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On the Monomial Birational Maps of the Projective Space

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ABSTRACT

We describe the group structure of monomial Cremona transformations. It follows that every element of this group is a product of quadratic monomial transformations, and geometric descriptions in terms of fans.

Key words: monomial birational maps, toric varieties.

1. INTRODUCTION

The best known birational map of $\mathbb{P}^r$ is maybe the *standard Cremona transformation* $S_r : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ defined by $S_r = (X_0^{-1} : \cdots : X_r^{-1})$. For $r = 2$ there is a geometric description in the classic references on this subject; moreover a Max Noether's famous theorem shows that every birational map of $\mathbb{P}^2$ is a composition of automorphisms and S_2, and then that every Cremona transformation is a composition of *quadratic* ones. If $r > 2$ there is no analogous to Noether's theorem (Hudson 1927, Katz 1992, Pan 1999). In this note we consider birational maps generalizing S_r. Our approach is based in a toric point of view and the property that S_r stabilizes the open set $X_0 \cdots X_r \neq 0$. We consider birational maps of $\mathbb{P}^r$ with this property. In the second paragraph of (Russo and Simis 2001) the birationality of these maps is characterized in terms of certain syzygies as an application of a more general criterion; see also (Simis and Villarreal 2002): compare their Proposition 1.1 with our Proposition 3.1.

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2. MONOMIAL CREMONA TRANSFORMATIONS

Let N be a rank r free $\mathbb{Z}$-module, $\mathbb{K}$ an algebraically closed field, $\mathbb{K}^* := \mathbb{K}\backslash\{0\}$. Let $\mathbb{T} = \mathbb{T}_N := N \otimes_{\mathbb{Z}} \mathbb{K}^*$ be the algebraic torus associated to N over $\mathbb{K}$. The action of $\mathbb{T}$ on itself induces a natural inclusion ι of the torus in the group $\mathrm{Aut}(\mathbb{T})$ of the algebraic variety's automorphisms of $\mathbb{T}$; we consider the subgroup $G_{\mathbb{T}}$ of $\mathrm{Aut}(\mathbb{T})$ given by the algebraic group's automorphisms of $\mathbb{T}$. Note that, with an appropriate choice of a basis of N, we may identify $\mathbb{T}$ to $(\mathbb{K}^*)^r$. Since $\mathbb{K}$ is algebraically closed an automorphism in $\mathrm{Aut}(\mathbb{T})$ may be written in the form $F = (F_1, \ldots, F_r)$ with

$$F_i = \lambda_i x_1^{a_{i1}} \cdots x_r^{a_{ir}} \tag{1}$$

where $\lambda_i \in \mathbb{K}^*$, $a_{ij} \in \mathbb{Z}$, $1 \leq i, j \leq r$.

LEMMA 2.1. *There is a split exact sequence of groups*

$$1 \to \mathbb{T} \xrightarrow{\iota} \mathrm{Aut}(\mathbb{T}) \xrightarrow{\phi} G_{\mathbb{T}} \to 1$$

where $\phi(F) := \iota(F(1_{\mathbb{T}})^{-1}) \circ F$, *for* $F \in \mathrm{Aut}(\mathbb{T})$.

PROOF. One has $\phi(F) \in G_{\mathbb{T}}$, because $\lambda_i^{-1} F_i$ is a character of the torus, $1 \leq i \leq r$; it is easily seen that the sequence is exact. The inclusion of $G_{\mathbb{T}}$ in $\mathrm{Aut}(\mathbb{T})$ is a section of ϕ, hence the sequence splits. □

In the following we consider the projective space $\mathbb{P}^r$ over $\mathbb{K}$ as a toric variety, *e.g.*, a compactification of the torus $\mathbb{T} = \mathbb{T}_N$ associated to a complete regular fan of $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ with $(r+1)$ cones of dimension 1. (One standard reference for toric varieties is (Oda 1988)).

DEFINITION 2.2. *A* monomial Cremona transformation *is a birational map* $F : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ *defined on* $\mathbb{T}$ *and such that* $F(\mathbb{T}) \subset \mathbb{T}$.

We note $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ the group of these transformations; there is a natural isomorphism $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{T})$.

PROPOSITION 2.3. *Fixing a basis of the lattice N, there exists an isomorphism*

$$\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r) \simeq \mathbb{T} \rtimes \mathrm{GL}_r(\mathbb{Z}).$$

PROOF. It follows from Lemma 2.1. Given $F \in \mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$, we associate $(F(1_{\mathbb{T}}), (a_{ij}))$, where (a_{ij}) is the $r \times r$-matrix corresponding to $\phi(F) \in G_{\mathbb{T}}$ via the equality (1) and the isomorphism $G_{\mathbb{T}} \xrightarrow{\sim} \mathrm{GL}_r(\mathbb{Z})$ induced by the choice of the basis of N. □

REMARK 2.4. The action of $\mathrm{GL}_r(\mathbb{Z})$ on $\mathbb{T}$ corresponding to the semidirect product in the proposition is

$$(a_{ij}) \cdot (\lambda_1, \ldots, \lambda_r) = \left(\prod_j \lambda_j^{a_{1j}}, \ldots, \prod_j \lambda_j^{a_{rj}} \right).$$

Corollary 2.5. *The center of* $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ *is trivial. The center of the subgroup* $\{F : F(1_{\mathbb{T}}) = 1_{\mathbb{T}}\}$ *is* $\langle S_r \rangle$.

As a consequence of Proposition 2.3 we obtain an analogous to the M. Noether's theorem (on the generators of the Cremona group of $\mathbb{P}^2$) in arbitrary dimension for the monomial case.

Theorem 2.6. *The group* $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ *is generated by a quadratic transformation and the linear monomial automorphisms. Consequently, every birational map is a product of quadratic transformations.*

Proof. It follows from Proposition 2.3 and the fact that $\mathrm{GL}_r(\mathbb{Z})$ is generated by a transvection and two permutations (Coxeter and Moser 1957, Trott 1962) which induce respectively a quadratic transformation and two linear maps in $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$. □

3. DEGREES AND MATRICES

Let $X_0, \ldots, X_r$ be homogeneous coordinates in $\mathbb{P}^r$, $x_i := X_i/X_0$, $1 \leq i \leq r$, the affine coordinates corresponding to the canonical basis of $N = \mathbb{Z}^r$. Every non-constant monomial *rational* map $F : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ satisfying $F(1_{\mathbb{T}}) = 1_{\mathbb{T}}$ may be written uniquely in the form

$$F = (X_0^{\alpha_{00}} \cdots X_r^{\alpha_{0r}} : \cdots : X_0^{\alpha_{r0}} \cdots X_r^{\alpha_{rr}}) \tag{2}$$

such that:

(i) $\alpha(e_j) \in \partial C_+,\ 0 \leq j \leq r$,

(ii) $\alpha(e) = \deg(F)e$, where $\alpha = \alpha_F = (\alpha_{ij})$, $e_j, 0 \leq j \leq r$, is the canonical basis of $\mathbb{Z}^{r+1}$, $e = \sum_j e_j$, C_+ is the cone $\sum_j \mathbb{R}_{\geq 0} e_j$ and ∂C_+ its boundary. The positive integer $\deg(F)$ is the *degree* of F, *e.g.* $\deg(F) = \sum_j \alpha_{ij}, \forall i$.

The map F is *birational* if and only if there exists an integer $(r+1) \times (r+1)$- matrix β, satisfying (i), (ii), and $v \in \mathbb{Z}^{r+1}$ such that:

(iii) $\alpha\beta = Id + e \cdot {}^t v$.

In the affine open set $X_0 \cdots X_r \neq 0$ one may write F as

$$F = (1 : x_1^{a_{11}} \cdots x_r^{a_{1r}} : \cdots : x_1^{a_{r1}} \cdots x_r^{a_{rr}}). \tag{3}$$

The matrix $A := (a_{ij})$ is the matrix of exponents of (1).

Proposition 3.1. *Let* $F \in \mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ *be such that* $F(1_{\mathbb{T}}) = 1_{\mathbb{T}}, \alpha = \alpha_F, d = \deg(F)$. *There exists a unique isomorphism* $\bar{\alpha}$ *such that the following diagram is commutative and with exact lines.*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "e"] \arrow[d, "d"] & \mathbb{Z}^{r+1} \arrow[r] \arrow[d, "\alpha"] & \mathbb{Z}^{r+1}/\mathbb{Z}e \arrow[r] \arrow[d, "\bar{\alpha}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "e"] & \mathbb{Z}^{r+1} \arrow[r] & \mathbb{Z}^{r+1}/\mathbb{Z}e \arrow[r] & 0\,.
\end{tikzcd}$$

One has $d = |\det(\alpha)|$ *and* A *is the matrix of* $\bar{\alpha}$ *in the basis* $\bar{e}_1, \dots, \bar{e}_r$.

PROOF. It follows from properties (ii) and (iii) of α and the form in which α is written in the basis $e, e_1, \dots, e_r$ because (2) and (3) imply $a_{ij} = \alpha_{ij} - \alpha_{0j}$, $1 \leq i, j \leq r$. □

REMARK 3.2.

(a) $\mathbb{Z}^{r+1}/\alpha(\mathbb{Z}^{r+1})$ is cyclic of order d.

(b) We obtain α from A as follows:

$$\alpha_{0j} = -\min(0, a_{ij}, 1 \leq i \leq r), \text{ for } 1 \leq j \leq r;$$

$$\alpha_{00} = \max(0, \sum_j a_{ij}, 1 \leq i \leq r);$$

$$\alpha_{ij} = a_{ij} + \alpha_{0j}, \quad \text{for } 1 \leq i, j \leq r;$$

$$\alpha_{i0} = \alpha_{00} - \sum_j a_{ij}, \quad \text{for } 1 \leq i \leq r.$$

EXAMPLE 3.3. Let A be the matrix whose lines are $(1, 0, 0)$, $(a, 1, 0)$, $(0, a, 1)$, $a \geq 1$. Then

$$F_A = (X_0^{a+1} : X_0^a X_1 : X_1^a X_2 : X_2^a X_3)$$

and

$$F_{A^{-1}} = (X_0^{a^2-a+1} X_1^a X_2^a : X_0^{a^2-a} X_1^{a+1} X_2^a : X_0^{a^2} X_2^{a+1} : X_1^{a^2+a} X_3).$$

EXAMPLE 3.4. Finite subgroup of $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ are obtained by this method. For example the Weyl's group of type $W(D_r)$, $r \geq 3$, and order $2^{r-1}r!$, as a Cremona subgroup may be represented as the subgroup generated by an involution of degree 3

$$F_{\alpha_r} = (X_0 X_{r-1} X_r : X_1 X_{r-1} X_r : \cdots : X_{r-2} X_{r-1} X_r : X_0^2 X_{r-1} : X_0^2 X_r)$$

and the linear automorphisms F_{α_i} permuting X_i with X_{i+1}, $1 \leq i < r$. In an analogous form one obtains a representation of the group $W(B_r)$, $r \geq 2$, of order $2^r r!$, generated by an involution of degree 2

$$F_{\beta_r} = (X_0 X_r : X_1 X_r : \cdots : X_{r-1} X_r : X_0^2)$$

and the preceding $r - 1$ permutations.

4. GEOMETRIC DESCRIPTION, EXAMPLES

Let $F \in \mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$; we denote by B_F its base-scheme. By composing F with an automorphism induced by an element of the torus we may assume that $F(1_{\mathbb{T}}) = 1_{\mathbb{T}}$. *e.g.* $F = (1_{\mathbb{T}}, A) \in \mathbb{T} \rtimes \mathrm{GL}_r(\mathbb{Z})$, we note $F_A = F$; the base-scheme is not changed.

The union of the fundamental hyperplanes $\mathbb{P}^r \backslash \mathbb{T}$ contains the base-points set.

Let $e_1, \dots, e_r$ the canonical basis for $N = \mathbb{Z}^r$, $e := e_1 + \cdots + e_r$, Δ the fan associated to the faces of the simplex $\delta = [e_1, \dots, e_r, -e]$ and $A(\Delta)$ the same with respect to the simplex $A(\delta)$. We consider $\mathbb{P}^r$ defined by Δ; let Σ be a fan that is a common subdivision for Δ and $A(\Delta)$.

Proposition 4.1. *One has a commutative diagram*

$$\begin{tikzcd} & X_\Sigma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^r \arrow[rr, dashed, "F"] & & \mathbb{P}^r \end{tikzcd}$$

where X_Σ is the toric variety associated to Σ, π_1 is equivariant and π_2 is equivariant relatively to the automorphism of the torus induced by A.

Proof. From (3), il follows that F induce $F^* : \mathbb{K}[\mathbb{T}] \to \mathbb{K}[\mathbb{T}]$ which corresponds, by duality, to the lattice's automorphism $A : N \to N$. □

Example 4.2. Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ be, $F_A = (X_0^2, X_0X_1, X_1X_2)$. One obtains a (regular) fan Σ that is a subdivision of Δ and $A(\Delta)$ by blowing-up the closed orbits associated to the cones $\langle e_1, e_2\rangle$ and $\langle e_1, -e\rangle$ and then $\langle -e_2, -e\rangle$.

Example 4.3. Let $A = -Id$, then $F_A = S_r$. One obtains a (regular) fan Σ by the elementary subdivisions of Δ (resp. of $-\Delta$), successively, of the cones of decreasing dimensions from r to 2. For example, if $r = 3$, let Σ_0 (resp. Σ_0') be the fan obtained by the elementary subdivisions of the 4 maximal cones of Δ (resp. $-\Delta$), and Σ_1 (resp. Σ_1'), and the following subdivisions corresponding to the 6 cones of dimension 2 of Δ (resp. $-\Delta$). One has $\Sigma = \Sigma_1 = \Sigma_1'$. The toric variety V_{Σ_0} (resp. $V_{\Sigma_0'}$) is the blowing-up of $\mathbb{P}^3$ in 4 points and V_{Σ_1} (resp. $V_{\Sigma_1'}$) is the blowing-up of these in the strict transforms of the 6 lines. The induced birational map $V_{\Sigma_0} \dashrightarrow V_{\Sigma_0'}$ is composition of 6 *flops, e.g.,* corresponding to small resolutions of singularities of type an affine cone over a smooth quadric, associated to the 6 faces of the convex polyhedron $P_\Delta := Conv(\Delta \cup (-\Delta))$.

On the other hand, the (normalized) blowing-up of the base-scheme B is the toric variety associated to the fan $\Sigma(B)$ defined by the faces of the polyhedron $P(B) = Conv(P_\Delta \cup C)$, where C is the set of sums of the vertices of a diagonal of each 2-face of P_Δ. The toric variety $V_{\Sigma(B)}$ has 12 singular points of type an affine cone over a smooth quadric. Finally, the fan Σ is a regular subdivision of $\Sigma(B)$ and the induced morphism $V_\Sigma \to V_{\Sigma(B)}$ is a minimal resolution of $V_{\Sigma(B)}$.

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RESUMO

Descrevemos a estrutura do grupo das transformações de Cremona monomiais. Concluímos que todo elemento deste grupo é um produto de aplicações monomiais quadráticas e damos descrições geométricas em termos de leques.

Palavras-chave: aplicações birracionais monomiais, variedades tóricas.

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