$) for some $j_0 \in \{1, \ldots, m-1\}$ from which we get $\mathbf{T}^{l-1}(x_0) = D_{j_0}(\alpha)$ (or $\mathbf{T}^{l-2}(x_0) = D_{j_0}(\alpha)$) or

$$\mathbf{S}^l(K) \bigcap \{D_i^\pi(\beta)\}_{i=1}^{m-1} = \emptyset$$

and in this case $\sup(\mathbf{S}^l(K)) = D_{j_0}^\pi(\beta)$ for some $j_0 \in \{1, \ldots, m-1\}$ and we conclude in a similar way that $\mathbf{T}^{l-1}(x_0) = D_{j_0}(\alpha)$ (or $\mathbf{T}^{l-2}(x_0) = D_{j_0}(\alpha)$). Summing up the two conclusions $\mathbf{T}^n(x_0) = D_{i_0}(\alpha)$ and $\mathbf{T}^{l-1}(x_0) = D_{j_0}(\alpha)$ (or $\mathbf{T}^{l-2}(x_0) = D_{j_0}(\alpha)$) for $0 \geq l$ and $n \geq 0$ and i_0 and $j_0 \in \{1, \ldots, m-1\}$ we get a contradiction with the fact that $\mathbf{T}$ satisfies i.d.o.c.. This proves the lemma.

Proof of Theorem 4.1: All we have to do is collect the above lemmas together: if the orbits of 0 under iteration by $\mathbf{T}$ and $\mathbf{S}$ hit corresponding intervals at the same time Lemma 4.2 shows that the map $\mathbf{T}^n(0) \mapsto \mathbf{S}^n(0)$ is order preserving and we can construct ϕ. The following lemmas show that ϕ is continuous so that all that remains to be proved is that ϕ is onto or, which is the same, that the intervals $[0, \inf \phi)$ and $[\sup \phi, 1)$ are empty. Now, π is irreducible, thus these intervals must be transposed by $\mathbf{S}$ but this is impossible for it would imply that $\mathbf{I}_1(\beta) = [0, \inf \phi)$ and $\mathbf{I}_m(\beta) = [\sup \phi, 1)$. The theorem follows.

Corollary 4.1 *Let $\mathbf{T}$ be an interval exchange map satisfying i.d.o.c. and v_n^k; $k = 1, \ldots, a_n$, be its n-th order approximants, $n \geq 0$, then, $\mathbf{T}$ is uniquely ergodic iff for every choice of approximants $v_n^{k_n}$ to $\mathbf{T}$, $n \geq 0$, we have $\lim_{n \to \infty} v_n^{k_n} = \alpha$.*

Proof: If, for each $n \geq 0$, we choose $v_n^{k_n}$ such that:

$$d_n := \sup\{\| \beta - \alpha \| \mid \beta \in \mathcal{F}_n\} = \| v_n^{k_n} - \alpha \|$$

it is clear that if $d_n \to 0$ as $n \to \infty$, $\{\mathbf{T}\} = \bigcap_n interior(\mathcal{F}_n)$ and $\mathbf{T}$ is uniquely ergodic. Suppose now that $d_{n_i} \geq \epsilon > 0$ for $n_i \to \infty$. We can suppose that a choice of $v_{n_i}^{k_{n_i}}$ satisfying $\| v_{n_i}^{k_{n_i}} - \alpha \| = d_{n_i}$ converges to $\beta \in closure(\mathcal{S}_m)$, $\beta \neq \alpha$. Let us show that $\gamma = \frac{\alpha+\beta}{2} \in$ conjugacy class of $\mathbf{T}$, thus proving that $\mathbf{T}$ is not uniquely ergodic and completing the proof of the Corollary. Suppose $\gamma \notin$ conjugacy class of $\mathbf{T}$. This means that there is a $n \geq 0$ such that γ such that $\gamma \notin \mathcal{F}_n$, but $\alpha \in$ conjugacy class of $\mathbf{T}$ and this means that α and β are in opposite sides of H, an hyperplane obtained by substituting one of the inequalities defining $\mathcal{F}_n$ by an equality. If we let i be great enough such that $k_{n_i} > n$ and $v_{n_i}^{k_{n_i}}$ and β are in the same side of H, we have a contradiction considering that H enters in the definition of $\mathcal{F}_{n_i} \subseteq \mathcal{F}_n$.

Fix now $x \in [0,1)$ and two sequences of integers n_k and N_k such that $\lim_{k\to\infty} N_k - n_k = \infty$ and consider the Borel probabilities μ_k given by:

$$\mu_k = \frac{1}{N_k - n_k} \sum_{i=n_k}^{N_k-1} \delta_{\mathbf{T}^i(x)}$$

where δ_y is the Dirac Borel probability concentrated at $y \in [0,1)$.

Lemma 4.7 *If $\mathbf{T}$ satisfies i.d.o.c. and is uniquely ergodic then μ_k converges weakly to the Lebesgue measure on $[0,1)$. In particular if we take $x = 0$, $n_k = k$-th $\mathbf{T}$-critical iterate and $N_k = k+1$-th $\mathbf{T}$-critical iterate we see that the sequence of the normalized distribution vectors between two consecutive critical iterates converge to α.*

Proof: Compactifying the interval $[0,1)$ by identifying the extremes of the interval $[0,1]$, we see that μ_k has a subsequence μ_{k_l} that converges weakly to a probability μ. We have to prove that μ is the Lebesgue measure. To keep the notation simple let us assume the sequence μ_k itself converges weakly to μ. This means:

$$\int f \, d\mu = \lim_{k\to\infty} \frac{1}{N_k - n_k} \sum_{i=n_k}^{N_k-1} f(\mathbf{T}^i(x))$$

for every real continuous function on $[0, 1)$ satisfying

$$f(0) = \lim_{t \uparrow 1} f(t) \tag{19}$$

An easy computation shows that $\int f \circ \mathbf{T}\, d\mu = \int f\, d\mu$ if f and $f \circ \mathbf{T}$ are continuous and both satisfy (19). Using this equality we see that $\mu(\mathbf{T}^{-1}(I)) = \mu(I)$ for every interval $I \subseteq [0, 1)$ whose real closure doesn't meet the set of discontinuities of $\mathbf{T}^{-1}$. In fact, it is easy to construct a uniformly bounded sequence f_l satisfying the above requirements such that $f_l \to \chi_I$, as $l \to \infty$, where χ_I is the characteristic function of I. We have then by Lebesgue's theorem:

$$\mu(\mathbf{T}^{-1}(I)) = \int \chi_I \circ \mathbf{T}\, d\mu = \lim_{l \to \infty} \int f_l \circ \mathbf{T}\, d\mu =$$

$$\lim_{l \to \infty} \int f_l\, d\mu = \int \chi_I\, d\mu = \mu(I)$$

If we bear in mind that $\mathbf{T}$ has no periodic points, it is easy to conclude that if μ has an atom p the backward $\mathbf{T}$-orbit of p must hit the set $\{D_i^{\pi}(\alpha)\}_{i=1}^{m-1}$. Thus if μ has atoms it has one atom, p, in the set $\{D_i^{\pi}(\alpha)\}_{i=1}^{m-1}$. Now, using an argument analogous to the one used to prove the $\mathbf{T}$-invariance of the μ-measure of intervals disjoint from $\{D_i^{\pi}(\alpha)\}_{i=1}^{m-1}$, we see that p is in a finite set F, $F \subseteq \{D_i^{\pi}(\alpha)\}_{i=1}^{m-1}$ such that $\mu(\mathbf{T}^{-1}(F)) = \mu(F)$ from which we conclude that μ has an atom also in the set $\mathbf{T}^{-1}(F) \subseteq \{D_i(\alpha)\}_{i=1}^{m-1}$. This atom, in its turn, as we saw above, by backward iteration using $\mathbf{T}$ must hit $\{D_i^{\pi}(\alpha)\}_{i=1}^{m-1}$ and this contradicts Keane's condition. This contradiction proves that $\mathbf{T}$ has no atoms and thus $\mu(\mathbf{T}^{-1}(I)) = \mu(I)$ for every interval $I \subseteq [0, 1)$. But this means that μ is $\mathbf{T}$-invariant and, by the unique ergodicity of $\mathbf{T}$, it follows that μ is the Lebesgue measure, proving the lemma.

Theorem 4.2 *The necessary and sufficient condition for an i.d.o. interval exchange map* $\mathbf{T}$ *to be uniquely ergodic is that the sequence of its normalized distribution vectors between two consecutive critical iterates converge to* α.

Proof: The preceeding lemma shows the necessity of the condition. To show the sufficiency, start by noting that if the sequence of normalized distribution vectors between two consecutive critical iterates converges to α then the sequence of normalized distribution vectors between two consecutive right (left) critical iterates also converges to α since the normalized distribution vector between two consecutive right (left) critical iterates n_1 and n_2 is a convex linear combination of the normalized distribution vectors between two consecutive critical iterates that lie between n_1 and n_2. But this means that the normalized column vectors of the distribution matrices of $\mathbf{T}$, (λ^n, ρ^n), go to α as $n \to \infty$ and from this we get that the approximants also go to α since these, by Lemma 3.5, are again convex linear combinations of the normalized columns of (λ^n, ρ^n). This finishes the proof of the theorem.

5 Gauss Maps

In this section we define the Gauss map $\mathcal{G} = \mathcal{G}(\pi): \mathcal{C} \to \mathcal{C}$ where $\mathcal{C} =$ disjoint union of $\mathcal{C}_\gamma$, $\gamma \in \mathcal{A}$ and show how it generates the approximants to an i.d.o. $\mathbf{T} \in \mathcal{S}_m(\pi)$.

We start by defining two maps $\mathcal{L}$ and $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ as follows $\mathcal{L}(\gamma) = \gamma^{\mathcal{L}}$ where $\gamma = (g, G)$ and $\gamma^{\mathcal{L}} = (g^{\mathcal{L}}, G^{\mathcal{L}})$ is given by:

$$g^{\mathcal{L}}(j) = \begin{cases} g(j), & \text{if } \#g^{-1}(g(j)) = 1 \text{ or } j = g^{m-1}(0); \\ g^2(j), & \text{otherwise.} \end{cases}$$

and $G^{\mathcal{L}} = g^{\mathcal{L}} \circ f^{-1}$. As to the definition of $\mathcal{R}$ we have $\mathcal{R}(\gamma) = \gamma^{\mathcal{R}}$ where $\gamma = (g, G)$ and $\gamma^{\mathcal{R}} = (g^{\mathcal{R}}, G^{\mathcal{R}})$ is given by :

$$G^{\mathcal{R}}(j) = \begin{cases} G(j), & \text{if } \#G^{-1}(G(j)) = 1 \text{ or } j = G^{m-1}(m); \\ G^2(j), & \text{otherwise.} \end{cases}$$

and $g^{\mathcal{R}} = G^{\mathcal{R}} \circ f$. It is easily seen that $\gamma^{\mathcal{L}}$ and $\gamma^{\mathcal{R}}$ satisfy (6) and (7) above.

Now, fix $\gamma \in \mathcal{A}$ and consider the hyperplane $R_{i_0} = L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}$ where i_0 is the type of γ. This hyperplane divides the polyhedron $\mathcal{C}_\gamma$ into two polyhedra:

$$\mathcal{C}_\gamma^{\mathcal{R}} = \{R_{i_0} \geq L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}\} \bigcap \mathcal{C}_\gamma$$

$$\mathcal{C}_\gamma^{\mathcal{L}} = \{R_{i_0} < L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}\} \bigcap \mathcal{C}_\gamma$$

with non-empty interiors.

In fact, using the notation of Lemma 3.5, it is clear that the vertices $\widetilde{X}^1$ and $\widetilde{X}^2$ of $\mathcal{C}_\gamma$ given by $R^1 = 0$ and $supp(L^1)$ is the g-orbit of i_0 and $L^2 = 0$

and $supp(R^2)$ is the G-orbit of i_0, are in opposite sides of the hyperplane $R_{i_0} = L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}$.

The set of equations (8), which define the support of the cone with vertex 0 spanned by $\mathcal{C}_\gamma$ can be written:

$$\begin{cases} L_i + R_i = L_{g^{-1}(i)} + R_{f(g^{-1}(i))}, & \text{if } i \neq i_0 \ , \ g(i_0); \\ L_{g(i_0)} + R_{g(i_0)} = L_{i_0} + R_{f(i_0)}, & \text{if } i = g(i_0); \\ L_{i_0} + R_{i_0} = & \\ L_{g^{-1}(i_0)} + R_{f(g^{-1}(i_0))} + L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}, & \text{if } i = i_0. \end{cases}$$

$i = 1, \ldots, m-1$, or

$$\begin{cases} L_i + R_i = L_{g^{-1}(i)} + R_{f(g^{-1}(i))}, & \text{if } i \neq i_0 \ , \ g(i_0); \\ L_{g(i_0)} + R_{g(i_0)} = L_{i_0} - (L_{g^{-1}(i_0)} + R_{f(g^{-1}(i_0))}) + & \\ R_{f(i_0)} + L_{g^{-1}(i_0)} + R_{f(g^{-1}(i_0))}, & \text{if } i = g(i_0); \\ L_{i_0} - (L_{g^{-1}(i_0)} + R_{f(g^{-1}(i_0))}) + R_{i_0} = & \\ L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}, & \text{if } i = i_0. \end{cases}$$

$i = 1, \ldots, m-1$.

Now, restricting ourselves to $(L, R) \in \mathcal{C}_\gamma^{\mathcal{L}}$ and defining $L_i^{\mathcal{L}}$ and $R_i^{\mathcal{L}}$ by $R^{\mathcal{L}} = R_i$ for $i = 1, \ldots, m$ and:

$$L_i^{\mathcal{L}} = \begin{cases} L_{i_0} - (L_{g^{-1}(i_0)} + R_{f(g^{-1}(i_0))}), & \text{if } i = i_0; \\ L_i, & \text{otherwise.} \end{cases}$$

and substituting in the above equalities we get:

$$\begin{cases} L_i^{\mathcal{L}} + R_i^{\mathcal{L}} = L_{g^{-1}(i)}^{\mathcal{L}} + R_{f(g^{-1}(i))}^{\mathcal{L}}, & \text{if } i \neq i_0 \ , \ g(i_0); \\ L_{g(i_0)}^{\mathcal{L}} + R_{g(i_0)}^{\mathcal{L}} = & \\ L_{i_0}^{\mathcal{L}} + R_{f(i_0)}^{\mathcal{L}} + L_{g^{-1}(i_0)}^{\mathcal{L}} + R_{f(g^{-1}(i_0))}^{\mathcal{L}}, & \text{if } i = g(i_0); \\ L_{i_0}^{\mathcal{L}} + R_{i_0}^{\mathcal{L}} = L_{g^{m-1}(0)}^{\mathcal{L}} + R_{f(g^{m-1}(0))}^{\mathcal{L}}, & \text{if } i = i_0. \end{cases}$$

$i = 1, \ldots, m-1$, which is precisely the set equations defining the support of the cone spanned by $\mathcal{C}_{\mathcal{L}(\gamma)}$. This shows that $\mathcal{L}(\gamma)$ is in $\mathcal{A}$ and that the projective map induced by $\mathbf{L}(\gamma): (L, R) \mapsto (L^{\mathcal{L}}, R^{\mathcal{L}})$ is an isomorphism between $\mathcal{C}_\gamma^{\mathcal{L}}$ and $\mathcal{C}_{\mathcal{L}(\gamma)}$. A similar argument shows that $\mathcal{R}(\gamma)$ is in $\mathcal{A}$ and that $\mathbf{R}(\gamma): (L, R) \mapsto (L^{\mathcal{R}}, R^{\mathcal{R}})$ given by:

$$L^{\mathcal{R}} = L_i \text{ for } i = 1, \ldots, m \tag{20}$$

and:
$$R_i^{\mathcal{R}} = \begin{cases} R_{i_0} - (L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}), & \text{if } i = i_0; \\ R_i, & \text{otherwise.} \end{cases} \tag{21}$$

induces an isomorphism between $\mathcal{C}_\gamma^{\mathcal{R}}$ and $\mathcal{C}_{\mathcal{R}(\gamma)}$. The Gauss map $\mathcal{G}$ is defined by $\mathcal{G}\mid_{\mathcal{C}_\gamma^{\mathcal{L}}} = \mathbf{L}(\gamma)$ and $\mathcal{G}\mid_{\mathcal{C}_\gamma^{\mathcal{R}}} = \mathbf{R}(\gamma)$ for $\gamma \in \mathcal{A}$. We sum up our conclusions in the next theorem.

Theorem 5.1 *The Gauss map $\mathcal{G}: \mathcal{C} \longrightarrow \mathcal{C}$ is a 2-1 map that establishes a projective isomorphism $\mathcal{C}_\gamma^{\mathcal{R}} \to \mathcal{C}_{\mathcal{R}(\gamma)}$ and $\mathcal{C}_\gamma^{\mathcal{L}} \to \mathcal{C}_{\mathcal{L}(\gamma)}$ for each $\gamma \in \mathcal{A}$. Let $\mathcal{F}_s \subseteq \mathcal{S}_m$ be a small Farey cell with s critical, n be the next critical iterate of $\mathcal{F}_s$, and $\mathbf{P}: \mathcal{F}_s \to \mathcal{C}_\gamma$ be the canonical isomorphism, where γ is the abstract type of $\mathcal{F}_s$. Take $\mathbf{T} \in \mathcal{F}_s$ and $r \in \mathcal{C}_\gamma$ given by $r = \mathbf{P}(\mathbf{T})$. Let δ be such that $\mathcal{G}(r) \in \mathcal{C}_\delta$ and denote by $\mathcal{G}_\delta^{-1}$ the branch of $\mathcal{G}$ defined on $\mathcal{C}_\delta$. We have then:*
a)$\mathbf{P}^{-1}(\mathcal{G}_\delta^{-1}(\mathcal{C}_\delta))$ is the next Farey cell around $\mathbf{T}$, $\mathcal{F}_n(\mathbf{T})$.
b)$\mathbf{Q} = \mathcal{G}_\delta \circ \mathbf{P}: \mathcal{F}_n(\mathbf{T}) \longrightarrow \mathcal{C}_\delta$ is the canonical isomorphism between $\mathcal{F}_n(\mathbf{T})$ and its abstract type $\mathcal{C}_\delta$.

Proof: We have already shown that $\mathcal{G}\mid_{\mathcal{C}_\gamma^{\mathcal{L}}}$ and $\mathcal{G}\mid_{\mathcal{C}_\gamma^{\mathcal{R}}}$ are projective isomorphisms.

Suppose $r \in \mathcal{C}_\gamma^{\mathcal{R}}$. The case $r \in \mathcal{C}_\gamma^{\mathcal{L}}$ is analogous. Using the notation of Theorem 3.1 we have $r = (\widetilde{L, R})^t$ where L and R are defined by (16) and (17). Using the observation that follows Lemma 3.4 we see that $\mathcal{F}_n(\mathbf{T}) = \mathbf{P}^{-1}(\mathcal{C}_\gamma^{\mathcal{R}}) = \mathbf{P}^{-1}(\mathcal{G}_\delta^{-1}(\mathcal{C}_\delta))$ which proves a) of the theorem. To show b) start by using (18) to write $\alpha = (\lambda^n\ \rho^n)\begin{pmatrix} L \\ R \end{pmatrix} = (\lambda^n\ \rho^n)M^{-1}M\begin{pmatrix} L \\ R \end{pmatrix}$ where $(\lambda^n\ \rho^n)$ is the distribution matrix of $\mathcal{F}_s$ and M is the matrix of the linear map defined by (20) and (21). The matrix $(\lambda\ \rho) = (\lambda^n\ \rho^n)M^{-1}$ is equal to the matrix $(\lambda^n\ \rho^n)$ but for two columns; the $g^{m-1}(0)$-th column of λ and the $f(g^{m-1}(0))$-th column of ρ for which we have:

$$\lambda_{g^{m-1}(0)} = \lambda^n_{g^{m-1}(0)} + \rho^n_{i_0} \text{ and } \rho_{f(g^{m-1}(0))} = \rho^n_{f(g^{m-1}(0))} + \rho^n_{i_0} \tag{22}$$

Now, this is precisely the distribution matrix of $\mathcal{F}_n(\mathbf{T})$ since the stacks associated to $\mathcal{F}_n(\mathbf{T})$ are the same as the ones associated to $\mathcal{F}_s(\mathbf{T})$ except for two stacks which will account for the changes (22) in the distribution matrix. To see this, observe that the stack $\mathcal{P}$ of $\mathcal{F}_s$ with botton $L^\flat_{g^{m-1}(0)} + R^\flat_{f(g^{m-1}(0))}$ should now be moved to a position bellow $R^\flat_{i_0}$ at the botton of the stack

$\mathcal{Q}$ of $\mathcal{F}_s(\mathbf{T})$ which has botton $L^{\flat}_{f^{-1}(i_0)} + R^{\flat}_{i_0}$. This because the top of $\mathcal{P}$ is contained in $R^{\sharp}_{i_0}$ which is mapped by $\mathbf{T}$ onto $R^{\flat}_{i_0}$. $\mathcal{P}$ with the right slice of $\mathcal{Q}$ of width $L_{g^{m-1}(0)} + R_{f(g^{m-1}(0))}$ on its top is a new stack of $\mathcal{F}_n(\mathbf{T})$; the other is the remaining of the stack $\mathcal{Q}$ after the slicing. This shows that $\mathbf{Q}^{-1} \circ \mathcal{G}_\delta \circ \mathbf{P}$ is the identity on $\mathcal{F}_n(\mathbf{T})$ where $\mathbf{Q}$ is the canonical isomorphism $\mathcal{F}_n(\mathbf{T}) \longrightarrow \mathcal{C}_\delta$ and completes the proof of the theorem.

Using this theorem we can construct the generalized continued fraction expansion of an i.d.o. $\mathbf{T} \in \mathcal{S}_m$ as described in the introduction.

We start with the integral cell around $\mathbf{T}$, $\mathcal{F}_{n_0}(\mathbf{T})$, and its integral part $\mathbf{P}_0^{-1}: \mathcal{C}_{\gamma_0} \to \mathcal{F}_{n_0}(\mathbf{T})$, where γ_0 is the integral type of $\mathbf{T}$. We can write $\mathbf{T} = \mathbf{P}_0^{-1}(r_0)$, where $r_0 \in \mathcal{C}_{\gamma_0}$ is the fractional part of $\mathbf{T}$. Now using Theorem 5.1 repeatedly we have:

$$\alpha = \mathbf{P}_0^{-1}(\mathcal{G}_{\gamma_1}^{-1}(\ldots\mathcal{G}_{\gamma_n}^{-1}(r_n)\ldots))$$

where the remainder r_n is equal to $\mathcal{G}^n(r_0)$ and $\mathcal{G}_{\gamma_i}^{-1}$ is the branch of $\mathcal{G}^{-1}$ taking $\mathcal{C}_{\gamma_i}$ into $\mathcal{C}_{\gamma_{i-1}}$. $\mathcal{C}_{\gamma_i}$, in its turn, is defined by requiring that $\mathcal{G}^i(r_0) \in \mathcal{C}_{\gamma_i}$; $i = 1, \ldots, n$.

We use the formal expansion

$$\alpha = \mathbf{P}_0^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ \ldots \mathcal{G}_{\gamma_n}^{-1} \circ \ldots$$

to indicate the above construction.

It is clear from Theorem 5.1 that we get the n-th order approximants to $\mathbf{T}$ by truncating the above expansion at level n, substituting the remainder r_n for the vertices of $\mathcal{C}_{\gamma_n}$ and carrying the indicated operations. Thus, bearing in mind Corollary 4.1, we see that just by looking at the generalized continued fraction expansion of an i.d.o. $\mathbf{T}$ we are able to decide if it is uniquely ergodic or not.

Theorem 5.2 *Let $\mathbf{T} = \mathbf{T}(\alpha, \pi)$ be an i.d.o.c. interval exchange map and*

$$\alpha = \mathbf{P}_0^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ \ldots \mathcal{G}_{\gamma_n}^{-1} \circ \ldots$$

its generalized continued fraction expansion associated to π, then $\mathbf{T}$ is uniquely ergodic iff its approximants, computed as described above, converge to α.

On the other hand, if we ask for the conditions under which an expansion

$$\mathbf{P}_0^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ \ldots \mathcal{G}_{\gamma_n}^{-1} \circ \ldots \qquad (23)$$

is the generalized continued fraction expansion of an uniquely ergodic $\mathbf{T}$ we must first construct the space of the one-sided subshift of finite type on the set of simbols $\mathcal{A}$ and with a transition γ_1 to γ_2 allowed iff $\mathcal{G}(\gamma_2)\bigcap\gamma_1 \neq \emptyset$ or, which is the same, iff there is a branch of $\mathcal{G}^{-1}$ mapping γ_1 into γ_2 .

This is the space $\mathbf{Sshift} = \mathbf{Sshift}(\pi)$ of sequences $(\gamma_n)_{n=1}^{\infty}$, $\gamma_n \in \mathcal{A}$ such that $\mathcal{G}(\gamma_{n+1})\bigcap\gamma_n \neq \emptyset$ for $n = 1, 2, \ldots$. We say that the allowed transition γ_1 to γ_2 in $\mathbf{Sshift}$ is a *left transition* if the branch of $\mathcal{G}^{-1}$ on γ_1 is given by $\mathbf{L}^{-1}(\gamma_2)$ and *right transition* if the branch of $\mathcal{G}^{-1}$ on γ_1 is given by $\mathbf{R}^{-1}(\gamma_2)$.

It is clear that the sequence of γ_n's in a generalized continued fraction expansion of an i.d.o. $\mathbf{T}$ is in the space of the subshift but that is not enough to guarantee the convergence of the approximants to an interval exchange map. Clearly the conditions that:

a) the limit of the diameters of the set of n-th order approximants goes to 0 as n goes to ∞ and

b) for every integer $n \geq 1$ there is $p > n$ such that every p-th order approximant, is in the interior of the convex hull of the n-th order approximants

are sufficient to guarantee the existence of an unique interval exchange map $\mathbf{T}$ with the given expansion (23), however this map can fail even to be minimal. We can get a condition sufficient for the minimality of the map by watching in the sequence $(\gamma_n)_{n=1}^{\infty}$, the types $i_0 \in \{1, 2, \ldots, m-1\}$ of the abstract cells γ_n which come from a right transition and are followed by a left transition or vice-versa. We call these cells the *transition cells* of $(\gamma_n)_{n=1}^{\infty}$. The condition we are seeking is that:

c) each type i_0 occurs infinitely often as a type of a transition cell in the sequence $(\gamma_n)_{n=1}^{\infty}$.

Before proving the sufficiency of the condition note that now the n-th order approximants can be found only by truncating the expansion (23) at level n, substituting the remainder for the vertices of the corresponding abstract Farey cell and carrying the indicated operations.

Theorem 5.3 *Let*

$$\mathbf{P}_0^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ \ldots \mathcal{G}_{\gamma_n}^{-1} \circ \ldots \tag{24}$$

be a generalized continued fraction expansion where $(\gamma_n)_{n=1}^{\infty}$ *is a sequence in the space* $\mathbf{Sshift}$*. then conditions a), b) and c) above are sufficient in order that the approximants defined by the expansion converge to an unique uniquely ergodic interval exchange map* $\mathbf{T} = \mathbf{T}(\pi, \alpha)$*. In this case*

$$\alpha = \mathbf{P}_0^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ \ldots \mathcal{G}_{\gamma_n}^{-1} \circ \ldots$$

is the generalized continued fraction expansion of $\mathbf{T}$.

Proof: It is obvious that the expansion (24) defines a unique interval exchange map $\mathbf{T} = \mathbf{T}(\pi, \alpha)$ as the limit of its approximants and that (24) is the generalized continued fraction expansion of $\mathbf{T}$. We claim that the $\mathbf{T}$-orbit of 0 is dense in $[0, 1)$. In fact, from the stacks parametrization of the Farey cells it is clear that, as we move forward with the Gauss map $\mathcal{G}$ starting at $\mathbf{T}$, each time we hit a transition cell γ_n of type i_0 the number of intervals in the stack $\mathcal{C}_{\gamma_n}$ containing D_{i_0} in its top increases. Thus, using c), we see that the number of intervals in the stacks of $\mathcal{C}_{\gamma_n}$ go to ∞ as $n \to \infty$ and this forces the width of these stacks go to 0 as $n \to \infty$ proving the claim.

Now, using Keane's minimality condition, Keane [2], we conclude that $\mathbf{T}$ is minimal and, since $\mathcal{C}(\mathbf{T}) \subseteq \bigcap_{n=0}^{\infty} interior(\mathcal{F}_n(\mathbf{T}))$ if $\mathbf{T}$ is minimal, we have $\mathbf{T}$ uniquely ergodic which proves the theorem.

We give now a combinatorial description of the distribution matrices (λ, ρ) associated to small Farey cells of a fixed type $\gamma = (g, G) \in \mathcal{A}$. Using this description we are able to construct the integral parts of interval exchange maps in $\mathcal{S}_m(\pi)$ that have integral type γ. To this end we will transform the problem of finding the distribution matrices into the problem of finding the cycles in the set of permutations $\Pi_\perp = \Pi_\perp(\pi, g, G, h_0, \dots, h_{m-1})$ constructed from g, G and m non-negative integral parameters, $h_0, \dots, h_{m-1}$, by a procedure to be defined bellow. To motivate the definition of $\Pi_\perp$ suppose we have $\mathcal{F}_s$ a small Farey cell of type γ where s is critical. Take n the next critical iterate of $\mathcal{F}_s$ and h_j; $j = 0, \dots, m-1$, the number of of intervals in the stack $\mathcal{P}_j$ of $\mathcal{F}_s$ where $\mathcal{P}_j$ is such that $b(\mathcal{P}_j) = L_j^b + R_{f(j)}^b$. Each iterate of $\{\mathbf{T}^k(0)\}_{k=0}^n$ except $0 = \mathbf{T}^0(0)$ and $\mathbf{T}^{\max}(0) = \max\{\mathbf{T}^k(0) \mid k = 0, \dots, n\}$ occurs twice as an extreme of the intervals in $\mathcal{P}_j$; first as a left and then as a right extreme. If we define $\Pi_\perp$ taking the order of iteration of a left extreme of an interval to the order of iteration of the right extreme of the same interval and $\max \mapsto 0$ we have a cycle on $\{0, \dots, n\}$.

Reversing the direction of our considerations we start with the h_j's, the heights of the stacks and use (g, G) to define $\Pi_\perp$ without reference to the interval exchange map. This is done as follows, take $h_0, \dots, h_{m-1}$, m non-negative integers and define $H_k = h_{f^{-1}(k)}$, $k = 1, \dots, m$ and $n = \sum_{j=0}^{m-1} h_j$. The permutation $\Pi_\perp$ on $\{0, 1, \dots, n\}$ is defined as follows:
$\Pi_\perp(\sum_{i=0}^{j} h_i) = 0$ for $j \in \{0, \dots, m-1\}$ such that $g^{j+1}(0) = \pi^{-1}(m)$ and, to define $\Pi_\perp$ on $\{0, \dots, n\} - \{\sum_{i=0}^{j} h_i\}$, subdivide this set into m sucessive

intervals $\mathbf{h}_0, \mathbf{h}_{g(0)}, \ldots, \mathbf{h}_{g^{m-1}(0)}$ of lengths $h_0, h_{g(0)}, \ldots, h_{g^{m-1}(0)}$ respectively and the set $\{1, \ldots, n\}$ into m sucessive intervals $\mathbf{H}_m, \mathbf{H}_{G(m)}, \ldots, \mathbf{H}_{G^{m-1}(m)}$ of lenghts $H_m, H_{G(m)}, \ldots, H_{G^{m-1}(m)}$ respectively; $\Pi_\perp$ maps the j-th interval, $\mathbf{h}_{g^j(0)}$, increasingly onto the k-th interval, $\mathbf{H}_{G^k(0)}$, where k is given by $G^k(m) = f(g^j(m))$.

As explained above, we must require that $\Pi_\perp$ so defined be a cycle which we will assume from now on.

Define the quantities r_k, for $k = 0, \ldots, m$ and l_j, for $j = 1, \ldots, m$ as $r_0 = 0$ and $r_{G^k(m)} = \max \mathbf{H}_{G^{k-1}(m)}$, for $k = 1, \ldots, m$, and $l_{g^j(0)} = \max \mathbf{h}_{g^{j-1}(0)}$. These quantities play the role of the order of the remaining critical iterates. For $i = 1, \ldots, m$, define the set $\mathbf{i}_i$, which will represent the set of iterates of the interval exchange map $\mathbf{T}$ in the interval $\mathbf{I}_i(\mathbf{T})$, as follows:
$\mathbf{i}_i$ is the set of iterates $\Pi_\perp^q(r_{i-1})$ for q running from 0 to the time just before $\Pi_\perp^q(r_{i-1})$ hits the set $\{r_k\}_{k=1}^{m-1}$ again.

With those representations in mind it is clear that, if the order of the points $l_{g^j(0)}$ and $r_{G^k(m)}$ in the cycle $\Pi_\perp$ starting at 0 is correct, we get a distribution matrix by taking:
$\lambda_{ij} =$ the number of points q in $\mathbf{i}_i$ as q runs from l_j up to the time just before q enters $\{l_j\}_{j=1}^m$ again, and
$\rho_{ik} =$ the number of points q in $\mathbf{i}_i$ as q runs from r_k up to the time just before q enters $\{r_k\}_{k=1}^m$ again, for j and $k = 1, \ldots, m-1$.

We finish this section and this paper using the above theory to construct an example of an uniquely ergodic interval exchange map. It is clear that the ideas used in this construction can be used to give a wide class of examples of uniquely and non-uniquely ergodic maps, a matter that will be pursued elsewhere.

As explained in the introduction, maps exchanging $m = 2$ intervals can be considered as rotations on the circle and i.d.o.c iff irrational rotation iff uniquely ergodic. The study of maps exchanging $m = 3$ intervals can also be reduced to the case of rotations by looking at the induced map on a suitable subinterval. Thus our theory is really useful for $m \geq 4$.

If $m = 4$ we have seven irreducible and discontinuous permutations:

$$(2,4,1,3),\ (2,4,3,1),\ (3,1,4,2),\ (3,2,4,1),$$

$$(4,1,3,2),\ (4,2,1,3),\ \text{and}\ (4,3,2,1)$$

here, and in what follows, we denote a map F defined on an interval of integers $k+1, k+2, \ldots, k+l$ by the l-th uple

$$(F(k+1), F(k+2), \ldots, F(k+l)).$$

We take $\pi = (2,4,3,1)$ to construct our example. In this case $f = f(\pi) = (3,2,4,1)$ and $\mathcal{A}$ has also seven elements:

$$\gamma_1 = ((1,3,1,2),(2,3,1,1)), \gamma_2 = ((1,2,3,2),(2,2,1,3)),$$

$$\gamma_3 = ((1,3,2,2),(2,3,1,2)), \gamma_4 = ((2,3,1,2),(2,3,2,1)),$$

$$\gamma_5 = ((1,3,3,2),(2,3,1,3)), \gamma_6 = ((2,3,1,3),(3,3,2,1)),$$

$$\text{and } \gamma_7 = ((3,3,1,2),(2,3,3,1))$$

where, as before, the first entry of the pair γ_i denotes g and the second G. The vertices of the corresponding abstract Farey cells $\mathcal{C}_\gamma$ are given, respectively, by the normalized columns of the matrices

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}, v_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

$$\text{and } v_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

The transition matrix

$$\mathcal{T}_{ij} = \begin{cases} 1, & \text{if } \mathcal{G}(\mathcal{C}_{\gamma_i}) \bigcap \mathcal{C}_{\gamma_j} \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

$i, j = 1, 2, \ldots, 7$, is given by:

$$\mathcal{T} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

and is the sum of the left and right transition matrices

$$\mathcal{LT} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathcal{RT} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrices inducing the branches of the inverse of the Gauss map are:

$$M(4,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, M(5,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M(2,2)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&1&1&1&0\\0&0&0&0&0&1\end{pmatrix},M(5,2)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\1&0&1&0&1&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$M(1,3)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&1&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},M(3,3)=\begin{pmatrix}1&0&0&0&0&0\\0&1&1&1&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$M(6,4)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&1&1&0&1\end{pmatrix},M(7,4)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$M(2,5)=\begin{pmatrix}1&0&0&0&0&0\\1&1&0&0&1&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},M(3,5)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&1&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$M(4.6)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&1&1&1&0\\0&0&0&0&0&1\end{pmatrix},M(6,6)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\1&0&1&0&1&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

and $M(1,7)=\begin{pmatrix}1&0&0&0&0&1\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},M(7,7)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\1&0&0&0&1&1\end{pmatrix},$

where $M(i,j)$ induces the branch of $\mathcal{G}^{-1}$ mapping γ_j into γ_i.

The first small Farey cell of type γ_1 has the distribution matrix

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and vertices given by normalizing the columns of

$$\mathcal{D}.v_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix},$$

Computing the product

$$M = M(1,3).M(3,5).M(5,2).M(2,5).M(5,1).$$

$$M(1,7).M(7,4).M(4,6).M(6,4).M(4,1),$$

which is a sequence allowed by T, we get

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 \\ 1 & 0 & 4 & 3 & 1 & 2 \\ 1 & 1 & 2 & 3 & 1 & 2 \\ 1 & 1 & 3 & 3 & 2 & 2 \\ 0 & 1 & 1 & 1 & 0 & 2 \end{pmatrix}.$$

This matrix has characteristic polinomial

$$1 - 13X + 44X^2 - 64X^3 + 44X^4 - 13X^5 + X^6$$

$$= (-1+X)^2(1 - 11X + 21X^2 - 11X^3 + X^4),$$

and exactly one eigenvalue with modulus greater then one

$$\frac{\frac{11}{2} + \frac{3\sqrt{5}}{2} + \sqrt{-4 + (\frac{11}{2} + \frac{3\sqrt{5}}{2})^2}}{2} = 8.7396813182\ldots,$$

This eigenvalue has multiplicity one, the associated eigenspace hits $\mathcal{C}_{\gamma_1}$ at

$$A = \begin{pmatrix} 0.0660453933\ldots \\ 0.1729090847\ldots \\ 0.2275346295\ldots \\ 0.1952496795\ldots \\ 0.2498752244\ldots \\ 0.0883859882\ldots \end{pmatrix}$$

and, it is easy to see, M contracts $\mathcal{C}_{\gamma_1}$ to A.

Using $\mathcal{D}$ as integral part we see by Theorem 5.3 that $\mathbf{T} = \mathbf{T}(\pi, \alpha)$, for

$$\alpha = \mathcal{D}.A = \begin{pmatrix} 0.0443606097\ldots \\ 0.4594745664\ldots \\ 0.3206611563\ldots \\ 0.1755036674\ldots \end{pmatrix},$$

is uniquely ergodic with periodic fractional expansion

$$\mathcal{G}_{\gamma_3}^{-1} \circ \mathcal{G}_{\gamma_5}^{-1} \circ \mathcal{G}_{\gamma_2}^{-1} \circ \mathcal{G}_{\gamma_5}^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_7}^{-1} \circ$$

$$\mathcal{G}_{\gamma_4}^{-1} \circ \mathcal{G}_{\gamma_6}^{-1} \circ \mathcal{G}_{\gamma_4}^{-1} \circ \mathcal{G}_{\gamma_1}^{-1} \circ \mathcal{G}_{\gamma_3}^{-1} \circ \mathcal{G}_{\gamma_5}^{-1} \circ \ldots$$

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