Domain-size heterogeneity in the Ising model: Geometrical and thermal transitions

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(Received 2 February 2015; published 13 April 2015)

A measure of cluster size heterogeneity (H), introduced by Lee *et al.* [Phys. Rev. E **84**, 020101 (2011)] in the context of explosive percolation, was recently applied to random percolation and to domains of parallel spins in the Ising and Potts models. It is defined as the average number of different domain sizes in a given configuration and a new exponent was introduced to explain its scaling with the size of the system. In thermal spin models, however, physical clusters take into account the temperature-dependent correlation between neighboring spins and encode the critical properties of the phase transition. We here extend the measure of H to these clusters and, moreover, present new results for the geometric domains for both $d = 2$ and 3. We show that the heterogeneity associated with geometric domains has a previously unnoticed double peak, thus being able to detect both the thermal and percolative transitions. An alternative interpretation for the scaling of H that does not introduce a new exponent is also proposed.

DOI: 10.1103/PhysRevE.91.042113 PACS number(s): 64.60.De, 68.35.Rh, 05.50.+q

I. INTRODUCTION

Cluster or domain size distributions are commonly used in statistical mechanics to unveil geometric properties and characterize both the equilibrium critical behavior [1] and the out-of-equilibrium dynamics [2,3] of several models. If the system is finite, not all possible domain sizes are present on a single configuration. These sample-to-sample fluctuations disappear and the distributions become dense in the infinite size limit or after ensemble averages are taken. In order to characterize these fluctuations, a quantity associated with how heterogeneously sized the equilibrium domains are, H, was recently proposed in the context of explosive percolation [4] and then applied to ordinary percolation [5] and to the Ising [6] and Potts models [7]. Differently from the cluster size distribution, the heterogeneity H only takes into account whether a given size is present in each configuration and gives the number of such distinct sizes. Although the number of equal-sized clusters and their actual size do not enter in the measure of H, both entropic and thermal effects are relevant. At high temperatures, thermal noise breaks large clusters and the fragments are small, and so is the diversity and H. On the other hand, for low temperature, the presence of a very large, probably percolating cluster leaves small room to the other clusters, also decreasing the diversity.

We here revisit the original formulation for the equilibrium scaling behavior of H for the Ising model and present new results for both geometric and physical clusters. While the former considers all nearest neighbor parallel spins belonging to the same domain, the latter only takes the fraction of spins effectively correlated [8,9]. This difference, although important for systems of interacting spins under thermal noise, obviously does not exist for percolation models. For random percolation [5], H has a clear peak and an excellent data collapse is obtained. For the Ising model, a kink but no peak has been observed for the system sizes considered in Ref. [6], although the authors had correctly argued for the existence of such a peak. Although in two dimensions (2d) both kinds of spin domains percolate at the same temperature, only the physical ones encode the critical properties (i.e., the critical exponents being the same, the percolative transition of the physical domains and the thermal transition are in the same universality class). It is thus interesting to see how their heterogeneity, H_p, differs from H_g measured with the geometric domains. As we show below, even for small sizes, H_p presents a clear peak and an excellent collapse is obtained. For the geometric domains, on the other hand, we observe two peaks for sufficiently large systems, none of which were observed in Ref. [6], where only an abrupt change in the derivative was present at the critical temperature T_c. The first peak is small, very close to T_c, while the second one is large, broad and appears far above the critical temperature. Although the very existence of such double peak is interesting enough, there are several further questions that can be asked. What is the origin of such peaks? Do they merge, leaving a single peak, as the system size diverges? Do they share the same exponents or the percolation exponents play a role? What happens in 3d where the critical temperature does not coincide with the percolation threshold of geometric domains? In order to try to answer these questions, we studied the equilibrium Ising model on square and cubic lattices with linear sizes up to 2560 and 250, respectively. Averages up to 200 samples were performed for the smaller systems while larger sizes required fewer samples. All measures were obtained after equilibrating the system through 500 Swendsen-Wang steps [10].

The paper is organized as follows. We start Sec. II by reviewing the definition of H and its scaling behavior in the context of a thermal system. Then, in Secs. II A and II B we present the results of extensive simulations for the 2d and 3d Ising model, in which we measure H for both geometric and physical clusters. We finally discuss our results and conclude in Sec. III.

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II. SIZE HETEROGENEITY

Differently from the susceptibility that corresponds, in percolation terms, to the mean cluster size [11], and thus correlates with both the size and frequency of fluctuations, for the measure of H it is only the presence of a given cluster size that is relevant. Both at too low and too high temperatures, the distribution of cluster sizes is concentrated at small domains, the latter because the large noise dissolves the clusters and the former because a giant cluster dominates the system leaving almost no room for the others. In both cases the size distribution has a small tail, domains are rather homogeneous, and H is not large. We thus expect a maximum of H between these two limits, close to the critical temperature because the fragmentation of the infinite cluster creates several small domains and the size distribution per site, $n(s)$, being broad (power law), will increase the probability of these clusters having different sizes and thus contribute to H. Moreover, as the linear size L increases, more and larger clusters are allowed when the average distribution is broad enough and H may also increase with system size. Notice, however, that having a broad distribution does not necessarily imply a large value of H: it depends on how many of these different sizes are present on a given state or, equivalently, the amount of vacancies in the distribution for a particular configuration of a finite size system (the average cluster size distribution, on the other hand, being an ensemble average, is dense).

The precise dependence of H on the linear size L can be understood with a scaling argument [4,5]. Let us start with the height of the peak close to the critical temperature. Small clusters are present with a high probability in a finite size system and we define s_0, which is a function of the temperature, as the (average) smallest size not present in a configuration. Below s_0 the distribution is dense; that is, the expected number of clusters whose sizes are smaller than s_0, $L^d\, n(s < s_0)$, is larger than 1. On the other hand, above s_0 there are some sizes without a realization, $L^d\, n(s > s_0) < 1$, and vacancies appear in the distribution. Thus, by definition, $L^d\, n(s_0) \sim 1$. As was shown in Ref. [5] and discussed below, the contributions to H from clusters in these two regions are of the same order [5]. Close to the transition, we write [11] $n(s) \simeq s^{-\tau} f(z)$ where $z = ts^{\sigma}$ and $t \equiv T/T_c - 1$ is the reduced temperature. Two critical exponents are introduced. The first one, σ, is associated with the extension of the power-law tail as the criticality is approached while the Fisher exponent τ is the exponent of such a power law. For the 2d cases considered here, we have $\tau_g = 379/187$ for geometric domains [12,13], $\tau_p = 31/15$ for physical clusters and $\tau_{perc} = 187/91$ for random percolation. The scaling function $f(z)$ approaches a constant for $|z| \ll 1$ and has a fast decay for $|z| \gg 1$. Close to the transition, $n(s_0) \sim s_0^{-\tau} \sim L^{-d}$, and thus s_0 scales as $s_0 \sim L^{d/\tau}$ [4]. The remaining contribution to H comes from the clusters with $s > s_0$. These clusters may be ranked by size, starting from the largest [5,14]. At the criticality, the largest cluster size lies in the interval $[s_1, \infty)$ such that the expected number of clusters in this interval is unity. That is, the integral of $n(s)$ from s_1 to ∞, multiplied by the size of the system, is 1. Analogously, the second largest cluster lies in the interval $[s_2, \infty)$ such that the integral of $n(s)$ from s_2 to ∞, multiplied for the rth cluster in the rank, at the critical temperature, we get

$$r = L^d \int_{s_r}^{\infty} ds\, s^{-\tau} f(0) \sim L^d\, s_r^{1-\tau}. \qquad (1)$$

In the Appendix we present a more detailed discussion of the convergence of this integral. We get that $s_r \sim (L^d/r)^{1/(\tau-1)}$ and the fractal dimension of these large clusters is $d/(\tau - 1)$ [11], larger than d/τ of s_0. As r increases, remembering that we are ranking the sizes in reverse order, $s_r \to s_0$. Since $s_0 \sim L^{d/\tau}$, for very large r_0 one has $L^{d/\tau} \sim (L^d/r_0)^{1/(\tau-1)}$, hence $r_0 \sim L^{d/\tau}$. Thus, the two contributions to H, $H \sim s_0 + r_0$, scale in the same way and $H \sim L^{d/\tau}$ [5]. Since H and s_0 share the same scaling behavior, measuring the first missing cluster size s_0 is enough to obtain the critical scaling behavior of H.

The point at which the distribution deviates from the power law grows as $s_* \sim |t|^{-1/\sigma}$, which defines the exponent σ. As pointed out in Ref. [6], the competing scales in this problem can be considered s_0 and s_* instead of the usual L and ξ. When $s_* \gg s_0$, the power law is well developed and the distribution for a single configuration has many holes (in the opposite limit, $s_* \ll s_0$, such distribution is dense and the system behaves as if its size was infinite). In this case, $H \sim s_0 \sim L^{d/\tau}$. Away from the transition, s_* decreases and the critical region extends up to the point in which $s_* \sim s_0$. Thus, $L^d s_*^{-\tau} \sim 1$ and, using the above definition, the size of the critical region is $|t| \sim L^{-d\sigma/\tau} \equiv L^{-1/\nu_H}$, where $\nu_H = \tau/\sigma d$ by the size of the system, is now 2. Repeating the process,

or, equivalently, $\nu_H = \nu t/(t-1)$. Consistently with this, Lee *et al.* [4] considered that the scaling behavior of H was given by $H(t,L) = L^{d/\tau} h(|t|L^{1/\nu_H})$. While close to the transition the scaling function $h(x)$ is a constant, away from T_c the behavior is $h(x) \sim x^{-1/\sigma}$.

A. 2d

The definition of H and its scaling were proposed and verified in Ref. [4] for explosive percolation and, later on, also in ordinary percolation [5] and some 2d spin models [6,7] for which there are several possible definitions of domains. For the Ising model, on which we focus here, a (geometric) domain is defined as a connected group of nearest-neighbor aligned spins. We show, in Fig. 1, the geometric domains heterogeneity H_g as a function of the temperature for several system sizes near the 2d Ising critical point. For the small sizes considered in Ref. [6], only a change in the derivative of H_g, very close to T_c, was observed. By considering larger sizes, interestingly, two well separated peaks develop. The first, small one [Fig. 1(a)], is very close to T_c and, as shown by the collapse in Fig. 1(d), grows as $L^{d/\tau}$. A good collapse was obtained with $\tau \approx 2.016(4)$ (keeping $\nu = 1$) and, accordingly, $\nu_H \approx 1.984$. These values are close to the exponents for the geometric domains in the Ising model [6], $\tau_g = 379/187 \approx 2.027$ and $\nu_H = 379/192 \approx 1.974$. On a broader range of temperatures [notice the logarithmic scale in Fig. 1(b)], there is a second peak that, together with the first one, was not previously reported. In the large temperature limit, $H \to H_g(\infty)$ and $H_g(\infty)$ can be independently measured on a random configuration with, on average, half of the spins up and the other half down. They are shown as horizontal lines whose heights slowly increase with L, $H_g(\infty) \sim \log L$ [5,6], and are approached as

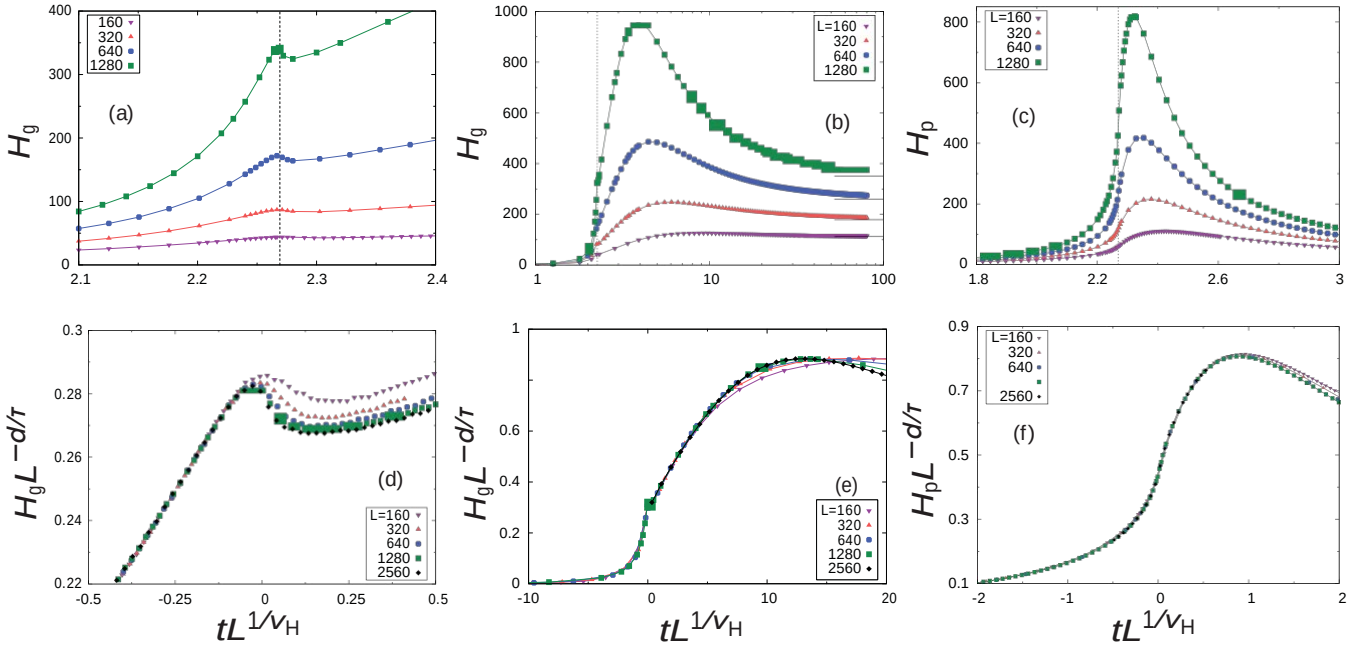

FIG. 1. (Color online) Size heterogeneity H for the $d = 2$ Ising model in equilibrium as a function of the temperature for several system sizes (top row) and the corresponding collapses (bottom row) using the reduced temperature $t \equiv T/T_c - 1$. The vertical dotted line locates the critical temperature $T_c = 2/\ln(1 + \sqrt{2})$. First (small) peak for the geometric domains (a), very close to T_c. The best collapse (d), keeping $\nu = 1$, was obtained with $\tau \approx 2.016(4)$. Second, large and broad peak for the geometric domains (b). The height of the peak grows with an effective $\tau \approx 2.048$ that is definitively different from the first peak. A good collapse (e) is obtained with $\nu \approx 1.25$. The horizontal lines correspond to an independent measure of H at $T \to \infty$ using a random configuration. Measure of H_p for the physical domains (c) whose exponents are $\tau_p = 31/15$ and $\nu_H = 31/16$ and give, again, an excellent collapse (f). In this case, there is a single peak close to T_c. Notice the different horizontal scales in all graphs and the logarithmic one in (b).

$H_g(T) - H_g(\infty) \sim T^{-1}$. This excess heterogeneity at large temperatures for the geometric domains is due to the entropic contribution of uncorrelated spins. Because of this size dependence, that persists even at very high temperatures where $\xi \to 0$, the scaling of this second peak turns to be rather complicated. Indeed, it does not collapse well with exponents obtained for the first peak and, instead, a satisfactory collapse was obtained with $\tau \approx 2.048$ and $\nu \approx 1.25$. These values are close to the random site percolation exponents, $\tau_{\text{perc}} = 187/91$ and $\nu = 4/3$ (thus, $\nu_H = 187/72$), but this may be just a crossover because of the large separation between the two peaks for finite systems (see the 3d case below). It seems that even larger systems would be necessary to clarify the nature of the critical exponents in this problem. Both peaks, besides increasing in height, move towards T_c in the thermodynamic limit. Indeed, both horizontal scalings in Figs. 1(d) and 1(e) considered the known value of T_c in $d = 2$. This is a consequence of the fact that, in $d = 2$, the geometric clusters percolative and thermal transitions occur at the same temperature. As the position of both peaks converge to T_c, the exponents obtained in both collapses also should be the same. In this way, as discussed in more detail in the conclusions, the exponents for the second peak represent just a crossover from the random percolation case to the exponents of the geometric domains.

Since H obviously depends on the nature of the clusters, further insight is gained by comparing with the behavior of physical clusters. If we remove from the geometric clusters the fraction of spins that, albeit parallel, are not effectively correlated, we obtain the so-called Coniglio-Klein droplets [9,15] whose physical properties are also related to the random cluster model [8]. Spins may be parallel even in the absence of correlations (for example, at infinite temperature). To obtain a correct mapping between geometry and correlation, starting from a geometric domain, for every pair of parallel spins in the domain we remove the bond between them with a temperature-dependent probability p_{CK} [9,15]. The physical cluster is thus defined by the parallel spins connected by the remaining bonds after this process. Fig. 1(c) shows H_p as a function of the temperature for several sizes and presents a clear peak close to the transition that gets larger as the size of the system increases. Notice that the temperatures of all the three peaks discussed here are well separated for the system sizes considered. Differently than $H_g(\infty)$, which is an increasing function of L, $H_p \to 1$ when $T \to \infty$ since all spins become uncorrelated and form clusters of unit size. The scaling proposed in Ref. [4] works very well also for H_p, Fig. 1(f), with the Ising thermal exponents ($\nu = 1$, $\tau_p = 31/15$, and $\nu_H = 31/16$). There is, however, another possible point of view for the scaling in this problem. Obviously, if $|t|L^{1/\nu_H}$ is a scaling variable, so is $(|t|L^{1/\nu_H})^{\tau/(\tau-1)}$. Indeed, using $|t|^{\tau/(\tau-1)}L^{1/\nu}$, collapses equivalent to those in Fig. 1 are obtained (an example is shown in Fig. 2), with no need to resort to a new exponent ν_H. In order to collapse the data for H, besides understanding how its peak scales with L, one must also know how the width of the critical region depends on the system size. Standard finite size scaling considers the region around T_c in which the measured correlation length ξ will differ from its infinite size limit behavior. In a finite system,

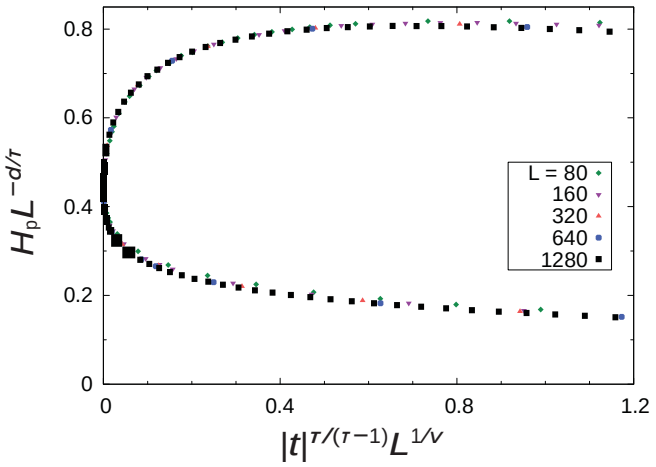

FIG. 2. (Color online) Collapsed H_p for $d = 2$, analogous to Fig. 1(f), but using the alternative (but equivalent) scaling with $\nu = 1$.

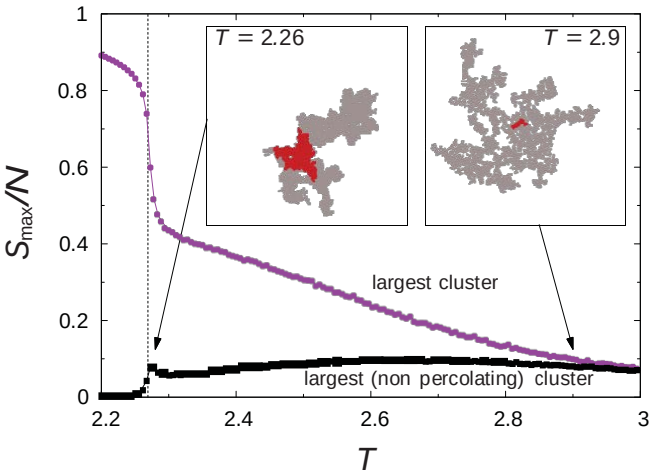

FIG. 3. (Color online) Size of the largest and the largest nonpercolating clusters, normalized by $N = L^d$, as a function of temperature for $L = 640$. A cluster is considered percolating when it touches all sides of the square lattice. Below T_c, indicated by a vertical dashed line, there is almost always a large, percolating cluster and the available space for the second largest cluster is small. For large temperatures, it is almost certain that there is no percolating cluster and both curves coincide. The insets show snapshots for $L = 320$ with the largest nonpercolating geometric cluster in light gray at two different temperatures (that roughly correspond to the local maxima of the bottom curve). At the critical temperature, left snapshot, there is a percolating cluster (not shown) and the second largest is compact. In red (dark gray) we show, among the several physical subclusters in which it divides, the largest one. The right snapshot shows that, at higher temperature (second maximum), although the largest nonpercolating geometric cluster has almost the same average size as at T_c, the physical subclusters are much smaller.

the divergence of ξ as $T \to T_c$, $\xi \sim |T - T_c|^{-\nu}$, is constrained by the linear size L, and the width of the critical region is thus $|T - T_c| \sim L^{-1/\nu}$. This interval is, in principle, different for other divergent quantities at T_c. If we consider H instead of ξ, since $1/\nu = d\sigma/(\tau - 1) > d\sigma/\tau$, the divergence of H, $H \sim L^{d/\tau}$, is weaker and the critical region wider. Indeed, as discussed in Sec. II, $|T - T_c| \sim L^{-(\tau-1)/\tau\nu}$.

Figure 3 shows the fraction of the whole system occupied by the largest cluster as a function of the temperature, distinguishing whether the largest cluster does or does not percolate. At very high temperatures both measures coincide since the noise is too high to allow a percolating cluster. As the temperature decreases, spins become more correlated, the clusters larger, and these measures, still coinciding, steadily increase. They become different when some samples percolate at temperatures roughly below the second peak of H_g. The average fraction size of the largest component (top curve) increases almost linearly from this point, as T decreases, up to T_c where there is an abrupt jump and it approaches unity below T_c. It is, on the other hand, the other measure—the fraction of the largest nonpercolating cluster—that seems to encode the relevant information for H and presents a similar double peak structure. When some samples start to percolate, the largest nonpercolating cluster is actually the second largest in these samples. As a consequence, its measure starts decreasing, and both measures diverge, leaving a peak close to the temperature of the second peak of H. The insets of Fig. 3 show, in light gray, one example of the largest nonpercolating cluster at the temperatures indicated: one close to T_c and the other at the point of separation of the two measures discussed above. These are geometric clusters formed by parallel spins. Each one is a collection of physical clusters; the largest one is shown in red (dark gray). Obviously, as the temperature decreases and the spins are more correlated, these clusters increase.

B. 3d

The heterogeneities can also be measured in the 3d case, where the percolative transition of the geometric domains does not coincide with the thermal transition, $T_{perc} < T_c$ ≈ 4.51. In 3d, the random site percolation threshold for a cubic lattice is 0.312 [16]. For all temperatures above T_c (even in the $T \to \infty$ limit), the fraction of each spin is 0.5, on average, and since both spins densities are above the threshold, both percolate. For temperatures below T_c, the system is magnetized and the majority spin obviously percolates. The percolation transition occurs when, coming from low temperature, the minority spin (the one opposite to the system magnetization), whose domains are small and isolated, first percolates. We obtain, from the best collapse of H (see below), T_{perc} ≈ $0.95T_c$, slightly larger than the estimate in Ref. [17], T_{perc} ≈ $0.92T_c$.

Fig. 4 (left) shows both H_g and H_p as a function of temperature. Again, H_p shows a single peak slightly above T_c ≈ 4.51. On the other hand, as in the 2d case, H_g is sensitive to both transitions. While there is a large peak very close to T_{perc}, at T_c there is a crossover to a different declivity, as shown in the inset of Fig. 4. While in the 2d case two peaks with the same infinite size limit were present, in 3d there is a single peak and a crossover, each one located in a different temperature for $L \to \infty$. Of course, it cannot be ruled out that for even larger system sizes, a small peak would appear at T_c. The high temperature limit of H_g seems to increase even slower than in the 2d case, and is barely visible in the figure.

The data collapses in 3d are shown in Fig. 4 (right). In the bottom part, the results for H_p are very well collapsed, as in the 2d case, with the thermal exponents of the 3d Ising model

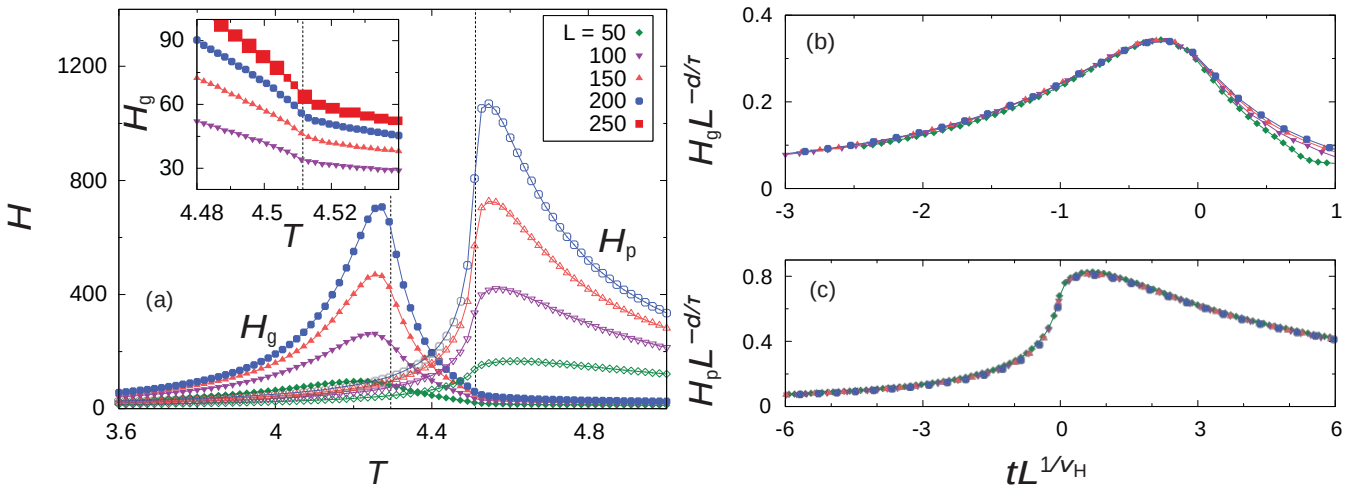

FIG. 4. (Color online) (a) Heterogeneity H for both geometric (left peak, filled symbols) and physical clusters (right peak, empty symbols) for the 3d Ising model. The vertical lines are located at T_{perc} ≈ $0.95T_c$ (pure geometric percolation) and T_c ≈ 4.511 (Ising critical temperature). Inset: the behavior of H_g at the thermal critical point, crossing over to a different declivity. Collapse of both H_g (b) and H_p (c). For H_g, $t \equiv (T - T_{perc})/T_{perc}$, and the best collapse was obtained with T_{perc} ≈ $0.95T_c$, τ ≈ 2.08, and ν ≈ 0.75. For H_p, on the other hand, $t \equiv (T - T_c)/T_c$, and an excellent collapse is obtained with 3d Ising model exponents: ν ≈ 0.63, τ ≈ 2.209.

and T_c ≈ 4.51: ν ≈ 0.63 and τ ≈ 2.209. In the top figure, for H_g, the best collapse was obtained with T_{perc} ≈ $0.95T_c$, τ ≈ 2.08, and ν ≈ 0.75. One would expect [18], in this specific case, the exponents of the 3d random site percolation problem, τ_{perc} ≈ 2.189 and ν_{perc} ≈ 0.875. The reason for this [18] is that, T_{perc} being different than T_c, the correlation length is finite at that temperature and clusters at distances larger than this length are uncorrelated. The small difference between this set of exponents and those that give the best collapse may be due to finite size effects [19].

III. CONCLUSIONS

The cluster size heterogeneity H is the number of distinct cluster sizes in a given sample configuration [4]. We presented here extensive simulations for the Ising model for both geometric (H_g) and physical (H_p) clusters in two and three dimensions. While for the latter, H has a single peak that diverges at the critical temperature of the Ising model, in the former case H presents a signature for both the percolative and the thermal transition. Indeed, finite 2d systems present two quite distinct peaks for H_g when the size is sufficiently large, but they merge as $L \to \infty$. On the other hand, in 3d there is a peak at T_{perc} and a sudden change in the declivity at $T_c > T_{perc}$. Moreover, since the peaks of H_g and H_p do not occur at the same temperature in 3d, in the region $T_{perc} < T < T_c$, while H_p increases, H_g decreases (a similar effect occurs in 2d, but it disappears in the limit $L \to \infty$). We also discussed the scaling of H and how its behavior correlates with the large clusters in the system. Interestingly, although the physical clusters collapse is clearly related to the Ising thermal exponents, the behavior of the largest peak of H_g is more subtle. In both 2d and 3d the best collapse was obtained with exponents that are close to, albeit different from, those of the respective random site percolation. Although in 3d this is expected (since $T_{perc} \neq T_c$, the correlation length is finite and distant clusters become uncorrelated [18]), in 2d both transitions coincide and the reason for the similarity with the random site exponents is still an open question. We conjecture that it may be just a crossover: since for large systems the two peaks get closer to each other and converge to T_c, the correlation length diverges and even clusters far apart are correlated. In this way, as the size increases, the exponents should converge to those of the geometric clusters, used in the collapse of the first peak.

Both in two and three dimensions the heterogeneity measured for geometric domains, H_g, is very sensitive to the thermodynamical transition, even if the critical properties are actually encoded in the physical domains. The small peak and the declivity change present in 2d and 3d, respectively, are very close to T_c even for very modest sizes, and their locations do not suffer from strong finite size effects. If proven general, H_g may provide a quite precise method to obtain a preliminary estimate for the thermal transition.

There are, nonetheless, several remaining questions. For example, although $T_{perc} < T_c$ in the Ising 3d model, when considering a 2d slice, the 2d clusters do start to percolate at T_c. It would be interesting to study the behavior of H not only for the 3d clusters as done here, but for the 2d clusters on the slices as well [19]. Another question is whether the above behavior of H_g occurs in other models as well. We are presently studying the Potts model, both for continuous and discontinuous transitions, in order to better understand which are the conditions for having a double peak. Finally, it is important to perform other geometric measures in order to better understand the underlying mechanism responsible for the behavior of H.

ACKNOWLEDGMENTS

J.J.A. deeply thanks Antonio Coniglio for very interesting discussions. J.J.A. and P.M.C.O. are partially supported by the INCT-Sistemas Complexos and the Brazilian agencies CNPq and CAPES. J.J.A. has also partial support from FAPERGS.

APPENDIX: ON THE CONVERGENCE OF EQ. (1)

When listing the domain sizes present on a particular configuration, we notice that all sizes up to a specific size appear, while above it several holes are present. After averaging over several configurations, we call this size s_0.

The ranking r_0 of the cluster size s_0 is

$$r_0 = L^d \sum_{s_0}^{\infty} n(s), \tag{A1}$$

not necessarily at the critical point. Near it and using the scaling relation $n(s) \sim s^{-\tau} f(z)$ where $z = ts^{\sigma}$, one could try to transform this discrete sum into an integral $\int_{s_0}^{\infty} ds\, s^{-\tau} f(z)$. Unfortunately, this integral does not converge inside the critical region $|t| \sim s_0^{-\sigma} \sim L^{-d\sigma/\tau}$. However, one can write the nth derivative of the quoted sum:

$$\frac{d^n r_0}{dt^n} \sim L^d \sum_{s_0}^{\infty} s^{n\sigma-\tau} f^{(n)}(z) = \frac{L^d}{\sigma}\, t^{-n+\frac{\tau-1}{\sigma}} \int_{z_0}^{\infty} dz\, z^{n-1-\frac{\tau-1}{\sigma}} f^{(n)}(z). \tag{A2}$$

Now, inside the critical region the bottom limit z_0 vanishes, and the resulting integral converges to a finite (not t-dependent) constant for some high enough integer n. Therefore, besides additional nonsingular terms (proportional to $t^0, t^1, \ldots, t^{n-1}$), the leading singular form for r_0 reads

$$r_0 \sim L^d\, t^{(\tau-1)/\sigma} \sim L^{d/\tau}. \tag{A3}$$

Notice that the particular mathematical form of the scaling function $f(z)$ does not enter in the argument. The above scaling relations are valid anyway, only the proportionality prefactors being dependent on $f(z)$. In Ref. [5], the authors assume $f(z)$ to be an exponential function and explicitly use this form to derive the above scaling relations. However, in three dimensions for instance, $f(z)$ is very well fitted to a (noncentered) Gaussian function [20]. In short, the nice and surprising theory introduced in Ref. [5] is correct, but the particular assumption adopted by the authors, in order to justify it, is unnecessary.

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