COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 67, No. 2 2019

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

A Cyclic Analogue of Multiple Zeta Values

by

Minoru HIROSE, Hideki MURAHARA and Takuya MURAKAMI

(Received January 21, 2019)
(Revised October 7, 2019)

Abstract. We consider a cyclic analogue of multiple zeta values (CMZVs), which has two kinds of expressions: series and integral expression. We prove an 'integral=series' type identity for CMZVs. By using this identity, we construct two classes of $\mathbb{Q}$-linear relations among CMZVs. One of them is a generalization of the cyclic sum formula for multiple zeta-star values. We also give an alternative proof of the derivation relation for multiple zeta values.

1. Introduction

For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$ with $k_r \geq 2$, the multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined by

$$\zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

and

$$\zeta^\star(k_1, \ldots, k_r) := \sum_{0<n_1\leq\cdots\leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

We say that an index $(k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$ is admissible if $k_r \geq 2$.

In [5], Nakasuji-Phuksuwan-Yamasaki gave an integral expression of ribbon type Schur multiple zeta values, which is a generalization of the 'integral=series' identity established by Kaneko-Yamamoto [3]. The first main result of this paper is a cyclic analogue of their results. Let $s \in \mathbb{Z}_{\geq 1}$, $r_1, \ldots, r_s \in \mathbb{Z}_{\geq 1}$, and $k_{1,1}, \ldots, k_{1,r_1}, \ldots, k_{s,1}, \ldots, k_{s,r_s} \in \mathbb{Z}_{\geq 1}$. A multi-index $[(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]$ is called an admissible multi-index if

- for all $1 \leq i \leq s$, the index $(k_{i,1}, \ldots, k_{i,r_i})$ is admissible or equal to (1),
- there exists $1 \leq i \leq s$ such that $(k_{i,1}, \ldots, k_{i,r_i}) \neq (1)$.

2010 Mathematics Subject Classification. 11M32.
Key words and phrases. Multiple zeta values, Multiple zeta-star values, Schur multiple zeta values, Cyclic sum formula, Derivation relation.

For an admissible multi-index $\Bbbk = [(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]$, we define the cyclic multiple zeta value (CMZV) by

$$\zeta_{\mathrm{cyc}}(\Bbbk) := \sum_{(n_{1,1},\ldots,n_{s,r_s})\in S} \prod_{i=1}^{s} \prod_{j=1}^{r_i} \frac{1}{n_{i,j}^{k_{i,j}}}, \tag{1.1}$$

where

$$S := \{(n_{1,1}, \ldots, n_{s,r_s}) \in \mathbb{Z}_{\geq 1}^{r_1+\cdots r_s} \mid n_{1,1} < \cdots < n_{1,r_1} \geq n_{2,1} < \cdots < n_{2,r_2} \geq \cdots \geq n_{s,1} < \cdots < n_{s,r_s} \geq n_{1,1}\}.$$

By definition, CMZVs satisfy the cyclic property

$$\zeta_{\mathrm{cyc}}(\Bbbk_1, \ldots, \Bbbk_s) = \zeta_{\mathrm{cyc}}(\Bbbk_i, \ldots, \Bbbk_s, \Bbbk_1, \ldots, \Bbbk_{i-1}) \quad (1 \leq i \leq s).$$

For the convergence of CMZVs, see Remark 4.

THEOREM 1 (Cyclic integral-series identity). *Let* $\Bbbk = [(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]$ *be an admissible multi-index. Put* $k_i := \sum_{j=1}^{r_i} k_{i,j}$. *Then we have*

$$\zeta_{\mathrm{cyc}}(\Bbbk) = \int_D \prod_{i=1}^{s} \prod_{j=1}^{k_i} a_{i,j} dt_{i,j}, \tag{1.2}$$

where

$$a_{i,j} := \begin{cases} \frac{1}{1-t_{i,j}} & j \in \{1, k_{i,1}+1, \ldots, k_{i,1}+\cdots+k_{i,r_i-1}+1\}, \\ \frac{1}{t_{i,j}} & \text{otherwise} \end{cases}$$

and

$$D := \{(t_{1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_s} \mid t_{1,1} < \cdots < t_{1,k_1} > t_{2,1} < \cdots < t_{2,k_2} > \cdots > t_{s,1} < \cdots < t_{s,k_s} > t_{1,1}\}.$$

We call (1.1) (resp. (1.2)) as series (resp. integral) expression of $\zeta_{\mathrm{cyc}}(\Bbbk)$.

The second and third main theorems (Theorems 2 and 3) are two classes of $\mathbb{Q}$-linear relations among CMZVs. Theorem 2 is a generalization of the cyclic sum formula for MZSVs which was proved by Ohno-Wakabayashi [6]. We recall Hoffman's algebraic setup with a slightly different convention (see [1]). We put $\mathfrak{h} := \mathbb{Q}\langle x, y\rangle$. We define subspaces $\mathfrak{h}_C$, $\mathfrak{h}^0$, $\mathfrak{h}^1$, $\mathfrak{h}_C^0$ and $\mathfrak{h}_C^1$ by $\mathfrak{h}_C := \mathfrak{h}x \oplus \mathfrak{h}y$, $\mathfrak{h}^0 := \mathbb{Q} \oplus y\mathfrak{h}x$, $\mathfrak{h}^1 := \mathbb{Q} \oplus y\mathfrak{h}$, $\mathfrak{h}_C^0 := \mathfrak{h}^0 \cap \mathfrak{h}_C$ and $\mathfrak{h}_C^1 := \mathfrak{h}^1 \cap \mathfrak{h}_C$. Put $z_k := yx^{k-1}$ for $k \in \mathbb{Z}_{\geq 1}$. We denote by $\mathfrak{h}^{\mathrm{cyc}}$ the subspace of $\oplus_{s=1}^{\infty} \mathfrak{h}^{\otimes s}$ spanned by

$$\bigcup_{s=1}^{\infty} \{u_1 \otimes \cdots \otimes u_s \in \mathfrak{h}^{\otimes s} \mid u_1, \ldots, u_s \in \mathfrak{h}_C^0 \cup \{y\} \text{ and there exists } j \text{ such that } u_j \neq y\}.$$

We define a $\mathbb{Q}$-linear map $Z_{\mathrm{cyc}} : \mathfrak{h}^{\mathrm{cyc}} \to \mathbb{R}$ by

$$Z_{\mathrm{cyc}}(z_{k_{1,1}} \cdots z_{k_{1,r_1}} \otimes \cdots \otimes z_{k_{s,1}} \cdots z_{k_{s,r_s}}) = \zeta_{\mathrm{cyc}}([(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]).$$

THEOREM 2. *For $u_1 \otimes \cdots \otimes u_s \in \mathfrak{h}^{\mathrm{cyc}}$, we have*

$$\sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (y \,\underline{\sqcup\!\sqcup}\, u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s)$$
$$= \sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_i \otimes y \otimes u_{i+1} \otimes \cdots \otimes u_s)\,,$$

where $y \,\underline{\sqcup\!\sqcup}\, u_i = y \sqcup\!\sqcup u_i - yu_i - u_i y$ (see Section 3.1 for the general definition of $\underline{\sqcup\!\sqcup}$).

THEOREM 3. *For $u_1 \otimes \cdots \otimes u_s \in \mathfrak{h}^{\mathrm{cyc}}$ and $k \in \mathbb{Z}_{\geq 1}$, we have*

$$\sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (z_k \,\underline{*}\, u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s)$$
$$= \sum_{i=1}^{s} Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_i \otimes z_k \otimes u_{i+1} \otimes \cdots \otimes u_s)\,,$$

where $z_k \,\underline{*}\, u_i = z_k * u_i - z_k u_i - u_i z_k$ (see Section 4.1 for the general definition of $\underline{*}$).

This paper is organized as follows. In Sections 2, 3 and 4, we give proofs of Theorems 1, 2 and 3, respectively. In Section 5, we give an alternative proof of the cyclic sum formula for MZSVs (see [6]), the derivation relation for MZVs (see [2]) and the sum formula for MZVs as applications of Theorems 1 and 2.

2. Proof of cyclic integral-series identity

2.1. Nakasuji-Phuksuwan-Yamasaki's integral-series identity for ribbon type Schur MZVs

For the proof of the cyclic integral-series identity, let us introduce the notion of ribbon type Schur MZVs. Let

$$\mathbb{k} = [(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]$$

be an admissible multi-index with $(k_{1,1}, \ldots, k_{1,r_1}) \neq (1)$. Put $k_i := \sum_{j=1}^{r_i} k_{i,j}$. Then the ribbon type Schur MZV $\zeta_{\mathrm{ribbon}}(\mathbb{k})$ is defined by

$$\sum_{(n_{1,1},\ldots,n_{s,r_s}) \in S'} \prod_{i=1}^{s} \prod_{j=1}^{r_i} \frac{1}{n_{i,j}^{k_{i,j}}}\,,$$

where

$$S' := \{(n_{1,1}, \ldots, n_{s,r_s}) \in \mathbb{Z}_{\geq 1}^{r_1+\cdots r_s} \mid n_{1,1} < \cdots < n_{1,r_1} \geq n_{2,1} < \cdots < n_{2,r_2} \geq \cdots \geq n_{s,1} < \cdots < n_{s,r_s}\}\,.$$

REMARK 4. The ribbon type Schur MZV $\zeta_{\mathrm{ribbon}}(\mathbb{k})$ converges for any admissible multi-index such that $(k_{1,1}, \ldots, k_{1,r_1}) \neq (1)$ (see [5, Lemma 2.1]). We can show the convergence of $\zeta_{\mathrm{cyc}}(\mathbb{k})$ for any admissible multi-index by the following way. First, without loss of generality we can assume $\mathbb{k}_1 \neq (1)$ by the cyclic property of CMZVs. Then, since the domain of the summation in $\zeta_{\mathrm{cyc}}(\Bbbk)$ is contained in the one of $\zeta_{\mathrm{ribbon}}(\Bbbk)$, $\zeta_{\mathrm{cyc}}(\Bbbk)$ also converges.

In [5, Section 6.1], Nakasuji-Phuksuwan-Yamasaki gave a following integral expression:

$$\zeta_{\mathrm{ribbon}}(\Bbbk) = \int_{D'} \prod_{i=1}^{s} \prod_{j=1}^{k_i} a_{i,j} dt_{i,j} , \tag{2.1}$$

where

$$a_{i,j} := \begin{cases} \frac{1}{1-t_{i,j}} & j \in \{1, k_{i,1}+1, \ldots, k_{i,1}+\cdots+k_{i,r_i-1}+1\} , \\ \frac{1}{t_{i,j}} & \text{otherwise} \end{cases}$$

and

$$D' :=\{(t_{1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_s} \mid t_{1,1} < \cdots < t_{1,k_1} > t_{2,1} < \cdots < t_{2,k_2} > \cdots > t_{s,1} < \cdots < t_{s,k_s}\} .$$

Note that $S = S' \cap \{n_{s,r_s} \geq n_{1,1}\}$ and $D = D' \cap \{t_{s,k_s} > t_{1,1}\}$.

2.2. Proof of cyclic integral-series identity

In this section, we prove Theorem 1. Let

$$\Bbbk = [(k_{1,1}, \ldots, k_{1,r_1}), \ldots, (k_{s,1}, \ldots, k_{s,r_s})]$$

be an admissible multi-index. Put $\Bbbk_i := (k_{i,1}, \ldots, k_{i,r_i})$ and $k_i := \sum_{j=1}^{r_i} k_{i,j}$. We denote by $\zeta_{\mathrm{cycint}}(\Bbbk)$ the integral expression appeared in Theorem 1. We prove $\zeta_{\mathrm{cyc}}(\Bbbk) = \zeta_{\mathrm{cycint}}(\Bbbk)$ by induction on s. By the cyclic property of CMZVs, without loss of generality, we can assume $\Bbbk_1 \neq (1)$. The case $s = 1$ is just a usual integral expression of a multiple zeta value. Note that we have

$$\begin{aligned} &\{(t_{1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_s} \mid t_{1,1} < \cdots < t_{1,k_1} > t_{2,1} < \cdots < t_{2,k_2} > \cdots > t_{s,1} \\ &\qquad < \cdots < t_{s,k_s}\} \\ =&\{(t_{1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_s} \mid t_{1,1} < \cdots < t_{1,k_1} > t_{2,1} < \cdots < t_{2,k_2} > \cdots > t_{s,1} \\ &\qquad < \cdots < t_{s,k_s} \geq t_{1,1}\} \\ &\sqcup \{(t_{1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_s} \mid t_{1,1} < \cdots < t_{1,k_1} > t_{2,1} < \cdots < t_{2,k_2} > \cdots > t_{s,1} \\ &\qquad < \cdots < t_{s,k_s} < t_{1,1}\} . \end{aligned}$$

Thus from (2.1),

$$\zeta_{\mathrm{ribbon}}([\Bbbk_1, \ldots, \Bbbk_s]) = \zeta_{\mathrm{cycint}}([\Bbbk_1, \ldots, \Bbbk_s]) + \zeta_{\mathrm{cycint}}([\Bbbk_s\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{s-1}]) . \tag{2.2}$$

Here we denote by $\Bbbk_s\Bbbk_1$ the concatenation of $\Bbbk_s$ and $\Bbbk_1$, i.e., $\Bbbk_s\Bbbk_1 := (k_{s,1}, \ldots, k_{s,r_s}, k_{1,1}, \ldots, k_{1,r_1})$. From

$$\begin{aligned} &\{(n_{1,1}, \ldots, n_{s,r_s}) \in \mathbb{Z}_{\geq 1}^{r_1+\cdots r_s} \mid n_{1,1} < \cdots < n_{1,r_1} \geq n_{2,1} < \cdots < n_{2,r_2} \geq \cdots \geq n_{s,1} \\ &\qquad < \cdots < n_{s,r_s}\} \\ =&\{(n_{1,1}, \ldots, n_{s,r_s}) \in \mathbb{Z}_{\geq 1}^{r_1+\cdots r_s} \mid n_{1,1} < \cdots < n_{1,r_1} \geq n_{2,1} < \cdots < n_{2,r_2} \geq \cdots \geq n_{s,1} \\ &\qquad < \cdots < n_{s,r_s} \geq n_{1,1}\} \end{aligned}$$

$$\sqcup \{(n_{1,1}, \dots, n_{s,r_s}) \in \mathbb{Z}_{\geq 1}^{r_1+\cdots r_s} \mid n_{1,1} < \cdots < n_{1,r_1} \geq n_{2,1} < \cdots < n_{2,r_2} \geq \cdots \geq n_{s,1} < \cdots < n_{s,r_s} < n_{1,1}\},$$

we have

$$\zeta_{\mathrm{ribbon}}([\mathbb{k}_1, \dots, \mathbb{k}_s]) = \zeta_{\mathrm{cyc}}([\mathbb{k}_1, \dots, \mathbb{k}_s]) + \zeta_{\mathrm{cyc}}([\mathbb{k}_s\mathbb{k}_1, \mathbb{k}_2, \dots, \mathbb{k}_{s-1}]) . \tag{2.3}$$

From the induction hypothesis, we have

$$\zeta_{\mathrm{cycint}}([\mathbb{k}_s\mathbb{k}_1, \mathbb{k}_2, \dots, \mathbb{k}_{s-1}]) = \zeta_{\mathrm{cyc}}([\mathbb{k}_s\mathbb{k}_1, \mathbb{k}_2, \dots, \mathbb{k}_{s-1}]) . \tag{2.4}$$

From (2.2), (2.3) and (2.4), we have

$$\zeta_{\mathrm{cycint}}([\mathbb{k}_1, \dots, \mathbb{k}_s]) = \zeta_{\mathrm{cyc}}([\mathbb{k}_1, \dots, \mathbb{k}_s]) . \tag{2.5}$$

Thus Theorem 1 is proved.

3. Proof of Theorem 2

3.1. Inner shuffle product

We define the shuffle product $\sqcup\!\sqcup : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ by

$$1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w ,$$
$$uw \sqcup\!\sqcup u'w' = u(w \sqcup\!\sqcup u'w') + u'(uw \sqcup\!\sqcup w') ,$$

where $u, u' \in \{x, y\}$ and $w, w' \in \mathfrak{h}$. We define the inner shuffle product $\underline{\sqcup\!\sqcup} : \mathfrak{h} \times \mathfrak{h}_C \to \mathfrak{h}_C$ by

$$w \,\underline{\sqcup\!\sqcup}\, x = w \,\underline{\sqcup\!\sqcup}\, y = 0 ,$$
$$w \,\underline{\sqcup\!\sqcup}\, uw'u' = u(w \sqcup\!\sqcup w')u' ,$$

where $u, u' \in \{x, y\}$ and $w, w' \in \mathfrak{h}$. Note that we have $y \,\underline{\sqcup\!\sqcup}\, w = y \sqcup\!\sqcup w - yw - wy$ for $w \in \mathfrak{h}_C$.

DEFINITION 5. For $0 < p < q < 1$, let $f_{p,q} : \mathfrak{h}_C \to \mathbb{R}$ be a $\mathbb{Q}$-linear map defined by $f_{p,q}(x) = f_{p,q}(y) = 0$ and

$$f_{p,q}(u_1 \cdots u_k) := \int_{p=t_1<t_2<\cdots<t_{k-1}<t_k=q} a_1 \cdots a_k dt_2 \cdots dt_{k-1}$$

for $k > 1$, where $u_1, \dots, u_k \in \{x, y\}$ and

$$a_i = \begin{cases} \frac{1}{t_i} & u_i = x , \\ \frac{1}{1-t_i} & u_i = y . \end{cases}$$

Here, for $k = 2$, we understand the right-hand side as a_1a_2.

LEMMA 6. *For $0 < p < q < 1$, $k \in \mathbb{Z}_{\geq 0}$, $u_1, \dots, u_k \in \{x, y\}$ and $w \in \mathfrak{h}_C$, we have*

$$f_{p,q}(u_1 \cdots u_k \,\underline{\sqcup\!\sqcup}\, w) = f_{p,q}(w) \int_{p<t_1<\cdots<t_k<q} \prod_{i=1}^{k} a_i dt_i ,$$

where

$$a_i = \begin{cases} \frac{1}{t_i} & u_i = x\,, \\ \frac{1}{1-t_i} & u_i = y\,. \end{cases}$$

Proof. Let $g_{p,q} : \mathfrak{h}_C \to \mathbb{R}$ be a $\mathbb{Q}$-linear map defined by

$$g_{p,q}(v_1 \cdots v_l) := \int_{p<t_1<\cdots<t_l<q} b_1 \cdots b_l dt_1 \cdots dt_l\,,$$

where $v_1, \dots, v_l \in \{x, y\}$ and

$$b_i = \begin{cases} \frac{1}{t_i} & v_i = x\,, \\ \frac{1}{1-t_i} & v_i = y\,. \end{cases}$$

Then, for all $A, B \in \mathfrak{h}$, we have

$$g_{p,q}(A \sqcup\!\sqcup B) = g_{p,q}(A) g_{p,q}(B)\,.$$

The case $w \in \{x, y\}$ is obvious since $u_1 \cdots u_k \,\underline{\sqcup\!\sqcup}\, w = 0$ and $f_{p,q}(w) = 0$. Put $w = vWv'$ $(v, v' \in \{x, y\})$ and

$$\beta := \begin{cases} \frac{1}{p} & v = x \\ \frac{1}{1-p} & v = y \end{cases} \times \begin{cases} \frac{1}{q} & v' = x\,, \\ \frac{1}{1-q} & v' = y\,. \end{cases}$$

Then from the definition, we have

$$f_{p,q}(vAv') = g_{p,q}(A) \cdot \beta$$

for all $A \in \mathfrak{h}$. Thus we get

$$\begin{aligned} f_{p,q}(u_1 \cdots u_k \,\underline{\sqcup\!\sqcup}\, w) &= f_{p,q}(v(u_1 \cdots u_k \sqcup\!\sqcup W)v') \\ &= g_{p,q}(u_1 \cdots u_k \sqcup\!\sqcup W) \cdot \beta \\ &= g_{p,q}(W) \cdot \beta \cdot g_{p,q}(u_1 \cdots u_k) \\ &= f_{p,q}(w) \cdot g_{p,q}(u_1 \cdots u_k)\,. \end{aligned}$$

Since $g_{p,q}(u_1 \cdots u_k) = \int_{p<t_1<\cdots<t_k<q} \prod_{i=1}^{k} a_i dt_i$, the lemma has proved. □

EXAMPLE 7. When $k = 2$, $u_1 = x$, $u_2 = x$ and $w = yx$, we have

$$\begin{aligned} f_{p,q}(x^2 \,\underline{\sqcup\!\sqcup}\, yx) &= f_{p,q}(yxxx) \\ &= \frac{1}{1-p} \cdot \frac{1}{q} \cdot \int_{p<t_2<t_3<q} \frac{dt_2}{t_2} \cdot \frac{dt_3}{t_3} \\ &= f_{p,q}(yx) \cdot \int_{p<t_2<t_3<q} \frac{dt_2}{t_2} \cdot \frac{dt_3}{t_3}\,. \end{aligned}$$

3.2. Proof of Theorem 2

For $s \leq s'$, we put

$$E(s, s', t) := \begin{cases} 1 & s \leq t \leq s', \\ 0 & \text{otherwise}. \end{cases}$$

Assume that $u_1, \ldots, u_s$ are monomials given by $u_i = u_{i,1} \cdots u_{i,k_i}$ with $u_{i,j} \in \{x, y\}$. By definition, we have

$$\begin{aligned} &Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (y \sqcup\!\sqcup u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s) \\ &= \int_{D'} f_{p,q}(y \sqcup\!\sqcup u_i) \left(\prod_{\substack{1 \leq c \leq s \\ c \neq i}} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) dp dq, \end{aligned}$$

where the integral domain $D' \subset \{(t_{1,1}, \ldots, t_{i-1,k_{i-1}}, p, q, t_{i+1,1}, \ldots, t_{s,k_s}) \in (0,1)^{k_1+\cdots+k_{i-1}+2+k_{i+1}+\cdots+k_s}\}$ is given by

$$D' := \{t_{1,1} < \cdots < t_{1,k_1} > \cdots < t_{i-1,k_{i-1}} > p < q > t_{i+1,1} < \cdots < t_{s,k_s} > t_{1,1}\},$$

and $a_{i,j}$ is given by

$$a_{i,j} = \begin{cases} \frac{1}{t_{i,j}} & u_{i,j} = x, \\ \frac{1}{1-t_{i,j}} & u_{i,j} = y. \end{cases}$$

By Lemma 6, we have

$$\begin{aligned} &\int_{D'} f_{p,q}(y \sqcup\!\sqcup u_i) \left(\prod_{\substack{1 \leq c \leq s \\ c \neq i}} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) dp dq \\ &= \int_{D'} f_{p,q}(u_i) \left(\prod_{\substack{1 \leq c \leq s \\ c \neq i}} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) dp dq \int_{0<t<1} E(p, q, t) \frac{dt}{1-t} \\ &= \int_D \left(\prod_{c=1}^{s} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) \int_{0<t<1} E(t_{i,1}, t_{i,k_i}, t) \frac{dt}{1-t}. \end{aligned}$$

Thus we have

$$\begin{aligned} &Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (y \sqcup\!\sqcup u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s) \\ &= \int_D \left(\prod_{c=1}^{s} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) \int_{0<t<1} E(t_{i,1}, t_{i,k_i}, t) \frac{dt}{1-t}. \end{aligned}$$

By definition, we have

$$Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_i \otimes y \otimes u_{i+1} \otimes \cdots \otimes u_s)$$

$$= \int_D \left(\prod_{c=1}^{s} \prod_{j=1}^{k_c} a_{c,j} dt_{c,j} \right) \int_{0<t<1} E(t_{(i+1 \bmod s),1}, t_{i,k_i}, t) \frac{dt}{1-t} ,$$

where $(i+1 \bmod s)$ means $i+1$ for $1 \leq i < s$ and 1 for $i = s$. Thus Theorem 2 follows from

$$\sum_{i=1}^{s} E(t_{i,1}, t_{i,k_i}, t) = \sum_{i=1}^{s} E(t_{(i+1 \bmod s),1}, t_{i,k_i}, t) \quad (t \in (0,1)) .$$

4. Proof of Theorem 3

4.1. Inner harmonic product

We define the harmonic product $* : \mathfrak{h}^1 \times \mathfrak{h}^1 \to \mathfrak{h}^1$ by

$$z_{k_1} \cdots z_{k_r} * z_{l_1} \cdots z_{l_s} := \sum_{d=\max(r,s)}^{r+s} \sum_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ f,g:\text{strictly increasing} \\ \mathrm{Im} f \cup \mathrm{Im} g = \{1,\dots,d\}}} z_{m_1} \cdots z_{m_d} ,$$

where

$$m_i = \begin{cases} k_{f^{-1}(i)} & i \in \mathrm{Im} f \setminus \mathrm{Im} g , \\ l_{g^{-1}(i)} & i \in \mathrm{Im} g \setminus \mathrm{Im} f , \\ k_{f^{-1}(i)} + l_{g^{-1}(i)} & i \in \mathrm{Im} f \cap \mathrm{Im} g . \end{cases}$$

Similarly, we define an inner harmonic product $\underline{*} : \mathfrak{h}^1 \times \mathfrak{h}^1_C \to \mathfrak{h}^1_C$ by

$$z_{k_1} \cdots z_{k_r} \underline{*} z_{l_1} \cdots z_{l_s} := \sum_{d=\max(r,s)}^{r+s} \sum_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ f,g:\text{strictly increasing} \\ \mathrm{Im} f \cup \mathrm{Im} g = \{1,\dots,d\} \\ g(1) \leq f(i) \leq g(s) \text{ for all } i}} z_{m_1} \cdots z_{m_d} ,$$

where the definition of m_i is same as the one in the previous definition. Note that we have $z_k \underline{*} w = z_k * w - z_k w - w z_k$ for $w \in \mathfrak{h}^1_C$ since

$$\begin{aligned} z_k * z_{l_1} \cdots z_{l_s} - z_k \underline{*} z_{l_1} \cdots z_{l_s} &= \sum_{d=\max(1,s)}^{1+s} \sum_{\substack{f:\{1\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ g:\text{strictly increasing} \\ \mathrm{Im} f \cup \mathrm{Im} g = \{1,\dots,d\} \\ f(1)<g(1) \text{ or } g(s)<f(1)}} z_{m_1} \cdots z_{m_d} \\ &= \sum_{\substack{f:\{1\}\to\{1,s+1\} \\ g:\{1,\dots,s\}\to\{1,\dots,s+1\} \\ g:\text{strictly increasing} \\ f(1)<g(1) \text{ or } g(s)<f(1)}} z_{m_1} \cdots z_{m_d} \\ &= z_k z_{l_1} \cdots z_{l_s} + z_{l_1} \cdots z_{l_s} z_k . \end{aligned}$$

Furthermore, we have $u_1 \underline{*} (u_2 \underline{*} u_3) = (u_1 * u_2) \underline{*} u_3$ for $u_1 \in \mathfrak{h}^1$ and $u_2, u_3 \in \mathfrak{h}^1_C$ since both $z_{k_1} \cdots z_{k_r} \underline{*} (z_{l_1} \cdots z_{l_s} \underline{*} z_{m_1} \cdots z_{m_t})$ and $(z_{k_1} \cdots z_{k_r} * z_{l_1} \cdots z_{l_s}) \underline{*} z_{m_1} \cdots z_{m_t}$ are equal to

$$\sum_{d=\max(r,s,t)}^{r+s+t} \sum_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ h:\{1,\dots,t\}\to\{1,\dots,d\} \\ f,g,h:\text{strictly increasing} \\ \mathrm{Im} f\cup \mathrm{Im} g \cup \mathrm{Im} h=\{1,\dots,d\} \\ h(1)\le f(i)\le h(t) \text{ for all } i \\ h(1)\le g(i)\le h(t) \text{ for all } i}} z_{n_1} \cdots z_{n_d} \,,$$

where

$$n_i = \begin{cases} k_{f^{-1}(i)} & i \in \mathrm{Im} f \\ 0 & i \notin \mathrm{Im} f \end{cases} + \begin{cases} l_{g^{-1}(i)} & i \in \mathrm{Im} g \\ 0 & i \notin \mathrm{Im} g \end{cases} + \begin{cases} m_{h^{-1}(i)} & i \in \mathrm{Im} h \,, \\ 0 & i \notin \mathrm{Im} h \,. \end{cases}$$

DEFINITION 8. For positive integers $p \le q$, define a $\mathbb{Q}$-linear map $\lambda_{p,q} : \mathfrak{h}^1_C \to \mathbb{R}$ by

$$\lambda_{p,q}(z_{k_1} \cdots z_{k_r}) := \sum_{p=n_1<\cdots<n_r=q} n_1^{-k_1} \cdots n_r^{-k_r} \,,$$

where $k_1, \dots, k_r \in \mathbb{Z}_{\ge 1}$. Here, for $q - p < r - 1$, we understand the right-hand side as 0, and for $r = 1$, we understand the right-hand side as

$$\sum_{p=n=q} n^{-k_1} = \begin{cases} p^{-k_1} & p = q \,, \\ 0 & p < q \,. \end{cases}$$

LEMMA 9. *For $w \in \mathfrak{h}^1_C$ and positive integers $p \le q, k_1, \dots, k_r$, we have*

$$\lambda_{p,q}(z_{k_1} \cdots z_{k_r} \underline{*} w) = \lambda_{p,q}(w) \sum_{p\le n_1<\cdots<n_r\le q} n_1^{-k_1} \cdots n_r^{-k_r} \,.$$

Proof. Put $w := z_{l_1} \cdots z_{l_s}$. Then we have

$$\lambda_{p,q}(w) \sum_{p\le n_1<\cdots<n_r\le q} n_1^{-k_1} \cdots n_r^{-k_r} = \sum_{(n_1,\dots,n_r,n'_1,\dots,n'_s)\in X} n_1^{-k_1} \cdots n_r^{-k_r} {n'_1}^{-l_1} \cdots {n'_s}^{-l_s}$$

where

$$X := \{(n_1, \dots, n_r, n'_1, \dots, n'_s) \in \mathbb{Z}^{r+s} \mid p \le n_1 < \cdots < n_r \le q,\, p = n'_1 < \cdots < n'_s = q\} \,.$$

Then we decompose X by the pattern of order of $(n_1, \dots, n_r, n'_1, \dots, n'_s)$, i.e.,

$$X = \bigsqcup_{d=\max(r,s)}^{r+s} \bigsqcup_{\substack{f:\{1,\dots,r\}\to\{1,\dots,d\} \\ g:\{1,\dots,s\}\to\{1,\dots,d\} \\ f,g:\text{strictly increasing} \\ \mathrm{Im} f\cup \mathrm{Im} g=\{1,\dots,d\} \\ g(1)\le f(i)\le g(s) \text{ for all } i}} X_{d,f,g} \,,$$

where $X_{d,f,g}$ is the set of $(n_1, \dots, n_r, n'_1, \dots, n'_s)$ such that $n_i = o_{f(i)}$ for $1 \le i \le r$ and $n'_j = o_{g(j)}$ for $1 \le j \le s$ where o_k is the k-th smallest element of $\{n_1, \dots, n_r\} \cup$

$\{n'_1, \ldots, n'_s\}$. Then we have

$$\sum_{(n_1,\ldots,n_r,n'_1,\ldots,n'_s)\in X_{d,f,g}} n_1^{-k_1}\cdots n_r^{-k_r} {n'_1}^{-l_1}\cdots {n'_s}^{-l_s}$$
$$= \sum_{p=o_1<\cdots<o_d=q} o_1^{-m_1}\cdots o_d^{-m_d} = \lambda_{p,q}(z_{m_1}\cdots z_{m_d}),$$

where

$$m_i = \begin{cases} k_{f^{-1}(i)} & i \in \mathrm{Im} f \setminus \mathrm{Im} g, \\ l_{g^{-1}(i)} & i \in \mathrm{Im} g \setminus \mathrm{Im} f, \\ k_{f^{-1}(i)} + l_{g^{-1}(i)} & i \in \mathrm{Im} f \cap \mathrm{Im} g. \end{cases}$$

Thus

$$\sum_{(n_1,\ldots,n_r,n'_1,\ldots,n'_s)\in X} n_1^{-k_1}\cdots n_r^{-k_r} {n'_1}^{-l_1}\cdots {n'_s}^{-l_s} = \lambda_{p,q}(z_{k_1}\cdots z_{k_r} \ast w).$$

Hence the lemma is proved. □

EXAMPLE 10. When $r = 1$ and $w = z_{l_1} z_{l_2}$, we have

$$\begin{aligned}\lambda_{p,q}(z_k \ast z_{l_1} z_{l_2}) &= \lambda_{p,q}(z_{l_1+k} z_{l_2}) + \lambda_{p,q}(z_{l_1} z_k z_{l_2}) + \lambda_{p,q}(z_{l_1} z_{l_2+k}) \\ &= \sum_{p=n_1<n_2=q} \frac{1}{n_1^{l_1+k} n_2^{l_2}} + \sum_{p=n_1<n_2<n_3=q} \frac{1}{n_1^{l_1} n_2^{k} n_3^{l_2}} + \sum_{p=n_1<n_2=q} \frac{1}{n_1^{l_1} n_2^{l_2+k}} \\ &= \sum_{p=n_1<n_2=q} \frac{1}{n_1^{l_1} n_2^{l_2}} \sum_{p\le n\le q} \frac{1}{n^k} \\ &= \lambda_{p,q}(z_{l_1} z_{l_2}) \sum_{p\le n\le q} \frac{1}{n^k}.\end{aligned}$$

4.2. Proof of Theorem 3

For $p \le q$, we put

$$E(p,q,n) := \begin{cases} 1 & p \le n \le q, \\ 0 & \text{otherwise}. \end{cases}$$

By the definition of Z_{cyc},

$$Z_{\mathrm{cyc}}(u_1 \otimes \cdots \otimes u_{i-1} \otimes (z_k \ast u_i) \otimes u_{i+1} \otimes \cdots \otimes u_s)$$
$$= \sum_{(n_{1,1},\ldots,n_{i-1,r_{i-1}},p,q,n_{i+1,1},\ldots,n_{s,r_s})\in S'} \left(\prod_{\substack{1\le c\le s \\ c\ne i}} \prod_{j=1}^{r_c} \frac{1}{n_{c,j}^{k_{c,j}}} \right) \lambda_{p,q}(z_k \ast u_i),$$

where the domain of the summation $S' \subset \mathbb{Z}_{\ge 1}^{r_1+\cdots+r_{i-1}+2+r_{i+1}+\cdots+r_s}$ of $n_{1,1}, \ldots, n_{i-1,r_{i-1}}$, $p, q, n_{i+1,1}, \ldots, n_{s,r_s}$ is defined by

$$\{n_{1,1} < \cdots < n_{1,r_1} \ge n_{2,1} < \cdots < n_{i-1,r_{i-1}} \ge p \le q \ge n_{i+1,1} < \cdots < n_{s,r_s} \ge n_{1,1}\}.$$

By Lemma 9,

$$\sum_{(n_{1,1},\dots,n_{i-1,r_{i-1}},p,q,n_{i+1,1},\dots,n_{s,r_s})\in S'} \left(\prod_{\substack{1\le c\le s\\ c\ne i}}\prod_{j=1}^{r_c}\frac{1}{n_{c,j}^{k_{c,j}}}\right)\lambda_{p,q}(z_k \mathbin{\underline{*}} u_i)$$
$$= \sum_{(n_{1,1},\dots,n_{i-1,r_{i-1}},p,q,n_{i+1,1},\dots,n_{s,r_s})\in S'} \left(\prod_{\substack{1\le c\le s\\ c\ne i}}\prod_{j=1}^{r_c}\frac{1}{n_{c,j}^{k_{c,j}}}\right)\lambda_{p,q}(z_k)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(p,q,n)}{n^k}$$
$$= \sum_{(n_{1,1},\dots,n_{s,r_s})\in S}\left(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}}\right)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(n_{i,1},n_{i,r_i},n)}{n^k}.$$

Thus, we have

$$Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_{i-1}\otimes(z_k \mathbin{\underline{*}} u_i)\otimes u_{i+1}\otimes\cdots\otimes u_s)$$
$$= \sum_{(n_{1,1},\dots,n_{s,r_s})\in S}\left(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}}\right)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(n_{i,1},n_{i,r_i},n)}{n^k}.$$

By definition, we have

$$Z_{\mathrm{cyc}}(u_1\otimes\cdots\otimes u_i\otimes z_k\otimes u_{i+1}\otimes\cdots\otimes u_s)$$
$$= \sum_{(n_{1,1},\dots,n_{s,r_s})\in S}\left(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\frac{1}{n_{i,j}^{k_{i,j}}}\right)\sum_{n\in\mathbb{Z}_{\ge1}}\frac{E(n_{(i+1 \bmod s),1},n_{i,r_i},n)}{n^k},$$

where $(i+1 \bmod s)$ means $i+1$ for $1\le i<s$ and 1 for $i=s$. Thus Theorem 3 follows from

$$\sum_{i=1}^{s}E(n_{i,1},n_{i,r_i},n)=\sum_{i=1}^{s}E(n_{(i+1 \bmod s),1},n_{i,r_i},n)\quad (n\in\mathbb{Z}_{\ge1}).$$

5. Applications of Theorems 1 and 2

5.1. Proof of cyclic sum formula for MZSVs

In this section, we give an alternative proof of the following theorem due to Ohno-Wakabayashi [6] as an application of Theorem 2.

THEOREM 11 ([6, Theorem 1], Cyclic sum formula for MZSVs). *For $k_1,\dots,k_s\in\mathbb{Z}_{\ge1}$ such that $k_1+\cdots+k_s>s$, we have*

$$\sum_{i=1}^{s}\sum_{j=1}^{k_i-1}\zeta^{\star}(k_i-j,k_{i+1},\dots,k_s,k_1,\dots,k_{i-1},j+1)=k\zeta(k+1),$$

where $k=k_1+\cdots+k_s$.

LEMMA 12. *For $k_1,\dots,k_s,l\in\mathbb{Z}_{\geq 1}$ such that $k_s>1$, we have*

$$\zeta_{\mathrm{cyc}}([(k_1),\dots,(k_s)])=\zeta(k_1+\cdots+k_s)\,,$$
$$\zeta_{\mathrm{cyc}}([(l,k_s),(k_{s-1}),\dots,(k_1)])=\zeta^\star(l,k_1,\dots,k_s)-\zeta(l+k_1+\cdots+k_s)\,.$$

Proof. This is an immediate consequence of the series expression of ζ_{cyc}. □

Proof of Theorem 11. Fix $k_1,\dots,k_s\in\mathbb{Z}_{\geq 1}$ such that $k_1+\cdots+k_s>s$. Put $k:=k_1+\cdots+k_s$. By Theorem 2, we have

$$\sum_{i=1}^{s}Z_{\mathrm{cyc}}(z_{k_s}\otimes\cdots\otimes z_{k_{i+1}}\otimes(y\sqcup\!\sqcup z_{k_i})\otimes z_{k_{i-1}}\otimes\cdots\otimes z_{k_1})$$
$$=\sum_{i=1}^{s}Z_{\mathrm{cyc}}(z_{k_s}\otimes\cdots\otimes z_{k_{i+1}}\otimes y\otimes z_{k_i}\otimes\cdots\otimes z_{k_1})\,.\tag{5.1}$$

By the previous lemma, we have

$$Z_{\mathrm{cyc}}(z_{k_s}\otimes\cdots\otimes z_{k_{i+1}}\otimes y\otimes z_{k_i}\otimes\cdots\otimes z_{k_1})=\zeta(k+1)\tag{5.2}$$

for $1\leq i\leq s$. Since

$$y\sqcup\!\sqcup z_l=\sum_{j=1}^{l-1}z_{l-j}z_{j+1}$$

for $l\in\mathbb{Z}_{\geq 1}$, we have

$$Z_{\mathrm{cyc}}(z_{k_s}\otimes\cdots\otimes z_{k_{i+1}}\otimes(y\sqcup\!\sqcup z_{k_i})\otimes z_{k_{i-1}}\otimes\cdots\otimes z_{k_1})$$
$$=\sum_{j=1}^{k_i-1}\zeta^\star(k_i-j,k_{i+1},\dots,k_s,k_1,\dots,k_{i-1},j+1)-(k_i-1)\zeta(k+1)\tag{5.3}$$

for $1\leq i\leq s$. From (5.1), (5.2) and (5.3), we have

$$\sum_{i=1}^{s}\sum_{j=1}^{k_i-1}\zeta^\star(k_i-j,k_{i+1},\dots,k_s,k_1,\dots,k_{i-1},j+1)=k\zeta(k+1)\,.$$

Thus the claim is proved. □

5.2. Algebraic preliminary

For $m\geq 1$, we define derivation maps ∂_m and δ_m on $\mathfrak{h}$ by

$$\delta_m(x)=0,\ \delta_m(y)=yx^{m-1}(x+y)\,,$$
$$\partial_m(x)=y(x+y)^{m-1}x,\ \partial_m(y)=-y(x+y)^{m-1}x\,.$$

By definition, we have

$$\delta_m(z_k)=\delta_m(yx^{k-1})=yx^{m-1}(x+y)x^{k-1}=z_mz_k+z_{m+k}\,.$$

Thus,

$$\delta_m(z_{k_1}\cdots z_{k_d})=\sum_{i=1}^{d}z_{k_1}\cdots z_{k_{i-1}}(z_mz_{k_i}+z_{m+k_i})z_{k_{i+1}}\cdots z_{k_d}\,.$$

Therefore,

$$\delta_m(w) = z_m * w - wz_m = z_m \underline{*} w + z_m w \tag{5.4}$$

for $w \in \mathfrak{h}^1_C$. We denote by $[A, B]$ a commutator $AB - BA$.

LEMMA 13. *For $m \geq 1$, we have*

$$\sum_{j=1}^{m-1} [\delta_j, \partial_{m-j}] = (m-1)(\partial_m + \delta_m).$$

Proof. Put $z := x + y$. we define a derivation $s : \mathfrak{h} \to \mathfrak{h}$ by

$$s(x) = x^2, \ s(z) = z^2.$$

Then we can easily check that

$$[s, \delta_m] = m\delta_{m+1},$$
$$[s, \partial_m] = m\partial_{m+1}.$$

We prove the lemma by induction on m. We can check the case $m \leq 2$ by direct calculation. Take $m \geq 3$ and assume that

$$\sum_{\substack{p+q=m-1 \\ 1\leq p,q\leq m-2}} [\delta_p, \partial_q] = (m-2)(\partial_{m-1} + \delta_{m-1}).$$

From Jacobi identity, we have

$$\begin{aligned} 0 = & \sum_{\substack{p+q=m-1 \\ 1\leq p,q\leq m-2}} \big([s, [\delta_p, \partial_q]] + [\partial_q, [s, \delta_p]] + [\delta_p, [\partial_q, s]]\big) \\ =& (m-2)(m-1)(\partial_m + \delta_m) \\ & - \sum_{\substack{p+q=m-1 \\ 1\leq p,q\leq m-2}} p[\delta_{p+1}, \partial_q] - \sum_{\substack{p+q=m-1 \\ 1\leq p,q\leq m-2}} q[\delta_p, \partial_{q+1}] \\ =& (m-2)(m-1)(\partial_m + \delta_m) - (m-2) \sum_{\substack{p+q=m \\ 1\leq p,q\leq m-1}} [\delta_p, \partial_q]. \end{aligned}$$

Since $m > 2$, we obtain

$$\sum_{\substack{p+q=m \\ 1\leq p,q\leq m-1}} [\delta_p, \partial_q] = (m-1)(\partial_m + \delta_m).$$

Thus the claim is proved. □

5.3. Proof of derivation relation

We define a $\mathbb{Q}$-linear map $Z : \mathfrak{h}^0 \to \mathbb{R}$ by

$$Z(z_{k_1} \cdots z_{k_r}) := \zeta(k_1, \ldots, k_r).$$

In this subsection, we give an alternative proof of the derivation relation

$$Z(\partial_m(w)) = 0 \qquad (m \in \mathbb{Z}_{\geq 1}, w \in \mathfrak{h}^0_C)$$

due to Ihara-Kaneko-Zagier [2]. We put $\{1\}^m := y^m$ and

$$\{1\}^m_\star := \begin{cases} 1 & m = 0\,, \\ y(x+y)^{m-1} & m > 0\,. \end{cases}$$

LEMMA 14. *For $m \geq 1$, we have*

$$m\{1\}^m_\star = \sum_{i=1}^{m} z_i * \{1\}^{m-i}_\star\,, \tag{5.5}$$

$$m\{1\}^m = \sum_{i=1}^{m} (-1)^{i-1} z_i * \{1\}^{m-i}\,, \tag{5.6}$$

$$\sum_{i=0}^{m} (-1)^i \{1\}^{m-i}_\star * \{1\}^i = \begin{cases} 1 & m = 0\,, \\ 0 & m > 0\,. \end{cases} \tag{5.7}$$

Proof. We prove (5.5), first. By definition,

$$\{1\}^n_\star = \sum_{d=1}^{n} \sum_{\substack{k_1+\cdots+k_d=n \\ k_i \geq 1}} z_{k_1} \cdots z_{k_d}$$

for $n \geq 1$. Thus, we have

$$\begin{aligned}
&\sum_{i=1}^{m} z_i * \{1\}^{m-i}_\star \\
&= \sum_{i=1}^{m-1} \sum_{d=1}^{m-i} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_i \geq 1}} z_i * z_{k_1} \ldots z_{k_d} + z_m \\
&= \sum_{i=1}^{m-1} \sum_{d=1}^{m-i} \left(\sum_{j=1}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j+i} \ldots z_{k_d} \right. \\
&\qquad \left. + \sum_{j=0}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j} z_i z_{k_{j+1}} \ldots z_{k_d} \right) + z_m
\end{aligned}$$

$$\begin{aligned}
&= \sum_{d=1}^{m-1} \sum_{i=1}^{m-d} \sum_{j=1}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j+i} \ldots z_{k_d} \\
&\qquad + \sum_{d=1}^{m-1} \sum_{i=1}^{m-d} \sum_{j=0}^{d} \sum_{\substack{k_1+\cdots+k_d=m-i \\ k_l \geq 1}} z_{k_1} \ldots z_{k_j} z_i z_{k_{j+1}} \ldots z_{k_d} + z_m\,.
\end{aligned} \tag{5.8}$$

Here we have

$$\text{The first term of (5.8)} = \sum_{d=1}^{m-1}\sum_{j=1}^{d}\sum_{i=1}^{m-d}\sum_{\substack{k_1+\cdots+k_d=m-i\\ k_l\geq 1}} z_{k_1}\ldots z_{k_j+i}\ldots z_{k_d}$$

$$= \sum_{d=1}^{m-1}\sum_{j=1}^{d}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} (k_j-1) z_{k_1}\ldots z_{k_d}$$

$$= \sum_{d=1}^{m-1}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} (m-d) z_{k_1}\ldots z_{k_d}\,,$$

$$\text{The second term of (5.8)} = \sum_{d=1}^{m-1}\sum_{j=0}^{d}\sum_{i=1}^{m-d}\sum_{\substack{k_1+\cdots+k_d=m-i\\ k_l\geq 1}} z_{k_1}\ldots z_{k_j} z_i z_{k_{j+1}}\ldots z_{k_d}$$

$$= \sum_{d=1}^{m-1}\sum_{\substack{k_1+\cdots+k_{d+1}=m\\ k_l\geq 1}} (d+1) z_{k_1}\ldots z_{k_{d+1}}$$

$$= \sum_{d=2}^{m}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} d z_{k_1}\ldots z_{k_d}$$

$$= \sum_{d=1}^{m}\sum_{\substack{k_1+\cdots+k_d=m\\ k_l\geq 1}} d z_{k_1}\ldots z_{k_d} - z_m\,.$$

Then we get (5.5).

We prove (5.6), next. We note that

$$z_1 * z_1^{m-1} = m z_1^m + \sum_{i=1}^{m-1} z_1^{i-1} z_2 z_1^{m-1-i}\,,$$

$$z_2 * z_1^{m-2} = \sum_{i=1}^{m-1} z_1^{i-1} z_2 z_1^{m-1-i} + \sum_{i=1}^{m-2} z_1^{i-1} z_3 z_1^{m-2-i}\,,$$

$$\cdots$$

$$z_{m-1} * z_1 = \sum_{i=1}^{2} z_1^{i-1} z_{m-1} z_1^{2-i} + z_m\,,$$

$$z_m * 1 = z_m$$

holds. By taking alternating sum, we have (5.6).

The equation (5.7) follows from [4, Proposition 7.1]. □

By Lemma 9 and the series expression of ζ_{cyc}, we have

$$Z_{\mathrm{cyc}}(w \otimes \overbrace{y \otimes \cdots \otimes y}^{m \text{ times}}) = Z(\{1\}_\star^m \underline{*} w)$$

for $w \in \mathfrak{h}_C^0$. By Theorem 2, we have

$$Z_{\mathrm{cyc}}(y \sqcup\!\sqcup w \otimes \overbrace{y \otimes \cdots \otimes y}^{m \text{ times}}) - (m+1) Z_{\mathrm{cyc}}(w \otimes \overbrace{y \otimes \cdots \otimes y}^{m+1 \text{ times}}) = 0\,.$$

Thus, we get the following corollary of Theorem 2.

COROLLARY 15. *For $w \in \mathfrak{h}_C^0$ and $m \geq 0$, we have*

$$Z(F(w,m)) = 0\,,$$

where $F(w,m) := \{1\}_\star^m \underline{*} (y \sqcup\!\sqcup w) - (m+1)\{1\}_\star^{m+1} \underline{*} w$.

In fact, Corollary 15 is essentially derivation relation. More precisely, the following theorem holds.

THEOREM 16. *For $w \in \mathfrak{h}_C^0$ and $m \geq 1$, we have*

$$\sum_{i=1}^{m} (-1)^{i-1} F(y^{i-1} \underline{*} w, m-i) = \partial_m(w)\,.$$

REMARK 17. By Theorem 16 and (5.7), we can obtain

$$F(w, m-1) = \sum_{i=1}^{m} \partial_i(\{1\}_\star^{m-i} \underline{*} w)\,.$$

Thus Theorem 16 implies that the family of relations obtaind by Corollary 15 essentially coincides with the one obtained by the derivation relations.

We prove this theorem in the rest of this section. We prepare some lemmas.

LEMMA 18. *For $m \geq 1$, we have*

$$\sum_{j=1}^{m-1} \partial_{m-j}(z_j) = -(m-1) z_m\,.$$

Proof. By Lemma 13, we have

$$\sum_{j=1}^{m-1} [\delta_j, \partial_{m-j}](x+y) = (m-1)(\partial_m + \delta_m)(x+y)\,.$$

Since

$$\begin{aligned}\sum_{j=1}^{m-1} [\delta_j, \partial_{m-j}](x+y) &= -\sum_{j=1}^{m-1} \partial_{m-j}(y x^{j-1}(x+y)) \\ &= -\sum_{j=1}^{m-1} \partial_{m-j}(z_j)(x+y)\end{aligned}$$

and

$$(\partial_m + \delta_m)(x + y) = z_m(x + y)\,,$$

we have

$$\sum_{j=1}^{m-1} \partial_{m-j}(z_j) = -(m-1)z_m.$$

□

LEMMA 19. *For $m \geq 1$, we have*

$$\sum_{i=0}^{m-1}(-1)^i(m-i)\{1\}_\star^{m-i} * \{1\}^i = z_m\,.$$

Proof. It follows from the following calculation:

$$\begin{aligned}
\sum_{i=0}^{m-1}(-1)^i(m-i)\{1\}_\star^{m-i} * \{1\}^i &= \sum_{i=0}^{m}(-1)^i(m-i)\{1\}_\star^{m-i} * \{1\}^i \\
&= \sum_{i=0}^{m}(-1)^i\left(\sum_{j=1}^{m-i} z_j * \{1\}_\star^{m-i-j}\right) * \{1\}^i \quad \text{(by (5.5))} \\
&= \sum_{j=1}^{m} z_j * \sum_{i=0}^{m-j}(-1)^i\{1\}_\star^{m-j-i} * \{1\}^i \\
&= \sum_{j=1}^{m} z_j * \begin{cases} 1 & j = m\,, \\ 0 & j \neq m \end{cases} \qquad \text{(by (5.7))} \\
&= z_m.
\end{aligned}$$

□

Proof of Theorem 16. Put

$$G_m(w) := \sum_{i=1}^{m}(-1)^{i-1}F(y^{i-1} \,\underline{*}\, w, m-i)\,.$$

By definition, we have

$$G_m(w) = G'_m(w) + G''_m(w)\,,$$

where

$$G'_m(w) = \sum_{i=1}^{m}(-1)^{i-1}\{1\}_\star^{m-i} \,\underline{*}\, (y \,\underline{ш}\, (\{1\}^{i-1} \,\underline{*}\, w))$$

and

$$G''_m(w) = -\sum_{i=1}^{m}(-1)^{i-1}(m-i+1)\{1\}_\star^{m-i+1} \,\underline{*}\, (\{1\}^{i-1} \,\underline{*}\, w)$$

$$= -\left(\sum_{i=1}^{m}(-1)^{i-1}(m-i+1)\{1\}_{\star}^{m-i+1} * \{1\}^{i-1}\right) \underline{*}\, w$$
$$= -z_m \underline{*}\, w\,.$$

Here the last equality follows from Lemma 19. Thus we have

$$G_m(w) = \sum_{i=1}^{m}(-1)^{i-1}\{1\}_{\star}^{m-i} \underline{*}\,(y \sqcup\!\sqcup (\{1\}^{i-1} \underline{*}\, w)) - z_m \underline{*}\, w\,. \tag{5.9}$$

Now we prove $G_m(w) = \partial_m(w)$ by induction on m.

We first prove the case $m = 1$. By the definition of G_1 and (5.4), we have

$$\begin{aligned} G_1(w) &= y \sqcup\!\sqcup\, w - z_1 \underline{*}\, w \\ &= y \sqcup\!\sqcup\, w + yw - \delta_1(w)\,. \end{aligned}$$

Here, by the definition of shuffle product,

$$y \sqcup\!\sqcup\, w + yw = \theta(w)\,,$$

where θ is the derivation map defined by $\theta(x) = yx$ and $\theta(y) = y^2$. Since $(\theta - \delta_1)(x) = \partial_1(x)$ and $(\theta - \delta_1)(y) = \partial_1(y)$, we have $\theta - \delta_1 = \partial_1$. Thus $G_1(w) = (\theta - \delta_1)(w) = \partial_1(w)$. Hence the case $m = 1$ is proved.

For $m \geq 2$, we assume that $G_{m-j}(w) = \partial_{m-j}(w)$ for all $1 \leq j \leq m-1$. By Lemma 13, we have

$$\begin{aligned} &(m-1)(\partial_m + \delta_m)(w) \\ &= \sum_{j=1}^{m-1}(\delta_j \partial_{m-j} - \partial_{m-j}\delta_j)(w) \\ &= \sum_{j=1}^{m-1}(\delta_j G_{m-j} - G_{m-j}\delta_j)(w) \\ &= \sum_{j=1}^{m-1}\big(z_j \underline{*}\, G_{m-j}(w) + z_j G_{m-j}(w) - G_{m-j}(z_j \underline{*}\, w) - G_{m-j}(z_j w)\big)\,. \end{aligned} \tag{5.10}$$

Here the last equality follows from (5.4). Now we have

$$\begin{aligned} &\sum_{j=1}^{m-1}\big(z_j \underline{*}\, G_{m-j}(w) - G_{m-j}(z_j \underline{*}\, w)\big) \\ &= \sum_{j=1}^{m-1}\sum_{i=1}^{m-j}(-1)^{i-1}(z_j * \{1\}_{\star}^{m-i-j}) \underline{*}\,(y \sqcup\!\sqcup (\{1\}^{i-1} \underline{*}\, w)) \\ &\quad - \sum_{j=1}^{m-1}\sum_{i=1}^{m-j}(-1)^{i-1}\{1\}_{\star}^{m-i-j} \underline{*}\,(y \sqcup\!\sqcup ((z_j * \{1\}^{i-1}) \underline{*}\, w)) \end{aligned}$$

$$-\sum_{j=1}^{m-1} z_j \,\underline{*}\, (z_{m-j} \,\underline{*}\, w) + \sum_{j=1}^{m-1} z_{m-j} \,\underline{*}\, (z_j \,\underline{*}\, w) \qquad \text{(by (5.9))}$$

$$= \sum_{i=1}^{m-1}\sum_{j=1}^{m-i} (-1)^{i-1} (z_j * \{1\}_\star^{m-i-j}) \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))$$

$$+ \sum_{k=2}^{m}\sum_{j=1}^{k-1} (-1)^{k-1} \{1\}_\star^{m-k} \,\underline{*}\, (y \sqcup\!\sqcup (((-1)^{j-1} z_j * \{1\}^{k-j-1}) \,\underline{*}\, w)) \quad (k := i + j)$$

$$= \sum_{i=1}^{m-1} (-1)^{i-1} (m-i) \{1\}_\star^{m-i} \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))$$

$$+ \sum_{k=2}^{m} (-1)^{k-1} (k-1) \{1\}_\star^{m-k} \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{k-1} \,\underline{*}\, w)) \qquad \text{(by (5.5), (5.6))}$$

$$= (m-1) \sum_{i=1}^{m} (-1)^{i-1} \{1\}_\star^{m-i} \,\underline{*}\, (y \sqcup\!\sqcup (\{1\}^{i-1} \,\underline{*}\, w))$$

(5.11)

$$= (m-1) G_m(w) + (m-1) z_m \,\underline{*}\, w\,.$$

We also have

$$\sum_{j=1}^{m-1} \big(z_j G_{m-j}(w) - G_{m-j}(z_j w)\big) = \sum_{j=1}^{m-1} \big(z_j \partial_{m-j}(w) - \partial_{m-j}(z_j w)\big)$$

$$= -\sum_{j=1}^{m-1} \partial_{m-j}(z_j) w$$

(5.12)

$$= (m-1) z_m w\,.$$

Here the last equality follows from Lemma 18. From (5.4), (5.10), (5.11) and (5.12), we have

$$(m-1)(\partial_m + \delta_m)(w) = (m-1) G_m(w) + (m-1) z_m \,\underline{*}\, w + (m-1) z_m w$$
$$= (m-1)(G_m(w) + \delta_m(w))\,.$$

Thus the claim $\partial_m(w) = G_m(w)$ is proved. □

5.4. Proof of sum formula

Let $k > r > 0$. Put $\mathbb{k} := [(k-r+1), \overbrace{(1), \ldots, (1)}^{r-1}]$. From the series expression (1.1), we have

$$\zeta_{\mathrm{cyc}}(\mathbb{k}) = \zeta(k)\,.$$

On the other hand, from the integral expression (1.2), we have

$$\zeta_{\mathrm{cyc}}(\mathbb{k}) = \int_{t_{1,1} < \cdots < t_{1,k-r+1} > t_{2,1} > \cdots > t_{r,1} > t_{1,1}} \frac{dt_{1,1}}{1 - t_{1,1}} \frac{dt_{1,2}}{t_{1,2}} \cdots \frac{dt_{1,k-r+1}}{t_{1,k-r+1}} \frac{dt_{2,1}}{1 - t_{2,1}} \cdots \frac{dt_{r,1}}{1 - t_{r,1}}$$

$$
\begin{aligned}
&= Z(y^{r-1} \sqcup\!\sqcup\, z_{k-r+1}) \\
&= \sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r \geq 2}} \zeta(k_1,\ldots,k_r)\,.
\end{aligned}
$$

Then we get the sum formula:

$$
\sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_{r-1}\geq 1, k_r \geq 2}} \zeta(k_1,\ldots,k_r) = \zeta(k)\,.
$$

Acknowledgements. This work was supported by JSPS KAKENHI Grant Numbers JP18J00982, JP18K13392.

References

[1] M. E. Hoffman, 'The algebra of multiple harmonic series', J. Algebra **194** (1997), 477–495.
[2] K. Ihara, M. Kaneko and D. Zagier, 'Derivation and double shuffle relations for multiple zeta values', Compositio Math. **142** (2006), 307–338.
[3] M. Kaneko and S. Yamamoto, 'A new integral-series identity of multiple zeta values and regularizations', Selecta Math. New Ser. **24** (2018), 2499–2521.
[4] G. Kawashima, 'A class of relations among multiple zeta values', J. Number Theory **129** (2009), 755–788.
[5] M. Nakasuji, O. Phuksuwan and Y. Yamasaki, 'On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions', Adv. Math. **333** (2018), 570–619.
[6] Y. Ohno and N. Wakabayashi, 'Cyclic sum of multiple zeta values', Acta Arith. **123** (2006), 289–295.

Minoru HIROSE
Faculty of Mathematics, Kyushu University 744,
Motooka, Nishi-ku, Fukuoka, 819–0395, Japan
e-mail: m-hirose@math.kyushu-u.ac.jp

Hideki MURAHARA
Nakamura Gakuen University Graduate School, 5–7–1,
Befu, Jonan-ku, Fukuoka, 814–0198, Japan
e-mail: hmurahara@nakamura-u.ac.jp

Takuya MURAKAMI
Graduate School of Mathematics, Kyushu University,
744, Motooka, Nishi-ku, Fukuoka, 819–0395, Japan
e-mail: tak_mrkm@icloud.com