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# Analytic Continuation of Double Polylogarithm by Means of Residue Calculus

by

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**Abstract.** Analytic continuations of the double polylogarithm function and Hurwitz-Lerch zeta function are studied. Functional relation formula for each function is derived again without any complicated setting nor theory. The main theoretical basement is Cauchy residue theory.

## 1. Introduction

The polylogarithm function is defined by a power series

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (k \in \mathbb{N}) \tag{1.1}$$

of a complex variable $z$ for $|z| < 1$. It is known that it can be extended to $|z| \geq 1$. For example, several integral representations are known that furnish the analytic continuation of the polylogarithm beyond the unit circle except for the real axis with $\Re z \geq 1$. Actually, the functional relation

$$\mathrm{Li}_k(z) = (-1)^{k+1}\mathrm{Li}_k\left(\frac{1}{z}\right) - \frac{(2\pi i)^k}{k!} B_k\left(\frac{\log z}{2\pi i}\right) \tag{1.2}$$

is known. D. S. Mitrinović and J. D. Kečkić had noticed in [5] a method for deriving the formula (1.2) in the study of Cauchy method of residue. In this paper, we focus on the method introduced in [5] and rederive a functional relation formula of analytic continuation of double polylogarithm

$$\mathrm{Li}_{k_1,k_2}(z) = \sum_{m_1>m_2>0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2}} \quad (k_1, k_2 \in \mathbb{N}) . \tag{1.3}$$

That is, one of our main results is as follows;

**Main Result 1** (Theorem 2.1)
*Let* $k_1, k_2 \in \mathbb{N}$ *and* $0 < \arg z < 2\pi$. *The double polylogarithm* $\mathrm{Li}_{k_1,k_2}(z)$ *is continued analytically for* $\mathbb{C}$ *and the functional relation*

$$\begin{aligned}\mathrm{Li}_{k_1,k_2}(z) &= (-1)^{k_1+k_2}\mathrm{Li}_{k_1,k_2}\left(\frac{1}{z}\right)\\ &+ (-1)^{k_1+k_2}\mathrm{Li}_{k_1+k_2}\left(\frac{1}{z}\right) + \frac{(2\pi i)^{k_1+k_2}}{(k_1+k_2)!}B_{k_1+k_2}\left(\frac{\log z}{2\pi i}\right)\\ &+ (-1)^{k_1}\sum_{m=0}^{k_2}\binom{k_1+m-1}{k_1-1}\mathrm{Li}_{k_1+m}\left(\frac{1}{z}\right)\frac{(2\pi i)^{k_2-m}}{(k_2-m)!}B_{k_2-m}\left(\frac{\log z}{2\pi i}\right)\\ &+ \sum_{m=1}^{k_1}(-1)^m\binom{k_2+m-1}{k_2-1}\zeta(k_2+m)\frac{(2\pi i)^{k_1-m}}{(k_1-m)!}B_{k_1-m}\left(\frac{\log z}{2\pi i}\right)\end{aligned}$$

*holds.*

Restricting the variable $z$ on the unit circle, we have the same formula (Theorem 3.1) with the result of, for example, T. Nakamura (Theorem 2.2 of [6]).

A functional relation of multiple polylogarithm of general depth had been given, for example, by J. Zhao in [9] upon interests of Hodge structure and our main result is the same with depth 2 case in [9] (cf. [8]). Though these previous researches need some complicated structure in general, our approach in this paper is simple, elementary and efficient also for general case. Actually, we have succeeded to derive functional relation with no complicated setting but residue calculus (cf. [3] and [2]).

We also have another result concerning Hurwitz-Lerch zeta function;

**Main Result 2** (Theorem 2.2)
*For a natural number* $k \in \mathbb{N}$ *greater than one,* $s \in \mathbb{C}\backslash\mathbb{Z}$ *and* $z \in \mathbb{C}$ *with* $0 < \arg z < 2\pi$, *Hurwitz-Lerch zeta function* $\Phi(z,k,s)$ *is continued analytically on* $\mathbb{C}$ *and the functional relation*

$$\begin{aligned}z\Phi(z,k,s+1) &= -\frac{1}{s^k} + \frac{(-1)^{k+1}}{z}\Phi(\frac{1}{z},k,1-s)\\ &- z^{-s}\sum_{\substack{m_1+m_2=k-1\\ m_1,m_2\geq 0}}\frac{(-1)^{m_1}\psi^{(m_1)}(1+s)-\psi^{(m_1)}(-s)-\pi i\delta_{m_1,0}}{m_1!}\frac{\log^{m_2} z}{m_2!}\end{aligned}$$

*holds.*

In section 2, functional relations for the double polylogarithm and Hurwitz-Lerch zeta function will be given based on the method introduced in [5]. In section 3, we illustrate the case that the variable $z$ is on the unit circle (cf. [1], [7], [8] and [9], etc.).

## 2. Analytic continuation of double polylogarithm

In this section, we study functional relations for analytic continuation of double polylogarithm and Hurwitz-Lerch zeta function. The main theoretical tools are Cauchy's residues and Painlevé's theorem.

### 2.1. Double polylogarithm

For $k_1, k_2 \in \mathbb{N}$, the power series

$$\mathrm{Li}_{k_1,k_2}(z) = \sum_{n_1>n_2>0} \frac{z^{n_1}}{{n_1}^{k_1}{n_2}^{k_2}} = \sum_{n=2}^{\infty} \frac{z^n}{n^{k_1}} \sum_{m=1}^{n-1} \frac{1}{m^{k_2}} \tag{2.1}$$

$$\mathrm{Li}^{\star}_{k_1,k_2}(z) = \sum_{n_1\geq n_2>0} \frac{z^{n_1}}{{n_1}^{k_1}{n_2}^{k_2}}$$

are continuous in $|z| \leq 1$ except for $z = 1$ if $k_1 = 1$, and holomorphic in $|z| < 1$. The function $\mathrm{Li}_{k_1,k_2}(z)$ is called the double polylogarithm.

Let $f_{k_1,k_2}(z; s)$ be a function defined by

$$f_{k_1,k_2}(z; s) = \frac{2\pi i}{e^{2\pi i s} - 1} \frac{z^s}{s^{k_1}} \frac{(-1)^{k_2-1}}{(k_2 - 1)!} \{\psi^{(k_2-1)}(s) - \psi^{(k_2-1)}(1)\} \tag{2.2}$$

where $\psi^{(k)}(s)$ is the polygamma function

$$\psi^{(k)}(s) = \frac{d^{k+1}}{ds^{k+1}} \log \Gamma(s)$$

with the gamma function $\Gamma(s)$ and a complex parameter $z$ satisfying $0 < \arg z < 2\pi$. Then, $f_{k_1,k_2}(z; s)$ is meromorphic on $\mathbb{C}$ having simple poles at positive integers greater than one, $(k_2 + 1)$-st poles at negative integers and a $k_1 + k_2 + 1$-st pole at the origin.

Using the difference formula

$$\psi^{(n)}(s + 1) = \psi^{(n)}(s) + \frac{(-1)^n n!}{s^{n+1}}$$

of the polygamma function, we have

$$\begin{aligned}\operatorname*{Res}_{s=n} f_{k_1,k_2}(z; s) &= \frac{z^n}{n^{k_1}} \frac{(-1)^{k_2-1}}{(k_2 - 1)!} \left\{\psi^{(k_2-1)}(n) - \psi^{(k_2-1)}(1)\right\} \\ &= \frac{z^n}{n^{k_1}} \sum_{m=1}^{n-1} \frac{1}{m^{k_2}}\end{aligned}$$

for $n = 2, 3, \ldots$. Thus, by the definition (2.1), we have the following;

PROPOSITION 2.1. *For* $|z| \leq 1$,

$$\mathrm{Li}_{k_1,k_2}(z) = \sum_{n=2}^{\infty} \operatorname*{Res}_{s=n} f_{k_1,k_2}(z; s)\,.$$

Since $s=0$ is the $k_1+k_2+1$-st pole, we have

$$\begin{aligned}
&\operatorname*{Res}_{s=0} f_{k_1,k_2}(z;s)\\
&= \frac{1}{(k_1+k_2)!}\lim_{s\to 0}\frac{d^{k_1+k_2}}{ds^{k_1+k_2}} s^{k_1+k_2+1} f_{k_1,k_2}(z;s)\\
&= -\frac{(2\pi i)^{k_1+k_2}}{(k_1+k_2)!} B_{k_1+k_2}\left(\frac{\log z}{2\pi i}\right)\\
&\quad + (-1)^{k_1}\sum_{m=0}^{k_1-1}(-1)^{m+1}\frac{(2\pi i)^m}{m!}B_m\left(\frac{\log z}{2\pi i}\right)\binom{k_1+k_2-m-1}{k_2-1}\zeta(k_1+k_2-m)
\end{aligned}$$

where $B_m(x)$ is Bernouilli polynomial defined by

$$\frac{se^{xs}}{e^s-1} = \sum_{m=0}^{\infty}\frac{B_m(x)}{m!}s^m .$$

Since any negative integer $s=-n$ $(n\in\mathbb{N})$ is $(k_2+1)$-st pole, we have

$$\begin{aligned}
&\operatorname*{Res}_{s=-n} f_{k_1,k_2}(z;s)\\
&= \frac{1}{k_2!}\lim_{s\to -n}\frac{d^{k_2}}{ds^{k_2}}\sum_{j=0}^{\infty}\frac{(2\pi i)^j}{j!}B_j\left(\frac{\log z}{2\pi i}\right)(s+n)^j\frac{1}{z^n}\frac{1}{s^{k_1}}\\
&\quad\times\left\{\frac{(-1)^{k_2-1}}{(k_2-1)!}(s+n)^{k_2}(\psi^{(k_2-1)}(s+n+1)-\psi^{(k_2-1)}(1)) - 1 - \sum_{m=1}^{n-1}\frac{(s+n)^{k_2}}{(s+m)^{k_2}}\right\}\\
&= \frac{1}{z^n}\left\{\frac{1}{n^{k_1}}(-1)^{k_1+k_2+1}\sum_{m=1}^{n}\frac{1}{m^{k_2}}\right.\\
&\quad\left. +(-1)^{k_1+1}\sum_{m=0}^{k_2}\frac{1}{n^{k_1+m}}\frac{(k_1)_m}{m!}\frac{(2\pi i)^{k_2-m}}{(k_2-m)!}B_{k_2-m}\left(\frac{\log z}{2\pi i}\right)\right\} .
\end{aligned}$$

Thus, on $|z|\geq 1$, we have the following;

LEMMA 2.1. *For* $|z|\geq 1$ *with* $0<\arg z<2\pi$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\operatorname*{Res}_{s=-n} f_{k_1,k_2}(z;s)\\
&= (-1)^{k_1+k_2+1}\mathrm{Li}^{\star}_{k_1,k_2}\left(\frac{1}{z}\right)\\
&\quad + (-1)^{k_1+1}\sum_{m=0}^{k_2}\binom{k_1+m-1}{k_1-1}\mathrm{Li}_{k_1+m}\left(\frac{1}{z}\right)\frac{(2\pi i)^{k_2-m}}{(k_2-m)!}B_{k_2-m}\left(\frac{\log z}{2\pi i}\right)
\end{aligned}$$

*holds.*

Let $l_\infty$ be a contour which comes from $+\infty$ on the real axis, loops around 1 in the positive sense along a circle with the small radius $\varepsilon>0$ and returns to $+\infty$, and $l_{-\infty}$ a contour which comes from $-\infty$ on the real axis, loops around 0 in the positive sense along a circle with the radius $\varepsilon$ and returns to $-\infty$. Let $c$ be a positive real number between 0 and 1 satisfying $\varepsilon < c < 1-\varepsilon$. For a large number $N \in \mathbb{N}$, let $l_N$ be a finite part of $l_\infty$ with $\Re s \le N+\frac{1}{2}$, and $l_{-N}$ a finite part of $l_{-\infty}$ with $\Re s \ge -N-\frac{1}{2}$.

By the residue theorem, we have

$$\frac{1}{2\pi i}\left(\int_{l_N}+\int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}+i\varepsilon}\right) f_{k_1,k_2}(z;s)ds = \sum_{n=2}^{N}\operatorname*{Res}_{s=n} f_{k_1,k_2}(z;s) = \sum_{n=2}^{N}\frac{z^n}{n^{k_1}}\sum_{m=1}^{n-1}\frac{1}{m^{k_2}} \tag{2.3}$$

and

$$\frac{1}{2\pi i}\left(\int_{l_{-N}}+\int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}-i\varepsilon}\right) f_{k_1,k_2}(z;s)ds = \sum_{n=0}^{N}\operatorname*{Res}_{s=-n} f_{k_1,k_2}(z;s) \tag{2.4}$$

where $\int_\alpha^\beta$ represents the line integral along the segment between $\alpha$ and $\beta$ for $\alpha, \beta \in \mathbb{C}$.

Since $0 < \arg z < 2\pi$,

$$e^{\Im s \arg z}|e^{2\pi i s}-1| \ge \begin{cases} |1-e^{2\pi i s}| & \text{for } \Im s \ge 0\,, \\ |1-e^{-2\pi i s}| & \text{for } \Im s < 0 \end{cases} \tag{2.5}$$

hold. For $|z| \le 1$, if $k_2 \ge 2$, by the asymptotic formula

$$\psi^{(n)}(s) = O(|s|^{-n}) \quad (|s| \to \infty, |\arg s| < \pi)$$

for any $n \in \mathbb{N}$, there exists a positive integer $C > 0$ so that

$$\frac{1}{(k_2-1)!}|\psi^{(k_2-1)}(s+1)-\psi^{(k_2-1)}(1)| \le C$$

with $|s|$ large enough and $|\arg s| < \pi$, and if $k_2 = 1$, by the asymptotic formula

$$\psi(s) = \log s + O(|s|^{-1}) \quad (|s| \to \infty, |\arg s| < \pi)\,,$$

there exists a positive integer $C > 0$ so that

$$|\psi(s)-\psi(1)| \le C|s|^\delta \tag{2.6}$$

for any $\delta > 0$ and any $s \in \mathbb{C}$ with $|s|$ large enough and $|\arg s| < \pi$. Then, we have the estimate

$$\begin{aligned} |f_{k_1,k_2}(z;s)| &= \frac{2\pi|z|^{\Re s}}{|s|^{k_1}e^{\Im s \arg z}|e^{2\pi i s}-1|}\frac{1}{(k_2-1)!}|\psi^{(k_2-1)}(s)-\psi^{(k_2-1)}(1)| \\ &\le \begin{cases} \dfrac{2\pi|z|^{\Re s}C}{|s|^{k_1-\delta}|1-e^{2\pi i s}|} & (\Im s \ge 0)\,, \\ \dfrac{2\pi|z|^{\Re s}C}{|s|^{k_1-\delta}|1-e^{-2\pi i s}|} & (\Im s < 0) \end{cases} \end{aligned} \tag{2.7}$$

with $C > 0$, and $\delta = 0$ if $k_2 \ge 2$ and $0 < \delta < 1$ if $k_2 = 1$.

Since

$$\int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}+i\varepsilon} |f_{k_1,k_2}(z;s)||ds|$$

$$= \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}} |f_{k_1,k_2}(z;s)||ds| + \int_{N+\frac{1}{2}}^{N+\frac{1}{2}+i\varepsilon} |f_{k_1,k_2}(z;s)||ds|$$

$$\leq \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}} \frac{2\pi|z|^{\Re s}}{|s|^{k_1-\delta}|1-e^{-2\pi is}|}|ds| + \int_{N+\frac{1}{2}}^{N+\frac{1}{2}+i\varepsilon} \frac{2\pi|z|^{\Re s}}{|s|^{k_1-\delta}|1-e^{2\pi is}|}|ds|$$

$$\leq \frac{2\pi|z|^{N+\frac{1}{2}}}{(N+\frac{1}{2})^{k_1-\delta}}\left(\int_{\varepsilon}^{0} \frac{|d\tau|}{|1-e^{2\pi i(N+\frac{1}{2}-i\tau)}|} + \int_{0}^{\varepsilon} \frac{|d\tau|}{|1-e^{2\pi i(N+\frac{1}{2}+i\tau)}|}\right)$$

$$\leq \frac{2\pi}{(N+\frac{1}{2})^{k_1-\delta}}\left(\int_{\varepsilon}^{0} \frac{|d\tau|}{|1+e^{2\pi\tau}|} + \int_{0}^{\varepsilon} \frac{|d\tau|}{|1+e^{-2\pi\tau}|}\right)$$

$$\leq \frac{4\pi\varepsilon}{(N+\frac{1}{2})^{k_1-\delta}} \to 0 \tag{2.8}$$

holds for $|z| \leq 1$ as $N \to +\infty$, we have

$$\frac{1}{2\pi i}\lim_{N\to+\infty}\left(\int_{l_N} + \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}+i\varepsilon}\right) f_{k_1,k_2}(z;s)ds = \frac{1}{2\pi i}\int_{l_\infty} f_{k_1,k_2}(z;s)ds\,.$$

By (2.3), we have a contour integral representation of the double polylogarithm.

LEMMA 2.2.

$$\mathrm{Li}_{k_1,k_2}(z) = \frac{1}{2\pi i}\int_{l_\infty} \frac{2\pi i}{e^{2\pi is}-1}\frac{z^s}{s^{k_1}}\frac{(-1)^{k_2-1}}{(k_2-1)!}\{\psi^{(k_2-1)}(s) - \psi^{(k_2-1)}(1)\}ds \tag{2.9}$$

*with* $0 < \arg z < 2\pi$.

Since

$$\int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}-i\varepsilon} |f_{k_1,k_2}(z;s)||ds|$$

$$= \int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}} |f_{k_1,k_2}(z;s)||ds| + \int_{-N-\frac{1}{2}}^{-N-\frac{1}{2}-i\varepsilon} |f_{k_1,k_2}(z;s)||ds|$$

$$\leq \int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}} \frac{2\pi|z|^{\Re s}}{|s|^{k_1-\delta}|1-e^{2\pi is}|}|ds| + \int_{-N-\frac{1}{2}}^{-N-\frac{1}{2}-i\varepsilon} \frac{2\pi|z|^{\Re s}}{|s|^{k_1-\delta}|1-e^{-2\pi is}|}|ds|$$

$$\leq \frac{2\pi|z|^{-N-\frac{1}{2}}}{(N+\frac{1}{2})^{k}}\left(\int_{\varepsilon}^{0} \frac{|d\tau|}{|1-e^{2\pi i(-N-\frac{1}{2}+i\tau)}|} + \int_{0}^{\varepsilon} \frac{|d\tau|}{|1-e^{2\pi i(-N-\frac{1}{2}-i\tau)}|}\right)$$

$$\leq \frac{2\pi}{(N+\frac{1}{2})^{k_1-\delta}}\left(\int_{\varepsilon}^{0} \frac{|d\tau|}{|1+e^{-2\pi\tau}|} + \int_{0}^{\varepsilon} \frac{|d\tau|}{|1+e^{2\pi\tau}|}\right)$$

$$\leq \frac{4\pi\varepsilon}{(N+\frac{1}{2})^{k_1-\delta}} \to 0 \tag{2.10}$$

for $|z| \geq 1$ as $N \to +\infty$, we have

$$\frac{1}{2\pi i} \lim_{N\to\infty} \left( \int_{l_{-N}} + \int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}-i\varepsilon} \right) f_{k_1,k_2}(z;s)ds = \frac{1}{2\pi i} \int_{l_{-\infty}} f_{k_1,k_2}(z;s)ds\,.$$

Thus, by Lemma 2.1 and (2.4), we have the following;

LEMMA 2.3.

$$\begin{aligned}\frac{1}{2\pi i} \int_{l_{-\infty}} f_{k_1,k_2}(z;s)ds &= \sum_{n=0}^{\infty} \operatorname*{Res}_{s=-n} f_{k_1,k_2}(z;s) \\ &= (-1)^{k_1+k_2+1} \mathrm{Li}^{\star}_{k_1,k_2}\left(\frac{1}{z}\right) \\ &+ (-1)^{k_1+1} \sum_{m=0}^{k_2} \binom{k_1+m-1}{k_1-1} \mathrm{Li}_{k_1+m}\left(\frac{1}{z}\right) \frac{(2\pi i)^{k_2-m}}{(k_2-m)!} B_{k_2-m}\left(\frac{\log z}{2\pi i}\right) \\ &- \frac{(2\pi i)^{k_1+k_2}}{(k_1+k_2)!} B_{k_1+k_2}\left(\frac{\log z}{2\pi i}\right) \\ &+ (-1)^{k_1} \sum_{m=0}^{k_1-1} (-1)^{m+1} \frac{(2\pi i)^m}{m!} B_m\left(\frac{\log z}{2\pi i}\right) \binom{k_1+k_2-m-1}{k_2-1} \zeta(k_1+k_2-m)\end{aligned} \tag{2.11}$$

*holds. It is continuous on* $|z| \geq 1$ *except for* $z = 1$ *and holomorphic in* $|z| > 1$.

Let $C_N$ be a rectangle, oriented clockwise, of vertices at $N + \frac{1}{2} - iN$, $N + \frac{1}{2} + iN$, $c + iN$ and $c - iN$ jumped between $N + \frac{1}{2} + i\varepsilon$ and $N + \frac{1}{2} - i\varepsilon$, and $C_{-N}$ a rectangle, oriented clockwise, of vertices at $-N - \frac{1}{2} + iN$, $-N - \frac{1}{2} - iN$, $c - iN$ and $c + iN$ jumped between $-N - \frac{1}{2} - i\varepsilon$ and $-N - \frac{1}{2} + i\varepsilon$. Put $s = \sigma + i\tau$. When $\sigma = N + \frac{1}{2}$,

$$\begin{aligned}|e^{2\pi i s} - 1|^2 &= e^{-4\pi\tau} - 2e^{-2\pi\tau}\cos 2\pi\sigma + 1 \\ &\geq 1 + e^{-4\pi\tau} \geq 1\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}|f_{k_1,k_2}(z;s)| &\leq C \frac{e^{(N+\frac{1}{2})\log|z| - \tau \arg z}}{(N+\frac{1}{2})^{k_1-\delta}} \\ &\leq C \frac{|z|^{(N+\frac{1}{2})} e^{-\tau \arg z}}{N^{k_1-\delta}}\end{aligned} \tag{2.13}$$

hold. Since $0 < \arg z < 2\pi$, we have

$$\left| \int_{N+\frac{1}{2}+iN}^{N+\frac{1}{2}+i\varepsilon} f_{k_1,k_2}(z;s)ds \right| \leq C \frac{|z|^{N+\frac{1}{2}}}{N^{k_1-\delta}} \int_N^{\varepsilon} e^{-\tau \arg z} d\tau$$

$$= C \frac{|z|^{N+\frac{1}{2}}}{N^{k_1-\delta}} \frac{e^{-\varepsilon \arg z} - e^{-N \arg z}}{\arg z}$$

$$\leq \frac{C}{N^{k_1-\delta}} \frac{e^{-\varepsilon \arg z} - e^{-N \arg z}}{\arg z} \to 0 \tag{2.14}$$

for $|z| \leq 1$ as $N$ tends to $+\infty$. When $s = N + \frac{1}{2} - i\tau$ $(\tau > 0)$, by the estimate

$$|e^{2\pi i s} - 1| \geq (1 + e^{4\pi\tau})^{\frac{1}{2}} \geq e^{2\pi\tau}, \tag{2.15}$$

$$\left| \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}-iN} f_{k_1,k_2}(z;s)ds \right| \leq C \frac{|z|^{N+\frac{1}{2}}}{N^{k_1-\delta}} \int_{\varepsilon}^{N} \frac{1}{e^{-\tau \arg z}|e^{2\pi i s} - 1|} d\tau$$

$$\leq \frac{C}{N^{k_1-\delta}} \int_{\varepsilon}^{N} e^{(\arg z - 2\pi)\tau} d\tau$$

$$= \frac{C}{N^{k_1-\delta}} \frac{e^{(\arg z - 2\pi)N} - e^{(\arg z - 2\pi)\varepsilon}}{\arg z - 2\pi}$$

$$\to 0 \tag{2.16}$$

for $|z| \leq 1$ as $N$ tends to $+\infty$. We also have

$$\int_{N+\frac{1}{2}-iN}^{c-iN} f_{k_1,k_2}(z;s)ds \to 0, \quad \int_{c+iN}^{N+\frac{1}{2}+iN} f_{k_1,k_2}(z;s)ds \to 0 \tag{2.17}$$

as $N \to \infty$. Since

$$\int_{C_N+l_N} f_{k_1,k_2}(z;s)ds = \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}-iN} f_{k_1,k_2}(z;s)ds + \int_{N+\frac{1}{2}-iN}^{c-iN} f_{k_1,k_2}(z;s)ds$$
$$+ \int_{c+iN}^{N+\frac{1}{2}+iN} f_{k_1,k_2}(z;s)ds + \int_{N+\frac{1}{2}+iN}^{N+\frac{1}{2}+i\varepsilon} f_{k_1,k_2}(z;s)ds$$
$$+ \int_{c-iN}^{c+iN} f_{k_1,k_2}(z;s)ds + \int_{l_N} f_{k_1,k_2}(z;s)ds,$$

$$\int_{c-i\infty}^{c+i\infty} f_{k_1,k_2}(z;s)ds + \int_{l_\infty} f_{k_1,k_2}(z;s)ds = 0 \tag{2.18}$$

holds by estimates (2.14), (2.16), (2.17) and Cauchy's integral theorem, where the integral $\int_{c-i\infty}^{c+i\infty}$ indicates the line integral along $\Re s = c$. By Lemma 2.2, we obtain an integral representation of the double polylogarithm.

PROPOSITION 2.2. *For* $|z| < 1$,

$$\mathrm{Li}_{k_1,k_2}(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2\pi i}{e^{2\pi i s} - 1} \frac{z^s}{s^{k_1}} \frac{(-1)^{k_2}}{(k_2-1)!} \{\psi^{(k_2-1)}(s) - \psi^{(k_2-1)}(1)\} ds \tag{2.19}$$

*with* $0 < \arg z < 2\pi$.

By the same argument, we have

$$\int_{c-iN}^{-N-\frac{1}{2}-iN} f_{k_1,k_2}(z;s)ds \to 0, \quad \int_{-N-\frac{1}{2}-iN}^{-N-\frac{1}{2}-i\varepsilon} f_{k_1,k_2}(z;s)ds \to 0\,,$$
$$\int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}+iN} f_{k_1,k_2}(z;s)ds \to 0, \quad \int_{-N-\frac{1}{2}+iN}^{c+iN} f_{k_1,k_2}(z;s)ds \to 0$$

and thus

$$\int_{c+i\infty}^{c-i\infty} f_{k_1,k_2}(z;s)ds + \int_{l_{-\infty}} f_{k_1,k_2}(z;s)ds = 0\,. \tag{2.20}$$

By (2.18) and (2.20), we have the following;

LEMMA 2.4. *On* $|z| = 1$ *except for* $z = 1$*,*

$$\int_{l_\infty} f_{k_1,k_2}(z;s)ds = -\int_{c-i\infty}^{c+i\infty} f_{k_1,k_2}(z;s)ds = -\int_{l_{-\infty}} f_{k_1,k_2}(z;s)ds$$

*hold.*

Let us recall the following theorem;

**Painlevé's theorem**
If $\Gamma$ is a rectifiable Jordan curve lying in a domain $D$ in the complex plane and if a function $f$ is continuous in $D$ and analytic in $D \setminus \Gamma$, then $f$ is an analytic function in the entire domain $D$.

By Lemma 2.2, Lemma 2.3, Lemma 2.4 and Painlevé's theorem, we have the following;

THEOREM 2.1. *Let* $k_1, k_2 \in \mathbb{N}$ *and* $0 < \arg z < 2\pi$*. The double polylogarithm* $\mathrm{Li}_{k_1,k_2}(z)$ *is continued analytically for* $\mathbb{C}$ *and the functional relation*

$$\begin{aligned}\mathrm{Li}_{k_1,k_2}(z) &= (-1)^{k_1+k_2}\mathrm{Li}_{k_1,k_2}\left(\frac{1}{z}\right)\\ &\quad + (-1)^{k_1+k_2}\mathrm{Li}_{k_1+k_2}\left(\frac{1}{z}\right) + \frac{(2\pi i)^{k_1+k_2}}{(k_1+k_2)!}B_{k_1+k_2}\left(\frac{\log z}{2\pi i}\right)\\ &\quad + (-1)^{k_1}\sum_{m=0}^{k_2}\binom{k_1+m-1}{k_1-1}\mathrm{Li}_{k_1+m}\left(\frac{1}{z}\right)\frac{(2\pi i)^{k_2-m}}{(k_2-m)!}B_{k_2-m}\left(\frac{\log z}{2\pi i}\right)\\ &\quad + \sum_{m=1}^{k_1}(-1)^m\binom{k_2+m-1}{k_2-1}\zeta(k_2+m)\frac{(2\pi i)^{k_1-m}}{(k_1-m)!}B_{k_1-m}\left(\frac{\log z}{2\pi i}\right)\end{aligned}$$

*holds.*

### 2.2. Hurwitz-Lerch zeta function

For any $\alpha, \beta \in \mathbb{C}$, the function

$$\Phi(z, \beta, \alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(\alpha + n)^{\beta}}$$

is called Hurwitz-Lerch zeta function, which is holomorphic on $|z| < 1$ under the condition $\alpha \neq 0, -1, -2, \ldots$. Especially for $\Re\beta > 1$, it is continuous on $|z| = 1$. In [4], C. Ferreira and J. L. López had studied analytic continuation of $\Phi(z, \beta, \alpha)$. They had given two kinds of considerations under conditions of $\alpha$. In this section, we give a functional relation of analytic continuation of Hurwitz-Lerch zeta function using the method in [5] under a different condition with [4].

For $k \in \mathbb{N}$, let $g_k(s, z; t)$ define a function of a variable $t$ with complex parameters $s, z \in \mathbb{C}$ $(s \notin \mathbb{Z}, 0 < \arg z < 2\pi)$ given by

$$g_k(s, z; t) = \frac{2\pi i}{e^{2\pi i t} - 1} \frac{z^t}{(s + t)^k}, \tag{2.21}$$

which is a holomorphic function on $\mathbb{C}$ excepting for simple poles at $t = n$ $(n \in \mathbb{Z})$ and a $k$-th pole at $t = -s$. By (2.5), we have

$$\begin{aligned} |g_k(s, z; t)| &= \frac{2\pi |z|^{\Re t}}{|s + t|^k e^{\Im t \arg z} |e^{2\pi i t} - 1|} \\ &\leq \begin{cases} \dfrac{2\pi |z|^{\Re t}}{|s + t|^k |e^{2\pi i t} - 1|} & (\Im t \geq 0), \\ \dfrac{2\pi |z|^{\Re t}}{|s + t|^k |e^{-2\pi i t} - 1|} & (\Im t < 0). \end{cases} \end{aligned} \tag{2.22}$$

Since the residue of $g_k(s, z; t)$ at $t = n \in \mathbb{Z}$ is given by

$$\operatorname*{Res}_{t=n} g_k(s, z; t) = \frac{z^n}{(s + n)^k},$$

we have, for $|z| \leq 1$,

$$\sum_{n=1}^{\infty} \operatorname*{Res}_{t=n} g_k(s, z; t) = \sum_{n=1}^{\infty} \frac{z^n}{(s + n)^k} = z\Phi(z, k, s + 1) \tag{2.23}$$

which is holomorphic in $|z| < 1$ with $s \notin \mathbb{Z}$, and, for $|z| \geq 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} \operatorname*{Res}_{t=-n} g_k(s, z; t) &= \sum_{n=0}^{\infty} \frac{z^{-n}}{(s - n)^k} \\ &= \frac{1}{s^k} + \frac{(-1)^k}{z} \sum_{n=0}^{\infty} \frac{z^{-n}}{(n + 1 - s)^k} \\ &= \frac{1}{s^k} + \frac{(-1)^k}{z} \Phi(\frac{1}{z}, k, 1 - s) \end{aligned} \tag{2.24}$$

which is holomorphic in $|z| > 1$ with $s \notin \mathbb{N}$. Both functions (2.23) and (2.24) of variable $z$ are continuous on the unit circle $|z| = 1$ if $k > 1$. The residue of $g_k(s, z; t)$ at $t = -s$ is

$$\begin{aligned}
&\operatorname*{Res}_{t=-s} g_k(s, z; t) \\
&= \frac{1}{(k-1)!} \lim_{t \to -s} \frac{d^{k-1}}{dt^{k-1}} \left[ \frac{2\pi i z^t}{e^{2\pi i t} - 1} \right] \\
&= \frac{1}{(k-1)!} \lim_{t \to -s} \frac{d^{k-1}}{dt^{k-1}} [(\pi \cot \pi t + 2\pi i B_1) z^t] \\
&= \frac{1}{(k-1)!} \lim_{t \to -s} \sum_{\substack{m_1+m_2=k-1 \\ m_1, m_2 \geq 0}} \frac{(k-1)!}{m_1! m_2!} \left[ \frac{d^{m_1}}{dt^{m_1}} (\pi \cot \pi t - \pi i) \right] \left[ \frac{d^{m_2}}{dt^{m_2}} z^t \right] \\
&= z^{-s} \sum_{\substack{m_1+m_2=k-1 \\ m_1, m_2 \geq 0}} \frac{(-1)^{m_1} \psi^{(m_1)}(1+s) - \psi^{(m_1)}(-s) - \pi i \delta_{m_1,0}}{m_1!} \frac{\log^{m_2} z}{m_2!} \qquad (2.25)
\end{aligned}$$

which is continuous in $|z| \geq 1$ and holomorphic in $|z| > 1$ where $\delta_{m,0}$ is the Kronecker delta symbol

$$\delta_{m,0} = \begin{cases} 1 & (m = 0), \\ 0 & (m \neq 0). \end{cases}$$

Let $N > \max\{|\Re s|, |\Im s|\}$. For $|z| \leq 1$, since

$$\begin{aligned}
&\int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}+i\varepsilon} |g_k(s, z; t)||dt| \\
&= \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}} |g_k(s, z; t)||dt| + \int_{N+\frac{1}{2}}^{N+\frac{1}{2}+i\varepsilon} |g_k(s, z; t)||dt| \\
&\leq \int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}} \frac{2\pi |z|^{\Re t}}{|s+t|^k |e^{-2\pi i t} - 1|} |dt| + \int_{N+\frac{1}{2}}^{N+\frac{1}{2}+i\varepsilon} \frac{2\pi |z|^{\Re t}}{|s+t|^k |e^{2\pi i t} - 1|} |dt| \\
&\leq \int_{-\varepsilon}^{0} \frac{2\pi |z|^{N+\frac{1}{2}}}{|\Re s + i\Im s + N + \frac{1}{2} + i\eta|^k |1 - e^{-2\pi i (N+\frac{1}{2}+i\eta)}|} |d\eta| \\
&\quad + \int_{0}^{\varepsilon} \frac{2\pi |z|^{N+\frac{1}{2}}}{|\Re s + i\Im s + N + \frac{1}{2} + i\eta|^k |e^{2\pi i (N+\frac{1}{2}+i\eta)} - 1|} |d\eta| \\
&\leq \int_{-\varepsilon}^{0} \frac{2\pi}{||N+\frac{1}{2}| - |\Re s||^k |1 + e^{2\pi\eta}|} |d\eta| + \int_{0}^{\varepsilon} \frac{2\pi}{||N+\frac{1}{2}| - |\Re s||^k |1 + e^{-2\pi\eta}|} |d\eta| \\
&\leq \frac{4\pi\varepsilon}{(N+\frac{1}{2} - |\Re s|)^k} \to 0 \ (N \to +\infty)
\end{aligned}$$

(2.26)

holds, we have

$$\frac{1}{2\pi i}\int_{l_\infty} g_k(s,z;t)dt = \sum_{n=1}^{\infty} \operatorname*{Res}_{t=n} g_k(s,z;t) = z\Phi(z,k,s+1)\,. \tag{2.27}$$

Similarly, we have

$$\frac{1}{2\pi i}\int_{l_{-\infty}} g_k(s,z;t)dt = \sum_{n=0}^{\infty} \operatorname*{Res}_{t=-n} g_k(s,z;t) = \frac{1}{s^k} + \frac{(-1)^k}{z}\Phi(\frac{1}{z},k,1-s) \tag{2.28}$$

Assume that $\Re s \geq 0$. Since there are no singular points in the domain bounded by $C_N + l_N$, and $t = -s$ is the only singularity in the domain bounded by $C_{-N} + l_{-N}$,

$$\begin{aligned} &\int_{C_N+l_N} g_k(s,z;t)dt = 0\,, \\ &\int_{C_{-N}+l_{-N}} g_k(s,z;t)dt = -2\pi i \operatorname*{Res}_{t=-s} g_k(s,z;t) \end{aligned} \tag{2.29}$$

hold. Using the estimate (2.22) of $g_k(s,z;t)$, we can see, by similar arguments with the above section, that

$$\int_{c+iN}^{N+\frac{1}{2}+iN} g_k(s,z;t)dt, \quad \int_{N+\frac{1}{2}+iN}^{N+\frac{1}{2}+i\varepsilon} g_k(s,z;t)dt\,,$$

$$\int_{N+\frac{1}{2}-i\varepsilon}^{N+\frac{1}{2}-iN} g_k(s,z;t)dt, \quad \int_{N+\frac{1}{2}-iN}^{c-iN} g_k(s,z;t)dt$$

tend to 0 for $|z| \leq 1$ as $N$ tends to $+\infty$, and

$$\int_{c-iN}^{-N-\frac{1}{2}-iN} g_k(s,z;t)dt, \quad \int_{-N-\frac{1}{2}-iN}^{-N-\frac{1}{2}-i\varepsilon} g_k(s,z;t)dt\,,$$

$$\int_{-N-\frac{1}{2}+i\varepsilon}^{-N-\frac{1}{2}+iN} g_k(s,z;t)dt, \quad \int_{-N-\frac{1}{2}+iN}^{c+iN} g_k(s,z;t)dt$$

tend to 0 for $|z| \geq 1$ as $N$ tends to $+\infty$. Then, we have

$$\begin{aligned} &\int_{C_N} g_k(s,z;t)dt \to \int_{c-i\infty}^{c+i\infty} g_k(s,z;t)dt\,, \\ &\int_{C_{-N}} g_k(s,z;t)dt \to \int_{c+i\infty}^{c-i\infty} g_k(s,z;t)dt \end{aligned} \tag{2.30}$$

as $N \to +\infty$. Thus by (2.29) and (2.30), we have

$$\begin{aligned} &\int_{l_\infty} g_k(s,z;t)dt = -\int_{c-i\infty}^{c+i\infty} g_k(s,z;t)dt\,, \\ &\int_{l_{-\infty}} g_k(s,z;t)dt = -\int_{c+i\infty}^{c-i\infty} g_k(s,z;t)dt - 2\pi i \operatorname*{Res}_{t=-s} g_k(s,z;t)\,. \end{aligned} \tag{2.31}$$

For the case $\Re s < 0$, taking $c$ satisfying $0 < c < \min\{\Re(-s), 1\}$ and redefine $l_\infty, l_{-\infty}$ and $C_N, C_{-N}, l_N, l_{-N}$ in the same manner with the above section. Then, for $N > \max\{|\Im s|, |\Re s|\}$, we have

$$\begin{aligned}\int_{C_N+l_N} g_k(s,z;t)dt &= -2\pi i \operatorname*{Res}_{t=-s} g_k(s,z;t)\,,\\ \int_{C_{-N}+l_{-N}} g_k(s,z;t)dt &= 0\,.\end{aligned} \tag{2.32}$$

Thus we have

$$\begin{aligned}\int_{l_\infty} g_k(s,z;t)dt &= -\int_{c-i\infty}^{c+i\infty} g_k(s,z;t)dt - 2\pi i \operatorname*{Res}_{t=-s} g_k(s,z;t)\,,\\ \int_{l_{-\infty}} g_k(s,z;t)dt &= -\int_{c+i\infty}^{c-i\infty} g_k(s,z;t)dt\,.\end{aligned} \tag{2.33}$$

For both cases, we have the following;

LEMMA 2.5. *On $|z| = 1$ except for $z = 1$,*

$$\int_{l_\infty} g_k(s,z;t)dt = -\int_{l_{-\infty}} g_k(s,z;t)dt - 2\pi i \operatorname*{Res}_{t=-s} g_k(s,z;t)$$

*holds.*

By (2.25), (2.27), (2.28), (2.31), (2.32) and (2.33) and Painlevé's theorem, we have the following;

THEOREM 2.2. *For a natural number $k \in \mathbb{N}$ greater than one, $s \in \mathbb{C}\backslash\mathbb{Z}$ and $z \in \mathbb{C}$ with $0 < \arg z < 2\pi$, Hurwitz-Lerch zeta function $\Phi(z,k,s)$ is continued analytically on $\mathbb{C}$ and the functional relation*

$$\begin{aligned} z\Phi(z,k,s+1) &= -\frac{1}{s^k} + \frac{(-1)^{k+1}}{z}\Phi\Big(\frac{1}{z},k,1-s\Big)\\ &\quad - z^{-s}\sum_{\substack{m_1+m_2=k-1\\ m_1,m_2\geq 0}} \frac{(-1)^{m_1}\psi^{(m_1)}(1+s)-\psi^{(m_1)}(-s)-\pi i\delta_{m_1,0}}{m_1!}\frac{\log^{m_2} z}{m_2!}\end{aligned}$$

*holds.*

## 3. On the unit circle case

For $\theta \in \mathbb{R}$, let us consider $z = e^{2\pi i\theta}$ case for $f_{k_1,k_2}(z;s)$ of (2.2) and $g_k(z;s)$ of (2.21). By (2.7), we have

$$|f_{k_1,k_2}(e^{2\pi i\theta};s)| \leq \begin{cases} \dfrac{2\pi C}{|s|^{k_1-\delta}|1-e^{2\pi is}|} & (\Im s \geq 0)\,,\\ \dfrac{2\pi C}{|s|^{k_1-\delta}|1-e^{-2\pi is}|} & (\Im s < 0)\end{cases}$$

with $C > 0$, and $\delta = 0$ if $k_1 \geq 2$ and $0 < \delta < 1$ if $k_1 = 1$. Thus, for $k_1 \geq 2$, by (2.12),

$$\int_{N+\frac{1}{2}+iN}^{N+\frac{1}{2}+i\varepsilon} |f_{k_1,k_2}(e^{2\pi i\theta}; s)||ds| \leq \int_{N+\frac{1}{2}+iN}^{N+\frac{1}{2}+i\varepsilon} \frac{2\pi C}{|s|^{k_1-\delta}|1-e^{2\pi is}|}|ds|$$
$$= \frac{2\pi C(N-\varepsilon)}{(N+\frac{1}{2})^{k_1-\delta}}$$

tends to zero as $N$ tends to $+\infty$. We can also see that all integrals along segments between $N+\frac{1}{2}+iN$ and $N+\frac{1}{2}+i\varepsilon$, $N+\frac{1}{2}-iN$ and $N+\frac{1}{2}-i\varepsilon$, $-N-\frac{1}{2}+iN$ and $-N-\frac{1}{2}+i\varepsilon$, and, $-N-\frac{1}{2}-iN$ and $-N-\frac{1}{2}-i\varepsilon$ tend to zero as $N \to \infty$. Thus, we have the following result;

THEOREM 3.1. *Let $k_1, k_2 \in \mathbb{N}$ ($k_1 \geq 2$). Then,*

$$\begin{aligned}\mathrm{Li}_{k_1,k_2}(e^{2\pi i\theta}) &= (-1)^{k_1+k_2}\mathrm{Li}_{k_1,k_2}(e^{-2\pi i\theta}) \\ &\quad + (-1)^{k_1+k_2}\mathrm{Li}_{k_1+k_2}(e^{-2\pi i\theta}) + \frac{(2\pi i)^{k_1+k_2}}{(k_1+k_2)!}B_{k_1+k_2}(\theta) \\ &\quad + (-1)^{k_1}\sum_{m=0}^{k_2}\binom{k_1+m-1}{k_1-1}\mathrm{Li}_{k_1+m}(-e^{2\pi i\theta})\frac{(2\pi i)^{k_2-m}}{(k_2-m)!}B_{k_2-m}(\theta) \\ &\quad + \sum_{m=1}^{k_1}(-1)^m\binom{k_2+m-1}{k_2-1}\zeta(k_2+m)\frac{(2\pi i)^{k_1-m}}{(k_1-m)!}B_{k_1-m}(\theta)\,.\end{aligned}$$

That is, restricting the variable $z$ on the unit circle, we have the same formula with Theorem 2.2 in [6].

Similarly, we have the following;

THEOREM 3.2. *Let $k \in \mathbb{N}$ ($k \geq 2$), and $s \in \mathbb{C}\backslash\mathbb{Z}$. Then,*

$$\begin{aligned}e^{2\pi i\theta}\Phi(e^{2\pi i\theta}, k, s+1) &= -\frac{1}{s^k} + (-1)^{k+1}e^{-2\pi i\theta}\Phi(e^{-2\pi i\theta}, k, 1-s) \\ &\quad - e^{-2\pi is\theta}\sum_{\substack{m_1+m_2=k-1\\ m_1,m_2\geq 0}}\frac{(-1)^{m_1}\psi^{(m_1)}(1+s)-\psi^{(m_1)}(-s)-\pi i\delta_{m_1,0}}{m_1!}\frac{(2\pi i)^{m_2}\theta^{m_2}}{m_2!}\,.\end{aligned}$$

Substituting $\theta = 0$ for the results of Theorem 3.1, we have the formula which is the same with that given in [1].

COROLLARY 3.1 (cf. [1]). *Assume that $k_1 + k_2$ is odd number. Then,*

$$\zeta(k_1, k_2) = -\frac{1}{2}\zeta(k_1 + k_2) + (-1)^{k_1+1} \sum_{m=0}^{[\frac{k_2}{2}]} \binom{k_1 + k_2 - 2m - 1}{k_1 - 1} \zeta(k_1 + k_2 - 2m)\zeta(2m) + (-1)^{k_1+1} \sum_{m=0}^{[\frac{k_1-1}{2}]} \binom{k_1 + k_2 - 2m - 1}{k_2 - 1} \zeta(k_1 + k_2 - 2m)\zeta(2m)$$

*holds.*

Substituting $\theta = 0$ for the result of Theorem 3.2, we have the obvious formula

$$\frac{(-1)^k \psi^{(k-1)}(1 + s) + \psi^{(k-1)}(-s)}{(k-1)!} = \sum_{j \in \mathbb{Z}} \frac{1}{(s + j)^k}$$

of the polygamma function.

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