Universidade Federal do Rio Grande do Sul
Instituto de Matemática
Programa de Pós Graduação em Matemática

Constant mean curvature hypersurfaces on symmetric spaces, minimal graphs on semidirect products and properly embedded surfaces in hyperbolic 3-manifolds.

Tese de doutorado

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Porto Alegre, 26 de maio de 2015

Tese submetida por Álvaro Krüger Ramos[1], como requisito parcial para a obtenção do grau de Doutor em Matemática pelo Programa de Pós Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio Grande do Sul.

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Agradecimentos

Jaime: quero te agradecer não só pelos vários anos de orientação matemática desde a gradução até o doutorado, mas também pelos inúmeros ensinamentos sobre a vida em geral. Espero um dia poder passá-los adiante, se possível com a mesma maestria e leveza que tu tens.

Pati e Miriam: minhas queridas amigas, muito obrigado por tudo que passamos juntos durante todos esses anos, pelo companheirismo e pela pura e simples *retardadice*, que é intrínseca a nós três. Obrigado também pelos comentários que vocês, como uma banca excepcional, fizeram sobre a tese.

Joaquín: muito obrigado pela orientação e hospitalidade na Universidad de Granada. Todas as tuas sugestões, conversas e idéias contribuiram forte e decisivamente com a tese e, mais geralmente, para a minha formação como pesquisador em matemática.

Bill: quero te deixar aqui um grande *thank you*, por tudo que me ajudaste, sobretudo nesse último ano: a tua disposição, disponibilidade e apoio foram fundamentais para me dar confiança na tese e também em mim mesmo, como entusiasta da pesquisa.

Rosa: o Vale só é o Vale por tua causa, pelo teu apoio quase de "tia da creche", sempre cuidando de uma forma ou de outra da gente, com muita amizade e carinho. Obrigado por tudo.

Ariane: minha linda, tu és a pessoa mais importante da minha vida, e o teu apoio e compreensão são essenciais para mim. Quero te agradecer em especial por todo o carinho e atenção desde a época de Granada até agora, que me fez sempre ir adiante e não desistir nesses tempos difíceis. O teu amor me dá forças e me faz uma pessoa melhor constantemente. Muito obrigado por todos esses anos ao meu lado, com tantos momentos maravilhosos que passamos e ainda passaremos juntos. **Te amo!**

Resumo

Provamos resultados sobre a geometria de hipersuperfícies em diferentes espaços ambiente. Primeiro, definimos uma aplicação de Gauss generalizada para uma hipersuperfície $M^{n-1} \subseteq N^n$, onde N é um espaço simétrico de dimensão $n \geq 3$. Em particular, generalizamos um resultado de Ruh-Vilms e apresentamos aplicações. Em seguida, estudamos superfícies em espaços de dimensão 3: estudamos a equação da curvatura média em um produto semidireto $\mathbb{R}^2 \rtimes_A \mathbb{R}$ e obtemos estimativas da altura e a existência de gráficos mínimos do tipo Scherk. Finalmente, no espaço ambiente de uma variedade hiperbólica de dimensão 3, nós apresentamos condições suficientes para que um mergulho completo de uma superfície Σ de topologia finita em N com curvatura média $|H_\Sigma| \leq 1$ seja próprio.

Palavras-chave e frases: Superfícies mínimas, curvatura média constante, aplicação de Gauss, espaços simétricos, variedades homogêneas, grupos de Lie métricos, produtos semidiretos, operadores elípticos quasilineares, variedade hiperbólica, função raio de injetividade, superfícies de topologia finita.

Abstract

We prove results concerning the geometry of hypersurfaces on different ambient spaces. First, we define a generalized Gauss map for a hypersurface $M^{n-1} \subseteq N^n$, where N is a symmetric space of dimension $n \geq 3$. In particular, we generalize a result due to Ruh-Vilms and make some applications. Then, we focus on surfaces on spaces of dimension 3: we study the mean curvature equation of a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ to obtain height estimates and the existence of a Scherk-like minimal graph. Finally, on the ambient space of a hyperbolic manifold N of dimension 3 we give sufficient conditions for a complete embedding of a finite topology surface Σ on N with mean curvature $|H_\Sigma| \leq 1$ to be proper.

Key words and phrases: Minimal surfaces, constant mean curvature, Gauss map, symmetric spaces, homogeneous manifolds, metric Lie groups, semidirect products, quasilinear elliptic operator, hyperbolic manifold, Calabi-Yau problem, injectivity radius function, finite topology surfaces.

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Prologue

This *Prologue* is to give the basic definitions and to fix the conventions used throughout the work, which is divided into three independent chapters as follows:

Chapter 1: *The generalized Gauss map on symmetric spaces* presents the results on the joint work of the author with his academic advisor J. B. Ripoll [54] for immersed hypersurfaces on symmetric spaces. It is defined a geometric Gauss map for a hypersurface M^{n-1} of a symmetric space N^n that takes values on the semi-Riemannian sphere of the Lie algebra of the isometry group of N. An extension to a well known theorem of Ruh-Vilms [58] in the Euclidean space is then obtained to symmetric spaces and this is applied to extend Hoffman-Osserman-Schoen Theorem ([35]) to $3-$dimensional symmetric spaces.

It is also shown that the holomorphic quadratic form induced by the Gauss map coincides (up to a sign) with the Hopf quadratic form when the ambient space is $\mathbb{H}^3$, $\mathbb{R}^3$ and $\mathbb{S}^3$ and with the Abresch-Rosenberg quadratic form when the ambient space is $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ providing, then, an unified way of relating Hopf's and Abresch-Rosenberg's quadratic form with the quadratic form induced by a harmonic Gauss map of a CMC surface in these 5 spaces.

Chapter 2: *The mean curvature equation on* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is based on [53], developed by the author under the supervision of J. Pérez, during his stay on the University of Granada. It concerns the mean curvature equation on a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$: by considering a domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ and vertical π-graphs over Ω, the partial differential equation that a function $u : \Omega \to \mathbb{R}$ must satisfy in order to have prescribed mean curvature H is deduced. Using techniques from quasilinear elliptic equations we prove that if a $\pi-$graph has non-negative mean curvature with respect to the upwards pointing orientation, then it satisfies some uniform height estimates that depend on diam(Ω) and on a parameter $\alpha \in \mathbb{R}$, given a priori and that is necessary. When trace(A) > 0, these estimates imply that the oscillation of a minimal graph assuming the same constant value n along the boundary tends to zero when $n \to +\infty$ and goes to $+\infty$ if $n \to -\infty$.

We also apply the results about the mean curvature operator, in conjunction with techniques from Killing graphs in order to generalize the result of A. Menezes [50] to obtain families of Scherk-like minimal graphs on the ambient space $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for any matrix $A \in M_2(\mathbb{R})$.

Chapter 3: *Finite topology surfaces on hyperbolic 3-manifolds* is based on the joint ongoing work of the author with W. Meeks III [46]: using the existence of short geodesic loops on surfaces with bounded injectivity radius and non positive sectional curvature, it is proved that every complete, non compact annulus E (properly or not) embedded in $\mathbb{H}^3$ with bounded absolute mean curvature $|H_E| \leq 1$ has unbounded injectivity radius function. As a consequence it is proved that a complete embedding of a surface Σ of finite topology in a hyperbolic 3-manifold whose injectivity radius function of each end goes to zero at infinity must be proper, provided its mean curvature function satisfies $|H_\Sigma| \leq 1$.

As a main bibliography, we recommend the book of M. do Carmo [22] for the basic concepts on Riemannian geometry, the work of W. Meeks and J. Pérez [44] to metric Lie groups of dimension 3, specially for a comprehensive approach to semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$. For more general Lie groups, homogeneous manifolds and symmetric spaces, we recommend the book of S. Helgason [34]. For a very general and reasonably up-to-date view on the classical theory of minimal surfaces, the book of T. Colding and W. Minicozzi [11] or the survey of Meeks and Pérez [45] are recommended. For the subject of elliptic partial differential equations the classic book of D. Gilbarg and N. Trudinger [32] is the main reference. The reference we suggest when dealing with hyperbolic manifolds is the book of R. Bendetti and C. Petronio, [4]. Finally, for aspects on semi-Riemannian geometry, we recommend the book of B. O'Neill [52].

Acknowledgements. I would like to thank Jaime Ripoll for all the support from the beginning, Joaquín Pérez for his valuable advice and hospitality during my stay at Granada, William Meeks for his shared enthusiasm and energy to solve many interesting questions, Harold Rosenberg for his important comments and suggestions and also for his invitation to spend time at IMPA, where the third part of this work took place. Finally, I thank CNPq, for the financial support both in Brazil and in Spain.

Notation

We use this section to give a brief overview about some basic concepts of Riemannian geometry, focusing on those who can be found in different ways on the literature.

Let N be a Riemannian manifold. We denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on N, and by $\nabla : \mathfrak{X}(N)\times\mathfrak{X}(N) \to \mathfrak{X}(N)$ the Riemannian connection on N. We define (cf. do Carmo [22]) the *curvature tensor* R of N as

$$R(X,\, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \tag{1}$$

and the sectional curvature of a plane $\sigma \subseteq T_pN$ generated by two orthogonal, unitary vectors $u,\, v \in T_pN$ as

$$K_N(u,\, v) = \langle R(u,\, v)u,\, v\rangle. \tag{2}$$

If $\{e_1,\, e_2,\, \dots,\, e_n\}$ is an orthogonal basis of T_pN, the *Ricci tensor* of N at p is defined as

$$\mathrm{Ric}(u,\, v) = \sum_{i=1}^{n} \langle R(u,\, e_i)v,\, e_i\rangle,\ \ u,\, v \in T_pN, \tag{3}$$

and the Ricci curvature of N on the direction of $v \in T_pN$ is

$$\mathrm{Ric}_p(v) = \mathrm{Ric}(v,\, v). \tag{4}$$

If $Y \in \mathfrak{X}(N)$ is a Killing field of N and $\varphi : \mathbb{R} \times N \to N$ is it flux, for each $t \in \mathbb{R}$ we consider the isometry $\varphi_t = \varphi(t, \cdot) : N \to N$. Then, given a subset $\Omega \subseteq N$ such that Y is everywhere transversal to Ω and a function $u : \Omega \to \mathbb{R}$, we let the $Y-$*Killing graph* of u be $Gr_Y(u)$ given by

$$Gr_Y(u) = \{\varphi_{u(p)}(p);\ p \in \Omega\}. \tag{5}$$

We also let the $Y-$*Killing cylinder* over Ω be

$$Cyl_Y(\Omega) = \{\varphi_t(p);\ p \in \Omega,\ t \in \mathbb{R}\}. \tag{6}$$

Concerning the geometry of submanifolds, if N is a manifold of dimension n and $\Sigma \subseteq N$ is a hypersurface[2], oriented with respect to an unitary vector field η normal to Σ, we let the *shape operator* of Σ at a point p be the map $A_\eta : T_p\Sigma \to T_p\Sigma$ given by

[2]Hypersurfaces are assumed to be *immersed* and *orientable*, unless otherwise stated.

$$A_\eta(v) = -\left(\nabla_v \eta\right)^T,$$

where $(\cdot)^T$ denotes the projection on $T_p\Sigma$. We let the *mean curvature* of Σ, H_Σ be given by

$$H_\Sigma = \frac{1}{n-1}\mathrm{trace}(A_\eta).$$

Whenever the function H_Σ is constant, we say that Σ has *constant mean curvature*, usually abbreviated as CMC, and on the particular case of $H_\Sigma \equiv 0$ vanishing identically we say that Σ is a *minimal surface* of N.

For the case of arbitrary codimension $\Sigma^m \subseteq N^n$, we let the *second fundamental form* B of Σ, at a point p, be the bilinear form $B : T_p\Sigma \times T_p\Sigma \to T_p\Sigma^\perp$ given by

$$B(x, y) = \left(\nabla_X Y\right)^\perp,$$

where X and Y are extensions of x and y, ∇ is the Riemannian connection of N and $(\cdot)^\perp$ is the projection on $(T_p\Sigma)^\perp$. It is not difficult to see that B does not depend on the extensions considered and that it is symmetric $B(x, y) = B(y, x)$ (see, for instance, Chapter 1 of [11]).

We let the trace of B be the *mean curvature vector of* Σ, $\vec{H}$, given by

$$\vec{H} = \sum_{i=1}^{m} B(e_i, e_i),$$

where $\{e_1, e_2, \ldots, e_m\}$ is an orthogonal basis to $T_p\Sigma$, and, we define the *mean curvature of* Σ by

$$H_\Sigma = \frac{1}{m}\left\|\vec{H}\right\|.$$

On the special case of $m = n-1$ (Σ is a hypersurface), the two definitions above for the mean curvature of Σ coincide. To see this, just notice that the following relation between B and A_η holds:

$$\langle A_\eta(u), v\rangle = \langle B(u, v), \eta\rangle.$$

CHAPTER 1

The generalized Gauss map on symmetric spaces

This chapter follows the joint work of the author and J. Ripoll [54] and has two main purposes: First, to extend a well known theorem of Ruh-Vilms [58] in the Euclidean space to symmetric spaces and, secondly, to apply this result to extend Hoffman-Osserman-Schoen Theorem (HOS Theorem, [35])) to 3−dimensional symmetric spaces.

1.1 Introduction

A well known theorem due to Ruh-Vilms [58] establishes that an orientable immersed hypersurface S in $\mathbb{R}^n$, $n \geq 3$, has constant mean curvature if and only if the Gauss map $\mathcal{N} : S \to \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ of S is harmonic[1]. This result also applies to submanifolds of arbitrary codimension, with the Gauss map assuming values in a Grassmannian manifold, but on the special case of hypersurfaces it can be easily obtained from the fact that $\mathcal{N}$ satisfies the equation

$$\Delta\mathcal{N} = -\text{grad}(H) - \|B\|^2 \mathcal{N}, \tag{1.1}$$

where B is the second fundamental form of S and H is its mean curvature. Here, as $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, the Laplacian of $\mathcal{N}$ is considered as taken on each coordinate of its image on $\mathbb{R}^n$,

[1]A smooth map $\varphi : M \to N$ between two Riemannian manifolds M and N (of any dimensions) is said to be *harmonic* if its tension field (which can be interpreted as the divergence of its differential) vanishes.

$$\Delta\mathcal{N} = \sum_{i=1}^{n} \Delta\left(\langle \mathcal{N},\, e_i \rangle\right), \tag{1.2}$$

then the result follows from the fact (see (4.13), [24]) that a map $\varphi : M \to \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ is harmonic if and only if $\Delta\varphi = \lambda\varphi$, for some $\lambda : M \to \mathbb{R}$.

In [5], F. Bitttencourt and J Ripoll defined a Gauss map of an orientable hypersurface on ambient spaces of the form $N = \mathbb{G}/\mathbb{K} \times \mathbb{R}^n$, $n \geq 0$, where $\mathbb{G}/\mathbb{K}$ is a compact symmetric space. The Gauss map is defined by taking the horizontal lift of the unit normal vector field of the hypersurface to $\mathbb{G} \times \mathbb{R}^n$ followed by a translation to the unit sphere in the Lie algebra of $\mathbb{G} \times \mathbb{R}^n$. Ruh-Vilms Theorem is then extended to hypersurfaces of N, that is, they prove that a hypersurface of N has CMC if and only if this Gauss map is harmonic (Corollary 3.4 of [5]).

In the present work we extend the construction of the Gauss map done in [5] to any symmetric space, not necessarily reducible nor compact and of any dimension, obtaining an extension of Ruh-Vilms Theorem to these spaces (Theorem 1.2.4 and Corollary 1.2.5).

We recall that an application of Ruh-Vilms theorem in the Euclidean 3-dimensional space is a theorem of D. Hoffman, R. Osserman and R. Shoen (HOS Theorem for short, [35]), which reads:

Theorem (Hoffman-Osserman-Schoen)**.** *Let S be a complete surface of constant mean curvature immersed in $\mathbb{R}^3$. If the image of the Gauss map of S lies in a hemisphere, then S is a plane or a cylinder.*

Sketch of the proof. Let $\mathcal{N}$ be the Gauss map of S. By hypothesis, there is $V \in \mathbb{S}^2$ such that $u := \langle \mathcal{N}, V \rangle \geq 0$; from (1.1) it follows that the lift $\widetilde{u}$ of u to the universal covering $\widetilde{S}$ of S is a bounded superharmonic function on $\widetilde{S}$. If $\widetilde{S}$ has the conformal type of the plane then u must be constant and then S is a plane or a cylinder. If $\widetilde{S}$ has the conformal type of the disk then, by the maximum principle, either $\widetilde{u} > 0$ everywhere or $\widetilde{u} \equiv 0$. But from (1.1) we see that $\widetilde{u}$ satisfies the PDE $\Delta\widetilde{u} - 2K\widetilde{u} + P = 0$ where K is the sectional curvature of S and $P = 4H^2 \geq 0$ which is in contradiction with Corollary 3 of [30] that asserts this PDE has no positive solutions if $\widetilde{S}$ is conformal to the disk. □

J. Espinar and H. Rosenberg in [26] remarked that in product spaces $M^2 \times \mathbb{R}$ the condition of the Gauss map being contained in a hemisphere can interpreted as the angle function $\nu = \langle \eta, \partial_t \rangle$ having a sign, where η is an unit vector field normal to the surface. They then classified all these CMC surfaces in terms of the infimum $c(S)$ of the sectional cuvature at the points of M that are in the projection of the surface S. Precisely, they proved that if $c(S) \geq 0$ and $H \neq 0$ then S is a cylinder over a complete curve with curvature $2H$. If $H = 0$ and $c(S) \geq 0$ then S must be either a vertical plane, a slice $M \times \{t\}$, or $M = \mathbb{R}^2$ with the flat metric. We note that when $M = \mathbb{R}^2$ these results recover HOS theorem. When $c(S) < 0$ and $H > \sqrt{-c(S)}/2$, then S is invariant under the group of isometries generated by the Killing field ∂_t and is a vertical cylinder over a complete curve on M of constant geodesic curvature $2H$.

In Bittencourt-Ripoll [5], using the extension of Ruh-Vilms theorem to $\mathbb{S}^3$ and to $\mathbb{S}^2 \times \mathbb{R}$ it is obtained an extension of HOS theorem to these ambient spaces. Here, with the extension of Ruh-Vilms theorem to any symmetric space, we are able to extend HOS theorem to include the ambient spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$ as well, this is Theorem 1.5.1 below. We note that an extension of HOS theorem to the hyperbolic space, despite all these previous results, had not been obtained via a geometrically defined Gauss map so far.

We think it is important at this point to remark that our work and the one of Espinar and Rosenberg both extend HOS theorem to $\mathbb{H}^2 \times \mathbb{R}$, and both require a lower bound for the mean curvature. In [26] it is $H > 1/2$ which is better than the one that follows from our result, namely, $H \geq 1/\sqrt{2}$. The lower bound of [26] is indeed optimal among CMC surfaces in ambient spaces of the form $M^2 \times \mathbb{R}$. Our case is optimal among CMC surfaces in 3-dimensional symmetric spaces since in $\mathbb{H}^3$ the lower bound is 1, which is optimal (see the last remark of this chapter, Remark 1.2).

Another application of Ruh-Vilms theorem in $\mathbb{R}^3$ is the well known classical Hopf Theorem ([37]), namely:

Theorem (Hopf Theorem)**.** *The round sphere is the only CMC topological sphere in* $\mathbb{R}^3$.

Sketch of the proof. If S is a CMC surface in $\mathbb{R}^3$, then Ruh-Vilms theorem implies that the Gauss map $\mathcal{N}$ of S is harmonic. In particular, it induces a quadratic holomorphic form q in S (see 10.5 of [24]) which coincides with the so called Hopf differential. Then, if S has zero genus, q must be zero everywhere which implies that S is totally umbilic and then a round sphere. □

Concerning Hopf's Theorem, U. Abresch and H. Rosenberg in [1] extended it to CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and in $\mathbb{H}^2 \times \mathbb{R}$, defining a quadratic form $\mathcal{Q}$ in these spaces, presently well known as Abresch-Rosenberg quadratic form:

$$\mathcal{Q} = 2H\mathcal{A} - \mathcal{T}, \text{ resp. } \mathcal{Q} = 2H\mathcal{A} + \mathcal{T}, \tag{1.3}$$

where H is the mean curvature of the surface, $\mathcal{A}$ is the Hopf differential and $\mathcal{T} = (dh \otimes dh)^{2,0}$, with h standing for the height function. They prove that $\mathcal{Q}$ is holomorphic when the surface is CMC. In particular, $\mathcal{Q} \equiv 0$ holds if S is a CMC topological sphere and, from this fact, they obtain that a CMC sphere is rotationally symmetric.

Abresch and Rosenberg result raised a natural question of whether their quadratic form $\mathcal{Q}$ could be induced by a geometric Gauss map for surfaces in the spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, this Gauss map having the property of being harmonic if (and only if hopefully) the surface has constant mean curvature.

This question has been answered in the affirmative first in the space $\mathbb{H}^2 \times \mathbb{R}$ and for CMC 1/2 surfaces by I. Fernandez and P. Mira in [27]. They introduced the *hyperbolic Gauss map* $G : S \to \mathbb{H}^2$ for any surface $S \subset \mathbb{H}^2 \times \mathbb{R}$ nowhere vertical and show that if S has CMC $H = 1/2$, then G is harmonic. Moreover, its induced holomorphic quadratic differential in the surface coincides (up to a sign) with the Abresch-Rosenberg form. But they go further and use these previous results to obtain another quite interesting part of their work: to prove the existence of CMC 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with prescribed hyperbolic Gauss map and to show that any holomorphic quadratic differential on an open simply connected Riemann surface can be realized as the Abresch-Rosenberg differential of some complete surface with $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$.

The above question in the space $\mathbb{S}^2 \times \mathbb{R}$, as far as the author is aware, was answered by M. L. Leite and J. Ripoll in [38], where it is proved that the quadratic form induced by the Gauss map defined in [5] coincides with the Abresch-Rosenberg form on CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$. Moreover, they used this Gauss map to motivate an ad hoc construction of a Gauss map in $\mathbb{H}^2 \times \mathbb{R}$ and obtain the same result.

With the Gauss map constructed here and with Theorem 1.2.4, we have the following unifying result: the quadratic form induced by the Gauss map in a surface immersed in a space of constant sectional curvature coincides with the Hopf's quadratic form and in a suface immersed in $\mathbb{H}^2 \times \mathbb{R}$ or in $\mathbb{S}^2 \times \mathbb{R}$ it coincides with the Abresch-Rosenberg quadratic form (this is Section 1.4); moreover, the surfaces have CMC and these forms are holomorphic *if and only if* their Gauss maps are harmonic.

To close this introduction, we observe that generalizations of the Gauss map have been defined in many different spaces and in many different ways. These generalizations have been proved to be particularly useful in describing and understanding CMC surfaces in the 8 models of Thurston's geometries and more recently in a broad class of 3-dimensional simply connected Lie groups endowed with a left invariant metric. Quite interesting and deep results have been obtained in a series of papers by B. Daniel [18], by B. Daniel, I. Fernández and P. Mira in [19], by B. Daniel and Mira [21] and its generalization by W. Meeks III in [40]. We finally mention joint works of W. Meeks III, P. Mira, J. Pérez and A. Ros [44, 41, 42], where using the left invariant Gauss map on a metric Lie group the authors are able to show strong results concerning CMC spheres on these ambient spaces.

Our results are organized as follows: in Section 1.2 it is introduced a Gauss map $\mathcal{N}$ for hypersurfaces of a symmetric space and it is proved that an orientable hypersurface $M \subseteq N$ has CMC if and only if $\mathcal{N}$ is harmonic (Corollary 1.2.5). In Section 1.3 we obtain explicit expressions for $\mathcal{N}$ when the ambient space is $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$.

In Section 1.4, we study the particular case when N has dimension 3 and we analyse the quadratic complex form induced by $\mathcal{N}$, denoted by $\mathcal{Q}_{\mathcal{N}}$. We then obtain that $\mathcal{Q}_{\mathcal{N}}$ coincides with the Hopf differential when N is $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$ and with the Abresch-Rosenberg quadratic form when N is $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.

Finally, in Section 1.5, we use the Gauss map $\mathcal{N}$ to extend HOS theorem when M is a surface immersed in a symmetric space of dimension 3.

1.2 The Gauss map of a hypersurface on a symmetric space

In this section we introduce and discuss some aspects of the Gauss map $\mathcal{N}$ of a hypersurface M^{n-1} immersed in a symmetric space N. We use the same construction of [5] for hypersurfaces in a homogeneous space $\mathbb{G}/\mathbb{H}$ but instead of asking for a bi-invariant *Riemannian* metric on $\mathbb{G}$ (which, up to an abelian factor, implies that $\mathbb{G}$ is compact), we show, in §1.2.1, that any symmetric space N is a quotient $\mathbb{G}/\mathbb{K}$ of a group $\mathbb{G}$ acting transitively on N via isometries and $\mathbb{K}$, the isotropy subgroup of $\mathbb{G}$ at a fixed point of N. Such $\mathbb{G}$ admits naturally a bi-invariant *semi-Riemannian* metric (Proposition 1.2.1).

We relate the Laplacian of $\mathcal{N}$ and the mean curvature of M and, as a consequence, we obtain that $\mathcal{N}$ is harmonic if and only if M has constant mean curvature. Throughout the text a hypersurface is always understood as being immersed and oriented, and we will refer to the generalized Gauss map simply as the Gauss map.

1.2.1 Symmetric spaces $N = \mathbb{G}/\mathbb{K}$

A *symmetric space* is a Riemannian manifold N such that the geodesic reflection at any point is an isometry. Precisely, N is a symmetric space if for every $x \in N$ there is an isometry $s_x : N \to N$ (called a *symmetry*) such that $s_x(x) = x$ and

$$d(s_x)_x(v) = -v, \ \forall v \in T_xN.$$

It is not difficult to see that every symmetric space is also a homogeneous space (thus it is complete) and, conversely, every homogeneous space that has a symmetry around a point is a symmetric space. For further details, see [34].

We use this section to prove that any symmetric space N is isometric to a quotient $\mathbb{G}/\mathbb{K}$, where $\mathbb{G}$ is endowed with a semi-Riemannian metric and the metric on $\mathbb{G}/\mathbb{K}$ is the push forward of the metric of $\mathbb{G}$ via the projection $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{K}$, which is a submersion.

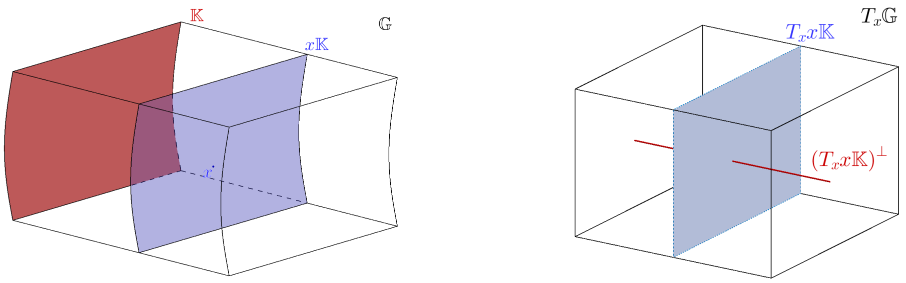
We recall that, if $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$ and $u \in \mathfrak{g}$ is a left invariant vector field, then the *adjoint action* related to u is the linear map $ad_u : \mathfrak{g} \to \mathfrak{g}$ given by $ad_u(v) = [u, v]$. From the composition of two adjoint actions it is defined the *Killing form* of $\mathbb{G}$, a bilinear symmetric form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

$$K(u, v) = \mathrm{trace}(ad_u \circ ad_v).$$

When the Killing form is non degenerate, it can be extended to the group $\mathbb{G}$ via left translations to induce a left invariant semi-Riemannian metric on $\mathbb{G}$. On this case, the Lie algebra $\mathfrak{g}$ is called *semisimple*. This is the basic element on the proof of next proposition, that states that the metric of a symmetric space N comes (up to a multiplicative constant) from the descent of the Killing form of the isometry group of N via the projection.

Proposition 1.2.1. *Let N be a Riemannian symmetric space. Then there is a Lie group $\mathbb{G}$, endowed with a bi-invariant semi-Riemannian metric and a subgroup $\mathbb{K} < \mathbb{G}$ such that N is isometric to que quotient $\mathbb{G}/\mathbb{K}$, where the metric in $\mathbb{G}/\mathbb{K}$ is the one induced by the projection $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{K}$ in such way it becomes a Riemmanian submersion.*

Proof. We follow S. Helgason, [34] specially Proposition 5.5: assume, at first, that N is an irreducible symmetric space. Let $\mathbb{G} = \mathrm{ISO}(N)^0$ be the connected component of the identity on the isometry group of N and set $\mathbb{K}$ as the isotropy group of some fixed point on N. Then N is isometric to $\mathbb{G}/\mathbb{K}$, where the metric on $\mathbb{G}/\mathbb{K}$ (up to a multiple factor) is the descent of the



(a) $\mathbb{G}$, $\mathbb{K}$ and $x\mathbb{K} = \pi^{-1}(\{\pi(x)\})$.   (b) $T_x\mathbb{G} = T_x x\mathbb{K} \oplus (T_x x\mathbb{K})^\perp$

Figure 1.1: $\mathbb{G}$ is foliated by left translates of $\mathbb{K}$

Killing form of the Lie algebra of $\mathbb{G}$, which is a bi-invariant semi-Riemannian metric.

Now, if N is not irreducible, it decomposes as the Riemannian product of irreducible symmetric spaces with a $\mathbb{R}^m$ factor

$$N = N_1 \times N_2 \times \ldots \times N_l \times \mathbb{R}^m,$$

so each $N_i = \mathbb{G}_i/\mathbb{K}_i$ can be written as above. Then, if we set $\mathbb{G} = \mathbb{G}_1 \times \ldots \times \mathbb{G}_l \times \mathbb{R}^m$ and $\mathbb{K} = \mathbb{K}_1 \times \ldots \times \mathbb{K}_l \times \{0\}$, it follows that N is isometric to the quotient $\mathbb{G}/\mathbb{K}$. Since the metric of $\mathbb{R}^m$ is bi-invariant and the Riemannian product of bi-invariant metrics is also bi-invariant, the proposition is proved. □

Herein we will assume that $\mathbb{G}$ is endowed with a bi-invariant semi-Riemannian metric that comes from the Killing form, and that this metric descends onto $\mathbb{G}/\mathbb{K}$ as a Riemannian metric via the projection π. We also assume that $\dim(\mathbb{G}) = n + k$ where $n = \dim(N)$ and $k = \dim(\mathbb{K})$ and denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$. These assumptions on $\mathbb{G}$ and $\mathbb{G}/\mathbb{K}$ will be assumed throughout the chapter.

1.2.2 Construction of the Gauss map

On this section, we follow the construction of [5] to obtain a Gauss map for a hypersurface M of a symmetric space $N = \mathbb{G}/\mathbb{K}$, by lifting tangent vectors of TN to $T\mathbb{G}$ and then right translating them to $\mathfrak{g}$. As the projection $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{K}$ is a submersion whose fibres are left translates of $\mathbb{K}$, we can make the following definition:

Given $x \in \mathbb{G}$, a vector $u \in T_x\mathbb{G}$ is called *vertical* if $u \in T_x x\mathbb{K}$ and it is called *horizontal* if $u \in (T_x x\mathbb{K})^\perp$ (see figure 1.1, right). It follows that a vector $u \in T_x\mathbb{G}$ is vertical if and only if its projection $d\pi_x(u)$ is 0. We set

$$\begin{array}{rccl} \ell_x := d\pi_x|_{(T_x(x\mathbb{K}))^\perp}: & (T_x x\mathbb{K})^\perp & \to & T_{\pi(x)}\mathbb{G}/\mathbb{K} \\ & v & \mapsto & d\pi_x(v). \end{array}$$

By definition, ℓ_x is a linear isometry between horizontal vectors of $T_x\mathbb{G}$ and $T_{\pi(x)}(\mathbb{G}/\mathbb{K})$. We then define $\Gamma : T(\mathbb{G}/\mathbb{K}) \to \mathfrak{g}$ by

$$\begin{array}{rccl} \Gamma_p: & T_p\mathbb{G}/\mathbb{K} & \to & \mathfrak{g} \\ & u & \mapsto & d(R_{x^{-1}})_x \, \ell_x^{-1}(u), \end{array} \tag{1.4}$$

where $p \in \mathbb{G}/\mathbb{K}$, x is any point on $\pi^{-1}(p)$ and, for $g \in \mathbb{G}$, R_g denotes the right translation by g, $R_g(h) = hg$.

Proposition 1.2.2. *For each $p \in \mathbb{G}/\mathbb{K}$, the map Γ_p is well-defined, is linear and preserves the metric.*

Proof. Consider $x, y \in \pi^{-1}(p)$. There exists $h \in \mathbb{K}$ such that $x = R_h(y)$. Then, for any $u \in T_p\mathbb{G}/\mathbb{K}$, we have

$$u = d\pi_y\ell_y^{-1}(u) = d(\pi \circ R_h)_y\ell_y^{-1}(u) = d\pi_x d(R_h)_y\ell_y^{-1}(u).$$

Since $h \in \mathbb{K}$ and the metric of $\mathbb{G}$ is bi-invariant, R_h is an isometry of $\mathbb{G}$ that additionally preserves horizontality, in particular $d(R_h)_y\ell_y^{-1}(u)$ is a horizontal vector, so, from the previous equation we obtain that $\ell_x^{-1}(u) = d(R_h)_y\ell_y^{-1}(u)$ and hence

$$\begin{aligned} d(R_{x^{-1}})_x\ell_x^{-1}(u) &= d(R_{x^{-1}})_x d(R_h)_y\ell_y^{-1}(u) \\ &= d(R_{x^{-1}} \circ R_h)_y\ell_y^{-1}(u) \\ &= d(R_{y^{-1}})_y\ell_y^{-1}(u), \end{aligned}$$

what proves that Γ_p is well defined. That it is linear and preserves the metric follows directly from the definition of ℓ_x and from the fact that the projection is a Riemannian submersion, so it preserves the metric for horizontal vectors. □

We may now define the Gauss map of an oriented hypersurface M of N by setting

$$\begin{array}{rccl} \mathcal{N}: & M & \to & \mathbb{S}^{n+k-1} \subseteq \mathfrak{g} \\ & p & \mapsto & \Gamma_p(\eta(p)), \end{array} \tag{1.5}$$

where η is a fixed unit normal vector field on M. In particular, we notice that $\mathcal{N}$ coincides with the usual Gauss map when M is a hypersurface of the euclidean space $\mathbb{R}^n$.

1.2.3 Orthogonality and invariancy

On this section we prove Proposition 1.2.3, which relates invariant hypersurfaces of $N = \mathbb{G}/\mathbb{K}$ with respect to a subgroup of isometries $\mathbb{H} < \mathbb{G}$ and the hypersurfaces of N whose image of the Gauss map is contained in a Lie subalgebra of $\mathfrak{g}$. This proposition will be the key to prove the analogous to HOS theorem, Theorem 1.5.1 on Section 1.5.

The group $\mathbb{G}$ acts transitively on $\mathbb{G}/\mathbb{K}$ via isometries: for each element $g \in \mathbb{G}$ and $p = \pi(x) \in \mathbb{G}/\mathbb{K}$, we let

$$g(p) = g(\pi(x)) = \pi(L_g(x)) = \pi(R_x(g)), \ x \in \mathbb{G}, \tag{1.6}$$

where L and R are the left and the right translations on $\mathbb{G}$. Using this action, we notice that any vector $V \in \mathfrak{g}$ defines a Killing vector field on $\mathbb{G}/\mathbb{K}$, here denoted by $\zeta(V)$, given by

$$\zeta(V)(p) = \left.\frac{d}{dt}\right|_{t=0} \Big[(\exp tV)(p) \Big], \, p \in \mathbb{G}/\mathbb{K}, \tag{1.7}$$

where $\exp : \mathfrak{g} \to \mathbb{G}$ is the Lie exponential map.

Let $p \in \mathbb{G}/\mathbb{K}$ and $x \in \pi^{-1}(p)$. By (1.6) we have

$$\exp(tV)\,(p) = \exp(tV)\,(\pi(x)) = \pi(R_x(\exp(tV)))$$

and then

$$\zeta(V)(p) = d\pi_x(d(R_x)_e(V)). \tag{1.8}$$

This equation allows us to relate ζ and Γ: let $p \in \mathbb{G}/\mathbb{K}$, $u \in T_p\mathbb{G}/\mathbb{K}$ and $V \in \mathfrak{g}$. If $x \in \pi^{-1}(p)$, we have

$$\begin{aligned} \langle \zeta(V)(p),\, u\rangle &= \langle d\pi_x(d(R_x)_e(V)),\, u\rangle \\ &= \langle d(R_x)_e(V),\, \ell_x^{-1}(u)\rangle \\ &= \langle V,\, \Gamma_p(u)\rangle. \end{aligned} \tag{1.9}$$

The next result gives a characterization of the Lie subgroups of $\mathbb{G}$ that preserve M in terms of the Gauss map of M. This proposition comes to generalize Proposition 3.4 of [5]:

Proposition 1.2.3. *Let M^{n-1} be an orientable hypersurface of $\mathbb{G}/\mathbb{K}$ and let $\mathcal{N} : M \to \mathbb{S}^{n+k-1} \subseteq \mathfrak{g}$ be its Gauss map. Then*

$$\mathfrak{h} = (\mathcal{N}(M))^{\perp} = \{w \in \mathfrak{g};\, \langle w,\, \mathcal{N}(p)\rangle = 0\,\forall\, p \in M\} \tag{1.10}$$

is a Lie subalgebra of $\mathfrak{g}$ *and* M *is invariant under the Lie subgroup* $\mathbb{H}$ *of* $\mathbb{G}$ *whose Lie algebra is* $\mathfrak{h}$*. Conversely, if* M *is invariant under a Lie subgroup* $\mathbb{H}$ *of* $\mathbb{G}$*, then* $\mathfrak{h} \subseteq (\mathcal{N}(M))^{\perp}$*, where* $\mathfrak{h}$ *is the Lie algebra of* $\mathbb{H}$*.*

Proof. First we notice that if $w \in (\mathcal{N}(M))^{\perp}$, (1.9) implies that, for all $p \in M$,

$$0 = \langle w, \mathcal{N}(p)\rangle = \langle \zeta(w)(p), \eta(p)\rangle,$$

so $\zeta(w)$ is a vector field tangent to M. Now if $v,\ w \in \mathcal{N}(M)^{\perp}$, then $\zeta(v),\ \zeta(w)$ are two vector fields on M, thus $[\zeta(v),\ \zeta(w)]$ is also a vector field on M. Since $[\zeta(v),\ \zeta(w)] = \zeta([v,\ w])$, for $p \in M$ we have that

$$0 = \langle \zeta([v,\ w])(p),\ \eta(p)\rangle = \langle [v,\ w],\ \mathcal{N}(p)\rangle, \tag{1.11}$$

proving that $[v, w] \in \mathcal{N}(M)^{\perp}$. Hence $\mathfrak{h}$, defined by (1.10), is a Lie subalgebra of $\mathfrak{g}$.

Now let $\mathbb{H}$ be a subgroup of $\mathbb{G}$ that leaves M invariant and let $\mathfrak{h}$ be the Lie algebra of $\mathbb{H}$. Then $\mathfrak{h}$ acts on M as Killing fields and therefore $\langle \zeta(\mathfrak{h}), \eta\rangle = 0$. It follows that

$$0 = \langle \zeta(\mathfrak{h}),\ \eta\rangle = \langle \mathfrak{h},\ \mathcal{N}\rangle,$$

proving that $\mathfrak{h} \subseteq \mathcal{N}(M)^{\perp}$. □

1.2.4 Harmonicity of $\mathcal{N}$ and the mean curvature of M

A hypersurface S on the Euclidean space $\mathbb{R}^n$ has its Gauss map defined via the translation of its normal vector field to the origin, thus it is a map $\mathcal{N} : S \to \mathbb{S}^{n-1}$, taking values on the $n-1$ sphere centred at the origin of $\mathbb{R}^n$. It is a result of Ruh-Vilms, [58], that S has constant mean curvature if and only if its Gauss map is harmonic. This is a direct consequence of the well known formula

$$\Delta\mathcal{N} = -\mathrm{grad}H - \|B\|^2\mathcal{N}, \tag{1.12}$$

where $\|B\|$ is the norm of the second fundamental form of S.

The formula (1.12) was extended to hypersurfaces in a Lie Group endowed with a bi invariant metric by N. Espírito-Santo, S. Fornari, K. Frensel and J. Ripoll on [25]. Then, F. Bittencourt and J. Ripoll [5] generalized the results to a homogeneous space $\mathbb{G}/\mathbb{H}$ where $\mathbb{G}$ admits a Riemannian bi-invariant metric and $\mathbb{H}$ is a closed subgroup.

We present a more general formula for the Laplacian of the Gauss map given by (1.5), that comes to extend the ones obtained on [25] and [5] to a broader class of ambient spaces, namely the symmetric spaces treated on Section 1.2.1.

Theorem 1.2.4. *Let M be an immersed orientable hypersurface of $\mathbb{G}/\mathbb{K}$ and let $\mathcal{N} : M \to \mathbb{S}^{n+k-1} \subseteq \mathfrak{g}$ be the Gauss map of M, where $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$. Then*

$$\Delta\mathcal{N}(p) = -n\Gamma_p(gradH) - \left(\|B\|^2 + \mathrm{Ric}(\eta)\right)\mathcal{N}(p) \tag{1.13}$$

for all $p \in M$, where η is an unitary vector field normal to M, $\mathrm{Ric}(\eta)$ is the Ricci curvature of $\mathbb{G}/\mathbb{K}$ with respect to η, $\|B\|$ is the norm of the second fundamental form B of M in $\mathbb{G}/\mathbb{K}$ and H is the mean curvature of M.

Proof. Fix $V \in \mathfrak{g}$ and define the function

$$\begin{array}{rccl} f_V: & M & \to & \mathbb{R} \\ & p & \mapsto & \langle \mathcal{N}(p), V\rangle. \end{array} \tag{1.14}$$

For any $p \in M$ we have $f_V(p) = \langle \mathcal{N}(p), V\rangle = \langle \eta(p), \zeta(V)(p)\rangle$. As $\zeta(V)$ is a Killing field on $\mathbb{G}/\mathbb{K}$, it follows from Proposition 1 of [31] that

$$\Delta f_V = -n\langle \mathrm{grad}H, \zeta(V)\rangle - \left(\|B\|^2 + \mathrm{Ric}(\eta)\right) f_V. \tag{1.15}$$

Using (1.9) we obtain $\langle \mathrm{grad}H, \zeta(V)\rangle = \langle \Gamma_p(\mathrm{grad}(H)), V\rangle$, and then

$$\langle \Delta\mathcal{N}(p), V\rangle = \Delta f_V = \left\langle -n\Gamma_p(\mathrm{grad}H) - \left(\|B\|^2 + \mathrm{Ric}(\eta)\right)\mathcal{N}(p), V\right\rangle. \tag{1.16}$$

As (1.16) holds for any $V \in \mathfrak{g}$ we have (1.13), proving the theorem. □

The main consequence of this Theorem is an equivalence, that comes to generalize the result of Ruh-Vilms and the ones on [25] and [5].

Corollary 1.2.5. *Let M be an orientable hypersurface of $\mathbb{G}/\mathbb{K}$ and let $\mathcal{N} : M \to \mathbb{S}^{n+k-1} \subseteq \mathfrak{g}$ be the Gauss map of M. Then the following alternatives are equivalent:*

i. *M has constant mean curvature.*

ii. *The Gauss map $\mathcal{N} : M \to \mathbb{S}^{n+k-1}$ is harmonic*

iii. *$\mathcal{N}$ satisfies the equation*

$$\Delta\mathcal{N}(p) = -\left(\|B\|^2 + \mathrm{Ric}(\eta)\right)\mathcal{N}(p). \tag{1.17}$$

1.3 The Gauss map on spaces of constant sectional curvature

On this section, we present explicit expressions for the Gauss map on the case of $N = \mathbb{G}/\mathbb{K}$ to be a space form. In the Euclidean case, our Gauss map coincides with the usual one, as the horizontal lift is simply the identity. We then pass to consider the spherical and hyperbolic cases.

The Gauss map of M^{n-1} immersed in $\mathbb{S}^n$.

Let $O(n+1)$ be the orthogonal group of isometries of $\mathbb{R}^{n+1}$ that fixes the origin, and consider it as the matrix group

$$O(n+1) = \left\{ x \in GL_{n+1}(\mathbb{R});\ x^T = x^{-1} \right\}.$$

The Lie algebra $\mathfrak{o}(n+1)$ of $O(n+1)$ is given by

$$\mathfrak{o}(n+1) = \left\{ u \in M_{n+1}(\mathbb{R});\ u + u^T = 0 \right\},$$

and we consider the inner product on $\mathfrak{o}(n+1)$

$$\langle u,\, v\rangle_0 = \frac{1}{2}\mathrm{trace}(uv^T).$$

We can then extend $\langle\ ,\ \rangle_0$ via left translations to $O(n+1)$ defining the following left invariant metric on $O(n+1)$: for $x \in O(n+1)$ and $u,\, v \in T_xO(n+1)$, we let

$$\langle u,\, v\rangle_x = \langle x^{-1}u,\, x^{-1}v\rangle_0 = \frac{1}{2}\mathrm{trace}(x^{-1}uv^Tx) = \frac{1}{2}\mathrm{trace}(uv^T), \tag{1.18}$$

and we have that the metric given by (1.18) is bi-invariant.

Let $\{e_1,\, e_2,\, \dots,\, e_{n+1}\}$ be the canonical base of $\mathbb{R}^{n+1}$ and let $O(n)$ be the subgroup that fixes e_1,

$$O(n) = \{x \in O(n+1);\ xe_1 = e_1\},$$

then the quotient $O(n+1)/O(n)$, endowed with the metric induced via the projection $\pi : O(n+1) \to O(n+1)/O(n)$ is isometric to the unit sphere $\mathbb{S}^n$ centred at the origin of $\mathbb{R}^{n+1}$.

Next, we obtain an explicit expression for $\Gamma : T\mathbb{S}^n \to \mathfrak{o}(n+1)$: choose $p = (x_1,\, x_2,\, \dots,\, x_{n+1}) \in \mathbb{S}^n$ and let $\{v_2,\, v_3,\, \dots,\, v_{n+1}\}$ be an orthogonal basis of $T_p\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, so the matrix $(p\, v_2\, v_3\, \dots\, v_{n+1})$ is in $O(n+1)$. Then we define

$$x = \begin{pmatrix} x_1 & v_{12} & \dots & v_{1n+1} \\ x_2 & v_{22} & \dots & v_{2n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+1} & v_{n+1\,2} & \dots & v_{n+1\,n+1} \end{pmatrix},$$

where $v_j = \sum_{i=1}^{n+1} v_{ij} e_i \in \mathbb{R}^{n+1}$, and it follows that $x \in O(n+1)$ and $\pi(x) = x(e_1) = p$.

Now, let $u = (u_1, u_2, \dots, u_{n+1}) \in T_p\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ and write $u = \sum_{i=2}^{n+1} (u \cdot v_i)v_i$ where $(\ \cdot\ )$ is the inner product of $\mathbb{R}^{n+1}$. Let $Z \in \mathfrak{o}(n)^{\perp}$ be given by

$$Z = \begin{pmatrix} 0 & -(u \cdot v_2) & \dots & -(u \cdot v_{n+1}) \\ (u \cdot v_2) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (u \cdot v_{n+1}) & 0 & \dots & 0 \end{pmatrix}$$

and set $\widetilde{u} = d(L_x)_e Z \in (T_x xO(n))^{\perp}$. In coordinates, $\widetilde{u} = x.Z$ is the usual matrix multiplication and is represented as

$$\widetilde{u} = \begin{pmatrix} U_1 & -x_1(u \cdot v_2) & \dots & -x_1(u \cdot v_{n+1}) \\ U_2 & -x_2(u \cdot v_2) & \dots & -x_2(u \cdot v_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ U_{n+1} & -x_{n+1}(u \cdot v_2) & \dots & -x_{n+1}(u \cdot v_{n+1}) \end{pmatrix}$$

where

$$U_i = \sum_{j=2}^{n+1} v_{ij}(u \cdot v_j).$$

We claim that $\widetilde{u} = \ell_x^{-1}(u)$ is the horizontal lift of u. To see this, just apply the projection:

$$\begin{aligned} d\pi_x(\widetilde{u}) = \sum_{i=1}^{n+1} U_i e_i = \sum_{i=1}^{n+1}\sum_{j=2}^{n+1} v_{ij}(u \cdot v_j)e_i &= \sum_{j=2}^{n+1}(u \cdot v_j)\sum_{i=1}^{n+1} v_{ij}e_i \\ &= \sum_{j=2}^{n+1}(u \cdot v_j)v_j = u. \end{aligned}$$

This equation shows not only that $\widetilde{u}$ is the horizontal lift of u on $T_xO(n+1)$, but also that $U_i = (u \cdot e_i) = u_i$, as it was expected. Then, it becomes simple to find an expression for $\Gamma_p(u) = d(R_{x^{-1}})_x(\widetilde{u}) = \widetilde{u}.x^{-1}$. As $x \in O(n+1)$ we have that $x^{-1} = x^T$. Using again that $U_i = u_i$, the matrix expression for $\Gamma_p(u)$ is

$$\Gamma_p(u) = \begin{pmatrix} 0 & u_1x_2 - u_2x_1 & \dots & u_1x_{n+1} - u_{n+1}x_1 \\ u_2x_1 - u_1x_2 & 0 & \dots & u_2x_{n+1} - u_{n+1}x_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n+1}x_1 - u_1x_{n+1} & u_{n+1}x_2 - u_2x_{n+1} & \dots & 0 \end{pmatrix}.$$

If we let $\Phi_1 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to M_{n+1}(\mathbb{R})$ be given by

$$\begin{aligned} \Phi_1(x,y) &= \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{n+1} \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} y_1x_1 & y_2x_1 & \dots & y_{n+1}x_1 \\ y_1x_2 & y_2x_2 & \dots & y_{n+1}x_2 \\ \vdots & \vdots & \ddots & \vdots \\ y_1x_{n+1} & y_2x_{n+1} & \dots & y_{n+1}x_{n+1} \end{pmatrix}, \end{aligned} \tag{1.19}$$

then we can write

$$\Gamma_p(u) = \Phi_1(u,p) - \Phi_1(p,u). \tag{1.20}$$

We then obtain an explicit matrix expression for the Gauss map of a hypersurface of $\mathbb{S}^n$:

Proposition 1.3.1. *Let M^{n-1} be an orientable hypersurface of $\mathbb{S}^n$ oriented with respect to a normal unit vector field η. Let $\mathcal{N} : M \to \mathbb{S}^{\frac{(n+1)n}{2}-1} \subseteq \mathfrak{o}(n+1)$ be the Gauss map of M. Then*

$$\mathcal{N}(p) = \Phi_1(\eta(p), p) - \Phi_1(p, \eta(p)) \tag{1.21}$$

where Φ_1 is given by (1.19).

Using such expression and some properties of the map Φ_1, we relate the derivative of the Gauss map of a surface M in $\mathbb{S}^n$ with the shape operator of M.

Proposition 1.3.2. *Let M be an orientable surface in $\mathbb{S}^n$ oriented with respect to a normal unitary vector field η and let $\mathcal{N} : M \to \mathbb{S}^{\frac{(n+1)n}{2}-1} \subseteq \mathfrak{o}(n+1)$ be its Gauss map. Then, for any $p \in M$ and X, $Y \in T_pM$ it holds*

$$\langle d\mathcal{N}_p(X),\, \Gamma_p(Y)\rangle = -\langle A_\eta(X),\, Y\rangle,$$

where A_η is the shape operator of M.

Proof. Let M be as above. Let $p \in M$ and $X,\, Y \in T_pM$ and let $\alpha : (-\varepsilon,\, \varepsilon) \to M$ be such that $\alpha(0) = p$ and $\alpha'(0) = X$. Set $\mathcal{N}(t) = \mathcal{N}(\alpha(t))$ and $\eta(t) = \eta(\alpha(t))$. From Proposition 1.3.1 we have

$$\mathcal{N}(t) = \Phi_1(\eta(t), \alpha(t)) - \Phi_1(\alpha(t), \eta(t)).$$

Hence

$$d\mathcal{N}_p(X) = -\Phi_1(A_\eta(X), p) + \Phi_1(\eta(p), X) - \Phi_1(X, \eta(p)) + \Phi_1(p, A_\eta(X)),$$

as $\eta'(0) = \nabla_X \eta = -A_\eta(X)$. On the other hand, we also have $\Gamma_p(Y) = \Phi_1(Y, p) - \Phi_1(p, Y)$. An useful (and easy to check) identity concerning Φ_1 is that, for every $x,\, y,\, u,\, v \in \mathbb{R}^{n+1}$ it holds

$$\mathrm{trace}\,(\Phi_1(x, u).\Phi_1(y, v)) = (x \cdot v)(y \cdot u). \tag{1.22}$$

Then, as the metric of $\mathbb{S}^n$ (and consequently the metric of M) is the one induced by $(\,\cdot\,)$, (1.22) implies the identities:

$$\begin{aligned}
\mathrm{trace}(\Phi_1(p, A_\eta(X))\Phi_1(Y, p)) = {} & \langle A_\eta(X),\, Y\rangle && = \mathrm{trace}(\Phi_1(A_\eta(X), p)\Phi_1(p, Y)) \\
\mathrm{trace}(\Phi_1(X, \eta(p))\Phi_1(Y, p)) = {} & 0 && = \mathrm{trace}(\Phi_1(\eta(p), X)\Phi_1(p, Y)) \\
\mathrm{trace}(\Phi_1(\eta(p), X)\Phi_1(Y, p)) = {} & 0 && = \mathrm{trace}(\Phi_1(X, \eta(p))\Phi_1(p, Y)) \\
\mathrm{trace}(\Phi_1(A_\eta(X), p)\Phi_1(Y, p)) = {} & 0 && = \mathrm{trace}(\Phi_1(p, A_\eta(X))\Phi_1(p, Y)).
\end{aligned}$$

Then, it follows from the expression (1.18) for the metric of $O(n+1)$ that

$$\begin{aligned}
\langle d\mathcal{N}_p(X),\, \Gamma_p(Y)\rangle &= -\frac{1}{2}\mathrm{trace}(d\mathcal{N}_p(X)\Gamma_p(Y)) \\
&= -\langle A_\eta(X),\, Y\rangle.
\end{aligned}$$

□

An immediate consequence of Proposition 1.3.2 is a generalization of the result for the classical Gauss map, which derivative coincides, up to a sign, with the shape operator: here it is shown that the projection of $\mathcal{N}$ back to the sphere coincides with the shape operator. Precisely, we have proved:

Corollary 1.3.3. *Let M be a hypersurface in $\mathbb{S}^n$ oriented with respect to η an unitary vector field normal to M and let $\mathcal{N} : M \to \mathbb{S}^{\frac{n(n+1)}{2}-1} \subseteq \mathfrak{o}(n+1)$ be its Gauss map. Then, for any $x \in O(n+1)$ such that $\pi(x) \in M$ it holds*

$$d\pi_x d(R_x)_e d\mathcal{N}_{\pi(x)} = -A_\eta.$$

The Gauss map of M^{n-1} immersed in $\mathbb{H}^n$.

Consider the pseudo inner product $(\,*\,)$ on $\mathbb{R}^{n+1}$ given by

$$(x * y) = -x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1},$$

and let us introduce the following notation: For $i = 1, 2, \ldots, n+1$, let $\xi_1 = -1$ and $\xi_i = 1$ otherwise. Then we can write $(\,*\,)$ as

$$(x * y) = \sum_{i=1}^{n+1} \xi_i x_i y_i.$$

In the Lorentz space $\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, (\,*\,))$,

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1};\ (x * x) = -1 \text{ and } x_1 > 0\},$$

endowed with the metric of $\mathbb{L}^{n+1}$ is the hyperbolic space with constant sectional curvature -1. We consider

$$O(1,n) = \{g \in M_{n+1}(\mathbb{R});\ (gx*gy) = (x*y),\ \forall x,\ y \in \mathbb{L}^{n+1} \text{ and } g\,(\mathbb{H}^n) = \mathbb{H}^n\}.$$

In terms of matrices, the property that characterizes $O(1,n)$ is

$$O(1,n) = \{x \in M_{n+1}(\mathbb{R});\ x^{-1} = \widetilde{I} x^T \widetilde{I}\},$$

where

$$\widetilde{I} = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

The Lie algebra of $O(1,n)$, denoted by $\mathfrak{o}(1,n)$, can be written as

$$\mathfrak{o}(1,n) = \left\{ \begin{pmatrix} 0 & a_1 & \ldots & a_n \\ a_1 & & & \\ \vdots & & A & \\ a_n & & & \end{pmatrix},\ A \in \mathfrak{o}(n),\ a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R} \right\},$$

and we notice that $u = (u_{ij})_{ij} \in \mathfrak{o}(1,n) \Leftrightarrow u_{ij} = -\xi_i\xi_j u_{ji}$.

We introduce a pseudo-Riemannian bi-invariant metric $\langle\ ,\ \rangle$ on $O(1,n)$ by extending the non degenerate bilinear form $\langle u,\, v\rangle = \frac{1}{2}\mathrm{trace}(uv)$ on $\mathfrak{o}(1,n)$ to $O(1,n)$ via left translations.

With such metric, setting $O(n) = \{x \in O(1,n);\, x(e_1) = e_1\}$, $\mathbb{H}^n$ is isometric to the quotient $O(1,n)/O(n)$. In the next result we obtain an explicit expression for $\Gamma : T\mathbb{H}^n \to \mathfrak{o}(1,n)$:

Lemma 1.3.4. *Let $p \in \mathbb{H}^n$. Then, if $u \in T_p\mathbb{H}^n$, it holds*

$$\Gamma_p(u) = \Phi_{-1}(p,u) - \Phi_{-1}(u,p), \tag{1.23}$$

where $\Phi_{-1} : \mathbb{L}^{n+1} \times \mathbb{L}^{n+1} \to M_{n+1}(\mathbb{R})$ is given by

$$\begin{aligned}\Phi_{-1}(x,y) &= \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{n+1} \end{pmatrix}\begin{pmatrix} -1 & 1 & \dots & 1 \\ -1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} -y_1x_1 & y_2x_1 & \dots & y_{n+1}x_1 \\ -y_1x_2 & y_2x_2 & \dots & y_{n+1}x_2 \\ \vdots & \vdots & \ddots & \vdots \\ -y_1x_{n+1} & y_2x_{n+1} & \dots & y_{n+1}x_{n+1} \end{pmatrix}. \end{aligned} \tag{1.24}$$

Proof. The proof is similar to the spherical case. We write down some steps of it. Set $p = (x_1,\, x_2,\, \dots,\, x_{n+1}) \in \mathbb{H}^n$ and $u = (u_1,\, u_2,\, \dots,\, u_{n+1}) \in T_p\mathbb{H}^n$. Let $\{v_2,\, v_3,\, \dots,\, v_{n+1}\}$ be an orthogonal basis of $T_p\mathbb{H}^n$ in such way that the matrix $(p\, v_2\, v_3\, \dots\, v_{n+1}) \in O(1,n)$. Write each v_j in coordinates as $v_j = (v_{1j},\, v_{2j},\, \dots,\, v_{n+1\,j})$ and define

$$x = \begin{pmatrix} x_1 & v_{12} & \dots & v_{1\,n+1} \\ x_2 & v_{22} & \dots & v_{2\,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+1} & v_{n+1\,2} & \dots & v_{n+1\,n+1} \end{pmatrix}.$$

Then we have $x \in O(1,n)$ and $\pi(x) = p$. As in the spherical case, define $Z \in \mathfrak{o}(n)^{\perp}$ by

$$Z = \begin{pmatrix} 0 & (u * v_2) & \dots & (u * v_{n+1}) \\ (u * v_2) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (u * v_{n+1}) & 0 & \dots & 0 \end{pmatrix}.$$

Then $d(L_x)_e Z \in (T_x xO(n))^\perp$, $d\pi_x(xZ) = u$ and hence $\ell_x^{-1}(u) = xZ$. It follows that $\Gamma_p(u) = xZx^{-1}$. In terms of matrices,

$$\Gamma_p(u) = \begin{pmatrix} 0 & u_2x_1 - u_1x_2 & \dots & u_{n+1}x_1 - u_1x_{n+1} \\ -u_1x_2 + u_2x_1 & 0 & \dots & u_{n+1}x_2 - u_2x_{n+1} \\ -u_1x_3 + u_3x_1 & u_2x_3 - u_3x_2 & \dots & u_{n+1}x_3 - u_3x_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ -u_1x_{n+1} + u_{n+1}x_1 & u_2x_{n+1} - u_{n+1}x_2 & \dots & 0 \end{pmatrix}$$

$$= \Phi_{-1}(p, u) - \Phi_{-1}(u, p).$$

□

Proposition 1.3.5. *Let M be a hypersurface of the hyperbolic space $\mathbb{H}^n$ oriented with respect to an unitary normal vector field η. Let $\mathcal{N} : M \to \mathbb{S}^{\frac{(n+1)n}{2}-1} \subseteq \mathfrak{o}(1, n)$ be the Gauss map of M. Then it holds*

$$\mathcal{N}(p) = \Phi_{-1}(p, \eta(p)) - \Phi_{-1}(\eta(p), p), \tag{1.25}$$

where Φ_{-1} is given on (1.24).

This explicit formula for the Gauss map of a hypersurface M on $\mathbb{H}^n$ also implies that the derivative of $\mathcal{N}$ on this case is the lift of the shape operator:

Proposition 1.3.6. *Let M be an orientable hypersurface in $\mathbb{H}^n$ oriented by a normal unitary vector field η and let $\mathcal{N} : M \to \mathbb{S}^{\frac{(n+1)n}{2}-1} \subseteq \mathfrak{o}(1, n)$ be its Gauss map. Then for any $p \in M$ and X, $Y \in T_pM$ it holds*

$$\langle d\mathcal{N}_p(X), \Gamma_p(Y)\rangle = -\langle A_\eta(X), Y\rangle.$$

Proof. The proof to this proposition is analogous to the proof of Proposition 1.3.2, with the difference that here one uses $(p * p) = -1$ and the equation

$$\text{trace}(\Phi_{-1}(x, u)\Phi_{-1}(y, v)) = (x * v)(y * u) \tag{1.26}$$

instead of (1.22). □

As a consequence, similarly to the spherical case, we reobtain the shape operator of M from the derivative of its Gauss map:

Corollary 1.3.7. *Let M be an orientable surface in $\mathbb{H}^n$ oriented by an unitary vector field η normal to M and let $\mathcal{N} : M \to \mathbb{S}^{\frac{(n+1)n}{2}-1} \subseteq \mathfrak{o}(1, n)$ be its Gauss map. Then, for any $x \in O(1, n)$ such that $\pi(x) \in M$ it holds*

$$d\pi_x d(R_x)_e d\mathcal{N}_{\pi(x)} = -A_\eta.$$

1.4 The quadratic form induced by $\mathcal{N}$ on surfaces immersed in symmetric spaces of dimension 3

It is a classic result due to Heinz Hopf [37] that in the Euclidean three space, the Hopf differential $\mathcal{A}$ of a surface M (that is, the complexification of the traceless part of the second fundamental form of M) is holomorphic if and only if M has constant mean curvature. This result is also true in $\mathbb{H}^3$ and $\mathbb{S}^3$ (S.-S. Chern, [8]), but it is false in general. In [1] U. Abresch and H. Rosenberg "perturbed"the Hopf differential and defined a quadratic differential form $\mathcal{Q} = 2H\mathcal{A} - c\mathcal{T}$ of a surface M immersed in $\mathcal{M}^2(c) \times \mathbb{R}$ (H is the mean curvature of M, $\mathcal{A}$ is the Hopf differential and $\mathcal{T} = (dh \otimes dh)^{2,0}$, h standing for the height function), and extended Hopf's theorem for CMC spheres to these ambient spaces using $\mathcal{Q}$ instead of $\mathcal{A}$.

In $\mathbb{R}^3$ the differential of the Gauss map $\mathcal{N} : M \to \mathbb{S}^2$ coincides (up to a sign) with the shape operator of the surface, and the complex quadratic form induced by g is, up to a constant, the Hopf differential $\mathcal{A}$. In [38], M. L. Leite and J. Ripoll used the Gauss map $\mathcal{N}$ of a surface M in $\mathbb{S}^2 \times \mathbb{R}$, as defined in [5], to show that the quadratic form induced by $\mathcal{N}$ was actually the Abresch-Rosenberg quadratic form $\mathcal{Q}$. They also defined an "ad hoc" Gauss map $\mathcal{N}$, which they called *twisted normal map,* for a surface M in $\mathbb{H}^2 \times \mathbb{R}$ and again obtained that the quadratic form induced by $\mathcal{N}$ coincided with the Abresch-Rosenberg quadratic form $\mathcal{Q}$ of M.

In this section we will consider a surface M immersed in a 3-dimensional symmetric space $N = \mathbb{G}/\mathbb{K}$ satisfying the assumptions of Section 1.2. It will be shown that the complex quadratic form induced by $\mathcal{N}$ on M is the Hopf differential when N is $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$. Moreover, we show that the Gauss map $\mathcal{N}$ coincides with the twisted normal map defined in [38], when $N = \mathbb{H}^2 \times \mathbb{R}$, in particular the quadratic form induced by $\mathcal{N}$ on the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ will coincide with the Abresch-Rosenberg quadratic form.

1.4.1 The quadratic form on $\mathbb{H}^3$ and on $\mathbb{S}^3$

Let $\mathcal{M} = \mathcal{M}^3(c)$ be $\mathbb{S}^3$ when $c = 1$ and $\mathbb{H}^3$ when $c = -1$. Consider the bilinear form on $\mathbb{R}^4$ given by

$$(x,\, y) = cx_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

When $c = 1$, we obtain $(\mathbb{R}^4, (\,,\,)) = \mathbb{E}^4$ is the Euclidean 4-space with its usual flat metric and when $c = -1$ $(\mathbb{R}^4, (\,,\,)) = \mathbb{L}^4$ is the Lorentz space. We will consider $\mathcal{M} \subseteq \mathbb{R}^4$ as the set

$$\mathcal{M}^3(c) = \{x \in \mathbb{R}^4;\ (x,\, x) = c\}.$$

With the metric induced from $(\mathbb{R}^4, (\,,\,))$, $\mathcal{M}^3(1)$ becomes isometric to $\mathbb{S}^3$ and $\mathcal{M}^3(-1)$ is isometric to $\mathbb{H}^3$, and herein we will treat both cases using such notation. The metric on $\mathcal{M}$ will be denoted by $\langle\,,\,\rangle$ and we let D the Riemannian connection of $(\mathbb{R}^4, (\,,\,))$ and ∇ the connection of $\mathcal{M}$. In order to ease the notation, we will denote $(\mathbb{R}^4, (\,,\,)) = \mathbb{R}^4$.

Let $F : \Sigma \to \mathcal{M}$ be a conformal immersion of a Riemann surface $\Sigma \subseteq \mathbb{C}$ into $\mathcal{M}$ and let $z = x + iy$ to be the conformal structure on Σ induced by F. Then, we have

$$\langle F_x,\, F_x\rangle = \langle F_y,\, F_y\rangle = \lambda > 0 \text{ and } \langle F_x,\, F_y\rangle = 0,$$

which implies that

$$\langle F_z,\, F_z\rangle = \langle F_{\overline{z}},\, F_{\overline{z}}\rangle = 0,\ \langle F_z,\, F_{\overline{z}}\rangle = \lambda/2,$$

where $2F_z = F_x - iF_y$ and $2F_{\overline{z}} = F_x + iF_y$ is the complexification of the basis $\{F_x,\, F_y\}$.

Let η be an unitary vector field normal to F on $\mathcal{M}$. We define the quadruple $\sigma = \{F_z,\, F_{\overline{z}},\, \eta,\, F\}$ and notice that in each point of $F(\Sigma)$, σ is a basis for $\mathbb{R}^4$ (σ is called a *moving frame* for $\mathbb{R}^4$), satisfying:

$$\begin{array}{cccc}
(F_z,\, F_z) = 0 & (F_z,\, F_{\overline{z}}) = \lambda/2 & (F_z,\, \eta) = 0 & (F_z,\, F) = 0\\
(F_{\overline{z}},\, F_z) = \lambda/2 & (F_{\overline{z}},\, F_{\overline{z}}) = 0 & (F_{\overline{z}},\, \eta) = 0 & (F_{\overline{z}},\, F) = 0\\
(\eta,\, F_z) = 0 & (\eta,\, F_{\overline{z}}) = 0 & (\eta,\, \eta) = 1 & (\eta,\, F) = 0\\
(F,\, F_z) = 0 & (F,\, F_{\overline{z}}) = 0 & (F,\, \eta) = 0 & (F,\, F) = c.
\end{array}$$

Using the same approach as the one of I. Fernández and P. Mira [27], we obtain the structure equations for σ:

Lemma 1.4.1. *The moving frame σ satisfies the differential equations:*

$$\sigma_z = \mathcal{U}\sigma,\ \sigma_{\overline{z}} = \mathcal{V}\sigma, \tag{1.27}$$

where $\mathcal{U}$ and $\mathcal{V}$ are matrices given by

$$\mathcal{U} = \begin{pmatrix} (\log(\lambda))_z & 0 & \alpha & 0\\ 0 & 0 & \frac{H\lambda}{2} & -\frac{c\lambda}{2}\\ -H & -\frac{2\alpha}{\lambda} & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix},\ \mathcal{V} = \begin{pmatrix} 0 & 0 & \frac{H\lambda}{2} & -\frac{c\lambda}{2}\\ 0 & (\log(\lambda))_{\overline{z}} & \overline{\alpha} & 0\\ -\frac{2\overline{\alpha}}{\lambda} & -H & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix},$$

H is the mean curvature of the immersion F and $\alpha = -\langle\nabla_{F_z}\eta,\, F_z\rangle$ is the coefficient of the Hopf differential of F.

Proof. On this section, the interest of this Lemma is simply to compute $D_{F_z}\eta = \eta_z$ in order to obtain an expression for the form induced by $\mathcal{N}$. As all other equations follow analogously, we will explicit only the computations to obtain η_z.

First, we write, for some constants A, $B \in \mathbb{C}$ and C, $D \in \mathbb{R}$

$$\eta_z = AF_z + BF_{\bar{z}} + C\eta + DF.$$

Now, we observe that

$$(\eta_z,\, F_z) = \langle \nabla_{F_z}\eta,\, F_z\rangle = -\alpha,$$

thus $B\frac{\lambda}{2} = -\alpha$, and then $B = -2\alpha/\lambda$. It also holds that

$$\begin{aligned}(\eta_z,\, F_{\bar{z}}) = -\langle \eta,\, \nabla_{F_z}F_{\bar{z}}\rangle &= -\frac{1}{4}\langle \eta,\, \nabla_{F_x}F_x + \nabla_{F_y}F_y\rangle \\ &= -H\frac{\lambda}{2},\end{aligned}$$

thus $A = -H$. The equation $(\eta,\, \eta) = 1$ implies that $(\eta_z,\, \eta) = 0$, so $C = 0$ and finally the equation $(\eta_z,\, F) = -(\eta,\, F_z) = 0$ implies that $D = 0$ and we obtain that

$$\eta_z = -HF_z - \frac{2\alpha}{\lambda}F_{\bar{z}}. \tag{1.28}$$

□

Now we will use equation (1.28) to obtain an expression to the quadratic form induced by $\mathcal{N}$, $Q_{\mathcal{N}} = \langle \mathcal{N}_z,\, \mathcal{N}_z\rangle dz^2$.

Proposition 1.4.2. *Let $\mathcal{A} = 2\alpha dz^2$ be the Hopf differential of F. Then the quadratic form induced by $\mathcal{N}$ satisfies*

$$Q_{\mathcal{N}} = H\mathcal{A},$$

where $H : \Sigma \to \mathbb{R}$ is the mean curvature of F.

Proof. Denote by $\mathfrak{o}_c = \mathfrak{o}(4)$ when $c = 1$ and $\mathfrak{o}(1,3)$ when $c = -1$. Then the metric on $\mathfrak{o}_c$ is given by $\langle X,\, Y\rangle = -\frac{c}{2}\mathrm{trace}(XY)$. Recall from (1.21), (1.25) that the Gauss map of a surface immersed on $\mathcal{M}$ is given by

$$\mathcal{N}(p) = \Phi_c(\eta(p),\, p) - \Phi_c(p,\, \eta(p)),$$

where $\Phi_c : \mathbb{R}^4 \to \mathfrak{o}_c$ is either Φ_1 of (1.19) or Φ_{-1} of (1.24), and satisfies the relation

$$\text{trace}(\Phi_c(x,u)\Phi_c(y,v)) = (x,\, v)(u,\, y). \tag{1.29}$$

In particular, for each $p \in \Sigma$, we have

$$\mathcal{N}(p) = \Phi_c(\eta(F(p)),\, F(p)) - \Phi_c(F(p),\, \eta(F(p))),$$

and it follows that

$$\mathcal{N}_z = \Phi_c(\eta_z,\, F) - \Phi_c(F,\, \eta_z) + \Phi_c(\eta,\, F_z) - \Phi_c(F_z,\, \eta),$$

thus

$$\begin{aligned}
\langle \mathcal{N}_z, \mathcal{N}_z \rangle &= -\frac{c}{2}\text{trace}(\mathcal{N}_z\mathcal{N}_z) \\
&= -\frac{c}{2}\text{trace}\Big[\,\Phi(\eta_z,\, F)\big(\Phi(\eta_z,\, F) - \Phi(F,\, \eta_z) + \Phi(\eta,\, F_z) - \Phi(F_z,\, \eta)\big) \\
&\qquad - \Phi(F,\, \eta_z)\big(\Phi(\eta_z,\, F) - \Phi(F,\, \eta_z) + \Phi(\eta,\, F_z) - \Phi(F_z,\, \eta)\big) \\
&\qquad + \Phi(\eta,\, F_z)\big(\Phi(\eta_z,\, F) - \Phi(F,\, \eta_z) + \Phi(\eta,\, F_z) - \Phi(F_z,\, \eta)\big) \\
&\qquad - \Phi(F_z,\, \eta)\big(\Phi(\eta_z,\, F) - \Phi(F,\, \eta_z) + \Phi(\eta,\, F_z) - \Phi(F_z,\, \eta)\big)\Big] \\
&= -\frac{c}{2}\left(2(F,\, \eta_z)^2 - 2(F,\, F)(\eta_z,\, \eta_z)\right).
\end{aligned}$$

But $(F,\, \eta_z) = 0$ and $(F,\, F) = c$, so last equation becomes $\langle \mathcal{N}_z, \mathcal{N}_z \rangle = (\eta_z,\, \eta_z)$. Now, recall the expression (1.28), to obtain that

$$(\eta_z,\, \eta_z) = 2H\frac{2\alpha}{\lambda}(F_z,\, F_{\bar{z}}) = 2H\alpha.$$

Finally, it follows that

$$Q_{\mathcal{N}} = \langle \mathcal{N}_z, \mathcal{N}_z \rangle dz^2 = 2H\alpha dz^2 = H\mathcal{A}.$$

□

We then have the following two theorems, 1.4.3 and 1.4.4:

Theorem 1.4.3. *Let M be a surface immersed in $\mathbb{S}^3$ and let $\mathcal{N} : M \to \mathbb{S}^5 \subseteq \mathfrak{o}(4)$ be its Gauss map. Then the following alternatives are equivalent:*

i. *M has constant mean curvature;*

ii. *$\mathcal{N}$ is harmonic;*

iii. *The complex quadratic form $\mathcal{Q}_{\mathcal{N}}$ induced by $\mathcal{N}$ on M is holomorphic.*

Theorem 1.4.4. *Let M be a surface immersed in $\mathbb{H}^3$ and let $\mathcal{N} : M \to \mathbb{S}^5 \subseteq \mathfrak{o}(1,3)$ be its Gauss map. Then the following alternatives are equivalent:*

i. *M has constant mean curvature;*

ii. *$\mathcal{N}$ is harmonic;*

iii. *The complex quadratic form $\mathcal{Q}_{\mathcal{N}}$ induced by $\mathcal{N}$ on M is holomorphic.*

Proof of Theorems 1.4.3 and 1.4.4. It follows from Proposition 1.4.2 that the quadratic differential form induced by $\mathcal{N}$, $\mathcal{Q}_{\mathcal{N}}$ coincides, up to a constant with the Hopf differential $\mathcal{A}$ of M, either on $\mathbb{S}^3$ or on $\mathbb{H}^3$. Therefore, the work of Chern [8] gives us that $\mathcal{Q}_{\mathcal{N}}$ is holomorphic if and only if M has constant mean curvature. The equivalence between CMC and harmonicity of the Gauss map had already been obtained in Corollary 1.2.5, and this proves Theorems 1.4.3 and 1.4.4. □

1.4.2 The quadratic form on $\mathbb{H}^2 \times \mathbb{R}$ and on $\mathbb{S}^2 \times \mathbb{R}$

In this section we prove a result analogous to Theorems 1.4.3 and 1.4.4 for a surface M immersed in a product space $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. We will prove that if M has constant mean curvature, then the quadratic form induced by $\mathcal{N}$ is holomorphic. In order to prove this result we will show that the complex quadratic form induced by the Gauss map of M coincides with the Abresch-Rosenberg quadratic form. When $N = \mathbb{S}^2 \times \mathbb{R}$, our construction of the Gauss map coincides with the one in [5], therefore Theorem 3.1 of [38] shows this result. Thus, we focus when M is a surface immersed in $\mathbb{H}^2 \times \mathbb{R}$, and we relate $\mathcal{N}$ with the *twisted normal map* of M, introduced in [38].

For an orientable surface M in $\mathbb{H}^2 \times \mathbb{R}$ oriented with a vector field (η, ν) normal to M, the twisted normal map of M is defined by (see [38]):

$$\begin{array}{rccl} N: & M & \to & \mathbb{S}^3 \subseteq \mathbb{L}^3 \times \mathbb{R} \\ & (p,\, t) & \mapsto & (J(\eta(p)),\, \nu), \end{array} \tag{1.30}$$

where J is the operator acting on tangent planes of $\mathbb{H}^2$ as the clockwise $\pi/2$ rotation. Next proposition is an analogous to Proposition 2.3 of [38] to the case of $\mathbb{H}^2 \times \mathbb{R}$ and shows that, if $p \in \mathbb{H}^2$, then $\Gamma_p = J$. In particular, the Gauss map given by the expression (1.5) coincides with the twisted normal map defined by (1.30).

Proposition 1.4.5. *Let $p \in \mathbb{H}^2$ and let $v \in T_p\mathbb{H}^2 \subseteq \mathbb{L}^3$. Let $\{v_2, v_3\}$ be an orthogonal basis of $T_p\mathbb{H}^2$. If $u = av_2 + bv_3$, then $\Gamma_p(u) = -bv_2 + av_3$, via the identification*

$$\begin{pmatrix} 0 & -r & s \\ -r & 0 & -t \\ s & t & 0 \end{pmatrix} \in \mathfrak{o}(1,2) \leftrightarrow (t,\, s,\, r) \in \mathbb{L}^3.$$

Proof. Let $p = (x_1, x_2, x_3) \in \mathbb{H}^2$ and $u = (u_1, u_2, u_3) \in T_p\mathbb{H}^2$. Then, by equation (1.23), it follows that

$$\Gamma_p(u) = \begin{pmatrix} 0 & u_2x_1 - u_1x_2 & u_3x_1 - u_1x_3 \\ u_2x_1 - u_1x_2 & 0 & u_3x_2 - u_2x_3 \\ u_3x_1 - u_1x_3 & u_2x_3 - u_3x_2 & 0 \end{pmatrix}.$$

Writing $v_j = (v_{1j}, v_{2j}, v_{3j})$ and making the substitution $u_i = av_{i2} + bv_{i3}$ on the previous equality it becomes

$$\begin{aligned}\Gamma_p(u) &= a\begin{pmatrix} 0 & v_{22}x_1 - v_{12}x_2 & v_{32}x_1 - v_{12}x_3 \\ v_{22}x_1 - v_{12}x_2 & 0 & v_{32}x_2 - v_{22}x_3 \\ v_{32}x_1 - v_{12}x_3 & v_{22}x_3 - v_{32}x_2 & 0 \end{pmatrix} \\ &+ b\begin{pmatrix} 0 & v_{23}x_1 - v_{13}x_2 & v_{33}x_1 - v_{13}x_3 \\ v_{23}x_1 - v_{13}x_2 & 0 & v_{33}x_2 - v_{23}x_3 \\ v_{33}x_1 - v_{13}x_3 & v_{23}x_3 - v_{33}x_2 & 0 \end{pmatrix} \\ &= a\begin{pmatrix} 0 & -v_{33} & v_{23} \\ -v_{33} & 0 & -v_{13} \\ v_{23} & v_{13} & 0 \end{pmatrix} + b\begin{pmatrix} 0 & v_{32} & -v_{22} \\ v_{32} & 0 & v_{12} \\ -v_{22} & -v_{12} & 0 \end{pmatrix} \\ &= av_3 - bv_2.\end{aligned}$$

□

Then, we obtain

Corollary 1.4.6. *On the ambient space $\mathbb{H}^2 \times \mathbb{R}$, the Gauss map defined by* (1.5) *coincides with the twisted normal map of [38] given by* (1.30).

This corollary implies (together with Theorems 3.1 and 3.3 of [38]) the following result:

Proposition 1.4.7. *Let M be an orientable surface in $\mathcal{M}_2(c) \times \mathbb{R}$ oriented with respect to an unitary vector field (η, ν) normal to M. Let $\mathcal{N}$ be the Gauss map of M and let $\mathcal{Q}_{\mathcal{N}}$ be the complex quadratic form induced by $\mathcal{N}$. Then*

$$\mathcal{Q}_{\mathcal{N}} = \mathcal{Q},$$

where $\mathcal{Q} = 2H\mathcal{A} - c\mathcal{T}$ is the Abresch-Rosenberg quadratic form of M ([1]).

Now, using Theorem 1 of Abresch-Rosenberg [1], we can use our construction of the Gauss map to obtain:

Theorem 1.4.8. *Let M be a surface immersed either in $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$. If $\mathcal{N}$ is the Gauss map of M, then there is an equivalence between*

i. *M has constant mean curvature;*

ii. *$\mathcal{N}$ is harmonic.*

Moreover, both imply

iii. *$\mathcal{Q}_{\mathcal{N}}$ is holomorphic on M.*

Remark 1.1. The converse of this theorem is false. It was shown by I. Fernández and P. Mira in [28] the existence of certain rotational surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with holomorphic Abresch-Rosenberg differential that fails to be CMC.

1.5 HOS theorem in symmetric spaces of dimension 3

On [5], Theorem 4.9 proves HOS theorem for a complete CMC surface M immersed in a 3-dimensional homogeneous space $\mathbb{G}/\mathbb{H}$ where $\mathbb{G}$, up to an abelian factor, is compact. In particular, this result apply for M immersed in $\mathbb{S}^3$ and in $\mathbb{S}^2 \times \mathbb{R}$. We now extend HOS theorem for surfaces immersed in a symmetric space $N = \mathbb{G}/\mathbb{K}$ as in the preliminaries of Section 1.2, extending Theorem 4.9 of [5] to other spaces with non-compact irreducible factors, such as $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 1.5.1. *Let $N = \mathbb{G}/\mathbb{K}$ be a 3-dimensional symmetric space as in Section 1.2. Let $H \geq 0$ be given and assume that $2H^2 + \mathrm{Ric}_N \geq 0$, where $\mathrm{Ric}_N = \min_{|v|=1} \mathrm{Ric}_N(v)$. Let M be a complete orientable surface immersed with CMC H in N. Assume that $\mathcal{N}(M)$ is contained in a hemisphere of the unit sphere in $\mathfrak{g}$ determined by a nonzero vector $V \in \mathfrak{g}$, that is, $\langle \mathcal{N}(p), V\rangle \leq 0$ for all $p \in M$. We have:*

a) If M has the conformal type of the disk, then M is invariant under the 1-parameter subgroup of isometries of N determined by V;

b) If M has the conformal type of the plane and $\zeta(V)$ is a bounded[2] *Killing field on M, then M is invariant under the 1-parameter subgroup of isometries of N determined by V or M is umbilical and $Ric(\eta) = Ric_N$.*

Proof. Suppose that $\mathcal{N}(M)$ is contained in a hemisphere of $\mathfrak{g}$ determined by V. Let $\pi : \hat{M} \to M$ be the universal covering of M and consider $\hat{M}$ as an immersed surface in N. Write f as $f \circ \pi$. Set $f(p) = \langle \zeta(V)(p), \eta(p) \rangle$, $p \in \hat{M}$, where $\zeta(V)$ is the Killing field on N defined on (1.7). Since $\langle \zeta(V)(p), \eta(p) \rangle = \langle \mathcal{N}(p), V \rangle \leq 0$, we have $f \leq 0$. Assume first that $\hat{M}$ is conformal to the disk. We will then show that f vanishes identically and thus Proposition 1.2.3 implies that $\hat{M}$ is invariant under the group of isometries generated by V.

As H is constant, we have that $\operatorname{grad}(H) = 0$, so we can compute the Laplacian of f as on the proof of Theorem 1.2.4 to obtain

$$\Delta f + \left(\|B\|^2 + \operatorname{Ric}(\eta) \right) f = 0. \tag{1.31}$$

Then, as $\|B\|^2 \geq 2H^2$ and $f \leq 0$, it follows from the hypothesis $2H^2 + \operatorname{Ric}_N \geq 0$ that

$$\begin{aligned} \Delta f &= -\left(\|B\|^2 + \operatorname{Ric}(\eta) \right) f \\ &\geq -\left(2H^2 + \operatorname{Ric}(\eta) \right) f \geq 0, \end{aligned} \tag{1.32}$$

Therefore, f is a subharmonic function on $\hat{M}$. If f vanishes at some point $p \in \hat{M}$ then, by the maximum principle, $f \equiv 0$ and the theorem is proved on this case. So, let us suppose $f < 0$ and get a contradiction. From the Gauss equation we have $\|B\|^2 = 4H^2 - 2(K_{\hat{M}} - K_N)$ where $K_{\hat{M}}$ is the sectional curvature of $\hat{M}$ and K_N is the sectional curvature of N on tangent planes of M. Using this equation on (1.31), we obtain

$$\Delta f - 2K_{\hat{M}} f + \left(4H^2 + 2K_N + \operatorname{Ric}(\eta) \right) f = 0. \tag{1.33}$$

Considering an orthonormal basis E_1, E_2 of $T\hat{M}$ we observe that

$$\begin{aligned} \operatorname{Ric}(\eta) + 2K_N &= \langle R(\eta, E_1)\eta, E_1 \rangle + \langle R(\eta, E_2)\eta, E_2 \rangle + 2\langle R(E_1, E_2)E_1, E_2 \rangle \\ &= \langle R(E_1, \eta)E_1, \eta \rangle + \langle R(E_1, E_2)E_1, E_2 \rangle \\ &\quad + \langle R(E_2, \eta)E_2, \eta \rangle + \langle R(E_2, E_1)E_2, E_1 \rangle \\ &= \operatorname{Ric}(E_1) + \operatorname{Ric}(E_2) \geq 2 \operatorname{Ric}_N . \end{aligned}$$

[2] On the sense that it has bounded norm $\|\zeta(V)\|$ on M

Then, it follows that

$$P = \mathrm{Ric}(\eta) + 2K_N + 4H^2 \geq 2\,\mathrm{Ric}_N + 4H^2 \geq 0,$$

thus f is a negative solution to the equation $\Delta f - 2K_{\hat{M}} f + Pf = 0$, with $P \geq 0$, which contradicts Corollary 3 of [30], as $\hat{M}$ has the conformal type of the disk. It follows that $f \equiv 0$ and the first part of the theorem is proved.

Assume now that $\hat{M}$ is conformal to the plane and that $\zeta(V)$ is bounded in M. Then, it follows that f is a bounded function on M. However, follows again from (1.32) that f is subharmonic, so it is constant. In particular, $\Delta f = 0$, and this, together with (1.31), implies that

$$\left(\|B\|^2 + \mathrm{Ric}(\eta)\right) f = 0.$$

It follows that either $f \equiv 0$ (and then M is invariant under the 1-parameter family of isometries given by V) or $(\|B\|^2 + \mathrm{Ric}(\eta)) \equiv 0$. On this case the inequality on (1.32) would be an equality, and we would have

$$\|B\|^2 = 2H^2 \text{ and } \mathrm{Ric}(\eta) = \mathrm{Ric}_N,$$

and from $\|B\|^2 = 2H^2$ it follows that M is umbilical as it is easy to see. □

Remark 1.2. Since an equidistant surface of $\mathbb{H}^3$, that is, a surface which is at a constant distance to a totally geodesic surface of $\mathbb{H}^3$, has the conformal type of the disk and is orthogonal to a hyperbolic Killing field (that is, the Killing field which orbits are hypercycles equidistant to a fixed geodesic) we see that the hypothesis $2H^2 + \mathrm{Ric}_N \geq 0$ which, in the hyperbolic space, is equivalent to $H \geq 1$, cannot be improved.

CHAPTER 2

The mean curvature equation on semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$

On this chapter we study the mean curvature equation of π-graphs on semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, following the paper by the author [53], under supervision of J. Pérez. The semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the set $\mathbb{R}^3$, endowed with a group operation and a left invariant metric, depending on a matrix $A \in M_2(\mathbb{R})$. Its precise definition in given on Section 2.2, and follows the construction of Meeks-Pérez [44].

2.1 Introduction

The subject of minimal surfaces is among the most beautiful – and also most studied – objects in Differential Geometry, and many times minimal graphs play an important role on the field. Many deep results may be obtained from looking to a minimal surface as a local graph and bringing techniques from partial differential equations to help solve geometric questions.

On a series of papers ([40, 42, 43, 44]), W. Meeks, P. Mira, J. Pérez and A. Ros studied constant mean curvature spheres on three-dimensional homogeneous spaces and semidirect products of the form $\mathbb{R}^2 \rtimes_A \mathbb{R}$ came to play a very important role on their proofs. This is because any simply connected metric Lie group of dimension 3 is either $SU(2)$ or $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ endowed with a left invariant metric or it is isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical left invariant metric, where

$A \in M_2(\mathbb{R})$ is some 2×2 square matrix (see the work of W. Meeks and J. Pérez [44] or Section 2.2 below).

Although a lot has been done in the last years on the ambient space of semidirect products (or, more generally, on simply connected homogeneous spaces of dimension 3), the theory of minimal $\pi-$graphs on semidirect products is still on development, and many interesting questions on this field remain open. For instance is it true that a minimal $\pi-$graph on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is stable?[1] This question is equivalent to Problem 1, stated on Section 2.4, which asks about the *uniqueness* of a minimal $\pi-$graph with prescribed boundary. Unfortunately, this (thus both) question remains open, although the results presented on this chapter help us to have a better understanding on the behaviour of the mean curvature operator.

There are two main difficulties when dealing with minimal $\pi-$graphs on semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$: the first one is that vertical translations $(x, y, z) \mapsto (x, y, z + t)$ are not isometries of the ambient space, and this affects the mean curvature operator so its high order coefficients depend on the solution (see Section 2.3), and the comparison principle (for instance Theorem 10.1 of [32] and its generalizations) does not apply. The second one is that, unless $\mathrm{trace}(A) = 0$, constant functions do not provide minimal graphs, so there is no maximum principle.

On this chapter, we consider a convex domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ with piecewise smooth boundary and exhibit the differential equation some function $u : \Omega \to \mathbb{R}$ must satisfy for its $\pi-$graph

$$\mathrm{graph}(u) = \{(x,\, y,\, u(x,y)) \in \mathbb{R}^2 \rtimes_A \mathbb{R};\ (x,\, y,\, 0) \in \Omega\}$$

to have prescribed mean curvature function. Depending on the trace and on the determinant of A such PDE has a different behaviour. For instance, when $\mathrm{trace}(A) = 0$, if $u : \Omega \to \mathbb{R}$ is a function whose $\pi-$graph has non negative mean curvature $H \geq 0$ with respect to the upwards orientation, then it satisfies the maximum principle

$$\sup_{\partial\Omega} u = \sup_{\Omega} u. \tag{2.1}$$

This property was first observed by W. Meeks, P. Mira, J. Pérez and A. Ros ([43], stated on its generality as Lemma 2.4.2 below), and we remark that (2.1) does not hold on the case of $\mathrm{trace}(A) > 0$, even for $H \equiv 0$: a minimal graph that is constant along its boundary necessarily assumes an interior maximum and it is not constant, as horizontal planes (representing

[1] More generally, if the left invariant Gauss map of a surface $\Sigma \subseteq \mathbb{R}^2 \rtimes_A \mathbb{R}$ is contained on a hemisphere, does it imply Σ is stable?

constant functions) are no longer minimal. It becomes a natural question to ask if there is a *maximal jump* these minimal graphs that are constant along the boundary can attain, and this question is answered via height estimates of partial differential equations.

Let us describe some of the main results of this chapter: Section 2.3 is where we deduce the mean curvature equation for a $\pi-$graph and define the *mean curvature operator* Q, on equation (2.22). Theorem 2.4.3 on Section 2.4 is to obtain the height estimates: given $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ and a parameter $\alpha \in \mathbb{R}$, we obtain a constant $C = C(\text{diam}(\Omega), \alpha)$ such that if $u : \Omega \to \mathbb{R}$ is a function satisfying

$$\begin{cases} Q(u) \geq 0 \text{ in } \Omega \\ u \leq \alpha \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

then u satisfies

$$u \leq \alpha + C(\alpha) \text{ in } \Omega. \tag{2.3}$$

Still on Section 2.4 it is proved that the dependence on α to the constant C of (2.3) is essential (Theorem 2.4.4) for the validity of the result, on the sense that it is not possible to obtain some constant $C = C(\Omega)$ such that every $u : \Omega \to \mathbb{R}$ with $Q(u) \geq 0$ satisfies the uniform height estimate

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C.$$

We also use the freedom on the parameter α in order to obtain that (on the case trace$(A) > 0$, see Theorem 2.4.6 for details) the oscillation of a family of solutions to the problem

$$\begin{cases} Q(u) = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = c \in \mathbb{R} \end{cases}$$

converges to zero when c approaches $+\infty$ and goes to infinite, if $c \to -\infty$.

We finish the chapter on Section 2.5, where we bring techniques from Killing graphs, in addition to the estimates on the coefficients of the mean curvature operator obtained on Section 2.4, to generalize an argument of A. Menezes [50] to any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, and obtain the existence of minimal $\pi-$graphs which are similar to the fundamental piece of the doubly periodic Scherk surface of $\mathbb{R}^3$, on Theorem 2.5.1.

2.2 Semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$

This section is to give a brief review on semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$. We follow the notation and construction of W. Meeks and J. Pérez, [44].

Let H, V be two groups and let $\varphi : V \to \mathrm{Aut}(H)$ a group homomorphism between V and the group of automorphisms of H. Then, the *semidirect product between* H *and* V *with respect to* φ, denoted by $G = H \rtimes_\varphi V$, is the cartesian product $H \times V$ endowed with the group operation $* : G \times G \to G$ given by

$$(h_1,\, v_1) * (h_2,\, v_2) = (h_1 \cdot \varphi_{v_1}(h_2),\, v_1 v_2).$$

With this group operation, then both H and V can be viewed as subgroups of G and moreover, $H \lhd G$ is a normal subgroup of G. This construction comes to generalize the notion of direct product of groups, where the operation on the cartesian product $H \times V$ would be the product operation $(h_1,\, v_1) * (h_2,\, v_2) = (h_1 h_2,\, v_1 v_2)$. It is clear that this notion can be recovered from the one of a semidirect product, when considering the automorphism $\varphi(v) = Id_H$ being the constant map into the identity of the group H.

Even on the particular case of $H = \mathbb{R}^2$ and $V = \mathbb{R}$ being two abelian groups, it possible to obtain a great variety of groups via the semidirect product of $\mathbb{R}^2$ and $\mathbb{R}$, depending uniquely on the choice of the (now 1-parameter) family of automorphisms of $\mathbb{R}^2$. Precisely, with the exceptions of $SU(2)$ (not diffeomorphic to $\mathbb{R}^3$) and $\widetilde{PSL}(2,\, \mathbb{R})$ (has no normal subgroup of dimension 2), it is possible to construct all three dimensional simply connected Lie groups using the following setting: fix a matrix $A \in M_2(\mathbb{R})$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.4}$$

and, for each $z \in \mathbb{R}$ we consider the automorphism of $\mathbb{R}^2$ generated by the exponential map of Az, $e^{Az} : \mathbb{R}^2 \to \mathbb{R}^2$. Then, we let

$$\begin{array}{rccc} \varphi : & \mathbb{R} & \to & \mathrm{Aut}(\mathbb{R}^2) \\ & z & \mapsto & e^{Az}, \end{array}$$

and we define $\mathbb{R}^2 \rtimes_A \mathbb{R} = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ as the semidirect product between $\mathbb{R}^2$ and $\mathbb{R}$ with respect to the automorphisms generated by e^{Az}. Explicitly, the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the set $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ endowed with the group operation $*$ given by

$$(x_1,\, y_1,\, z_1) * (x_2,\, y_2,\, z_2) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + e^{Az} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},\, z_1 + z_2 \right). \tag{2.5}$$

Using the properties of the exponential map, it is easy to see that if A and B are similar matrices (that is, $A = PBP^{-1}$ for some $P \in GL_2(\mathbb{R})$), then the group structure of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is isomorphic to the one of $\mathbb{R}^2 \rtimes_B \mathbb{R}$. On the next section we will introduce the canonical left invariant metric of a semidirect product, and a similar result is going to hold: if P as before is an orthogonal matrix, then $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{R}^2 \rtimes_B \mathbb{R}$ will also be isometric.

2.2.1 The canonical left invariant metric

Using the notation of [44], we denote the exponential map e^{Az} by

$$e^{Az} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}, \tag{2.6}$$

and observe that the vector fields defined by

$$E_1(x,y,z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y,\ E_2(x,y,z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y,\ E_3 = \partial_z \tag{2.7}$$

are left invariant and extend the canonical basis $\{\partial_x(0),\, \partial_y(0),\, \partial_z(0)\}$ at the origin of $\mathbb{R}^3$. Analogously, if we let

$$F_1 = \partial_x,\ F_2 = \partial_y,\ F_3(x,y,z) = (ax+by)\partial_x + (cx+dy)\partial_y + \partial_z, \tag{2.8}$$

they are right invariant vector fields (therefore they are Killing fields with respect to left invariant metrics) of $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

The metric to be considered on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the *canonical left invariant metric* (cf. [44]), that is the one given by stating that $\{E_1,\, E_2,\, E_3\}$ are unitary and orthogonal to each other everywhere. In particular, as it holds

$$\begin{aligned} \partial_x(x,\, y,\, z) &= a_{11}(-z)E_1 + a_{21}(-z)E_2 \\ \partial_y(x,\, y,\, z) &= a_{12}(-z)E_1 + a_{22}(-z)E_2, \end{aligned}$$

we can express the metric of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ in coordinates as

$$\begin{aligned} ds^2 &= \left[a_{11}(-z)^2 + a_{21}(-z)^2\right] dx^2 + \left[a_{12}(-z)^2 + a_{22}(-z)^2\right] dy^2 + dz^2 \\ &\quad + \left[a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)\right] (dx \otimes dy + dy \otimes dx). \end{aligned}$$

Now, we remark that $e^{-Az} = \left(e^{Az}\right)^{-1}$, so, as $\det(e^{Az}) = e^{z\,\mathrm{trace}(A)}$, we have

$$\begin{pmatrix} a_{11}(-z) & a_{12}(-z) \\ a_{21}(-z) & a_{22}(-z) \end{pmatrix} = e^{-z\,\mathrm{trace}(A)} \begin{pmatrix} a_{22}(z) & -a_{12}(z) \\ -a_{21}(z) & a_{11}(z) \end{pmatrix}.$$

Finally, we introduce the notation

$$\begin{array}{rclcl} Q_{11}(z) & = & \langle \partial_x,\, \partial_x \rangle & = & e^{-2z\,\mathrm{trace}(A)} \left[a_{21}(z)^2 + a_{22}(z)^2 \right] \\ Q_{22}(z) & = & \langle \partial_y,\, \partial_y \rangle & = & e^{-2z\,\mathrm{trace}(A)} \left[a_{11}(z)^2 + a_{12}(z)^2 \right] \\ Q_{12}(z) & = & \langle \partial_x,\, \partial_y \rangle & = & -e^{-2z\,\mathrm{trace}(A)} \left[a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z) \right] \end{array} \tag{2.9}$$

and obtain that the metric ds^2 is expressed by

$$ds^2 = Q_{11}(z)dx^2 + Q_{22}(z)dy^2 + dz^2 + Q_{12}(z)(dx \otimes dy + dy \otimes dx). \tag{2.10}$$

We notice that, if $A,\, B \in M_2(\mathbb{R})$ are two congruent matrices, on the sense that there is some orthogonal matrix $P \in O(2)$ such that $B = PAP^{-1}$, then the groups $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{R}^2 \rtimes_B \mathbb{R}$, endowed with the canonical left invariant metrics are *isomorphic and isometric*, and the map that makes the identification is a simple rotation on horizontal planes induced by P:

$$\begin{array}{rccl} \varphi : & \mathbb{R}^2 \rtimes_A \mathbb{R} & \to & \mathbb{R}^2 \rtimes_B \mathbb{R} \\ & (x, y, z) & \mapsto & (P(x, y), z). \end{array} \tag{2.11}$$

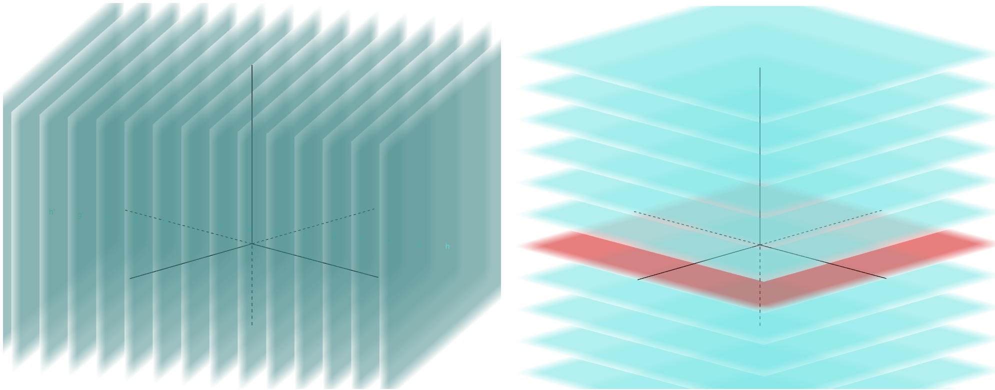
We also remark that the Lie brackets of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ satisfy

$$[E_1,\, E_2] = 0,\; [E_3,\, E_1] = aE_1 + cE_2,\; [E_3,\, E_2] = bE_1 + dE_2, \tag{2.12}$$

then Levi-Civita equation implies that the Riemannian connection of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is given by

$$\begin{array}{rcl|rcl|rcl} \nabla_{E_1} E_1 & = & aE_3 & \nabla_{E_1} E_2 & = & \frac{b+c}{2} E_3 & \nabla_{E_1} E_3 & = & -aE_1 - \frac{b+c}{2} E_2 \\ \nabla_{E_2} E_1 & = & \frac{b+c}{2} E_3 & \nabla_{E_2} E_2 & = & dE_3 & \nabla_{E_2} E_3 & = & -\frac{b+c}{2} E_1 - dE_2 \\ \nabla_{E_3} E_1 & = & \frac{c-b}{2} E_2 & \nabla_{E_3} E_2 & = & \frac{b-c}{2} E_1 & \nabla_{E_3} E_3 & = & 0. \end{array}$$

It is important to notice two properties of planes on $\mathbb{R}^2 \rtimes_A \mathbb{R}$: first, we observe that the metric ds^2 is invariant by rotations by angle π around the vertical lines $\{(x_0,\, y_0,\, z);\; z \in \mathbb{R}\}$, so vertical planes are minimal surfaces of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Moreover, horizontal planes $\{z = c\}$ have E_3 as an unitary normal vector field, so they have constant mean curvature (with respect to the upward orientation) given by $H = \mathrm{trace}(A)/2$. In particular, horizontal planes of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are minimal if and only if $\mathrm{trace}(A) = 0$.



(a) A foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by vertical (minimal) planes

(b) The foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by horizontal (CMC) planes

Figure 2.1: On semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$ every *vertical* plane is a minimal surface. Horizontal planes have constant mean curvature $H = \mathrm{trace}(A)/2$ and the subgroup $\mathbb{H} = \mathbb{R}^2 \rtimes_A \{0\}$ (highlighted on the figure above) is normal on $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

The difference between the cases $\mathrm{trace}(A) = 0$ and $\mathrm{trace}(A) \neq 0$ go further than horizontal planes being minimal: concerning the classification of simply connected Lie groups of dimension 3, we notice that W. Meeks and J. Pérez, [44] proved that any *non unimodular*[2] Lie group of dimension 3 is isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, endowed with its left invariant metric, where $A \in M_2(\mathbb{R})$ is such that $\mathrm{trace}(A) \neq 0$ (Lemma 2.11, [44]). Moreover, they also prove that, with the exceptions of $SU(2)$, which is not diffeomorphic to $\mathbb{R}^3$, and $\widetilde{PSL}(2, \mathbb{R})$, which has no normal subgroup of dimension 2, all other unimodular metric Lie groups are isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, with $\mathrm{trace}(A) = 0$ (Section 2.6 and Theorem 2.15, [44]). Herein, we shall make references to the cases $\mathrm{trace}(A) = 0$ or $\mathrm{trace}(A) \neq 0$ respectively as the *unimodular* and *non unimodular* case.

Let us give some examples of semidirect products in order to illustrate the manifold of groups that are isometric and isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric. The main examples are $\mathbb{R}^3$, Sol_3, Nil_3, $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$, which we resume on the table below:

$\mathbb{R}^2 \rtimes_A \mathbb{R}$	$\mathbb{R}^3$	Sol_3	Nil_3	$\mathbb{H}^3$	$\mathbb{H}^2 \times \mathbb{R}$
A	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

[2] A group G is said to be unimodular if $\det(\mathrm{Ad}_g) = 1$ for all $g \in G$

2.3 The mean curvature equation

On this section, we consider a smooth open domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ and a function $u : \Omega \to \mathbb{R}$. We define the $\pi-$graph[3] of u by

$$\Sigma = \text{graph}(u) = \{(x,\, y,\, u(x,y)) \in \mathbb{R}^2 \rtimes_A \mathbb{R};\, (x,\, y,\, 0) \in \Omega\},$$

and deduce the partial differential equation that u must satisfy for Σ to be a minimal surface (Theorem 2.3.2). This equation is a quasilinear elliptic PDE, and it will be used on Section 2.4, together with some techniques from [32], to obtain the main results of this chapter.

We begin with some preliminary calculations with the left invariant vectors E_i. We extend u to a function $\widetilde{u} : \Omega \rtimes_A \mathbb{R} \to \mathbb{R}$ given by $\widetilde{u}(x,y,z) = u(x,y,0)$. In order to ease the notation on next proposition, we let

$$u_i = E_i(\widetilde{u}),\, u_{ij} = E_j(E_i(\widetilde{u})).$$

Then, we orient Σ with respect to the unitary normal vector field η pointing upwards given by

$$\eta = \frac{E_3 - \text{grad}(\widetilde{u})}{\sqrt{1+u_1^2+u_2^2}}, \tag{2.13}$$

and on this setting, we prove the following:

Proposition 2.3.1. *Let $A \in M_2(\mathbb{R})$ be given by* (2.4) *and let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be a semidirect product with its canonical metric. Then if $u : \Omega \to \mathbb{R}$ is a smooth function and Σ is its $\pi-$graph, oriented with respect to η as in* (2.13)*, its mean curvature H_Σ is given by*

$$\begin{aligned} H_\Sigma \;=\; \frac{1}{2W^3}\Big[& u_{11}(1+u_2^2) + u_{22}(1+u_1^2) - 2u_{12}u_1u_2 \\ &+ (2a+d)u_1^2 + (a+2d)u_2^2 + (b+c)u_1u_2 + \text{trace}(A)\Big], \end{aligned} \tag{2.14}$$

where

$$W = \sqrt{1+u_1^2+u_2^2}.$$

[3]The nomenclature of $\pi-$graphs comes from the projection $\pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$, $\pi(x,y,z) = (x,y,0)$, and is used on this work in order to avoid any conflict with the notion of Killing graphs, that are studied on Section 2.5.

Proof. First, we remark that $\{E_1,\ E_2\ E_3\}$ are such that $\mathrm{Div}(E_1) = \mathrm{Div}(E_2) = 0$ and $\mathrm{Div}(E_3) = -\operatorname{trace}(A)$, thus

$$\begin{aligned}\mathrm{Div}(\eta) &= \mathrm{Div}\left(\frac{1}{W}E_3\right) - \mathrm{Div}\left(\frac{u_2}{W}E_2\right) - \mathrm{Div}\left(\frac{u_1}{W}E_1\right)\\ &= E_3\left(\frac{1}{W}\right) - \frac{1}{W}\operatorname{trace}(A) - E_2\left(\frac{u_2}{W}\right) - E_1\left(\frac{u_1}{W}\right).\end{aligned}$$

Now, it is just a simple computation. We begin with, for $i = 1,\ 2$

$$\begin{aligned}E_i\left(\frac{u_i}{W}\right) &= \frac{1}{W^2}\Big[u_{ii}W - u_iE_i(W)\Big] \qquad (2.15)\\ &= \frac{1}{W^3}\Big[u_{ii}(1+u_1^2+u_2^2) - u_i(u_{1i}u_1 + u_{2i}u_2)\Big].\end{aligned}$$

For $i = 3$, although $\widetilde{u}$ does not depend on the $z-$coordinate, its derivatives do. Precisely, it follows from the Lie brackets (2.12) that $u_{13} = au_1 + cu_2$ and $u_{23} = bu_1 + du_2$, so we have that

$$\begin{aligned}E_3\left(\frac{1}{W}\right) = -\frac{1}{2W^3}E_3\left(W^2\right) &= -\frac{1}{W^3}\Big[u_{13}u_1 + u_{23}u_2\Big] \qquad (2.16)\\ &= -\frac{1}{W^3}\Big[au_1^2 + (b+c)u_1u_2 + du_2^2\Big].\end{aligned}$$

From (2.15) and (2.16), it follows that

$$\begin{aligned}2H = -\mathrm{Div}(\eta) &= -E_3\left(\frac{1}{W}\right) + \frac{1}{W}\operatorname{trace}(A) + E_2\left(\frac{u_2}{W}\right) + E_1\left(\frac{u_1}{W}\right)\\ &= \frac{1}{W^3}\Big[u_{11}(1+u_2^2) - 2u_{12}u_1u_2 + u_{22}(1+u_1^2)\\ &\quad + (2a+d)u_1^2 + (b+c)u_1u_2 + (a+2d)u_2^2 + \operatorname{trace}(A)\Big]\end{aligned}$$

□

Now, we write each E_i in coordinates using (2.7) to obtain the main equation:

Theorem 2.3.2. *Let $A \in M_2(\mathbb{R})$ be a matrix given by* (2.4) *and let X be the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric. Then, for $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ a smooth open domain, the $\pi-$graph of a function $u : \Omega \to \mathbb{R}$, if oriented upwards, has mean curvature H given by*

$$H = \frac{e^{2u\,\mathrm{trace}(A)}}{2W^3}\Bigg[u_{xx}\left(Q_{22}(u)+u_y^2\right)+u_{yy}\left(Q_{11}(u)+u_x^2\right)-2u_{xy}\left(Q_{12}(u)+u_xu_y\right) + G_1(u)u_x^2+G_2(u)u_y^2+G_3(u)u_xu_y+(a+d)e^{-2u\,\mathrm{trace}(A)}\Bigg],$$

where Q_{ij} are defined as on (2.9)*, $G_i : \mathbb{R} \to \mathbb{R}$ are the functions given by*

$$\begin{aligned} G_1(z) &= e^{-2z\,\mathrm{trace}(A)}\big((2a+d)a_{11}(z)^2+(a+2d)a_{12}(z)^2+(b+c)a_{11}(z)a_{12}(z)\big)\\ G_2(z) &= e^{-2z\,\mathrm{trace}(A)}\big((2a+d)a_{21}(z)^2+(a+2d)a_{22}(z)^2+(b+c)a_{21}(z)a_{22}(z)\big)\\ G_3(z) &= e^{-2z\,\mathrm{trace}(A)}\big((4a+2d)a_{11}(z)a_{21}(z)+(2a+4d)a_{12}(z)a_{22}(z)\\ &\qquad +(b+c)(a_{11}(z)a_{22}(z)+a_{12}(z)a_{21}(z))\big), \end{aligned} \tag{2.17}$$

and W is

$$\begin{aligned} W(z,p) &= \sqrt{1+(a_{11}(z)p_1+a_{21}(z)p_2)^2+(a_{12}(z)p_1+a_{22}(z)p_2)^2}\\ &= \sqrt{1+e^{2z\,\mathrm{trace}(A)}\left(Q_{22}(z)p_1^2-2Q_{12}(z)p_1p_2+Q_{11}(z)p_2^2\right)}. \end{aligned}$$

Proof. From (2.7), it follows that

$$\begin{aligned} u_1 &= a_{11}u_x+a_{21}u_y\\ u_2 &= a_{12}u_x+a_{22}u_y\\ u_{11} &= a_{11}^2u_{xx}+2a_{11}a_{21}u_{xy}+a_{21}^2u_{yy}\\ u_{22} &= a_{12}^2u_{xx}+2a_{12}a_{22}u_{xy}+a_{22}^2u_{yy}\\ u_{12} &= a_{11}a_{12}u_{xx}+(a_{11}a_{22}+a_{12}a_{21})u_{xy}+a_{22}a_{21}u_{yy} \end{aligned} \tag{2.18}$$

where we are denoting $a_{ij} = a_{ij}(z)$, and $z = u(x, y)$. Now we divide (2.14) in first and second order terms, by setting

$$H_f = (2a+d)u_1^2+(a+2d)u_2^2+(b+c)u_1u_2+\mathrm{trace}(A), \tag{2.19}$$
$$H_s = u_{11}(1+u_2^2)+u_{22}(1+u_1^2)-2u_{12}u_1u_2 \tag{2.20}$$

thus $H = \frac{1}{2W^3}(H_s + H_f)$. Now using (2.18) on (2.20) we obtain directly that

$$\begin{aligned} H_s &= u_{xx}\left(a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})^2 u_y^2\right) \\ &\quad + u_{yy}\left(a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})^2 u_x^2\right) \\ &\quad + 2u_{xy}\left(a_{11}a_{21} + a_{12}a_{22} - (a_{11}a_{22} - a_{12}a_{21})^2 u_x u_y\right), \end{aligned}$$

and now we just point out that $a_{11}(z)a_{22}(z) - a_{12}(z)a_{21}(z) = \det\left(e^{Az}\right) = e^{z\,\mathrm{trace}(A)}$, then

$$\begin{aligned} H_s &= e^{2u\,\mathrm{trace}(A)} u_{xx}\left(e^{-2u\,\mathrm{trace}(A)}(a_{11}^2 + a_{12}^2) + u_y^2\right) \\ &\quad + e^{2u\,\mathrm{trace}(A)} u_{yy}\left(e^{-2u\,\mathrm{trace}(A)}(a_{21}^2 + a_{22}^2) + u_x^2\right) \\ &\quad + e^{2u\,\mathrm{trace}(A)} 2u_{xy}\left(e^{-2u\,\mathrm{trace}(A)}(a_{11}a_{21} + a_{12}a_{22}) - u_x u_y\right), \end{aligned}$$

and we can use the functions Q_{ij} of (2.9) to obtain that

$$H_s = e^{2u\,\mathrm{trace}(A)}\left(u_{xx}\left(Q_{22}(u) + u_y^2\right) + u_{yy}\left(Q_{11}(u) + u_x^2\right) - 2u_{xy}\left(Q_{12}(u) + u_x u_y\right)\right). \tag{2.21}$$

Now, to complete the proof, simply use the expression (2.18) on (2.19) to obtain the first order terms

$$\begin{aligned} H_f &= (2a + d)(a_{11}u_x + a_{21}u_y)^2 + (a + 2d)(a_{12}u_x + a_{22}u_y)^2 \\ &\quad + (b + c)(a_{11}u_x + a_{21}u_y)(a_{12}u_x + a_{22}u_y) + \mathrm{trace}(A) \\ &= \left((2a + d)a_{11}^2 + (a + 2d)a_{12}^2 + (b + c)a_{11}a_{12}\right) u_x^2 \\ &\quad + \left((2a + d)a_{21}^2 + (a + 2d)a_{22}^2 + (b + c)a_{21}a_{22}\right) u_y^2 \\ &\quad + \left((4a + 2d)a_{11}a_{21} + (2a + 4d)a_{12}a_{22} + (b + c)(a_{11}a_{22} + a_{12}a_{21})\right) u_x u_y \\ &\quad + \mathrm{trace}(A). \end{aligned}$$

By defining G_i as in (2.17), we find from last equation that

$$H_f = e^{2z\,\mathrm{trace}(A)}\left(G_1(u)u_x^2 + G_2(u)u_y^2 + G_3(u)u_x u_y + (a + d)e^{-2z\,\mathrm{trace}(A)}\right),$$

obtaining the promised expression for the mean curvature of Σ. □

With the mean curvature equation of last theorem, we finally define the *mean curvature operator*:

$$Q(u) = u_{xx}\left(Q_{22}(u) + u_y^2\right) + u_{yy}\left(Q_{11}(u) + u_x^2\right) + 2u_{xy}\left(Q_{12}(u) - u_x u_y\right)$$
$$+G_1(u)u_x^2 + G_2(u)u_y^2 + G_3(u)u_x u_y + (a+d)e^{-2u\,\mathrm{trace}(A)}, \tag{2.22}$$

and we notice that Q is a quasilinear elliptic operator, as the matrix

$$\mathcal{Q} = \begin{pmatrix} Q_{22}(z) + p_2^2 & -Q_{12}(z) - p_1 p_2 \\ -Q_{12}(z) - p_1 p_2 & Q_{11}(z) + p_1^2 \end{pmatrix} \tag{2.23}$$

is positive definite for every $z \in \mathbb{R}$ and $p = (p_1,\, p_2) \in \mathbb{R}^2$, as it is easy to see using the relation

$$Q_{11}(z)Q_{22}(z) - Q_{12}(z)^2 = e^{-2z\,\mathrm{trace}(A)}.$$

We would like to remark that the minimal graphs equation $Q(u) = 0$ admits a divergence form:

Proposition 2.3.3. *Let Q be the quasilinear elliptic operator given by* (2.22)*. Then the equation $Q(u) = 0$ admits an equivalent divergence form*

$$\frac{\partial}{\partial x}\bigg(A_1(u,\, Du)\bigg) + \frac{\partial}{\partial y}\bigg(A_2(u,\, Du)\bigg) + B(u, Du) = 0, \tag{2.24}$$

where $Du = (u_x,\, u_y)$ is the Euclidean gradient of u and A_1, A_2 are given by

$$A_1(z,p) = \frac{Q_{22}(z)p_1 - Q_{12}(z)p_2}{W(z,p)},$$

$$A_2(z,p) = \frac{-Q_{12}(z)p_1 + Q_{11}(z)p_2}{W(z,p)},$$

and W is

$$W(z,p) = e^{-2z\,\mathrm{trace}(A)}\sqrt{1 + e^{2z\,\mathrm{trace}(A)}(Q_{22}(z)p_1^2 - 2Q_{12}(z)p_1p_2 + Q_{11}(z)p_2^2)}.$$

2.4 Height estimates and lack of height estimates

On this section, we study some properties of minimal graphs on semidirect products. The main problem we would like to solve on this subject is the uniqueness of minimal graphs given a prescribed boundary:

Problem 1. *Let $A \in M_2(\mathbb{R})$ be a matrix and consider the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical left invariant metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$, be a bounded, convex and smooth domain and let Γ be a simple closed curve such that the projection $\pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ monotonically parametrizes*[4] *$\partial\Omega$. Show that there is an* ***unique*** *function $u : \Omega \to \mathbb{R}$ such that $\Sigma =$ graph(u) is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\partial\Sigma = \Gamma$.*

The question concerning the *existence* of a function u as on Problem 1 was already solved by W. Meeks, P. Mira, J. Pérez and A. Ros on [43]. Using the existence of many foliations of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by vertical planes, which are minimal, they solve the Plateau's problem and prove the solution is a graph over Ω. Precisely, they prove:

Theorem 2.4.1 (Theorem 15.1, [43])**.** *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product with its canonical metric and let $\pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ denote the projection $\pi(x, y, z) = (x, y, 0)$. Suppose Ω is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$, $C = \partial\Omega$ and $\Gamma \subseteq \pi^{-1}(C)$ a continuous simple closed curve such that $\pi : \Gamma \to C$ monotonically parametrizes C. Then:*

(1) Γ is the boundary of a compact embedded disk D of finite least area.

(2) The interior of D is a smooth $\pi-$ graph over the interior of Ω.

However, the *uniqueness* part of Problem 1 still remains open, even for some simple cases. For instance, if trace$(A) \neq 0$, even on the case of u being constant along the boundary the uniqueness is not known, although it is expected.

We remark that an approach for proving uniqueness that could be considered is to prove that every minimal graph is *stable*, and then it would follow that every minimal graph is *strictly stable*, from where we could obtain uniqueness with the usual techniques. However, proving stability of a minimal graph is not a trivial matter on this ambient space, as there is no *good candidate* for a Jacobi function (there is no ambient Killing field Y with the guarantee that the function $\langle Y, \eta\rangle$ has a sign, η being the normal unit field to the graph).

Some work has been developed in order to understand minimal graphs on semidirect products. We notice that the fact that $\mathbb{R}^2 \rtimes_A \mathbb{R}$ admits a foliation by parallel horizontal planes of constant mean curvature $H = \text{trace}(A)/2$ determines much of the structure of those graphs. For instance, using this property and the mean curvature comparison principle, Meeks, Mira, Pérez and Ros prove that

[4] This means that $\pi(\Gamma) \subset \partial\Omega$ and $\pi^{-1}(\{p\}) \cap \Gamma$ is either a single point or a compact interval for every $p \in \partial\Omega$.

Lemma 2.4.2 (Assertion 15.5, [43]). *Let $D \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a convex compact disk and let $C = \partial D$ be its boundary. Consider $\pi(x, y, z) = (x, y, 0)$ the vertical projection. If $\Gamma \subseteq \pi^{-1}(C)$ is a closed simple curve such that the projection $\pi : \Gamma \to C$ monotonically parameterizes C and $h : \Gamma \to \mathbb{R}$ is the height function, let $c_0 = \inf_\Gamma h$ and $c_1 = \sup_\Gamma h$. If Σ is a compact minimal surface with $\partial\Sigma = \Gamma$, it follows:*

1. *If* $\operatorname{trace}(A) \geq 0$, *then* $\Sigma \subseteq \pi^{-1}(D) \cap \{z \geq c_0\}$;
2. *If* $\operatorname{trace}(A) \leq 0$, *then* $\Sigma \subseteq \pi^{-1}(D) \cap \{z \leq c_1\}$.

On the particular case of graphs, Lemma 2.4.2 implies that a minimal graph over some smooth domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$, compact and convex, satisfy the maximum principle if $\operatorname{trace}(A) \leq 0$ and satisfy the minimum principle if $\operatorname{trace}(A) \geq 0$, satisfying both only on the unimodular case. However, when $\operatorname{trace}(A) > 0$ no uniform upper bound is obtained, neither a lower bound when $\operatorname{trace}(A) < 0$. This motivates the search for height estimates for minimal graphs, obtained on the next result. Perhaps, the proof of Theorem 2.4.3 is as important as the result itself, as it gives some light on the behaviour of the operator Q, given by (2.22), on the many possible settings for the matrix A. Such properties will be used on the proof of Theorem 2.4.6, and also on Section 2.5 to obtain the existence of minimal Killing graphs that converge to the Scherk-like fundamental piece of Theorem 2.5.1.

Theorem 2.4.3. *Let $A \in M_2(\mathbb{R})$ be a matrix as in (2.4) and let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be a semidirect product endowed with its canonical left invariant metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a bounded, convex domain and let $\alpha \in \mathbb{R}$ be any given constant. Then, there exists a constant $C = C(diam(\Omega), \alpha)$ such that for every u satisfying*

$$\begin{cases} Q(u) \geq 0 \text{ in } \Omega \\ u \leq \alpha \text{ on } \partial\Omega, \end{cases}$$

it holds that

$$\sup_\Omega u \leq \alpha + C(\alpha). \tag{2.25}$$

The proof of Theorem 2.4.3 uses techniques from quasilinear elliptic partial differential equations, mainly the comparison principle. However, the operator Q given on by (2.22) does not satisfy the hypothesis of the comparison principle (for instance Theorem 10.1 of [32] and its generalizations therein), so we define a quasilinear operator R related to Q, for which it holds the comparison principle. Then, we find an *ad hoc* positive function

$v : \Omega \to \mathbb{R}$, whose construction will depend only on Ω and α such that $R(v+\alpha) \leq R(u)$, and our constant C will be simply given by $C = \sup_\Omega v$.

Proof of Theorem 2.4.3. First, we notice that when $\mathrm{trace}(A) \leq 0$, the result is trivial with $C = 0$ and without the need for an α, by Lemma 2.4.2. Thus we will suppose that $\mathrm{trace}(A) > 0$ and focus on the non unimodular case. Without loss of generality, after a homothety of the metric we may assume that $\mathrm{trace}(A) = 2$ and that A is given by

$$A = \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix}, \tag{2.26}$$

for some a, b, $c \in \mathbb{R}$. Now, we will divide the proof in two cases, starting when A is not a diagonal matrix:

Case 1. First, we suppose that A is not a diagonal matrix. We begin by proving the following key claim:

Claim 2.1. *Let the functions Q_{ij} be the ones defined on* (2.9) *with respect to the matrix A of* (2.26)*, where either $b \neq 0$ or $c \neq 0$. Then, there is some $\lambda > 0$ such that at least one of the following holds, for every $z \in \mathbb{R}$:*

i. $Q_{22}(z)e^{2z} > \lambda$;

ii. $Q_{11}(z)e^{2z} > \lambda$.

Moreover, if $a^2 + bc \leq 0$, both i. and ii. hold, and if $a^2 + bc > 0$, then $b \neq 0$ is equivalent to i. and $c \neq 0$ is equivalent to ii.

Proof of Claim 2.1. We prove Claim 2.1 in each of three (family of) possibilities to the exponential of A. First, we write $A = I + A_0$, where I is the identity matrix and A_0 is the traceless part of A given by

$$A_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

As I and A_0 commute, we obtain that $e^{Az} = e^{Iz + A_0 z} = e^{Iz}e^{A_0 z}$, thus

$$e^{Az} = e^z \begin{pmatrix} a^0_{11}(z) & a^0_{12}(z) \\ a^0_{21}(z) & a^0_{22}(z) \end{pmatrix},$$

where we denote by $a^0_{ij}(z)$ the coefficients of the exponential $e^{A_0 z}$. In particular, we obtain that $a_{ij}(z) = e^z a^0_{ij}(z)$, and it follows that

$$Q_{11}(z)e^{2z} = e^{-4z}\left[a_{21}(z)^2 + a_{22}(z)^2\right]e^{2z} = a^0_{21}(z)^2 + a^0_{22}(z)^2,$$

and analogously

$$Q_{22}(z)e^{2z} = a^0_{11}(z)^2 + a^0_{12}(z)^2.$$

Now, we just observe that the characteristic equation of A_0 is given by $0 = \det(A_0 - tI) = t^2 - (a^2 + bc)$, so if we denote by $d = \sqrt{|a^2 + bc|}$, the exponential of A_0 is given by[5]

$$e^{A_0 z} = \begin{pmatrix} \cos(dz) + \frac{a}{d}\sin(dz) & \frac{b}{d}\sin(dz) \\ \frac{c}{d}\sin(dz) & \cos(dz) - \frac{a}{d}\sin(dz) \end{pmatrix}, \text{ when } a^2 + bc < 0, \tag{2.27}$$

$$e^{A_0 z} = \begin{pmatrix} 1 + az & bz \\ cz & 1 - az \end{pmatrix}, \text{ when } a^2 + bc = 0, \tag{2.28}$$

$$e^{A_0 z} = \begin{pmatrix} \cosh(dz) + \frac{a}{d}\sinh(dz) & \frac{b}{d}\sinh(dz) \\ \frac{c}{d}\sinh(dz) & \cosh(dz) - \frac{a}{d}\sinh(dz) \end{pmatrix}, \text{ when } a^2 + bc > 0. \tag{2.29}$$

Now we let $f(z) = a^0_{11}(z)^2 + a^0_{12}(z)^2$ and $g(z) = a^0_{21}(z)^2 + a^0_{22}(z)^2$ and prove that there is some $\lambda > 0$ such that either $f(z) > \lambda$ or $g(z) > \lambda$. First, we notice that both $f(z)$ and $g(z)$ are always positive, as the existence of some $z_0 \in \mathbb{R}$ such that $f(z_0) = 0$ or $g(z_0) = 0$ would imply that $\det(e^{A_0 z_0}) = 0$, an absurdity. Then, we just need to check the behaviour of f and g at $\pm\infty$.

First, if $a^2 + bc < 0$, the existence of λ as claimed follows directly from the fact that both f and g are periodic and positive, by (2.27). If $a^2 + bc = 0$, then we have that f and g are given by

$$\begin{aligned} f(z) &= (1 + az)^2 + (bz)^2 = (a^2 + b^2)z^2 + 2az + 1 \\ g(z) &= (1 - az)^2 + (cz)^2 = (a^2 + c^2)z^2 - 2az + 1, \end{aligned}$$

both strictly positive at infinity for any choice of a, b, c, so we also have the existence of λ on this case. Finally, if $a^2 + bc > 0$, f and g would be given by

$$\begin{aligned} f(z) &= \Big(\cosh(dz) + \frac{a}{d}\sinh(dz)\Big)^2 + \Big(\frac{b}{d}\sinh(dz)\Big)^2 \\ g(z) &= \Big(\cosh(dz) - \frac{a}{d}\sinh(dz)\Big)^2 + \Big(\frac{c}{d}\sinh(dz)\Big)^2. \end{aligned}$$

[5]We remark that the constant $a^2 + bc$ is linked with the Milnor $D-$invariant of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, which is (following [44]) defined by $D = \det(A) = 1 - (a^2 + bc)$. So each case $a^2 + bc > 0$, $a^2 + bc = 0$ and $a^2 + bc < 0$ is in correspondence with $D < 1$, $D = 1$ and $D > 1$, respectively.

If $i.$ was not true, either $\lim_{z\to-\infty} f(z) = 0$ or $\lim_{z\to+\infty} f(z) = 0$, so it would follow that $b = 0$. Analogously if $\lim_{z\to-\infty} g(z) = 0$ or $\lim_{z\to+\infty} g(z) = 0$, we would have $c = 0$. As A is not a diagonal matrix, at least one between $i.$ and $ii.$ is true, finishing the proof of the claim. $\diamondsuit$

Now we proceed with the proof of the theorem by proving the existence of $\Lambda > 0$ such that $G_1(z) \le \Lambda Q_{22}(z)$ and $G_2(z) \le \Lambda Q_{11}(z)$. Just observe that, by definition,

$$\begin{aligned}\frac{G_1(z)}{Q_{22}(z)} &= \frac{e^{-4z}\left[(3+a)a_{11}(z)^2 + (3-a)a_{12}(z)^2 + (b+c)a_{11}(z)a_{12}(z)\right]}{e^{-4z}\left[a_{11}(z)^2 + a_{12}(z)^2\right]}\\ &= 3 + a\frac{a_{11}(z)^2 - a_{12}(z)^2}{a_{11}(z)^2 + a_{12}(z)^2} + (b+c)\frac{a_{11}(z)a_{12}(z)}{a_{11}(z)^2 + a_{12}(z)^2}\\ &\le 3 + |a| + \frac{|b+c|}{2} = \Lambda, \end{aligned} \tag{2.30}$$

and, mutatis mutandis, the same estimate holds for the quotient $G_2(z)/Q_{11}(z)$.

Using the existence of λ and Λ, we can finish the proof of the theorem on this first case. First, we assume that i. holds and let $u : \Omega \to \mathbb{R}$ be any function that satisfy $Q(u) \ge 0$. Then we define the quasilinear elliptic operator R as

$$\begin{aligned}R(w) = w_{xx}\left(\frac{Q_{22}(u) + w_y^2}{Q_{22}(u)}\right) + w_{yy}\left(\frac{Q_{11}(u) + w_x^2}{Q_{22}(u)}\right) + 2w_{xy}\left(\frac{Q_{12}(u) - w_x w_y}{Q_{22}(u)}\right)\\ + \frac{G_1(u)}{Q_{22}(u)}w_x^2 + \frac{G_2(u)}{Q_{22}(u)}w_y^2 + \frac{G_3(u)}{Q_{22}(u)}w_x w_y + 2\frac{e^{-2u}}{Q_{22}(u)}e^{-2w}, \end{aligned} \tag{2.31}$$

and notice that $R(u) = Q(u)/Q_{22}(u) \ge 0$.

Now, as Ω is a bounded domain, after a horizontal translation (which is an isometry of the ambient space) we may suppose without loss of generality that it is contained in a strip

$$\Omega \subseteq \{(x, y, 0) \in \mathbb{R}^2 \rtimes_A \mathbb{R};\ 1 < x < M\},$$

for some $M > 1$. We let $v(x, y) = \ln(lx)/L$, where l, $L > 0$ are constants yet to be defined. Then, if $\alpha \in \mathbb{R}$ is any a priori chosen number, we have that

$$\begin{aligned}R(v+\alpha) &= v_{xx} + \frac{G_1(u)}{Q_{22}(u)}v_x^2 + 2\frac{e^{-2u}}{Q_{22}(u)}e^{-2(v+\alpha)}\\ &< v_{xx} + \Lambda v_x^2 + \frac{2}{\lambda}e^{-2v}e^{-2\alpha}.\end{aligned}$$

Then, using that $v_x = \frac{1}{Lx}$ and $v_{xx} = \frac{-1}{Lx^2}$, we obtain

$$\begin{aligned} R(v+\alpha) &< -\frac{1}{Lx^2} + \Lambda\frac{1}{L^2x^2} + \frac{2}{\lambda e^{2\alpha}}(lx)^{-2/L} \\ &= \frac{1}{Lx^2}\left[-1+\frac{\Lambda}{L}+\frac{2L}{\lambda e^{2\alpha} l^{2/L}}x^{(2L-2)/L}\right]. \end{aligned} \tag{2.32}$$

Now, take $L = 1+\Lambda$. As $1 < x < M$, follows that

$$R(v+\alpha) < \frac{1}{(1+\Lambda)x^2}\left[-\frac{1}{1+\Lambda} + 2\frac{1+\Lambda}{\lambda e^{2\alpha} l^{\frac{2}{1+\Lambda}}} M^{\frac{2\Lambda}{1+\Lambda}}\right], \tag{2.33}$$

and then we just choose l big enough (in particular we may assume $l \geq 1$, so $v > 0$) such that

$$-\frac{1}{1+\Lambda} + 2\frac{1+\Lambda}{\lambda e^{2\alpha} l^{\frac{2}{1+\Lambda}}} M^{\frac{2\Lambda}{1+\Lambda}} < 0, \tag{2.34}$$

so $R(v+\alpha) < 0$. We remark that the choice of l and L as above depends only on λ, Λ, α and M, but not on u. Now, assume $u \leq \alpha$ on $\partial\Omega$ and let

$$v_0 = v + \alpha.$$

It follows from the definition of R (and from the fact that $v_0 - v = \alpha$ is constant) that

$$R(v_0) = R(v+\alpha) < 0 \leq R(u),$$

then, as R satisfies the hypothesis of the comparison principle (Theorem 10.1 of [32]) and $u \leq v_0$ on $\partial\Omega$, follows that $\sup_\Omega u \leq \sup_\Omega v_0$. Finally, we set $C = \sup_\Omega v$, and the theorem follows when A is not diagonal and i. holds.

If i. was not true, then we assume ii. and define

$$\begin{aligned} R(w) = {} & w_{xx}\left(\frac{Q_{22}(u)+w_y^2}{Q_{11}(u)}\right) + w_{yy}\left(\frac{Q_{11}(u)+w_x^2}{Q_{11}(u)}\right) + 2w_{xy}\left(\frac{Q_{12}(u)-w_xw_y}{Q_{11}(u)}\right) \\ & + \frac{G_1(u)}{Q_{11}(u)}w_x^2 + \frac{G_2(u)}{Q_{11}(u)}w_y^2 + \frac{G_3(u)}{Q_{11}(u)}w_xw_y + 2\frac{e^{-2u}}{Q_{11}(u)}e^{-2w}. \end{aligned} \tag{2.35}$$

From here, the proof follows analogously as above, by letting $v(x,y) = \ln(ly)/L$.

Case 2. Now, let us assume that A is a diagonal matrix (then neither i. nor ii. hold), and we set

$$A = \begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix},$$

then $a_{11}(z) = e^{(1+a)z}$, $a_{22}(z) = e^{(1-a)z}$ and $a_{12}(z) = a_{21}(z) = 0$, and the operator Q is given by

$$\begin{aligned} Q(u) &= u_{xx}\left(e^{-2(1-a)u} + u_y^2\right) + u_{yy}\left(e^{-2(1+a)u} + u_x^2\right) - 2u_{xy}\left(u_x u_y\right) \\ &\quad +(3+a)e^{-2(1-a)u}u_x^2 + (3-a)e^{-2(1+a)u}u_y^2 + 2e^{-4u}. \end{aligned}$$

If $a \geq 0$ we define R as the operator

$$\begin{aligned} R(w) = {} & w_{xx}\left(1 + e^{2(1-a)u}w_y^2\right) + w_{yy}\left(e^{-4au} + e^{2(1-a)z}u_x^2\right) - 2w_{xy}\left(e^{2(1-a)z}w_x w_y\right) \\ & +(3+a)w_x^2 + (3-a)e^{-4au}w_y^2 + 2e^{-2(1+a)w}, \end{aligned} \tag{2.36}$$

and, if $a < 0$, R is defined as

$$\begin{aligned} R(w) = {} & w_{xx}\left(e^{4au} + e^{2(1+a)u}w_y^2\right) + w_{yy}\left(1 + e^{2(1+a)u}w_x^2\right) - 2w_{xy}\left(e^{2(1+a)u}w_x w_y\right) \\ & +(3+a)e^{4au}w_x^2 + (3-a)w_y^2 + 2e^{-2(1-a)w}. \end{aligned} \tag{2.37}$$

Now, we just set v to be again $v(x,y) = \ln(lx)/L$ when $a \geq 0$ and $v(x,y) = \ln(ly)/L$ when $a < 0$ and, as the term on both operators which contains no derivative terms in decreasing on u, the proof follows as in the previous case, using $\Lambda = 3 + |a|$ and $\lambda = 1$. □

On the next theorem we prove that the dependence on α cannot be removed, on the sense that the existence of a constant which does not depend on α is not possible. Precisely, we prove:

Theorem 2.4.4. *Let A be a matrix as in* (2.26) *and let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be the non-unimodular semidirect product endowed with its canonical left invariant metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a bounded, convex domain. Then, for every constant $C > 0$ there exists some function $u : \Omega \to \mathbb{R}$ satisfying $Q(u) = 0$ and also*

$$\sup_{\Omega} u > \sup_{\partial\Omega} u + C. \tag{2.38}$$

The proof of Theorem 2.4.4 above is by contradiction and consists in using the vertical translation that rises from the group structure to translate a family of solutions tending to $-\infty$, all to height 0. We prove that if Theorem 2.4.4 was false, such family would be uniformly bounded, and this would generate a contradiction with the following theorem, due to Meeks, Mira, Pérez and Ros [43]:

Theorem 2.4.5 (Theorem 15.4, [43]). *Let X be a non-unimodular metric Lie group which is isomorphic and isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, $A \in M_2(\mathbb{R})$. Suppose that $\Gamma(n) \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ is a sequence of C^2 simple closed convex curves with $e = (0, 0, 0) \in \Gamma(n)$ such that the geodesic curvatures of $\Gamma(n)$ converge uniformly to 0 and the curves $\Gamma(n)$ converge on compact subsets to a line L with $e \in L$ as $n \to \infty$. Then, for any sequence $M(n)$ of compact branched minimal disks with $\partial M(n) = \Gamma(n)$, the surfaces $M(n)$ converge C^2 on compact subsets as $n \to \infty$ to the vertical half plane $\pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$.*

Proof of Theorem 2.4.4. We begin by proving the following claim:

Claim 2.2. *Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ be the unit circle centred on the origin of $\mathbb{R}^2$. Let $A \in M_2(\mathbb{R})$ be a matrix with* $\operatorname{trace}(A) = 2$*, as in (2.26), and let e^{Az} be its exponential map. Then there is a point $p \in \mathbb{S}^1$ and an increasing sequence $z_n \in (0, +\infty)$ such that $\Gamma_n = e^{Az_n}\big(\mathbb{S}^1 - p\big)$ satisfies the hypothesis of Theorem 2.4.5 at the origin, i.e., as $n \to +\infty$ the geodesic curvature of Γ_n at 0 converges to zero and Γ_n converges to a line L on compact sets, with $0 \in L$.*

Proof of Claim 2.2. We again denote by A_0 the traceless part of A and observe that $e^{Az} = e^z e^{A_0 z}$. Then we have that $e^{Az}\mathbb{S}^1 = e^z \left(e^{A_0 z}\mathbb{S}^1\right)$ is a homothety by e^z of the curve $e^{A_0 z}\mathbb{S}^1$. Now we let $d = \sqrt{|a^2 + bc|}$ and divide the proof on the three aforementioned cases given by equations (2.27), (2.28) and (2.29).

First, if $a^2 + bc < 0$, we let $p \in \mathbb{S}^1$ be any point and define $z_n = \frac{2n\pi}{d}$. Then $e^{A_0 z_n} = \mathrm{Id}$, so $e^{Az_n}\mathbb{S}^1$ is a circle of radius e^{2z_n} centred at the origin, and $\Gamma_n = e^{Az_n}(\mathbb{S}^1 - p)$ is a circle through the origin with radius e^{2z_n}. As $z_n \to \infty$, Γ_n will converge to a line L through 0 and the claim is proved on this case.

Secondly, if $a^2 + bc = 0$, then $e^{A_0 z}$ is given by (2.28) and $e^{A_0 z}\mathbb{S}^1$ is an ellipse and the homotheties of an ellipse by e^n admits a point where its geodesic curvature converges to zero and, after a translation, it converges to a line on compact sets, proving the claim on the second case.

Finally, if $a^2 + bc > 0$, $e^{A_0 z}$ is given by (2.29). If $bc \neq 0$, then $d \neq |a|$, and if z is big enough we have that $\cosh(dz) \simeq e^{dz}/2$ and $\sinh(dz) \simeq e^{dz}/2$, so

$$e^{A_0 z} \simeq \frac{e^{dz}}{2d} \left(\begin{array}{cc} d+a & b \\ c & d-a \end{array} \right),$$

and $e^{Az}\mathbb{S}^1$ is asymptotic to a homothety of $e^{(d+2)z}$ of an ellipse, which has the desired properties. The last case to be treated is when $d^2 = a^2 + bc = a^2 > 0$, then

$$e^{A_0 z} = \left(\begin{array}{cc} e^{dz} & \frac{b}{d}\sinh(dz) \\ \frac{c}{d}\sinh(dz) & e^{-dz} \end{array} \right) \simeq \frac{e^{dz}}{d} \left(\begin{array}{cc} d & b \\ c & de^{-2dz} \end{array} \right),$$

and, for z large enough it follows that $e^{A_0 z}\mathbb{S}^1$ is asymptotic to a line segment, with multiplicity 2. Now, it depends on the two possible cases $0 < d \leq 1$ or $d > 1$ to understand what is the convergence of $e^{Az}\mathbb{S}^1$: if $d \leq 1$, then the homothety of e^z on $e^{A_0 z}$ will open the segment and make it asymptotic to an ellipse again, which again admits a point p as claimed. If $d > 1$, then the action of e^z still makes $e^{Az}\mathbb{S}^1$ converge to a line. ♢

Now we continue the proof of Theorem 2.4.4, arguing by contradiction. Suppose that for some smooth bounded domain $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ there is $C > 0$ such that for every solution of $Q(u) = 0$, it holds that

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C. \tag{2.39}$$

In particular, the same estimate holds for any bounded, smooth domain contained in Ω. Now, let $r > 0$ be such that an euclidean ball B_r with radius r is contained on Ω. Let $\mathbb{S}^1(r) = \partial B_r$ be the circle that bounds B_r and let $p \in \mathbb{S}^1(r)$ and $(z_n)_{n\in\mathbb{N}}$ be the ones given on by Claim 2. We consider, for each $n \in \mathbb{N}$, the problem

$$\begin{cases} Q(u) = 0 & \text{in } B_r \\ u = -z_n & \text{on } \partial B_r. \end{cases} \tag{2.40}$$

Theorem 2.4.1 ([43]) implies that (2.40) has a solution $u_n : B_r \to \mathbb{R}$, and, from equation (2.39), follows that, for every $n \in \mathbb{N}$, the function u_n satisfies

$$\sup_{B_r} u_n \leq -z_n + C.$$

Now, we will translate the functions u_n vertically using the left translation of the group $L_{(0,0,z_n)}$. If $\Sigma_n = \mathrm{graph}(u_n)$, we notice that

$$\begin{aligned} L_{(0,0,z_n)}\Sigma_n &= \left\{ L_{(0,0,z_n)}\left(x,y,u_n(x,y)\right); \, (x,\, y) \in B_r \right\} \\ &= \left\{ \left(e^{Az_n} \begin{pmatrix} x \\ y \end{pmatrix}, u_n(x,y) + z_n \right), \, (x,\, y) \in B_r \right\} \\ &= \left\{ \left(\widetilde{x},\, \widetilde{y}, u_n \left(e^{-Az_n} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} \right) + z_n \right), \, (\widetilde{x},\, \widetilde{y}) \in e^{Az_n} B_r \right\}. \end{aligned}$$

If we let $v_n : e^{Az_n}B_r \to \mathbb{R}$ be the function given by

$$v_n(x,y) = u_n\left(e^{-Az_n}\begin{pmatrix} x \\ y \end{pmatrix}\right) + z_n,$$

it follows that the graph of v_n is a left translate of the graph of u_n, in particular it is a minimal graph $\overline{\Sigma}_n = L_{(0,0,z_n)}\Sigma_n$. But we notice that these graphs $\overline{\Sigma}_n$ satisfy the hypothesis of Theorem 2.4.5, thus they should converge, in compact sets, to a vertical half plane. However

$$\sup_{e^{Az_n}B_r} v_n = \sup_{B_r} u_n + z_n \leq C,$$

so the sequence v_n is uniformly bounded, generating a contradiction. □

We notice that, on last proof we showed more than the existence of *a* function u as on (2.38) for a fixed constant C. We actually proved that *any* sequence of functions with values along the boundary converging to $-\infty$ should have unlimited oscillation. We state this, together with a consequence of the proof of Theorem 2.4.3 on the following result.

Theorem 2.4.6. *Let $A \in M_2(\mathbb{R})$ and let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be the semidirect product endowed with its left invariant canonical metric. Let $\Omega \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be some open, bounded, smooth domain, $k \in \mathbb{Z}$ be given and let u_k denote a solution to the problem*

$$\begin{cases} Q(u) = 0 & in\ \Omega \\ u = k & on\ \partial\Omega. \end{cases} \qquad (P_k)$$

Then:

- *if* $\operatorname{trace}(A) = 0$*, $u_k \equiv k$ is the constant function;*
- *if* $\operatorname{trace}(A) > 0$*, $u_k > k$ in Ω, $\lim\limits_{k\to-\infty} osc_\Omega(u_k) = +\infty$ and $\lim\limits_{k\to+\infty} osc_\Omega(u_k) =$ 0;*

- *if* $\text{trace}(A) < 0$*,* $u_k < k$ *in* Ω*,* $\lim_{k\to-\infty} osc_\Omega(u_k) = 0$ *and* $\lim_{k\to+\infty} osc_\Omega(u_k) = +\infty$*,*

where $osc_\Omega(u) = \sup_\Omega(u) - \inf_\Omega(u)$ *denotes the oscillation of a function* u *in* Ω*.*

Proof. If $\text{trace}(A) = 0$, it is clear that $u_k \equiv k$ is the unique solution to (P_k), by Lemma 2.4.2. Also, as the change $A \mapsto -A$ corresponds to a simple change of orientation $z \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mapsto -z \in \mathbb{R}^2 \rtimes_{-A} \mathbb{R}$, we can simply prove the case of $\text{trace}(A) > 0$, and, as previous, it is without loss of generality that we assume that $\text{trace}(A) = 2$, so A is written as on (2.26).

From Lemma 2.4.2, it follows that $u_k \geq k$ on Ω, and, if at an interior point $x \in \Omega$ the function u_k attains its minimum $u_k(x) = k$, then the mean curvature comparison principle, applied to $\Sigma_k = \text{graph}(u_k)$ and to the plane $\{z = k\}$ will imply that the mean curvature of Σ_k is at least as big as the one of the plane, which is $1 > 0$, a contradiction that proves that, on the interior of Ω, it holds $u_k > k$.

The second part of the claim follows as on the proof of Theorem 2.4.4: if the oscillation of u_k was not going to $+\infty$ when $k \to -\infty$ then we could translate all the minimal surfaces $\Sigma_k = \text{graph}(u_k)$ to height zero and obtain a contradiction with Theorem 2.4.5.

In order to obtain the last part of the Theorem, that the oscillation of u_k goes to zero when k approaches $+\infty$, we recall the proof of Theorem 2.4.3: we obtained a constant C depending on many parameters, $C = C(l, L, \lambda, \Lambda, M, \alpha)$. However, the constants λ and Λ depend only on the ambient space, as they come from estimates of the coefficients of the operator Q. The constant M depends uniquely on the diameter of Ω, so it was fixed from the beginning, together with Ω. The free parameters we could work with were l and L, depending on the previous ones and on the a priori constant α. Using an appropriate choice of l and L, we obtained that the constant claimed on the Theorem was

$$C = \frac{\ln(lM)}{L}.$$

The key steps to chose l and L were between equations (2.32), (2.33) and (2.34), but the way we proceeded was thinking on the worst case, where the number α was a *negatively large* number, so we began by choosing L and then got to the definition of a l big enough, in order to compensate $e^{2\alpha}$, close to zero. Now, we are taking $\alpha_k = k$ to be *positive* and *very large*, so we follow a different approach. We begin letting $L = \Lambda + j$, where $j \in \mathbb{N}$ is yet to be chosen, and take immediately $l=1$, to obtain, similarly to (2.33), the inequality

$$R(v+k) < \frac{1}{(\Lambda+j)x^2}\left[-\frac{j}{\Lambda+j}+2\frac{\Lambda+j}{\lambda e^{2k}}M^{\left(2-\frac{2}{\Lambda+j}\right)}\right]. \tag{2.41}$$

Then, we proceed as before, and try to find some $j\in\mathbb{N}$ such that the right hand side of (2.41) becomes negative. Such j exists if and only if

$$\frac{(\Lambda+j)^2}{jM^{\frac{2}{\Lambda+j}}} < \frac{\lambda}{2M^2}e^{2k}. \tag{2.42}$$

If k is small, maybe there is no $j\in\mathbb{N}$ such that (2.42) holds, but for some $k_0\in\mathbb{N}$ big enough it is possible to find some $j\in\mathbb{N}$ satisfying (2.42) (therefore also (2.41)). As the right hand side grows with k, for every $k\geq k_0$ there will exist such j, and we denote $j(k)$ the *largest* $j\in\mathbb{N}$ such that (2.42) holds (as the left hand side is unbounded with j this is well defined). By taking $L=\Lambda+j(k)$, we use (2.41) to obtain the existence of a constant $C(k)=C(\Omega,k)$ given by

$$C(k)=\frac{\ln(M)}{\Lambda+j(k)}$$

such that $\sup_\Omega(u)\leq\max\{\sup_{\partial\Omega}u,k\}+C(k)$, for every $u:\Omega\to\mathbb{R}$ with $Q(u)\geq 0$, the same result as on Theorem 2.4.3 but for a different constant C, and only for $k\geq k_0$. It follows, in particular, that the functions u_k satisfy, for k large enough, that

$$\sup_\Omega u_k\leq k+C(k),$$

so, as $\inf_\Omega u_k=k$, we obtain that $\mathrm{osc}_\Omega(u_k)\leq C(k)$.

Finally, as the right hand side of (2.42) is unbounded with respect to k and the left hand side is also unbounded with j, follows that $\lim_{k\to\infty}j(k)=\infty$, so $C(k)\to 0$ when $k\to\infty$, and so it does the oscillation of u_k. □

We can apply the same argument as above for functions which are not constant along the boundary, to estimate the maximum height it can attain with respect to its maximum along the boundary:

Corollary 2.4.7. *Let* $\mathbb{R}^2\rtimes_A\mathbb{R}$ *be a non unimodular semidirect product with* $\mathrm{trace}(A)>0$. *Let, for* $L>0$, $u_L:\Omega\to\mathbb{R}$ *be a function satisfying*

$$\begin{cases} Q(u)\geq 0 \text{ on } \Omega,\\ \sup_{\partial\Omega}u=L.\end{cases}$$

Then

$$\lim_{L\to+\infty}\left(\sup_{\Omega} u_L - L\right) = 0. \tag{2.43}$$

2.5 Scherk-like fundamental pieces

On this section, we use the tools developed on this chapter together with some Killing graphs techniques to obtain an existence result of what we call *Scherk-like fundamental pieces*, which are minimal π-graphs on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ assuming the value 0 along a piecewise smooth curve $\gamma \subset \mathbb{R}^2 \rtimes_A \{0\}$ and having $\gamma \cup (\{p_1\} \times [0, \infty)) \cup (\{p_2\} \times [0, \infty))$ as boundary, where p_1 and p_2 are the endpoints of γ.

On the ambient space of an *unimodular* group $\mathbb{R}^2 \rtimes_A \mathbb{R}$ A. Menezes [50] proved the existence of *complete* (without boundary) minimal surfaces, similar to the singly and to the doubly periodic Scherk minimal surfaces of $\mathbb{R}^3$. We would like to take a moment to give the main steps of the proof of Menezes to the existence of a doubly periodic example:

Sketch of the proof of Theorem 2, [50]. Let $\Delta \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ be a triangle with vertexes

$$o = (0,\, 0,\, 0),\;\; p_1 = (a,\, 0,\, 0),\;\; p_2 = (0,\, a,\, 0),$$

for some $a > 0$. Let P_c be the polygon given by the union of segments

$$P_c = \overline{op_1} \cup \overline{p_1 p_1(c)} \cup \overline{p_1(c)p_2(c)} \cup \overline{p_2(c)p_2} \cup \overline{p_2 o},$$

where $p_1(c) = (a, 0, c)$ and $p_2(c) = (0, a, c)$. Then, use Theorem 15.1 of [43] (here stated on Section 2.4 as Theorem 2.4.1) to obtain the existence of a minimal $\pi-$graph Σ_c with $\partial\Sigma_c = P_c$.

Then, one key property was observed: Σ_c is a *Killing graph* over the vertical domain $\Omega_c = \{(t, a-t, s); 0 \le t \le a,\; 0 \le s \le c\}$ with respect to the horizontal Killing field $\partial_x + \partial_y$, thus it is *unique*. This implies it is *stable* and also that the variation $c \mapsto \Sigma_c$ is continuous. Then, making $c \to \infty$, and using curvature estimates due to Rosenberg, Souam and Toubiana [56] for stable surfaces on homogeneous manifolds, it is possible to show the convergence of Σ_c to some surface Σ_∞, nowhere vertical and with boundary

$$\partial\Sigma_\infty = P_\infty = \overline{op_1} \cup \big(\{p_1\} \times [0,\, \infty)\big) \cup \overline{op_2} \cup \big(\{p_2\} \times [0,\, \infty)\big).$$

Finally, use the ambient isometries to rotate Σ_∞ along the two segments $\overline{op_1}$ and $\overline{op_2}$ to obtain a complete minimal $\pi-$graph on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, which can be extended periodically by horizontal translations. □

On this subject, our contribution is an extension of the above result to any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Although on the general case it is not possible to find examples with no boundary, on the above special case treated by A. Menezes we reobtain the same result with a different technique when taking limits. Precisely, we prove:

Theorem 2.5.1. *Let $\mathbb{R}^2 \rtimes_A \mathbb{R}$ be a semidirect product, where $A \in M_2(\mathbb{R})$ is any matrix with* $\text{trace}(A) \geq 0$. *Then, there is $L_0 = L_0(\text{trace}(A), \det(A)) > 0$ (and $L_0 = \infty$ when $\text{trace}(A) = 0$) such that if $p_1, p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$ satisfy $d(p_1, p_2) \leq L_0$, then for any piecewise smooth curve $\gamma \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ with endpoints p_1, p_2 which is a convex graph over the segment $\alpha = \overline{p_1 p_2}$ and meets α on angles less than $\pi/2$, there exists a minimal surface Σ which is a $\pi-$graph and with boundary*

$$\partial\Sigma = \gamma \cup (\{p_1\} \times [0, +\infty)) \cup (\{p_2\} \times [0, +\infty)).$$

Moreover, Σ is nowhere vertical, it is the unique minimal surface in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with such boundary and it is a Killing graph over the vertical domain $\Omega_\infty = \alpha \times [0, +\infty)$.

Remark 2.1. Our construction works in particular for the product space $\mathbb{H}^2 \times \mathbb{R}$, which is isometric and isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, when we choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

On this space, Scherk-like graphs have been already studied, and even more general results were obtained (for instance, on the work of B. Nelli and H. Rosenberg [51] and on the work of L. Hauswirth, H. Rosenberg and J. Spruck [33]). However, the isometry between $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ maps $\mathbb{R}^2 \rtimes_A \{0\}$ not to $\mathbb{H}^2 \times \{0\}$, as it could look on the first sight, but to a horocylinder (that is, the product of a horocycle of $\mathbb{H}^2$ with $\mathbb{R}$).

The proof of Theorem 2.5.1 is given on Section 2.5.2. If $\text{trace}(A) > 0$, when considering polygons as P_c above, there is a minimal $\pi-$graph Σ_c with boundary P_c. However, as the maximum principle does not hold, there is no reason for it to be a Killing graph over Ω_c and we do not have the tools to ensure the continuity of the family Σ_c, which makes it impossible to use geometric barriers. It becomes clear that, when $\text{trace}(A) \neq 0$, another sequence of surfaces Σ_c should be constructed, or other tools (such as stability of minimal $\pi-$graphs) developed.

Our approach will be as follows: instead of considering minimal $\pi-$graphs over a domain on $\mathbb{R}^2 \rtimes_A \{0\}$, we will look to the problem *horizontally*, and

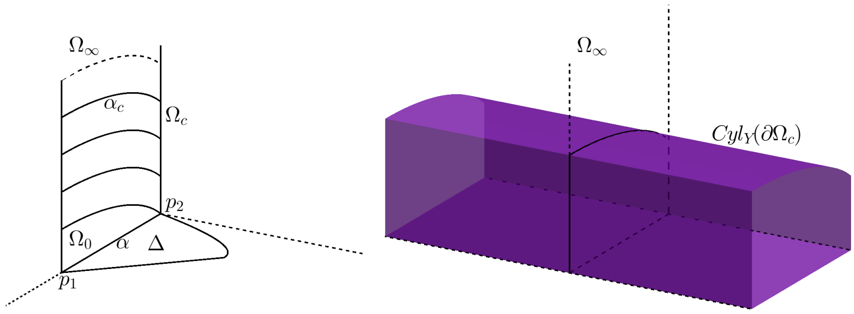


Figure 2.2: The horizontal domain Δ and the exhaustion of Ω_∞ by subdomains Ω_c whose Killing cylinder have mean curvature vector pointing inwards.

consider an exhaustion of the *half-strip* $\Omega_\infty = \alpha \times [0, +\infty)$ by subdomains Ω_c in which it is possible to find a family of minimal Killing graphs with prescribed boundary. Then, we use techniques from Killing graphs and elliptic partial differential equations to ensure the convergence of such family to another minimal Killing graph Σ_∞. Then, we go back to the problem *vertically* (as the intermediate Killing graphs are also $\pi-$graphs, by a result of Meeks, Mira, Pérez and Ros), and then we apply the geometric barriers developed by A. Menezes to see that the surface Σ_∞ is, as claimed, a $\pi-$graph, nowhere vertical.

2.5.1 A good exhaustion of Ω_∞

The next proposition will be of fundamental importance on the construction described on last section, as it will give the exhaustion of Ω_∞ by domains Ω_c where it is possible to solve the existence of minimal Killing graphs with prescribed boundary (see Figure 2.2).

Proposition 2.5.2. *Let* $\mathbb{R}^2 \rtimes_A \mathbb{R}$ *be a semidirect product where* $\operatorname{trace}(A) \geq 0$. *Let* $p_1, p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$ *and* $\alpha = \overline{p_1 p_2}$ *be the segment joining* p_1 *and* p_2. *We define the vertical domain*

$$\Omega_\infty = \alpha \times [0, +\infty). \tag{2.44}$$

Then, there exists some $L_0 = L_0(A) > 0$ *such that if* $L = length(\alpha) < L_0$, Ω_∞ *admits a continuous exhaustion* $\{\Omega_c\}_{c>0}$ *by domains* Ω_c *with boundary given by* α*, a graph over* α*, called* α_c*, and the two vertical segments joining the endpoints of* α *and* α_c *and this exhaustion is such that the Killing cylinder over* $\partial\Omega_c$ *with respect to the horizontal Killing field* $Y_\theta = \sin(\theta)\partial_x + \cos(\theta)\partial_y$

has mean curvature vector pointing inwards, where θ is such that Y_θ is perpendicular to Ω_∞ at $z = 0$.

Proof. First, we notice that, after a rotation on A as on (2.11) and a horizontal translation on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, which is an ambient isometry, we assume without loss of generality that $p_1 = (0, 0, 0)$ and $p_2 = (L, 0, 0)$ for some $L > 0$, then α becomes the segment $\alpha = \{(x,\ 0,\ 0);\ 0 \leq x \leq L\}$ and Ω_∞ is the half-strip

$$\Omega_\infty = \{(x,\ 0,\ z) \in \mathbb{R}^2 \rtimes_A \mathbb{R};\ 0 \leq x \leq L,\ z \geq 0\}, \tag{2.45}$$

transversal to the Killing field $Y = \partial_y$. Such assumptions will be kept until the end of the chapter.

If trace$(A) = 0$, then the result is trivial (and without the need for an upper bound L_0) by taking α_c to be the translate of α to height c, $\alpha_c = \{(x, 0, c); 0 \leq x \leq L\}$, as horizontal planes are minimal. Then, until the end of the proof we will treat the non-unimodular case and again we assume without loss of generality that trace$(A) = 2$, so A is a matrix as on (2.26). We will exhibit the curves α_c explicitly, then we prove they have the desired properties.

First, we treat the case where A is not diagonal and either $a^2 + bc \leq 0$ or $b \neq 0$: let λ, Λ the constants related with the matrix A via *i.* of Claim 2.1 and (2.30). We let

$$L_0 = \sqrt{\frac{\lambda}{2\Lambda}}\frac{\pi}{2} \tag{2.46}$$

and let $f : [0,\ L] \to \mathbb{R}$ to be given by

$$f(x) = \frac{1}{\Lambda} \ln\left(\frac{\cos\left(\sqrt{\frac{2\Lambda}{\lambda}}x\right)}{\cos\left(\sqrt{\frac{2\Lambda}{\lambda}}L\right)}\right). \tag{2.47}$$

First, we notice that f is well defined, as $0 \leq x \leq L < L_0$ implies

$$\cos\left(\sqrt{\frac{2\Lambda}{\lambda}}x\right) \geq \cos\left(\sqrt{\frac{2\Lambda}{\lambda}}L\right) > 0,$$

so the quotient on (2.47) is larger than (or equal to) 1, in particular $f \geq 0$, with $f(x) = 0 \iff x = L$. We let, for $c > 0$, $f_c = f + c$ and let $\alpha_c = \text{graph}(f_c) \subseteq \Omega_\infty$. When we set

$$\Omega_c = \left\{(x, 0, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R};\ 0 \leq x \leq L, 0 \leq z \leq f_c(x)\right\}, \tag{2.48}$$

it follows that $\{\Omega_c\}_{c>0}$ is an exhaustion of Ω_∞. Now we show that the Killing cylinder of the boundary of Ω_c with respect to ∂_y has mean curvature vector pointing inwards.

The ∂_y-Killing cylinder of $\partial\Omega_c$ has four smooth components (see Figure 2.2, right): one is a subdomain of a horizontal plane, so it has mean curvature 1 pointing upwards, two are contained on vertical planes, thus are minimal. The last component is the one corresponding to α_c, and it is a $\pi-$graph of the function $u_c(x,y) = f_c(x)$, and Theorem 2.3.2 will imply that its mean curvature is given by

$$H = \frac{e^{4f_c}}{2W^3}\left[Q_{22}(f_c)f_c'' + G_1(f_c)\left(f_c'\right)^2 + 2e^{-4f_c}\right], \tag{2.49}$$

when oriented upwards.

Now, we follow the steps on the proof of Theorem 2.4.3: as $b \neq 0$, Claim 2.1 implies that $Q_{22}(z) > \lambda e^{-2z}$, and $G_1/Q_{22} \leq \Lambda$. These relations imply that (as the derivatives of f_c coincide with the ones of f)

$$H \leq \frac{e^{4f_c}}{2W^3}Q_{22}(f_c)\left[f'' + \Lambda\left(f'\right)^2 + 2\frac{e^{-2f_c}}{\lambda}\right], \tag{2.50}$$

whenever A is not diagonal and satisfies either $b \neq 0$ or $a^2 + bc \leq 0$. In particular, since $f_c \geq 0$ we have

$$H \leq \frac{e^{4f_c}}{2W^3}Q_{22}(f_c)\left[f'' + \Lambda\left(f'\right)^2 + \frac{2}{\lambda}\right]. \tag{2.51}$$

Now, we observe that f was chosen in such a way it satisfies the ODE

$$f'' + \Lambda\left(f'\right)^2 + \frac{2}{\lambda} = 0, \tag{2.52}$$

so, by applying (2.52) on (2.51), we obtain that $H \leq 0$, with respect to the upward orientation, so the mean curvature vector of the Killing cylinder around α_c is mean convex, as promised.

This finishes the proof on the case where A is not diagonal and either $a^2 + bc \leq 0$ or $b \neq 0$. Now, we treat the simpler case of A being given by

$$A = \begin{pmatrix} 1+a & 0 \\ c & 1-a \end{pmatrix}. \tag{2.53}$$

On this case, we have that $Q_{22}(z) = e^{2(a-1)z}$ and $G_1(z) = (3+a)e^{2(a-1)z}$ (as, from (2.29) we obtain $a_{11}(z) = e^{(1+a)z}$ and $a_{12}(z) = 0$). Thus, the mean curvature of a $\pi-$graph to a function $u(x,y) = f(x)$ is given by

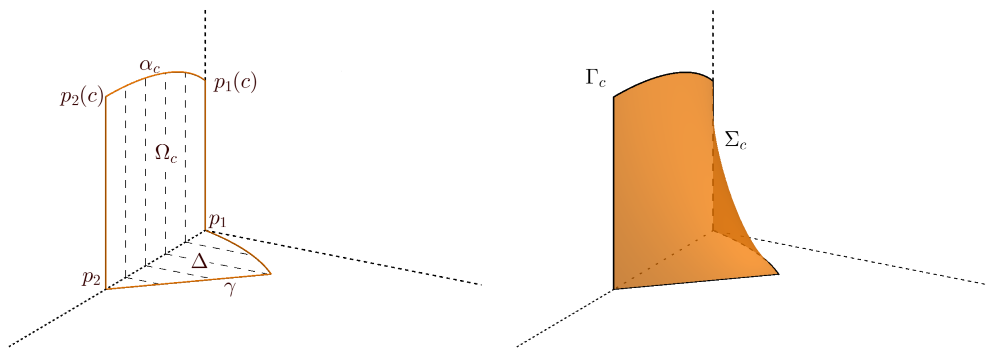


Figure 2.3: The surface Σ_c (on the right) is both a $\pi-$graph over Δ and a ∂_y-Killing graph over Ω_c, with $\partial\Sigma = \Gamma_c$ (the curve the left).

$$H = \frac{e^{2(a+1)f}}{2W^3}\left[f'' + (3+a)\,(f')^2 + 2e^{-2(1+a)f} \right],$$

and we can proceed the proof as done on the previous case. □

2.5.2 Existence of Scherk-like graphs: Proof of Theorem 2.5.1

On this section we prove Theorem 2.5.1. The proof is an standard argument of convergence, with the difference that we are look at the graphs sometimes vertically (as $\pi-$graphs), to have geometrically defined barriers, and sometimes horizontally (as Killing graphs), so we can use techniques of Killing graphs and elliptic partial differential equations.

Proof of Theorem 2.5.1. Let $A \in M_2(\mathbb{R})$ be any matrix with trace$(A) \geq 0$ and let $L_0 > 0$ be the one given by Proposition 2.5.2. Let $p_1, p_2 \in \mathbb{R}^2 \rtimes_A \{0\}$ be such that $d(p_1, p_2) = L < L_0$ and without loss of generality assume $p_1 = (0,0,0)$ and $p_2 = (L, 0, 0)$.

Let $\alpha = \{(x, 0, 0); 0 \leq x \leq L\}$ be the segment joining p_1 and p_2 and let $g : [0, L] \to \mathbb{R}$ be a convex, piecewise smooth function, with $g(0) = g(L) = 0$, meeting α on angles smaller than $\pi/2$ at 0 and L, defining a curve $\gamma \subseteq \mathbb{R}^2 \rtimes_A \{0\}$,

$$\gamma = \{(x,\, g(x),\, 0) \in \mathbb{R}^2 \rtimes_A \{0\};\ 0 \leq x \leq L\},$$

a curve smooth by parts with endpoints p_1, p_2 such that $\alpha \cup \gamma$ bounds a convex domain $\Delta \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ (as on Figure 2.3, left).

We also let $\Omega_\infty = \alpha \times [0, +\infty)$ and, following the notation of Proposition 2.5.2, let, for each $c \geq 0$,

$$\Omega_c = \{(x,\, 0,\, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R};\; 0 \leq x \leq L,\; 0 \leq z \leq f_c(x)\},$$

and $\alpha_c = \{(x,\, 0,\, f_c(x));\; 0 \leq x \leq L\}$ to be the graph of f_c, in such a way that its ∂_y-Killing cylinder

$$Cyl_{\partial_y}(\alpha_c) = \{(x,\, y,\, f_c(x));\; 0 \leq x \leq L,\, y \in \mathbb{R}\}$$

has mean curvature vector pointing downwards. We also denote by

$$p_1(c) = (0, 0, f_c(0)),\; p_2(c) = (L,\, 0,\, f_c(L))$$

the endpoints of α_c, and we let, for $c \geq 0$, Γ_c be a simple closed curve on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given by (see Figure 2.3, left)

$$\Gamma_c = \gamma \cup \overline{p_1 p_1(c)} \cup \alpha_c \cup \overline{p_2 p_2(c)}. \tag{2.54}$$

Claim 2.3. *The curve* Γ_c *as above bounds an* unique *minimal* π*-graph* Σ_c *over* Δ*, which is also a* ∂_y-*Killing graph over* Ω_c*.*

Proof of Claim 2.3. First, we notice that Γ_c monotonically parametrizes $\partial\Delta$, then we can use Theorem 2.4.1 to obtain a minimal, least area, $\pi-$graph Σ_c with boundary $\partial\Sigma_c = \Gamma_c$.

Next, we show that Σ_c is a ∂_y-Killing graph, on the sense that there will exist a function $g_c : \overline{\Omega}_c \to \mathbb{R}$, smooth up to the boundary, such that $R(g_c) = 0$ (here R will stand for the elliptic operator of minimal ∂_y-Killing graphs) and

$$\Sigma_c = Gr_{\partial_y}(g_c) = \{(x,\, g_c(x,z),\, z);\; (x,0,z) \in \Omega_c\}. \tag{2.55}$$

To begin with, as Σ_c is a $\pi-$graph, there exists a function $u_c : \Delta \to \mathbb{R}$ such that

$$\Sigma_c = \text{graph}(u) = \{(x,y,u_c(x,y));\; (x,y,0) \in \Delta\}. \tag{2.56}$$

We remark that Σ_c is contained on the ∂_y-Killing cylinder over Ω_c, so $0 \leq u_c(x,y) \leq f_c(x)$. Indeed, that $u > 0$ on the interior of Δ follows directly from the maximum principle. If there was an interior point $(x_0,\, y_0,\, 0) \in \Delta$ such that $u_c(x_0,\, y_0) > f_c(x_0)$, then we could consider the family $Cyl_{\partial_y}(\alpha_t)$, for $t > c$, and obtain a last contact point, interior for both Σ_c and $Cyl_{\partial_y}(\alpha_t)$, so the mean curvature of $Cyl_{\partial_y}(\alpha_t)$ would point upwards, in contradiction with Proposition 2.5.2.

Let $q = (x, 0, z) \in \Omega_c$ be an interior point and consider $\mathcal{O}(q)$ to be the orbit of q with respect to the flux φ_t of the Killing field ∂_y, which is the horizontal line $\mathcal{O}(x, 0, z) = \{(x, y, z); y \in \mathbb{R}\}$. Then we notice that $\mathcal{O}(x, 0, z) \cap \Sigma_c$ is never empty, otherwise Σ_c would not be simply connected and then it could not be a $\pi-$graph over Δ.

Moreover, the intersection $\mathcal{O}(x, 0, z) \cap \Sigma_c$ must be always a single point: if it contained two (or more) points $q_i = \varphi_{t_i}(q) \in \Sigma_c$, with $0 < t_1 < t_2$, then for $t_0 = t_2 - t_1 > 0$, $\varphi_{t_0}(\Sigma_c) \cap \Sigma_c \neq \emptyset$. Now, as $\varphi_t(\partial\Sigma_c) \cap \Sigma_c = \emptyset$ for all $t \neq 0$ by construction, we can consider the last contact point between $\varphi_t(\Sigma_c) \cap \Sigma_c$, and it will be interior for both Σ_c and $\varphi_t(\Sigma_c)$, a contradiction with the maximum principle.

We denote $(x, g_c(x, z), z) = \Sigma_c \cap \mathcal{O}(x, 0, z)$. This implies that Σ_c can be written as (2.55), but we still do not have the regularity on g_c. In order to prove that g_c is smooth, we begin by proving that $\text{grad}(g_c)$ is bounded.

Let $q \in \Omega_c$ be any interior point and consider a small ball $B = B_{\Omega_c}(q, r) \subseteq \text{int}(\Omega_c)$ such that $Cyl_{\partial_y}(\partial B)$ has mean curvature vector pointing inwards. Consider the following problem on B:

$$\begin{cases} R(w) = 0, \text{ on } \text{int}(B) \\ w|_{\partial B} = g_c|_{\partial B}, \end{cases} \tag{2.57}$$

where R is the mean curvature operator for ∂_y-Killing graphs. In other words, we are looking for a minimal ∂_y-Killing graph over a small ball on Ω_c that coincides with Σ_c on its boundary.

If $\Phi = g_c|_{\partial B}$ was of class $C^{2,\alpha}$, we could simply use the existence result due to M. Dajczer and J. H. de Lira, Theorem 1 of [16][6] to obtain a solution to (2.57). However, at this point we can only guarantee that Φ is of class C^0, so we need to use an approximation argument. Let $(\Phi_n^{\pm})_{n\in\mathbb{N}} \subseteq C^{2,\alpha}(\partial B)$ be two sequences of $C^{2,\alpha}$ functions, converging to Φ and such that

$$\Phi_n^- \leq \Phi_{n+1}^- \leq \Phi \leq \Phi_{n+1}^+ \leq \Phi_n^+, \tag{2.58}$$

for every $n \in \mathbb{N}$. By Theorem 1 of [16], there are functions $w_n^{\pm} \in C^{2,\alpha}(\overline{B})$ with minimal ∂_y-Killing graph and such that $w_n^{\pm}|_{\partial B} = \Phi_n^{\pm}$. From (2.58) we obtain that the sequences $w_n^{\pm}$ also are monotone, w_n^- is non-decreasing and w_n^+ is non-increasing, both uniformly bounded. To obtain the convergence of the sequences $w_n^{\pm}$ to a solution of (2.57), we use some recent gradient estimates for Killing graphs obtained by J.-B. Casteras and J. Ripoll on [6].

[6] We note that the hypothesis on the Ricci curvature on [16] is used uniquely to obtain an a priori estimate for the height of the graph, which is satisfied on our setting by the maximum principle.

Theorem 2.5.3 (Theorem 4, [6])**.** *Let M be a Riemannian manifold and let Y be a Killing field. Let Ω be a Killing domain in M and let $o \in \Omega$ and $r > 0$ such that the open geodesic ball $B_\Omega(o,r)$ is contained in Ω. Let $u \in C^3(B_\Omega(o,r))$ be a negative function whose $Y-$Killing graph has constant mean curvature H. Then there is a constant L depending only on $u(o)$, r, $|Y|$ and H such that $\|grad(u)(o)\| \leq L$.*

All functions $w_n^{\pm}$ have uniform bounds on the C^0 norm, thus Theorem 2.5.3 above implies that there are uniform gradient estimates on compact subsets of B. This implies that both sequences will converge on the C^2-norm to a function $w \in C^2(B) \cap C^0(\overline{B})$, which is a solution of (2.57). Now, just use the flux of ∂_y and the same translation argument as before to obtain that w coincides with g_c on B, in particular the gradient of g_c is bounded on interior points of Ω_c, as claimed.

Now we use the relation $(x, g_c(x,z), z) = (x, y, u_c(x,y))$ to prove that g_c is actually smooth *up to the boundary*, with the unique exceptions of p_1, p_2, $p_1(c)$, $p_2(c)$, where $\partial\Omega_c$ is not smooth, and the finite number of points where g is not differentiable. Just notice that u_c is smooth up to the boundary (except on the points where $\partial\Delta$ is not differentiable) and that the gradient of u_c is never horizontal on $\partial\Delta$, by the boundary maximum principle. Moreover, it follows from last argument that $\mathrm{grad}(u_c)$ never vanishes on interior points of Δ, so g_c is also smooth up to the boundary, with the exceptions above.

Finally, we remark that this argument proves that *any* minimal $\pi-$graph S with $\partial S = \Gamma_c$ is a Killing graph, then we obtain that Σ_c is unique, which proves Claim 2.3. $\diamondsuit$

Now, we notice that the uniqueness of Σ_c, given by Claim 2.3, implies that the correspondence $c \mapsto g_c$ is continuous, and, as defined, the functions g_c satisfied, on the boundary of Ω_c:

$$g_c(0,z) = g_c(L,z) = 0,\ g_c|_{\alpha_c} = 0,\ g_c(x,0) = g(x).$$

Again, as Σ_c is a $\pi-$graph over Δ, it is contained on the $\pi-$cylinder over Δ, and this can be translated to the horizontal setting as the inequality

$$0 \leq g_c(x,z) \leq g(x), \tag{2.59}$$

for every $(x,\ 0,\ z) \in \Omega_c$. Moreover, the usual argument using the translations given by the flux of ∂_y shows that the sequence g_c is monotonically increasing on c, that is $g_c(x_0,\ z_0) \leq g_{c'}(x_0,\ z_0)$, for every $(x_0,\ 0,\ z_0) \in \Omega_c$ and for every $c' \geq c$. In particular the sequence will converge (as it is bounded) pointwise for some function $g_\infty : \Omega_\infty \to \mathbb{R}$, such that $0 \leq g_\infty \leq g$. Next claim will show that the convergence is actually on the C^2-norm, so $Gr_{\partial_y}(g_\infty) = \Sigma_\infty$ is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

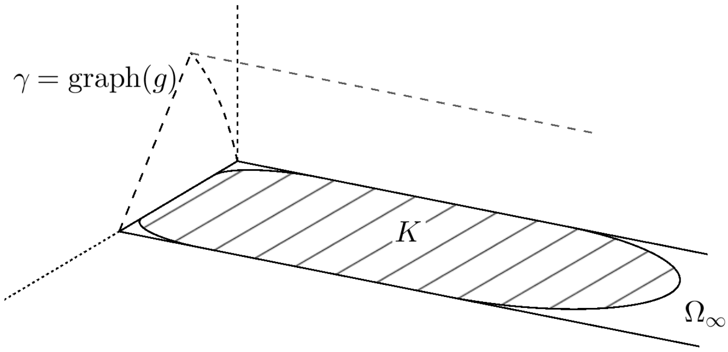


Figure 2.4: $\Omega_\infty \subseteq P$ viewed horizontally: on each compact set $K \subseteq \Omega_\infty$ there are uniform gradient estimates.

Claim 2.4. *When $c \to \infty$, the functions g_c converge on the $C^{2,\alpha}-$norm to $g_\infty : \Omega_\infty \to \mathbb{R}$.*

Proof of Claim 2.4. We use the same argument of Claim 2.3, via gradient and height estimates for Killing graphs. Let $K \subseteq \Omega_\infty$ be a compact set contained on $\overline{\Omega_\infty}$ with $C^{2,\alpha}$ boundary, as on Figure 2.4. As it holds $g_c(x, z) \leq g(x)$, follows that the height of g_c is uniformly bounded on K, so we can use Theorem 2.5.3 to obtain an uniform bound for the norm of the gradient of every g_c on interior points of K.

Now, we remark that (2.59), together with the assumption that the angle γ makes with α at p_1 and p_2 is less than $\pi/2$, implies that every g_c satisfy a uniform gradient estimate also along the boundary of K, as $g(0) = g(L) = 0$. As $g_c|_K \in C^{2,\alpha}(K)$ is smooth up to the boundary, this implies an uniform (not depending on c) estimate for the gradient of g_c on K.

Now, we just take an exhaustion of Ω_∞ by compact sets and, by using an standard argument via the theory of partial elliptic equations, we obtain that a subsequence of the g_c converges to g_∞ on the C^2-norm. In particular, as the sequence is monotone and converges pointwise, follows that the convergence is smooth on the whole Ω_∞. ♢

From this claim we obtain that Σ_∞ is a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, and that its boundary is

$$\partial\Sigma_\infty = \Gamma_\infty = \gamma \cup \big(\{p_1\} \times [0, \infty)\big) \cup \big(\{p_2\} \times [0, \infty)\big),$$

as, on the convergence of g_c we had $g_c = 0$ on the horizontal segments of $\partial\Omega_c$ and $g_c = g$ on α, with g_c smooth up to the boundary.

Now, in order to finish the proof of the theorem, it remains to show that Σ_∞ is nowhere vertical and that it is unique. The uniqueness comes directly from the fact that it was built as a Killing graph, and that every other surface with such boundary is contained on the ∂_y-Killing cylinder over Ω_∞.

To show that Σ_∞ is nowhere vertical, we go back to analyse the problem using $\pi-$graphs. First, if there was an *interior* point $p \in \Sigma_\infty$ such that $T_p\Sigma_\infty$ was vertical, we observe that Σ_∞ and $T_p\Sigma_\infty$ would be two minimal surfaces of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ tangent to each other at p. Then, there are at least two curves, meeting transversely at p on the intersection $T_p\Sigma_\infty \cap \Sigma_\infty$, so Σ_∞ cannot be a $\pi-$graph on a neighbourhood of p, so it is a $\pi-$cylinder over some line segment[7] β contained on $\partial\Delta$. Also if the point $p \in \partial\Sigma_\infty$ was a boundary point, then the boundary maximum principle would give the same conclusion. Next claim is to show that Σ_∞ meets $\pi^{-1}(\gamma)$ uniquely on γ, so $\Sigma_\infty \supseteq \big(\beta \times [0, \infty)\big)$ is a contradiction.

Claim 2.5. $\Sigma_\infty \cap \pi^{-1}(\gamma) = \gamma$.

Proof of Claim 2.5. We use the same barrier technique that A. Menezes, [50]: Let γ_i be a smooth component of γ and let $p \in \gamma_i$ be any point. Consider L the vertical plane of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ containing the tangent line to γ_i at p (this is well defined even for $p \in \partial\gamma_i$, as γ_i is smooth). As γ is convex, this implies that Δ is contained on the same connected component of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ defined by L, so also does Σ_∞.

Recall the functions $u_c : \Delta \to \mathbb{R}$ defined via (2.56). Let $c_0 = \sup_\Delta u_0$ and let $c_2 > c_1 > c_0$ be any two numbers. We consider a rectangle $R \subseteq L$ with boundary $\partial R = r_1 \cup r_2 \cup s_1 \cup s_2$ given by two parallel horizontal segments r_1 and r_2 and two vertical segments s_1 and s_2, such that $s_1 \subseteq \{z = c_1\}$ and $s_2 \subseteq \{z = c_2\}$ (see Figure 2.5). Moreover, we assume that R projects into $\mathbb{R}^2 \rtimes_A \{0\}$ in a compact segment $r \ni p$ with endpoints $q_1 = \pi(s_1)$ and $q_2 = \pi(s_2)$, contained on the same half-space determined by $\{y = 0\}$ (the vertical plane containing α) and with q_2 outside Δ.

Let $q_3 \in \pi(R)$ be a point interior to the projection of R but that lies outside Δ. Then, $\widetilde{q}_3 = \pi^{-1}(q_3) \cap r_2$ divides r_2 into two compact segments $r_3 \cup r_4$, r_3 projecting entirely outside of Δ and with $p \in \pi(r_4)$.

We remark that L is stable, as it is transversal to a (horizontal) Killing field, and in particular, it follows from the useful criteria due to D. Fischer-Colbrie and R. Schoen, Theorem 1 of [30] (also proved on Proposition 1.32 of the book by T. Colding and W. Minicozzi, [11]) that R is strictly stable, thus

[7] If $\beta \subseteq \mathbb{R}^2 \rtimes_A \{0\}$ is a smooth curve, the $\pi-$cylinder $\beta \times [0, \infty)$ is minimal if and only if β is a line segment: to see this, just use the foliation of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by vertical planes which are parallel to the vertical plane generated by the endpoints of β. It also follows from the more general formula $H(x, y, z) = k_g(x, y)e^{-z\,\mathrm{trace}(A)}$, where $k_g(x, y)$ denotes the geodesic curvature of β on the point $(x, y, 0)$. The proof of this formula is a simple computation.

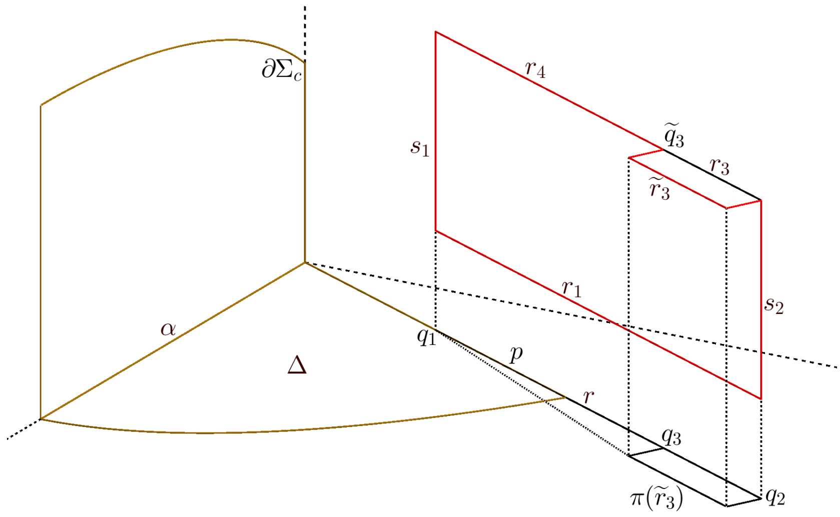


Figure 2.5: The construction of the barrier $\widetilde{R}$, by deforming the boundary of R over r_3.

small perturbations of ∂R give rise to minimal surfaces with the perturbed boundary.

We change r_2 by making a parallel translation of r_3 on the direction of the half-space that contains Σ_∞, whose projection still does not intersect Δ, joined by two small segments and denote such curve $\widetilde{r}_3$, in such a way that $r_3 \cup \widetilde{r}_3$ bounds a small rectangle on the horizontal plane $\{z = c_2\}$. We assume this perturbation is small in such way that its projection does not intersect Δ. Let $\widetilde{R}$ be a minimal surface of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ whose boundary is the perturbed rectangle $r_1 \cup \widetilde{r}_3 \cup r_4 \cup s_1 \cup s_2$. Such surface is nowhere vertical and its contained on the convex hull of its boundary, in particular it is contained on $\{z \geq c_1\}$ and on the same half space that Σ_∞ with respect to the plane L.

Now, it is easy to see that $\pi(\widetilde{R}) \cap \Delta \neq \emptyset$, as otherwise $\widetilde{R} \cap R$ would have an interior contact point. Moreover, $\widetilde{R}$ is above u_0 in $\pi(\widetilde{R}) \cap \Delta$, by the construction of $\widetilde{R}$. Then, if $\Sigma_\infty \cap \pi^{-1}(\gamma_i) \neq \gamma_i$, we would have that $\Sigma_\infty \cap \widetilde{R} \neq \emptyset$, thus, for some $\ell > 0$ there would be a first contact point between Σ_ℓ and $\widetilde{R}$. As $\partial\Sigma_\ell$ does not intersect the convex hull of $\partial\widetilde{R}$, it does not intersect $\widetilde{R}$. Moreover, neither $\partial\widetilde{R}$ can intersect Σ_ℓ, as this would imply such point would be on the plane L, so Σ_ℓ would have a vertical tangent plane. Then this contact point is going to be interior for both, reaching to a contradiction that proves the claim. $\diamondsuit$

From this claim and from the argument previously done, we obtain that

Σ_∞ is actually more than a Killing graph, it is also a $\pi-$graph, nowhere vertical, which finishes the proof of the theorem. $\square$

CHAPTER 3

Finite topology surfaces on hyperbolic 3-manifolds

This chapter is based on the joint (ongoing) work of the author with W. H. Meeks III, [46]. Using the existence of short geodesic loops on surfaces with bounded injectivity radius and non positive sectional curvature, we prove that a complete embedding of an annulus $E \equiv \mathbb{S}^1 \times [0, +\infty)$ in the hyperbolic space $\mathbb{H}^3$ has unbounded injectivity radius function, provided $|H_E| \leq 1$. As a consequence, we obtain that any complete surface Σ of finite topology embedded on a hyperbolic 3-manifold whose injectivity radius of each end goes to zero must be proper if its mean curvature function is bounded $|H_\Sigma| \leq 1$.

3.1 Introduction

To understand under which circumstances the embedding of a surface Σ into a Riemannian 3-manifold N is proper[1] is a fundamental question in surface geometry and has been a source of many interesting and deep results, especially when the ambient space is the Euclidean space $\mathbb{R}^3$. In a celebrated paper [10], T. Colding and W. Minicozzi proved that complete minimal surfaces of finite topology embedded in $\mathbb{R}^3$ are proper. Based on the proof of this result, W. Meeks and H. Rosenberg proved on [47] that complete minimal surfaces embedded in $\mathbb{R}^3$ with positive injectivity radius are proper. Recently W. Meeks and G. Tinaglia [49] extended both results by proving

[1] that is, for every compact set $\mathcal{C} \subseteq N$, the intersection $\mathcal{C} \cap \Sigma$ is also compact on Σ.

that complete surfaces embedded in $\mathbb{R}^3$ with constant mean curvature are proper if they have finite topology or positive injectivity radius.

It is natural try to answer the same question for more general ambient spaces, but it turns out that this result is not always true on such generality. On [14, 55], examples of minimal (topological) planes embedded but not properly embedded respectively in $\mathbb{H}^3$ (by B. Coskunuzer) and $\mathbb{H}^2 \times \mathbb{R}$ (by M. Rodríguez and G. Tinaglia) are presented. Also, on [15] it is shown by B. Coskunuzer, W. Meeks and G. Tinaglia that for each $H \in [0, 1)$ there is a plane embedded in $\mathbb{H}^3$ but not properly embedded with constant mean curvature H. As the plane is the simplest non compact topological type, it becomes clear that, at least on the ambient space $\mathbb{H}^3$, some other hypothesis is needed in order to ensure properness.

In this work we find sufficient conditions for an embedding of a finite topology surface Σ into a hyperbolic 3-manifold N (that is a complete manifold with constant sectional curvature $K_N = -1$) to be proper, under an assumption on the injectivity radius on each end of Σ. Precisely, we state our main theorem as follows:

Theorem 3.1.1. *Let N be a complete hyperbolic 3-manifold and let $\Sigma \hookrightarrow N$ be a complete embedded surface on N, orientable and with finite topology whose mean curvature function H_Σ satisfies $|H_\Sigma| \leq 1$. Then:*

A *If N is simply connected (i.e. $N = \mathbb{H}^3$), then each end of Σ has unbounded injectivity radius function.*

B *If N has positive injectivity radius $I_N = \delta > 0$, Σ has positive injectivity radius.*

C *If Σ has bounded injectivity radius function, then it has finite total curvature and $i^* : \pi_1(\Sigma) \to \pi_1(N)$ is non trivial. In particular, if Σ has genus zero, it has at least two ends and is π_1- injective.*

D *If the injectivity radius function of Σ converges to zero at infinity, then the embedding is proper.*

We would like to make a few remarks about Theorem 3.1.1 and some related results. First, we notice that our hypothesis on the mean curvature of Σ being $|H_\Sigma| \leq 1$ (not necessarily constant), on the ambient space of a hyperbolic manifold implies that the *intrinsic* sectional curvature of Σ is non-positive (see Section 3.3), which is of fundamental importance on the application of our technique. Also, $|H_\Sigma| \leq 1$ allows us to use geometric barriers of $\mathbb{H}^3$ such as distant spheres, cylinders and hyperbolic Delaunay surfaces, all with mean curvature larger than 1.

Concerning proper immersions of surfaces of finite topology into finite volume hyperbolic 3-manifolds, we would like to mention the work of P. Collin, L. Hauswirth and H. Rosenberg [13], where it is proved (Theorem 1.1) that a minimal surface Σ of finite topology, properly immersed in a hyperbolic manifold of finite volume N (i.e. a *hyperbolic cusp manifold*) has finite total curvature

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma), \tag{3.1}$$

where $\chi(\Sigma) = 2 - 2g - n$ is the Euler characteristic of Σ (g is the genus of Σ and n is the number of ends). Moreover, they prove that if Σ is a proper, minimal immersed surface of finite topology in N, then each end of Σ is asymptotic to a totally geodesic 2-cusp end in an end C of N. In particular, it follows that any minimal surface of finite topology, properly immersed on a hyperbolic cusp manifold has injectivity radius function converging to zero at infinity. Such remark, together with item D of Theorem 3.1.1, allows us to obtain

Corollary 3.1.2. *Let N be a hyperbolic manifold of finite volume. Then a complete minimal surface Σ embedded on N with finite topology is proper if and only if the injectivity radius function of Σ converges to zero at infinity.*

Moreover, on [12], P. Collin, L. Hauswirth, L. Mazet and H. Rosenberg prove the existence both of a compact, min-max minimal surface on N and of a non compact, properly embedded minimal surface of finite topology[2] Σ on N. Using Theorem 1.1 of [13], it follows that the injectivity radius function of such Σ is bounded, and we obtain that item A of Theorem 3.1.1 above does not hold on hyperbolic cusp manifolds.

We also remark that (3.1), together with Gauss formula, implies some topological restrictions for the existence of proper minimal embeddings of finite topology surfaces on a hyperbolic manifold of finite volume N. For instance, if a minimal surface Σ properly immersed on N is homeomorphic to a sphere ($g = 0$) with n punctures, then necessarily $n \geq 3$,[3] as it is easy to see. In particular, this implies that there are no minimal planes properly immersed on a hyperbolic cusp manifold. Item C above shows that any

[2]To obtain the non compact example they extend - with a different proof -Ruberman's minimization result ([57], every properly embedded, non compact, incompressible, non separating surface in N of finite topology is isotopic to a least area embedded minimal surface).

[3]On the $n = 3$ case, P. Collin, L. Hauswirth, L. Mazet and H. Rosenberg prove, on [12], that a minimal 3-punctured sphere properly immersed on N is totally geodesic.

(non proper) example of a minimal plane embedded on N necessarily has unbounded injectivity radius function.

We prove Theorem 3.1.1 in Section 3.3. The proof is via two propositions that analyse properties of embedded annuli on hyperbolic manifolds with absolute mean curvature bounded by 1, Propositions 3.3.3 and 3.3.5. We use Section 3.2 to give basic definitions and fix the notation used on this chapter, finishing on 3.2.1 with the statement of two theorems used on the proof of Proposition 3.3.3: Theorem 3.2.1 is a result of D. Chen [7] and gives an isoperimetric inequality for surfaces of bounded curvature on $\mathbb{H}^3$ which lie on a bounded domain of $\mathbb{H}^3$, while Theorem 3.2.2 can be found on [9] (in a more general context, here we present the statement adapted to our setting) and is an area growth estimate. The chapter is then finished on Section 3.4, where some generalizations of the results of this chapter are obtained to the case of a ambient space with negatively bounded sectional curvature.

3.2 Preliminaries

This section is to give some basic definitions and fix the notation used on the chapter and to present some results needed for the proof of Theorem 3.1.1. We begin with some intrinsic properties of surfaces, as the notion of cut locus and injectivity radius. Henceforth, Σ will always denote a finite topology surface, on the sense that it is homeomorphic to a compact Riemann surface S_g of (finite) genus g with a finite quantity of punctures,

$$\Sigma \equiv S_g \setminus \{p_1, p_2, \ldots, p_n\}.$$

Each one of the points $\{p_1, p_2, \ldots, p_n\}$ is in correspondence with an *end* of Σ, which has the topology of an annulus $E \equiv \mathbb{S}^1 \times [0, +\infty)$. We assume that Σ inherits the metric induced by the ambient space N, via the embedding, and that it is complete with respect to it.

For a Riemannian manifold M and for $p \in M$ we let the *injectivity radius of M at p* (denoted $I_M(p)$) be the maximal (possibly infinite) radius such that the exponential map restricted to the open ball $B(0, r) \subseteq T_pM$ is a diffeomorphism.

For instance, if M has non-empty boundary, we notice that, if $d(p, \partial M) = r_0$ and $r > r_0$, the exponential map is not defined on the whole $B(0, r)$, thus $I_M(p) \leq r_0$. This remark allows us to extend I_M continuously to the boundary points as being null and we obtain a function $I_M : M \to [0, +\infty]$, called the *injectivity radius function of M*.

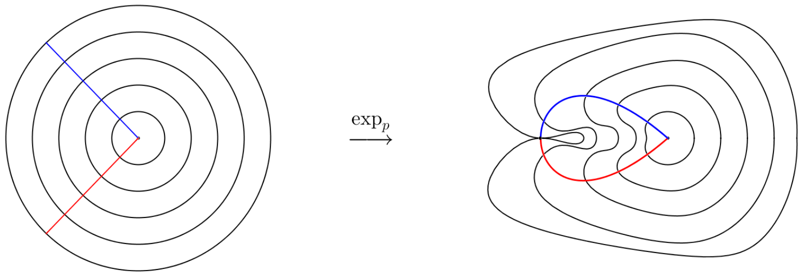


Figure 3.1: Injectivity radius is the maximum radius where $\exp_p |_{B(0,r)}$ is a diffeomorphism

When M is complete and has no boundary, we let the *injectivity radius of* M to be the infimum of the function $I_M : M \to (0, +\infty]$. By an abuse of notation, we also denote the injectivity radius of M by $I_M \in [0, +\infty]$. If M is not complete, or has non empty boundary, then automatically $I_M = 0$, but the converse does not hold, a complete surface M with empty boundary could be complete and have $I_M = 0$, as it happens, for instance, on hyperbolic cusps.

If $p \in M$ and γ is a radial geodesic parametrized by arc length $\gamma(t) = \exp_p(tv)$ for some unitary vector $v \in T_pM$, then, at least locally it holds that $d(p, \gamma(t)) = t$. If this property does not hold for some $t_1 > 0$, then the same happens for every $t > t_1$. By continuity it is clear that the set

$$\{t \geq 0; d(p, \gamma(t)) = t\}$$

is either a closed interval $[0, t_0]$ or the half line $[0, +\infty)$. Whenever the first case happens, the point $\gamma(t_0)$ is called a *minimal point with respect to* p *along* γ. We denote by $\text{Cut}(p)$ the *cut locus* of p the set of all minimal points with respect to p. It follows that a point p has empty cut locus if and only if its injectivity radius is infinite. Moreover, if $I_M(p) = \delta \in (0, +\infty)$, then there is at least a point $q \in \text{Cut}_p$ with $d(q, p) = \delta$ and this point minimizes the distance between Cut_p and p. It is a well known result (for instance Proposition 2.12, Chapter 13 of [22]) that if $p, q \in M$ are two points such that $q \in \text{Cut}(p)$ and $d(p, \text{Cut}(p)) = d(p, q)$, then either p and q are conjugate points or there are two geodesics $\gamma_1, \gamma_2 : [0, \ell] \to M$ from p with the same length $\ell = d(p, q)$ such that $\gamma_1(\ell) = \gamma_2(\ell) = q$ and $\gamma_1'(\ell) = -\gamma_2'(\ell)$.

This previous observation is of particular interest when M has non-positive sectional curvature, as in this case M has no conjugate points. If $p \in M$ is a point where the injectivity radius $I_M(p) = \delta \in (0, +\infty)$, then there is a closed geodesic loop[4] around p with length 2δ (which is the case depicted on Figure 3.1). The existence of such geodesic loops is used on this work to prove properties of annuli of non positive sectional curvature and bounded injectivity radius on Section 3.3.

3.2.1 Isoperimetric inequality and area growth

The proof of Theorem 3.1.1, given along Section 3.3 via Propositions 3.3.3 and 3.3.5, uses area estimates to obtain properness. We would like to recall two results for surfaces embedded in $\mathbb{H}^3$, the first one (Theorem 3.2.1) is an isoperimetric inequality for surfaces with absolute mean curvature function bounded by 1, obtained by Dechang Chen on his doctoral thesis [7], Theorem 3.2.3.

Theorem 3.2.1 ([7]). *Let $\Omega \subseteq \mathbb{H}^3$ be a bounded open subdomain of the hyperbolic 3-space $\mathbb{H}^3$. Then there is a constant $C = C(\mathrm{diam}(\Omega))$ such that for every compact surface $\Sigma \subseteq \Omega$ with compact boundary and bounded mean curvature function $|H_\Sigma| \leq 1$ it holds*

$$Area(\Sigma) \leq C\, length(\partial\Sigma).$$

We will present the proof of a generalization of this result on Theorem 3.4.1 of Section 3.4.

The second result we want to present on this section is an area growth estimate obtained by L. Cheung and P. Leung, [9], Corollary 2.2. The authors obtain actually more than what stated on Theorem 3.2.2, under the same hypothesis they show that the area growth of a surface as below is at least linear, but this is not used on this work. For the reader's convenience we write the main steps to the proof of this particular result.

Theorem 3.2.2 ([9]). *Let $\Sigma \subseteq N$ be a complete, non compact, oriented surface on a Hadamard manifold N of dimension 3 with $K_N \leq -a^2 \leq 0$. Then, if the absolute mean curvature function of Σ is bounded, Σ has infinite area.*

Proof. Let us assume that $\Sigma \subseteq N$ is a complete, non compact, oriented surface on N with bounded mean curvature $|H_\Sigma| \leq \alpha$, for some $\alpha > 0$. Let $q \in \Sigma$ be a given point and let Γ be a geodesic ray of Σ from q[5]. Then

[4] Let us not mistake *geodesic loops* with *closed geodesics*: the first one is not smooth at a single point, called the base point of the loop, while the second one is smooth everywhere.

[5] We denote a geodesic ray as a complete geodesic $\gamma : [0, +\infty) \to N$ that minimizes the distance between any two points $\gamma(t)$, $\gamma(s)$.

the isoperimetric inequality of Hoffman-Spruck, Theorem 2.2 of [36], gives us that, for every compactly contained subset Ω of Σ, it holds

$$\mathrm{Vol}(\Omega)^{\frac{1}{2}} \leq C(\mathrm{Vol}(\partial\Omega) + \alpha\mathrm{Vol}(\Omega)). \tag{3.2}$$

In particular, if we assume that

$$\mathrm{Vol}(\Omega) \leq \left(\frac{1}{2C\alpha}\right)^2, \tag{3.3}$$

then (3.2) can be rewritten as

$$\frac{1}{2}\mathrm{Vol}(\Omega)^{\frac{1}{2}} \leq C\mathrm{Vol}(\partial\Omega). \tag{3.4}$$

Now, we observe that, for some small radius $r_q \leq 2/\alpha$, we have that the ball centred at q with radius r_q has small volume $\mathrm{Vol}(B_\Sigma(q, r_q)) \leq (1/2C\alpha)^2$, satisfying (3.3). Let $q_1 \in \gamma$ be a point such that $d_\Sigma(q_1, q) = 2r_q$ and consider $B_1 = B_\Sigma(q_1, r_q)$. We claim that $\mathrm{Vol}(B_1) \geq (r_q/4C)^2$. First, either one of the two hold:

a. $\mathrm{Vol}(B_1) > \left(\frac{1}{2C\alpha}\right)^2$;

b. $\mathrm{Vol}(B_1) \leq \left(\frac{1}{2C\alpha}\right)^2$.

If it was a. to hold, then the hypothesis on r_q implies directly the claim on $\mathrm{Vol}(B_1)$. If it was b., then we can apply (3.4) for balls centred at q_1 with radii $r \leq r_q$ B_r to obtain

$$\frac{1}{2}\mathrm{Vol}(B_r)^{\frac{1}{2}} \leq C\mathrm{Vol}(\partial B_r).$$

Observing that $\mathrm{Vol}(\partial B_r) = \frac{d}{dr}\mathrm{Vol}(B_r)$ and defining $f(r) = \mathrm{Vol}(B_\Sigma(q_1, r))$, we arrive in a differential inequality

$$\frac{f'(r)}{f(r)^{\frac{1}{2}}} \geq \frac{1}{2C},$$

which we can integrate from 0 to r_q (observing that $f(0) = 0$) to obtain

$$2f(r_q)^{\frac{1}{2}} = \int_0^{r_q} \frac{f'}{f^{\frac{1}{2}}}dr \geq \frac{1}{2C}\int_0^{r_q} 1dr = \frac{r_q}{2C}$$

from where it follows the claim. Now, we can proceed inductively with this process, by choosing points $q_n \in \Gamma$ with $d_\Sigma(q_n, q) = 2nr_q$ and noticing that the balls $B_n = B_\Sigma(q_n, r_q)$ are pairwise disjoint from each other and have volume bounded by below by an uniform constant $(r_q/4C)^2$, which implies Σ has infinite volume. $\square$

3.3 Finite topology surfaces in hyperbolic 3-manifolds

On this section, we consider N to be a hyperbolic 3-manifold, that is a complete Riemannian manifold with constant sectional curvature $K_N = -1$, and let $\Sigma \hookrightarrow N$ be an embedding of a finite topology surface Σ on N with bounded mean curvature $|H_\Sigma| \leq 1$. We prove that if the injectivity radius function of each end of Σ converges to zero at infinity, then the embedding is proper. In order to do so, we begin by studying some properties of embeddings of $\mathbb{S}^1 \times [0, +\infty)$ in N.

Let $E \equiv \mathbb{S}^1 \times [0, +\infty)$ be an annulus embedded on N with bounded mean curvature function $|H_E| \leq 1$. Then it follows from Gauss equation that E has non positive *intrinsic* sectional curvature K_E. Indeed, it holds that

$$K_E - K_N = \lambda_1 \lambda_2, \tag{3.5}$$

where λ_1, λ_2 are the eigenvalues of the shape operator of E in some point $p \in E$. Then, as $K_N = -1$, we have that

$$K_E = \lambda_1\lambda_2 - 1 = \lambda_1(2H_E - \lambda_1) - 1 = -(\lambda_1 - H_E)^2 + (H_E)^2 - 1 \leq 0.$$

This property is important mainly to obtain the existence of geodesic loops on points away from the boundary of E, $\partial E \equiv \mathbb{S}^1 \times \{0\}$, where the injectivity radius is bounded, as on Section 3.2. If $p \in E$ has intrinsic distance to the boundary $\ell = d(p, \partial E)$ and injectivity radius $I_E(p) = L < \ell$, then there is a geodesic loop γ on E with base point p and length $2L$.

First, we notice that geodesic loops on surfaces of non positive curvature cannot be trivial, as it is easy to see using Gauss-Bonnet equation. In particular a geodesic loop $\gamma \subseteq E$ as above must generate the fundamental group of E, $\pi_1(E)$. Furthermore, it divides E into two components, one compact (with the topology of $\mathbb{S}^1 \times [0, 1]$) and one non compact (with the topology of $\mathbb{S}^1 \times [0, +\infty)$). We are use such property to prove the two next results, Lemma 3.3.1 and Lemma 3.3.2. The first one was obtained by T. Colding and W. Minicozzi on the proof of Lemma 4.2 of [10], but we present here its proof due to its simplicity and also for the sake of completeness.

Lemma 3.3.1. *Let $E \equiv \mathbb{S}^1 \times [0, +\infty)$ be a complete annulus with non positive sectional curvature $K_E \leq 0$ and bounded injectivity radius function $I_E \leq L$. Then E has finite total curvature.*

Proof. Take a divergent sequence of points $(p_n)_{n\in\mathbb{N}} \subseteq E$ such that $d(p_1, \partial E) > L$, $d(p_{n+1}, p_n) > 2L$, for every $n \in \mathbb{N}$ and that the sequence of distances to

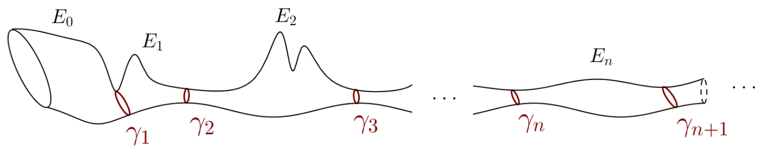


Figure 3.2: Short loops on $E \equiv \mathbb{S}^1 \times [0, +\infty)$ with $K_E \leq 0$.

the boundary $d(p_n, \partial E)$ is increasing. Such sequence exists, as E is complete and non compact. By assumption we have that $K_E \leq 0$, so at each p_n there exists a geodesic loop γ_n with length $2I_E(p_n) \leq 2L$, forming an exterior angle $\theta_n \in (-\pi, \pi)$ at p_n.

As previously stated, each loop γ_n is non trivial and divides E in two annuli, one compact and one non compact. We let E_0 to be the region bounded between ∂E and γ_1 and, for $n \in \mathbb{N}$, E_n the one bounded between γ_n and γ_{n+1}, as in Figure 3.2.

To prove that E has finite total curvature, we apply Gauss-Bonnet theorem on each region E_n, obtaining that

$$\int_{E_n} K_E + \theta_n - \theta_{n+1} = 0,$$

and, by fixing a $n_0 \in \mathbb{N}$ and making the sum for all $n \leq n_0$, we have

$$\int_{E \setminus \left(\cup_{n > n_0} E_n\right)} K_E = \sum_{n=0}^{n_0} \int_{E_n} K_E = \int_{E_0} K_E + \theta_{n_0+1} - \theta_1, \tag{3.6}$$

so the left hand side of (3.6) is uniformly bounded for every n_0, what implies that E has finite total curvature. □

Another interesting application of the short loops to annuli with non positive sectional curvature is that the convergence of the injectivity radius function at infinity must be uniform. Although it seems like a very natural result, its proof, as far as the author is concerned, had not been obtained yet.

Lemma 3.3.2. *Let $E \equiv \mathbb{S}^1 \times [0, +\infty)$ be a complete annulus with non positive sectional curvature $K_E \leq 0$ and let $I_E : E \to [0, +\infty)$ be the injectivity radius function of E. Admit there is a sequence of points $(p_n)_{n\in\mathbb{N}} \subseteq E$ diverging on the intrinsic distance of E such that*

$$\lim_{n\to+\infty} I_E(p_n) = L \in [0, +\infty].$$

Then every other sequence of points q_n that diverges on E will also satisfy $\lim_{n\to+\infty} I_E(q_n) = L$.

Proof. Every geodesic loop $\gamma \subseteq E$ divides E not only in one compact and one non compact part, but it also divides the annulus E in one *convex* part and a *non convex* part, except when γ is smooth (i.e. it is a closed geodesic), when it divides E into two convex pieces.

Fix a point $p \in E$ and let γ be a geodesic loop with base point p. Let α be a geodesic segment with endpoints $p_1, p_2 \in \gamma$ but not contained in γ, so it is contained in one convex part of E with respect to γ, which we denote by E_0 (see Figure 3.3).

We claim that $p \neq p_1, p_2$. In particular, $\{p_1, p_2, p\}$ separate γ into three components, which we denote as γ_1, γ_2 the geodesic segments joining p and p_1, p_2, respectively and γ_3 the segment joining p_1 and p_2. Moreover, we also claim that

i. $\alpha \cup \gamma_3$ generates the fundamental group of E;

ii. $\alpha \cup \gamma_1 \cup \gamma_2$ bounds a disk D, contained in E_0;

iii. There is a closed curve $\widetilde{\alpha}$, contained on $E_0 \setminus D$ and homotopic to $\alpha \cup \gamma_3$, such that $\text{length}(\widetilde{\alpha}) < \text{length}(\alpha \cup \gamma_3)$.

Let us prove the claim: first, we prove that $p \neq p_1, p_2$. Assume, for instance, that $p = p_1$. Then we would have $\gamma = \gamma_2 \cup \gamma_3$ as previously described, so either $\alpha \cup \gamma_2$ or $\alpha \cup \gamma_3$ would be trivial on E, bounding a disk. But there is no disk bounded by two geodesic segments on surfaces of non positive sectional curvature, then $\gamma \subseteq \alpha$. As we are assuming this does not happen, we obtain that $p \neq p_1, p_2$. The same argument also gives us that $\alpha \cup \gamma_3$ must generate $\pi_1(E)$, proving i.

As γ also generates $\pi_1(E)$, follows that $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ is homotopic to $\alpha \cup \gamma_3$, in particular $\alpha \cup \gamma_1 \cup \gamma_2$ shall be trivial, bounding a disk $D \subseteq E_0$, and this proves ii.

Now, in order to construct the curve $\widetilde{\alpha}$ of iii., we take any interior points $x \in \alpha$ and $y \in \gamma_3$ and consider the geodesic segment β joining x and y. By the convexity of E_0 it follows that $\beta \subseteq E_0$. If $\beta \cap \text{int}(D) \neq \emptyset$, the intersection between β and ∂D should have at least two points x and $\widetilde{x}$. This second point $\widetilde{x}$ could not lie in α, as we would find a disk bounded by two geodesic segments. But neither $\widetilde{x} \in \gamma_1 \cup \gamma_2$ is possible, because it would imply transversality between β and ∂E_0, so β would leave E_0, a contradiction that shows that $\beta \subseteq E_0 \setminus D$.

In order to ease the notation, we let, for any two points $q_1, q_2 \in E_0$, $\overline{q_1 q_2}$ to be the geodesic segment joining q_1 and q_2, for instance $\overline{xy} = \beta$, $\overline{pp_1} = \gamma_1$,

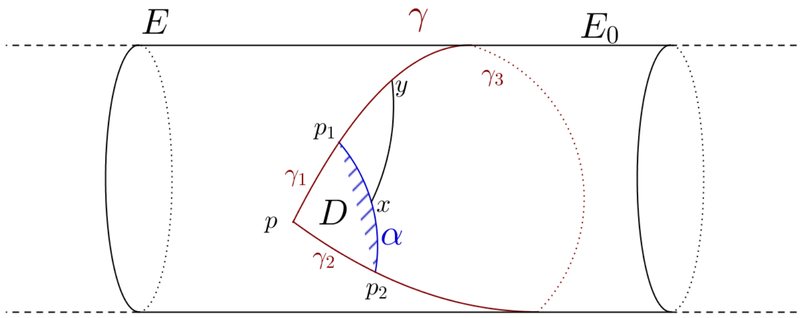


Figure 3.3: If the geodesic segment α meets the geodesic loop γ, the length of the non trivial curve formed by α and γ_3 may be decreased.

etc. Now, we will consider two closed curves, each one given by three geodesic segments. We let

$$\begin{aligned}\widetilde{\alpha}_1 &= \overline{xy} \cup \overline{yp_1} \cup \overline{p_1x}\\ \widetilde{\alpha}_2 &= \overline{xy} \cup \overline{yp_2} \cup \overline{p_2x}.\end{aligned}$$

One between $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ is homotopically trivial, and the other generates the fundamental group of E. Let us assume $\widetilde{\alpha}_1$ bounds a disc. Then, the triangle inequality implies that

$$\text{length}(\overline{xy}) < \text{length}(\overline{yp_1}) + \text{length}(\overline{p_1x}),$$

and in particular $\text{length}(\widetilde{\alpha}_2) < \text{length}(\alpha)$, obtaining the desired curve on iii. and proving the claim.

Now, the proof of the lemma follows easily. Let us assume, by contradiction, that there are two divergent sequences of points in E, (p_n) and (q_n) with $\lim I_E(p_n) = L \in [0, +\infty)$ and $\lim I_E(q_n) = \widetilde{L} \in (L, +\infty]$.

For n sufficiently large, over each point p_n there is a geodesic loop γ_n of length $2I_E(p_n) \simeq 2L$, and also a large geodesic loop Γ_n with base point q_n, on the sense it has length $2I_E(q_n) > 2L$. Without loss of generality we may pass to a subsequence and assume that every Γ_n is contained in between the region bounded by γ_n and γ_{n+1}. We let E_n be the region of E bounded by Γ_{n-1} and Γ_n, so $\gamma_n \subseteq E_n$.

Now, in each region E_n we claim it is possible to obtain a closed geodesic α_n contained on the interior of E_n, then Gauss-Bonnet Theorem implies that the region between every two such geodesics is flat, so the injectivity radius function is constant, which is a contradiction.

To find such α_n we use a standard minimization technique, together with the claim proved above. We let $\Lambda = \{\alpha \subseteq E_n;\ \alpha \in [\gamma_n]\}$ to be the homotopy class of γ_n on the annulus E_n. We define

$$\ell = \inf\{\text{length}(\alpha);\ \alpha \in \Lambda\} > 0$$

and take $(\alpha_k)_{k\in\mathbb{N}}$ a sequence of curves on Λ whose lengths converge to ℓ. By the compactness of E_n it follows that α_k converges to a curve α with length ℓ on the homotopy class of γ_n. It is clear that $\alpha \neq \Gamma_{n-1}, \Gamma_n$ and it is a composition of geodesic segments, and the possible points where it is not smooth are the ones where α meets the boundary of E_n. But the above claim shows us that in such setting, the length of α could be decreased keeping the homotopy class, so it would not be a minimizing curve. This implies that the limit curve α lies on the interior of E_n, and then it is smooth everywhere, a closed geodesic. □

So far, the properties obtained about annuli with non-positive sectional curvature are intrinsic. Now, we begin using geometric properties of $\mathbb{H}^3$ together with the short loops technique presented to prove next proposition, that gives item A of Theorem 3.1.1 and is of fundamental importance on the proof of its remaining items.

Proposition 3.3.3. *Let $E \equiv \mathbb{S}^1 \times [0, +\infty)$ be a complete embedding of an annulus in $\mathbb{H}^3$ with bounded mean curvature function $|H_E| \leq 1$. Then there is a sequence of points $(p_n) \subseteq E$ such that $I_E(p_n) \to +\infty$. In particular,* $\lim_{x\to\infty} I_E(x) = +\infty$.

Proof. We divide this proof into the two cases of whether the embedding of E is proper or not, as the proofs use different approaches. On the proper case the proof is mainly geometric, and when dealing with non proper embeddings we use analytical tools, namely the results on Section 3.2.1.

Claim 3.1. *Let $E \hookrightarrow \mathbb{H}^3$ be a complete embedding of an annulus on $\mathbb{H}^3$ with bounded mean curvature $|H_E| \leq 1$. Then, if E is not properly embedded, the injectivity radius function of E must be unbounded.*

Claim 3.2. *Let $E \hookrightarrow \mathbb{H}^3$ be a complete annulus properly embedded in $\mathbb{H}^3$ with bounded mean curvature $|H_E| \leq 1$. Then the injectivity radius of E must be unbounded.*

Proof of Claim 3.1. Assume that E is not properly embedded in $\mathbb{H}^3$ but has bounded injectivity radius function $I_E \leq L$. Then there is an intrinsically divergent sequence $(q_n)_{n\in\mathbb{N}} \subseteq E$ such that $q_n \to q \in \mathbb{H}^3$ and geodesic loops γ_n over q_n with length strictly less than $2L$. In particular, there is some

$n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $q_n \in B_{\mathbb{H}^3}(q, L)$, in such a way that $\gamma_n \subseteq B_{\mathbb{H}^3}(q, 2L)$.

For $m > n_0$ we let Σ_m be the region of E bounded between γ_{n_0} and γ_m, so $\partial\Sigma_m = \gamma_{n_0} \cup \gamma_m \subseteq B_{\mathbb{H}^3}(q, 2L)$. As the absolute mean curvature of Σ_m is bounded by 1 and the mean curvature of geodesic spheres of radius r in $\mathbb{H}^3$ is $H_r = \coth(r) > 1$, it follows that $\Sigma_m \subseteq B_{\mathbb{H}^3}(q, 2L)$. Then we can apply Theorem 3.2.1 to obtain a constant $C > 0$ such that every compact surface $\Sigma \subseteq B_{\mathbb{H}^3}(q, 2L)$ with $|H_\Sigma| \leq 1$ satisfy $\text{Area}(\Sigma) \leq C\text{length}(\partial\Sigma)$, and in particular it holds for every $m > n_0$ that

$$\text{Area}(\Sigma_m) \leq C \,\text{length}(\partial\Sigma_m) \leq 4CL, \tag{3.7}$$

where last inequality follows from

$$\text{length}(\partial\Sigma_m) = \text{length}(\gamma_{n_0}) + \text{length}(\gamma_m) \leq 4L.$$

From (3.7) we obtain that the area of Σ_m is uniformly bounded for every $m > n_0$, which implies that E has finite area. On the other hand, we can use Theorem 3.2.2 to obtain the contradiction that E has infinite area, proving Claim 3.1. ♢

Now, we recall [23], Theorem 4.1, where R. Sa Earp and H. Rosenberg use the existence of families of Delaunay surfaces in $\mathbb{R}^3$ to obtain *a maximum principle inside the Delaunay surface* $\mathcal{D}$. Precisely, let $\mathcal{D}$ be a Delaunay surface of $\mathbb{R}^3$ with mean curvature $1/2a > 0$ and let Σ be a properly embedded surface, complete and non compact, with compact boundary $\partial\Sigma$ that lies on the region of $\mathbb{R}^3$ inside $\mathcal{D}$ and with mean curvature $|H_\Sigma| \leq 1/2a$, then $\Sigma = \mathcal{D}$.

The same proof of [23] is used on [39], Lemma 6.3, using families of *hyperbolic* Delaunay surfaces, to obtain that if Σ is complete and non compact, properly embedded in $\mathbb{H}^3$ with constant mean curvature and compact boundary, then, if Σ is cylindrically bounded, its mean curvature is at least as big as the one of the cylinder containing it. Here, we adapt these proofs in order to show that a cylindrically bounded surface with mean curvature less than 1 cannot extend beyond the maximal period of a family of hyperbolic Delaunay surfaces (see Lemma 3.3.4 below). In particular, if E satisfies the hypothesis of Proposition 3.3.3 and has bounded injectivity radius function, it cannot be properly embedded, and this will prove Claim 3.2.

Proof of Claim 3.2. This proof is also by contradiction, so we assume that there is a divergent sequence of points $(p_n) \subseteq E$ with bounded injectivity radius $I_E(p_n) < L$. As the embedding is proper, by passing to a subsequence we may assume the extrinsic distance between p_n and ∂E is increasing, so

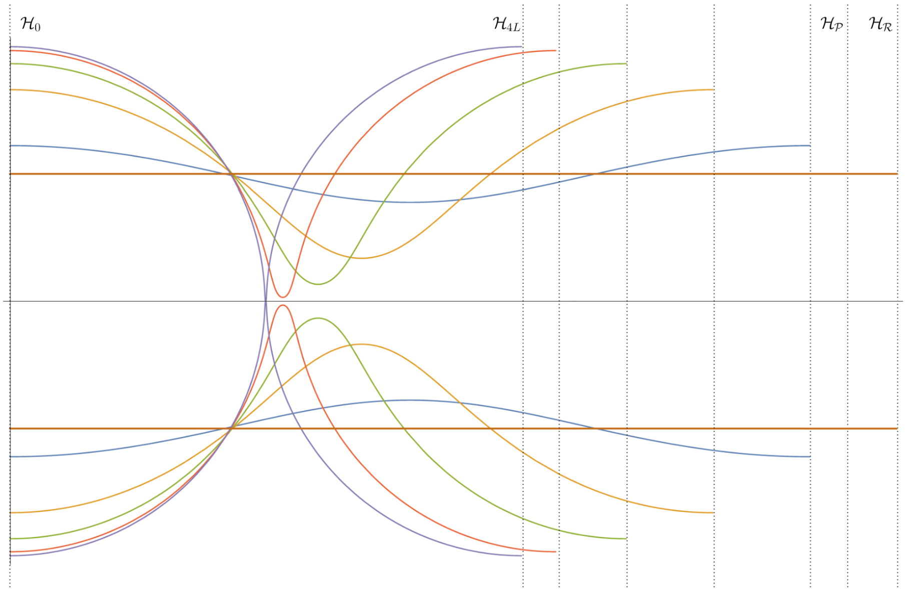


Figure 3.4: The profile of the family $\widetilde{\mathcal{D}}_t$ and the planes $\mathcal{H}_t$

(p_n) diverges uniformly on $\mathbb{H}^3$. As $K_E \leq 0$ and the points p_n are away from the boundary of E, there is some $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there are geodesic loops by p_n with length less than $2L$.

Let Γ be a geodesic line of $\mathbb{H}^3$ (to be specified later on) and let $H = \coth(2L) > 1$ be the mean curvature of the geodesic cylinder of radius L around Γ, which we call $\mathcal{D}_0$. There is a 1-parameter deformation of $\mathcal{D}_0$ by H-surfaces $\mathcal{D}_t$, converging to a chain of geodesic spheres with radius $2L$, $\mathcal{D}_\infty$, all centered at points of Γ, when $t \to \infty$. Each one of the surfaces $\mathcal{D}_t$ is of revolution around Γ and is also periodic with respect to hyperbolic translations along Γ, with period $P(t)$ depending uniquely on L and on t, and uniformly bounded.

As the sequence (p_n) is divergent and the embedding is proper, there is a $n_1 > n_0$ such that the extrinsic distance between γ_{n_0} and γ_{n_1} is bigger than the maximal period of the family $\mathcal{D}_t$, $\mathcal{P} = \sup\{P(t)\}$. We let $q_0 \in \gamma_{n_0}$ and $q_1 \in \gamma_{n_1}$ be a pair of points such that $d(q_0, q_1) = d(\gamma_{n_0}, \gamma_{n_1}) > \mathcal{P}$ and we fix Γ as the geodesic line of $\mathbb{H}^3$ passing through q_0 and q_1.

If we let $C = C(\Gamma, L)$ be the open solid cylinder around Γ of radius L (so $\partial C = \mathcal{D}_0$), it follows that both γ_{n_0} and γ_{n_1} are contained in C, as otherwise their lengths would be at least $2L$. Moreover, as the mean curvature of a geodesic cylinder of radius r around a geodesic is $\coth(2r) > 1$, it follows that the whole region bounded between γ_{n_0} and γ_{n_1}, which we denote by Σ, is contained on C.

We consider the deformation of the boundary of C, $\mathcal{D}_0$ into the chain of spheres $\mathcal{D}_\infty$ to obtain an interior point of contact between some $\mathcal{D}_{t_0}$ and Σ, which will contradict the mean curvature comparison principle, but first we need to have some control over the boundary of Σ. We consider the foliation $\mathcal{F}$ of $\mathbb{H}^3$ by totally geodesic parallel planes perpendicular to Γ, $\mathcal{F} = \{\mathcal{H}_t;\, t \in \mathbb{R}\}$, indexed in such a way that $q_0 \in \mathcal{H}_0$ and $d(\mathcal{H}_0, \mathcal{H}_t) = |t|$, for every $t \in \mathbb{R}$. We denote $\mathcal{R} = d(q_0, q_1)$, in such a way that we have $q_1 \in \mathcal{H}_\mathcal{R}$. We let

$$\widetilde{\Sigma} = E \cap \left(\cup_{t\in[0,\mathcal{R}]}\mathcal{H}_t\right)$$

be the region of the annulus E bounded in between the planes $\mathcal{H}_0$ and $\mathcal{H}_\mathcal{R}$, so $\widetilde{\Sigma} \subseteq \Sigma \subseteq C(\Gamma, L-\varepsilon)$.

Without loss of generality we may assume that all the surfaces $\mathcal{D}_t$ have a maximum bulge precisely in $\mathcal{H}_0$, so all the other maximal bulges of each $\mathcal{D}_t$ are contained in a sequence of parallel planes $\mathcal{H}_{iP(t)}$, for $i \in \mathbb{Z}$.

We will take $\widetilde{\mathcal{D}}_t$ as the portion of $\mathcal{D}_t$ contained between $\mathcal{H}_0$ and $\mathcal{H}_{P(t)}$, i.e. a fundamental piece of $\mathcal{D}_t$. We have that $\widetilde{\mathcal{D}}_t$ is a compact surface whose boundary lies outside of the cylinder C, as the maximal bulge radius increases with t, we have that $\partial\widetilde{\mathcal{D}}_t \cap \widetilde{\Sigma} = \emptyset$. Furthermore, the boundary of $\widetilde{\Sigma}$ lies on the two planes $\mathcal{H}_0$ and $\mathcal{H}_\mathcal{R}$, with $\mathcal{R} > \mathcal{P} > P(t)$, so $\widetilde{\mathcal{D}}_t$ does not intersect $\partial\widetilde{\Sigma}$ for any $t > 0$.

When $t \to +\infty$, the sequence $\widetilde{\mathcal{D}}_t$ converges to two half-spheres, intersecting tangentially at a single point of Γ, so these half spheres divide C, hence they must intersect $\widetilde{\Sigma}$. As $\mathcal{D}_0 = \partial C$ do not intersect $\widetilde{\Sigma}$, there will be a first contact point between $\widetilde{\Sigma}$ and some $\widetilde{\mathcal{D}}_t$, necessarily interior for both $\widetilde{\mathcal{D}}_t$ and $\widetilde{\Sigma}$. This implies that the mean curvature of $\widetilde{\Sigma}$ at such point is at least $H > 1$, a contradiction with the hypothesis on the mean curvature of E. $\diamondsuit$

Together, the two claims above prove that the injectivity radius function of E is unbounded, so there is a intrinsically divergent sequence of points where I_E diverges to $+\infty$. Then we can apply Lemma 3.3.2 to obtain that $\lim_{x\to\infty} I_E(x) = +\infty$. $\square$

We remark that the absurdity on the proof of Claim 3.2 above came from assuming the existence a compact surface Σ (the topology of Σ being the one of $\mathbb{S}^1 \times [0,1]$ was not used) whose boundary is given by two Jordan curves $\partial\Sigma = \gamma_0 \cup \gamma_1$ farther from each other than the maximal period of the family of Delaunay surfaces $\mathcal{D}_t$ whose $\partial\Sigma \Subset \mathcal{D}_0$. Precisely, it becomes proved the following

Lemma 3.3.4. *Let $\Sigma \subseteq \mathbb{H}^3$ be a compact surface embedded in $\mathbb{H}^3$ with two boundary curves $\partial\Sigma = \gamma_1 \cup \gamma_2$ and with bounded mean curvature $|H_\Sigma| \leq 1$.*

Let $2L = \max\{length(\gamma_1),\ length(\gamma_2)\}$*, so, if* $\mathcal{P} = \mathcal{P}(L)$ *is the maximal period of the family* $\mathcal{D}_t$ *of Delaunay surfaces of* $\mathbb{H}^3$ *with* $\mathcal{D}_0$ *being a geodesic cylinder of* $\mathbb{H}^3$ *of radius* L *around a geodesic line, then* $d_{\mathbb{H}^3}(\gamma_1,\ \gamma_2) < \mathcal{P}$*.*

We also remark that Proposition 3.3.3 does not hold for more general hyperbolic manifolds. Following the notation of [12], let C be a cusp-end of a hyperbolic manifold N: it is isometric to the quotient of a horoball M of $\mathbb{H}^3$ by a $\mathbb{Z}^2$ group of isometries G of $\mathbb{H}^3$ leaving M invariant. On the upper half-space model for $\mathbb{H}^3$, we can assume M is

$$M = \{(x, y, z) \in \mathbb{R}^3;\ z \geq 1/2\},$$

and that $G = G(v_1,\ v_2)$ is the group of isometries generated by translations of linearly independent horizontal vectors $v_1,\ v_2 \in \mathbb{R}^2 \times \{0\}$. Take $\mathbb{H}$ as a totally geodesic vertical plane of $\mathbb{H}^3$ (thus $\mathbb{H}$ is isometric to $\mathbb{H}^2$), parallel to v_1. Then the descend of $\mathbb{H} \cap M$ to C via the quotient $E = (\mathbb{H} \cap M)/G \hookrightarrow C \subseteq N$ is a minimal annulus on N with injectivity radius function not just bounded, but also converging to zero at infinity.

Next proposition gives us B, C and D on Theorem 3.1.1. We consider N to be a hyperbolic 3-manifold and let $E \equiv \mathbb{S}^1 \times [0,\ +\infty) \hookrightarrow N$ be a complete embedding of an annulus in N with bounded mean curvature $|H_E| \leq 1$. Proposition 3.3.3 implies that, if E has bounded injectivity radius function, then the embedding does not lift isometrically to $\mathbb{H}^3$, the universal cover of N. This is the key ingredient that allows us to prove:

Proposition 3.3.5. *Let* N *be a hyperbolic 3-manifold and let* $E \hookrightarrow N$ *be a complete embedding of an annulus* $E \equiv \mathbb{S}^1 \times [0,\ +\infty)$ *with mean curvature satisfying* $|H_E| \leq 1$*. Then*

(I) *If the injectivity radius function of* E *is bounded, the generator of the fundamental group of* E *is non trivial on* N*. Moreover,* E *has finite total curvature.*

(II) *If* N *has positive injectivity radius* $I_N = \delta$*, then, away from* ∂E*, the injectivity radius function of* E *satisfies* $I_E \geq \delta$*.*

(III) *If there is a sequence of points* $(p_n)_{n\in\mathbb{N}} \subseteq E$*, diverging intrinsically on* E *such that* $\lim_{n\to\infty} I_E(p_n) = 0$*, then* E *is properly embedded on* N*.*

Proof. Let us assume that the injectivity radius function I_E is bounded. Then, as the bound on the mean curvature of E implies it has non positive sectional curvature, it follows from Lemma 3.3.1 that E has finite total curvature. Furthermore, if the generator of $\pi_1(E)$ was trivial on N, then the embedding $\mathbb{S}^1 \times [0, +\infty) \hookrightarrow E \subseteq N$ would lift isometrically to the universal cover of N, $\mathbb{H}^3$, as a complete annulus with bounded injectivity radius function and absolute mean curvature bounded by 1, contradicting Proposition 3.3.3, and this proves (I).

To prove (II), we simply use (I): if at some point $p \in E$ away from ∂E we had $I_E(p) < \delta$, then there would be a geodesic loop around p with length smaller than 2δ, which implies such loop lies inside an extrinsic geodesic ball of radius δ, which is simply connected by the assumption $I_N = \delta$, contradicting (I).

Finally the proof of (III) follows from (I) analogously to the proof of (II): assume there is a sequence of points $(p_n)_{n\in\mathbb{N}} \subseteq E$ where the injectivity radius function of E converges to zero. By Lemma 3.3.2, the same will happen for every other intrinsically divergent sequence of points.

Now, by contradiction, assume that E is not properly embedded on N, so there is a sequence of points $q_n \in E$, diverging intrinsically on E but converging to a point $q \in N$. In particular, if we let $r_n = I_E(q_n)$ be the injectivity radius function of E at the points q_n, follows that $r_n \to 0$ at infinity. Moreover, there are geodesic loops γ_n over each q_n with length $2r_n$ and such loops generate $\pi_1(E)$. At $q \in N$ we have that $I_N(q) = \ell > 0$, and as the points q_n approach q and the geodesic loops have length going to zero, after some $n_0 \in \mathbb{N}$ all the curves γ_n lie in the same normal neighborhood of N with center at q, which is simply connected, and in particular they must be trivial on N, which is not possible by (I), showing that E is properly embedded on N and proving the proposition. □

We would like to remark that item (I) on the proposition above implies not only that N must have some topology in order to admit a complete annulus with bounded injectivity radius and absolute mean curvature bounded by 1, but also such annulus must *involve* the topology of N, on the sense that the inclusion map $i^* : \pi_1(E) \to \pi_1(N)$ is not trivial, so E is topologically incompressible on N. We apply this simple observation to the topological type of the plane to obtain the following corollary:

Corollary 3.3.6. *Let $P \hookrightarrow N$ be a complete embedding of a topological plane P into a hyperbolic manifold N whose absolute mean curvature is bounded $|H_P| \leq 1$. Then the set of points of P where the injectivity radius of P is infinite is unbounded on the intrinsic distance of P.*

Proof. Let us assume that the set $I_\infty = \{p \in P;\, I_P(p) = +\infty\}$ is bounded on the intrinsic distance of P. Then, we can choose $R > 0$ and $q \in P$ such that $I_\infty \subseteq B_P(q, R)$. Then, if we let $E = P \setminus B_P(q, R)$, it is a complete annulus with bounded injectivity radius function, embedded on N with absolute mean curvature bounded by 1. Then, Proposition 3.3.5, (I), implies that the generator of the fundamental group of E is non trivial on N, but this generator is trivial on P, as it is the boundary of a simply connected set, so it is also trivial on N. $\square$

Now we can prove Theorem 3.1.1, using Propositions 3.3.3 and 3.3.5:

Proof of Theorem 3.1.1. In order to prove the theorem, we notice that, as Σ has finite topology, then there is a compact set $K \subseteq \Sigma$ such that

$$\Sigma \setminus K = E_1 \cup E_2 \cup \ldots \cup E_n, \tag{3.8}$$

where each E_i is an annular end of Σ.

If N is simply connected, then Proposition 3.3.3 applied to each E_i implies that each end of Σ has unbounded injectivity radius function, so we obtain A.

Let us prove B, so we assume $I_N = \delta > 0$. It follows from item (II) of Proposition 3.3.5 that each E_i has positive injectivity radius function $I_{E_i} \geq \delta$ away from its boundary, on the sense that there is some $r > 0$ (which can be chosen to be uniform) such that every $p \in E_i$ with $d(p, \partial E_i) > r$ satisfy $I_\Sigma(p) = I_{E_i}(p) \geq \delta$.

We let $\widetilde{K} = K \cup \{x \in E_i;\ d_\Sigma(x, \partial E_i) \leq r\}$, and we restrict the function $I_\Sigma|_{\widetilde{K}}$ to the compact $\widetilde{K}$, so it attains a minimum $\widetilde{\delta} = \min_{\widetilde{K}} I_\Sigma > 0$, and we obtain that $I_\Sigma \geq \min\{\widetilde{\delta}, \delta\}$, so it is positive, and B follows.

The third part of Theorem 3.1.1, item C, is given by (I) of Proposition 3.3.5, using again the decomposition on (3.8): each end E_i will have finite total curvature and there is a finite number of ends. As K is compact, it also has finite total curvature, so the whole surface Σ will have finite total curvature. Moreover, each E_i has as boundary a simple closed curve, non trivial on N by (I), so, in particular $i^* : \pi_1(\Sigma) \to \pi_1(N)$ is non trivial.

Finally, to prove that the embedding is proper when the injectivity radius function of Σ converges to zero at infinity, item D, we let $\mathcal{C} \subseteq N$ be a compact set of N. Then $\mathcal{C} \cap K$ is compact, and (III) implies that $\mathcal{C} \cap E_i$ is also compact for every $i \in \{1, 2, \ldots, n\}$. It follows that

$$\mathcal{C} \cap \Sigma = (\mathcal{C} \cap K) \cup (\mathcal{C} \cap E_1) \cup \ldots \cup (\mathcal{C} \cap E_n)$$

is a finite union of compact sets, hence compact, and this proves D, finishing the proof of Theorem 3.1.1. $\square$

3.4 Generalizations to manifolds with negatively bounded sectional curvature

On this section, we give some partial results obtained on the intent to generalize Theorem 3.1.1 to a broader class of manifolds, namely manifolds N with negatively bounded sectional curvature $K_N \leq -a^2 < 0$. The universal cover of such manifolds is a Hadamard manifold, so the first step is to obtain a generalization of Theorem 3.2.1, due to D. Chen [7], for a Hadamard manifold, on Section 3.4.1.

Then, on Section 3.4.2 we prove results that are analogous to the ones obtained on Section 3.3, but under a stronger assumption on the mean curvature function: instead of the natural hypothesis $|H_\Sigma| \leq a$ (which implies $K_\Sigma \leq 0$ and the existence of short loops that seems to be the key property to our main results) we ask for a technical hypothesis $|H_\Sigma| \leq a(1-\varepsilon)$ for some $\varepsilon > 0$, so we can obtain that annuli embedded on a Hadamard manifold with such curvature condition must have unbounded injectivity radius function.

3.4.1 Isoperimetric inequalities on Hadamard manifolds

In order to obtain the results on this section, we make use of a well known tool, namely the Hessian comparison theorem. Its proof can be found, for instance, on Chapter 1 of the book by R. Schoen and S.-T. Yau [59].

Theorem (Hessian comparison principle)**.** *Let N a Hadamard $3-$manifold with sectional curvature bounded by above $K_N \leq -a^2 \leq 0$. Then if $p \in N$ is any point and we denote by $R = d_N(\cdot, p)$ the distance to the point p on N, then, for every unitary vector X tangent to the level sets of R, it holds*

$$Hess(R)(X,X) \geq \mu_a(R) = \begin{cases} a\coth(aR), & \text{if } a > 0; \\ \frac{1}{R} & \text{if } a = 0. \end{cases} \tag{3.9}$$

An important geometric application of the Hessian comparison principle is to compare the mean curvature of geodesic spheres and geodesic cylinders of a Hadamard manifold with the mean curvature of the same objects on a space of constant sectional curvature. Precisely, let N be a Hadamard manifold with sectional curvature $K_N \leq -a^2 < 0$, let $p \in N$ and Γ be a geodesic line of N through p. Fixed $L > 0$, we denote by $S_L = S_L(p)$ the geodesic sphere of N around p with radius L and $C_L = C_L(\Gamma)$ the geodesic cylinder around Γ, also with radius L, that is

$$\begin{aligned} S_L(p) &= \{x \in N;\ d_N(x,p) = L\}; \\ C_L(\Gamma) &= \{x \in N;\ d_N(x,\Gamma) = L\}. \end{aligned}$$

Then, their mean curvature function satisfies, in every point:

$$H_{S_L} \geq a \coth(aL) \tag{3.10}$$
$$H_{C_L} \geq a \coth(2aL), \tag{3.11}$$

where the right hand side of (3.10), (3.11) are respectively the mean curvature of a geodesic sphere and of a geodesic cylinder on a space of constant sectional curvature $-a^2$, $a > 0$, and an analogous inequality can be stated for the case $a = 0$. Using (3.9), (3.10) and following the proof of Chen [7] to Theorem 3.2.1, we obtain:

Theorem 3.4.1. *Let N be a Hadamard $3-$manifold with sectional curvature $K_N \leq -a^2 \leq 0$, with $a \geq 0$. Then, for each geodesic ball $B_N(p_0, R_0)$ there is a constant $C = C(a, R_0)$ such that every orientable compact surface $\Sigma \subseteq B_N(p_0, R_0)$ with compact boundary $\partial\Sigma$ and mean curvature $|H_\Sigma| \leq a$ satisfies*

$$\text{Area}(\Sigma) \leq C\,\text{length}(\partial\Sigma).$$

Proof. Let $p_0 \in N \setminus \Sigma$ be given and consider the foliation $\mathcal{F}$ of $N \setminus \{p_0\}$ by concentric geodesic spheres. We denote by $\mathcal{N}$ the normal unitary vector field to the leaves pointing inwards and let $R = d_N(\cdot, p_0)$ be the ambient distance function to p_0. Then, at a point $p \in \Sigma$, we can relate the ambient divergence of the field $\mathcal{N}$ with the intrinsic divergence of Σ of the projected field $\mathcal{N}^t$ (see Figure 3.5):

$$\text{Div}_\Sigma(\mathcal{N}^t) = \text{Div}_N\mathcal{N} + 2H_\Sigma\langle\mathcal{N},\, \eta\rangle - \langle\nabla_\eta\mathcal{N},\, \eta\rangle. \tag{3.12}$$

Integrating both sides of (3.12) over Σ and using the divergence theorem, we obtain that

$$\int_\Sigma \left(-\text{Div}_N(\mathcal{N}) - 2H_\Sigma\langle\mathcal{N},\, \eta\rangle + \langle\nabla_\eta\mathcal{N},\, \eta\rangle\right) = -\int_{\partial\Sigma}\langle\mathcal{N},\, \nu\rangle \tag{3.13}$$

where ν is the outward pointing conormal vector of $\partial\Sigma$. The right hand side of (3.13) is related with the length of $\partial\Sigma$ via

$$-\int_{\partial\Sigma}\langle\mathcal{N},\, \nu\rangle \leq \text{length}(\partial\Sigma), \tag{3.14}$$

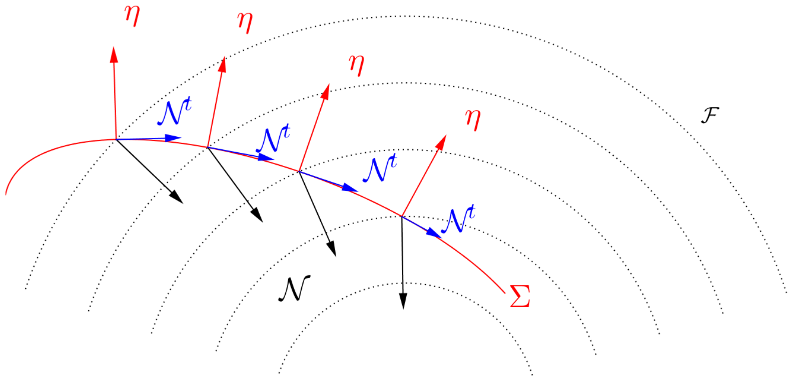


Figure 3.5: Representation of the foliation $\mathcal{F}$ of $N \setminus \{p_0\}$ and the fields $\mathcal{N}$, $\mathcal{N}^t$.

so, in order to obtain an isoperimetric inequality, we just need to estimate the function $\varphi = -\mathrm{Div}_N(\mathcal{N}) - 2H_\Sigma\langle\mathcal{N},\, \eta\rangle + \langle\nabla_\eta\mathcal{N},\, \eta\rangle$ by an uniform constant away from zero.

Claim 3.3. *It holds that* $\varphi \geq \xi_a(R)$*, where* $R = d_N(\cdot,\, p_0)$ *is the distance function to* p_0 *and* $\xi_a : (0,\, +\infty) \to (0,\, +\infty)$ *is given by*

$$\xi_a(r) = \begin{cases} ae^{-2ar}, & \text{if } a > 0, \\ \frac{1}{r}, & \text{if } a = 0. \end{cases} \tag{3.15}$$

Proof of the claim. Decompose $\eta = \eta^\top + \eta^\perp$, where $\eta^\top$ is tangent to the geodesic spheres and $\eta^\perp$ is on the direction of $\mathcal{N}$. We have that $\nabla_{\eta^\perp}\mathcal{N} = 0$, as curves normal to the foliation are geodesics, and it also holds that $\langle\nabla_X\mathcal{N},\, \mathcal{N}\rangle = 0$ for every X tangent to the foliation, so $\langle\nabla_{\eta^\top}N,\, \eta^\perp\rangle = 0$ and we have that

$$\langle\nabla_\eta\mathcal{N},\, \eta\rangle = \langle\nabla_{\eta^\top}\mathcal{N},\, \eta^\top\rangle. \tag{3.16}$$

Now, if it was $\eta^\top = 0$, we would have $\mathcal{N} = \pm\eta$, so

$$-\mathrm{Div}_N\mathcal{N} - 2H_\Sigma\langle\mathcal{N},\, \eta\rangle + \langle\nabla_\eta\mathcal{N},\, \eta\rangle = -\mathrm{Div}_N\mathcal{N} \mp 2H_\Sigma = 2H_r \mp 2H_\Sigma, \tag{3.17}$$

where H_r is the mean curvature of the geodesic sphere centred at p_0 with radius $r = d(p,\, p_0)$. The Hessian comparison principle implies that $H_r \geq \mu_a(r)$, and then

$$2H_r \mp 2H_\Sigma \geq 2(\mu_a(r) - |H_\Sigma|). \tag{3.18}$$

We will prove that $\mu_a(r) - |H_\Sigma| \geq \xi_a(r)$. Indeed, if it was $a = 0$, then $H_\Sigma = 0$ and $\mu_a(r) - |H_\Sigma| = \frac{1}{r} = \xi_a(r)$, so the claim holds. If $a > 0$, we would have that $|H_\Sigma| \leq a$ and $\mu_a(r) = a\coth(ar)$, so

$$\mu_a(r) - |H_\Sigma| \geq a\,(\coth(ar) - 1) > ae^{-2ar} = \xi_a(r), \tag{3.19}$$

where we used on the last inequality that $\coth(x) > 1 + e^{-2x}$ for every $x > 0$, proving that $\mu_a(r) - |H_\Sigma| \geq \xi_a(r)$, and the claim follows on the case where $\eta^\top = 0$.

Now, we let $p \in \Sigma$ be such that $\eta^\top \neq 0$ at p, and we can consider $\{E_1,\, E_2,\, \mathcal{N}\}$ a local orthogonal frame of TN with $E_1 = \eta^\top / \left\|\eta^\top\right\|$ on points of Σ. We let $\cos(\theta) = \langle \mathcal{N},\, \eta\rangle$, so $\sin(\theta) = \left\|\eta^\top\right\|$, and (3.16) implies that

$$\langle \nabla_\eta \mathcal{N},\, \eta\rangle = \sin^2(\theta)\langle \nabla_{E_1}\mathcal{N},\, E_1\rangle = \langle \nabla_{E_1}\mathcal{N},\, E_1\rangle - \cos^2(\theta)\langle \nabla_{E_1}\mathcal{N},\, E_1\rangle. \tag{3.20}$$

Moreover, as $\langle \nabla_{\mathcal{N}}\mathcal{N},\, \mathcal{N}\rangle = 0$, we obtain

$$\mathrm{Div}_N(\mathcal{N}) = \langle \nabla_{E_1}\mathcal{N},\, E_1\rangle + \langle \nabla_{E_2}\mathcal{N},\, E_2\rangle. \tag{3.21}$$

Using (3.20) and (3.21), we can find the following expression to φ:

$$\varphi = -\langle \nabla_{E_2}\mathcal{N},\, E_2\rangle - 2H_\Sigma\cos(\theta) - \cos^2(\theta)\langle \nabla_{E_1}\mathcal{N},\, E_1\rangle. \tag{3.22}$$

By simplicity, if $X \in T_pN$ is unitary and perpendicular to $\mathcal{N}$, we denote $f(X) = -\langle \nabla_X\mathcal{N},\, X\rangle$, and notice that, as $\mathcal{N} = -\mathrm{grad}(R)$, the Hessian comparison principle gives us again that

$$f(X) = \mathrm{Hess}(R)(X, X) \geq \mu_a(r) > 0, \tag{3.23}$$

and we may rewrite (3.22) as

$$\begin{aligned}\varphi &= f(E_2) - 2H_\Sigma\cos(\theta) + \cos^2(\theta)f(E_1)\\ &= f(E_1)\left(\cos(\theta) - \frac{H_\Sigma}{f(E_1)}\right)^2 - \frac{H_\Sigma^2}{f(E_1)} + f(E_2)\end{aligned}$$

and using that $f(E_i) \geq \mu_a(r) > 0$, we find the inequality

$$\varphi \geq f(E_2) - \frac{H_\Sigma^2}{f(E_1)} \geq \mu_a(r) - \frac{H_\Sigma^2}{\mu_a(r)} = \frac{\mu_a(r)^2 - H_\Sigma^2}{\mu_a(r)},$$

from where it follows that

$$\varphi \geq \frac{\mu_a(r) + |H_\Sigma|}{\mu_a(r)}(\mu_a(r) - |H_\Sigma|). \tag{3.24}$$

Finally, we can proceed as previously and prove that $\mu_a(r) - |H_\Sigma| \geq \xi_a(r)$, obtaining, as claimed, that $\varphi \geq \xi_a(r)$. $\diamondsuit$

In order to finish the proof of the theorem, we consider a geodesic ball $B_N(p_0, R_0) \supset \Sigma$, so the distance function R satisfies $R \leq R_0$ on Σ. From Claim 3.3, we obtain that the left hand side of (3.13) satisfies

$$\int_\Sigma \big(-\mathrm{Div}_N(\mathcal{N}) - 2H_\Sigma\langle \mathcal{N}, \eta\rangle + \langle \nabla_\eta \mathcal{N}, \eta\rangle\big) \geq \int_\Sigma \xi_a(R) \geq \xi_a(R_0)\mathrm{Area}(\Sigma),$$

as ξ_a is a decreasing function. Applying this inequality on (3.13), together with (3.14), we obtain that

$$\xi_a(R_0)\mathrm{Area}(\Sigma) \leq \mathrm{length}(\partial\Sigma),$$

so we may define $C = 1/\xi_a(R_0)$ to obtain the theorem. $\square$

We notice that the main tool for proving the isoperimetric inequality on Theorem 3.4.1 was actually the estimate obtained on Claim 3.3. We can modify its proof to obtain another isoperimetric inequality, that holds for any Σ on N without the assumption on the uniform bound to the diameter, but with a stronger assumption on the mean curvature. We prove:

Theorem 3.4.2. *Let N be a Hadamard $3-$manifold with sectional curvature $K_N \leq -a^2 < 0$. Let Σ be a compact surface with compact boundary $\partial\Sigma$ and mean curvature function H_Σ satisfying $|H_\Sigma| \leq a(1-\varepsilon)$, for some $\varepsilon > 0$. Then there is a constant $C = C(a, \varepsilon)$ such that*

$$\mathit{Area}(\Sigma) \leq C\mathit{length}(\partial\Sigma).$$

Proof. This proof is analogous to the one of Theorem 3.4.1, with a slight modification when using Claim 3.3. Consider again the foliation of $N \setminus \{p_0\}$ by concentric geodesic spheres, oriented with respect to $\mathcal{N}$ a inward pointing vector field that is normal to the geodesic spheres. If η is an unitary vector field orienting Σ, we can obtain, as in (3.13), (3.14), that

$$\int_\Sigma \big(-\mathrm{Div}_N(\mathcal{N}) - 2H\langle \mathcal{N}, \eta\rangle + \langle \nabla_\eta \mathcal{N}, \eta\rangle\big) \leq \mathrm{length}(\partial\Sigma). \tag{3.25}$$

Now we can restart the proof of Claim 3.3, observing that, on (3.19), we can obtain a stronger estimate,

$$\mu_a(r) - |H_\Sigma| \geq a - |H_\Sigma| \geq a\varepsilon, \tag{3.26}$$

as $\mu_a(r) > a$ and $|H_\Sigma| \leq a(1-\varepsilon)$. As this estimate does not depend on r, we can proceed with the proof as previously done and obtain the result with constant $C = 1/a\varepsilon$. □

3.4.2 Finite topology surfaces in manifolds with negatively bounded sectional curvature

On this section, we find an analogous to Theorem 3.1.1 on the ambient space of a 3-dimensional manifold N with negatively bounded sectional curvature $K_N \leq -a^2 < 0$. The universal cover of such spaces are Hadamard manifolds, so we begin by proving an analogous to Proposition 3.3.3 on this setting:

Proposition 3.4.3. *Let N be a Hadamard manifold with negatively bounded sectional curvature $K_N \leq -a^2 < 0$ and let $E \simeq \mathbb{S}^1 \times [0, +\infty)$ be a complete annulus embedded in N with mean curvature function satisfying $|H_E| \leq a(1-\varepsilon)$, for some $\varepsilon > 0$. Then every divergent sequence of points $(p_n) \subseteq E$ must satisfy $I_E(p_n) \to +\infty$.*

Remark 3.1. Here, the assumption of $|H_E| \leq a(1-\varepsilon)$ for $\varepsilon > 0$ is necessary only if E is properly embedded, as on this case there is no geometric barriers on Hadamard manifolds that come to play the role of Delaunay surfaces as on the proof of Claim 3.2. If the embedding was assumed to be not proper, we could just proceed as on the proof of Claim 3.1 with the assumption $|H_E| \leq a$, using the isoperimetric inequality of Theorem 3.4.1.

Proof of Proposition 3.4.3. As on Section 3.3, using the Gauss equation 3.5, follows from the hypothesis on the mean curvature of E that E has non positive sectional curvature. Then, if the injectivity radius function of E was bounded $I_E \leq L$, this would generate (as on Figure 3.2, Lemma 3.3.1) short geodesic loops γ_n with length length$(\gamma_n) \leq 2L$. Proceeding as on the proof of Claim 3.1 and using the isoperimetric inequality of Theorem 3.4.2, we obtain that E has finite area, which contradicts Theorem 3.2.2. □

With Proposition 3.4.3, we can proceed to manifolds with negatively bounded sectional curvature. As the proof of the next result is completely analogous to the one of Proposition 3.3.5, it will be omitted.

Proposition 3.4.4. *Let N be a 3-manifold with sectional curvature $K_N \leq -a^2 < 0$ and let $E \hookrightarrow N$ be a complete embedding of an annulus $E \equiv \mathbb{S}^1 \times [0, +\infty)$ with mean curvature satisfying $|H_E| \leq a(1-\varepsilon)$, for some $\varepsilon > 0$. Then*

I If the injectivity radius function of E is bounded, the generator of the fundamental group of E is non trivial on N. Moreover, E has finite total curvature.

II If N has positive (possibly infinite) injectivity radius $I_N = \delta$, then, away from ∂E, the injectivity radius function of E satisfies $I_E \geq \delta$.

III If there is a sequence of points $(p_n)_{n\in\mathbb{N}} \subseteq E$, diverging intrinsically on E such that $\lim_{n\to\infty} I_E(p_n) = 0$, then E is properly embedded on N.

Together, Propositions 3.4.3 and 3.4.4 imply a generalization of Theorem 3.1.1:

Theorem 3.4.5. *Let N be a complete 3-manifold with negatively bounded sectional curvature $K_N \leq -a^2 < 0$ and let $\Sigma \hookrightarrow N$ be a complete embedded surface on N, orientable and with finite topology whose mean curvature function H_Σ satisfy $|H_\Sigma| \leq a(1-\varepsilon)$, for some $\varepsilon > 0$. Then:*

A. *If N is simply connected, then each end of Σ has unbounded injectivity radius function.*

B. *If N has positive injectivity radius $I_N = \delta > 0$, Σ has positive injectivity radius.*

C. *If Σ has bounded injectivity radius function, then it has finite total curvature and $i^* : \pi_1(\Sigma) \to \pi_1(N)$ is non trivial. In particular, if Σ has genus zero, it has at least two ends and is π_1-injective.*

D. *If the injectivity radius function of Σ converges to zero at infinity, then the embedding is proper.*

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