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Polytope Duality for Families of $K3$ Surfaces Associated to Transpose Duality

by

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Abstract. We consider whether or not transpose-dual pairs, which is a Berglund–Hübsch mirror studied by Ebeling and Ploog [3], extend to a polytope duality that has a potential to be lattice dual.

1. Introduction

Isolated singularities in $\mathbb{C}^3$ are classified by Arnold [1] among which there are classes called *bi*modal and *uni*modal. Our notation follows that of Arnold's. Not only the classification, Arnold also finds that there is a duality among unimodal singularities that is called *Arnold's strange duality*. The duality is also related to toric geometry and lattice theory. Ebeling and Ploog [3] find an analogous duality concerning bimodal and other singularities, which is actually a Berglund–Hübsch mirror.

Batyrev's proposal [2] of polar duality of *reflexive* polytopes gives a breakthrough in a construction of mirror partner for *toric* Calabi-Yau hypersurfaces and later complete intersections.

Being origined in physics, there appear a numerical meanings of "mirror" such as cohomological mirror, among which in this article we focus on a relation between Ebeling and Ploog's transpose duality and Batyrev's polytope duality associating with bimodal singularities in some manner.

In a series of recent studies, it is concluded that transpose-dual pairs (Q_{12}, E_{18}), $(Z_{1,0}, E_{19})$, (E_{20}, E_{20}), $(Q_{2,0}, Z_{17})$, (E_{25}, Z_{19}), (Q_{18}, E_{30}) of singularities can extend to a lattice duality by the author [6] following an extension to polytope duality by the author and Ueda [5]. However, those pairs in the list $(*)$ below fail to extend to a lattice duality in spite of the fact that they are polytope dual.

$$(*)\quad \begin{array}{l}(Z_{13}, J_{3,0}), (Z_{1,0}, Z_{1,0}), (Z_{17}, Q_{2,0}), (U_{1,0}, U_{1,0}), (U_{16}, U_{16}),\\ (Q_{17}, Z_{2,0}), (W_{1,0}, W_{1,0}), (W_{17}, S_{1,0}), (W_{18}, W_{18}), (S_{17}, X_{2,0}).\end{array}$$

More precisely, for each pair one obtains in [5] reflexive polytopes $\Delta_{[MU]}$ and $\Delta'_{[MU]}$ satisfying that the polar dual of $\Delta_{[MU]}$ is isomorphic to $\Delta'_{[MU]}$ and that $\Delta_{[MU]}$ and $\Delta'_{[MU]}$

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respectively contains the Newton polytope of a compactification polynomial of the defining polynomial of singularities. Despite this fact it is concluded in [6] that the corresponding pairs of families $\mathcal{F}_{\Delta_{[MU]}}$ and $\mathcal{F}_{\Delta'_{[MU]}}$ of $K3$ surfaces are not lattice dual, that is, the Picard lattices $\mathrm{Pic}(\Delta_{[MU]})$ and $\mathrm{Pic}(\Delta'_{[MU]})$ of these families do not satisfy an isometry $\mathrm{Pic}(\Delta_{[MU]})^{\perp}_{\Lambda_{K3}} \simeq U \oplus \mathrm{Pic}(\Delta'_{[MU]})$. Moreover, for these pairs we can observe that the restriction map $H^{1,1}(\widetilde{\mathbb{P}_{\Delta_{[MU]}}}, \mathbb{Z}) \to H^{1,1}(\widetilde{Z}, \mathbb{Z})$ for the minimal model of any generic member $Z \in \mathcal{F}_{\Delta_{[MU]}}$ is not surjective.

The aim of the study is to consider the following problem arisen by Professor Ashikaga's question:

PROBLEM. *Let $((B, f), (B', f'))$ be a transpose-dual pair in the list $(*)$ together with their defining polynomials f and f'. Determine whether or not it is possible to take polynomials F and F' that are respectively compactifications of f and f', and a reflexive polytope Δ such that the following condition $(**)$ holds*:

$$(**) \qquad \Delta_F \subset \Delta,\ \Delta_{F'} \subset \Delta^*, \quad \textit{and} \quad L_0(\Delta) = 0\,.$$

Here, Δ_F and $\Delta_{F'}$ denote respectively the Newton polytopes of F and of F', and Δ^* is the polar dual polytope of Δ.

The main theorem of this paper is stated as follows:

MAIN THEOREM (Theorem 3.1). For each of the following pairs

$$(Z_{1,0},\ Z_{1,0}),\ (U_{1,0},\ U_{1,0}),\ (Q_{17},\ Z_{2,0}),\ (W_{1,0},\ W_{1,0})\,,$$

there exist compactifications $F,\ F'$ and reflexive polytopes Δ and Δ' such that

$$(**) \qquad \Delta^* \simeq \Delta',\ \Delta_F \subset \Delta,\ \Delta_{F'} \subset \Delta', \quad \textit{and} \quad \operatorname{rank} L_0(\Delta) = 0$$

hold. Moreover, $\rho(\Delta) + \rho(\Delta') = 20$.

It can be conjectured that there do not exist reflexive polytopes for pairs $(Z_{13},\ J_{3,0})$, $(Z_{17},\ Q_{2,0})$, $(U_{16},\ U_{16})$, $(W_{17},\ S_{1,0})$, $(W_{18},\ W_{18})$, $(S_{17},\ X_{2,0})$ of singularities satisfying the condition $(**)$. We leave the judgement about this conjecture to a further study in the furure.

Section 2 is devoted to recall some facts as to a polytope duality associated to singularities. The proof of the main theorem is given in Theorem 3.1 in section 3, where we explicitely give compactifications and reflexive polytopes for these pairs.

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2. Preliminary

Recall that a *Gorenstein* $K3$ *surface* is a compact complex connected 2-dimensional algebraic variety S with at most ADE singularities satisfying $K_S \sim 0$ and $H^1(S, \mathcal{O}_S) = 0$. If a Gorenstein $K3$ surface is nonsingular, we simply call it a $K3$ *surface*.

Let $M \simeq \mathbb{Z}^3$ be a 3-dimensional lattice and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \simeq \mathbb{Z}^3$ the dual of M with a natural pairing $\langle\ ,\ \rangle : N \times M \to \mathbb{Z}$. Let Δ be a 3-dimensional polytope, that is, Δ is a convex hull of finitely-many points in $M \otimes_{\mathbb{Z}} \mathbb{R}$. The associated toric 3-fold is denoted by $\mathbb{P}_\Delta$. The *polar dual* Δ^* of Δ is defined by

$$\Delta^* = \{y \in N \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle y, x\rangle \geq -1 \quad \text{for all} \quad x \in \Delta\}\ .$$

Let us recall a toric description of weighted projective spaces. Let $\boldsymbol{a} = (a_0, a_1, a_2, a_3)$ be a well-posed quadruple of natural numbers and $d = a_0 + a_1 + a_2 + a_3$. Define a 3-dimensional lattice $\tilde{M}$ by

$$\tilde{M} := \left\{(i, j, k, l) \in \mathbb{Z}^4 \mid a_0 i + a_1 j + a_2 k + a_3 l \equiv 0 \mod d\right\} \simeq \mathbb{Z}^3\ .$$

Note that the lattice $\tilde{M}$ is one-to-one corresponding to the set of monomials of weighted degree d: indeed, for each $(i, j, k, l) \in \tilde{M}$, a monomial $X_0^i X_1^j X_2^k X_3^l$ is of weighted degree d. Here, the weight of X_i is a_i for $i = 0, 1, 2, 3$. Besides, by letting $\Delta_{\boldsymbol{a}}$ be a convex hull of all primitive lattice vectors in $\tilde{M}$, the associated projective toric 3-fold is the weighted projective space of weight $\boldsymbol{a}$.

The introduction of *reflexive polytope* in [2] is motivated by mirror symmetry.

DEFINITION 2.1 ([2]). Let Δ be an integral polytope that contains the origin in its interior. The polytope Δ is called reflexive if its polar dual Δ^* is also integral.

Not only in a context of mirror, this notion is basically friendly with $K3$ surfaces as follows:

THEOREM 2.1 ([2]). *Let Δ be a* 3*-dimensional polytope.*

(1) *The followings are equivalent*:

  (i) *The polytope Δ is reflexive.*

  (ii) *The toric* 3*-fold $\mathbb{P}_\Delta$ is Fano*, *in particular*, *general anticanonical members of $\mathbb{P}_\Delta$ are Gorenstein $K3$.*

(2) *General anticanonical members of $\mathbb{P}_\Delta$ are simultaneously resolved by a toric (crepant) desingularization of $\mathbb{P}_\Delta$ to be $K3$ surfaces.*

Denote for a reflexive polytope Δ by $\mathcal{F}_\Delta$ a family of (Gorenstein) $K3$ surfaces parametrised by the complete anticanonical linear system $|-K_{\mathbb{P}_\Delta}|$. For a member Z in $\mathcal{F}_\Delta$, denote by $\tilde{Z}$ and $\widetilde{\mathbb{P}_\Delta}$ the minimal models in a cause of the simultaneous resolution.

In the article, we define that a member $Z \in \mathcal{F}_\Delta$ is *generic* if the following two conditions are satisfied:

(1) Z is Δ-regular. (See [2] for detail)

(2) The Picard group of $\widetilde{Z}$ is generated by irreducible components of the restrictions of the generator of the Picard group of $\widetilde{\mathbb{P}_\Delta}$.

It is proved in [2] that Δ-regularity is a general condition. The condition (2) is also a general condition. Note that all Picard lattices of the minimal models of any generic members are isometric.

DEFINITION 2.2. (1) The Picard lattice $\mathrm{Pic}(\Delta)$ of the family $\mathcal{F}_\Delta$ is the Picard lattice of the minimal model of a generic member.
(2) $\rho(\Delta) := \mathrm{rank}\,\mathrm{Pic}(\Delta)$ is called the Picard number of the family $\mathcal{F}_\Delta$.
(3) Let $r : H^{1,1}(\widetilde{\mathbb{P}_\Delta}, \mathbb{Z}) \to H^{1,1}(\tilde{Z}, \mathbb{Z})$ be the restriction mapping of the cohomology group. The cokernel of r is denoted by $L_0(\Delta)$.

In [5], a notion of transpose duality [3] for singularities is extended to a *polytope duality* in the sense of the following theorem :

THEOREM 2.2 ([5]). *Let $((B, f), (B', f'))$ be a transpose-dual pair together with their defining polynomials f and f' that are respectively compactified to polynomials F and F'. Then, there exist reflexive polytopes $\Delta_{[MU]}$ and $\Delta'_{[MU]}$ such that*

$$\Delta^*_{[MU]} \simeq \Delta'_{[MU]}, \quad \Delta_F \subset \Delta_{[MU]}, \quad \textit{and} \quad \Delta_{F'} \subset \Delta'_{[MU]}\,.$$

However, it depends on the pairs that whether or not $\mathrm{rank}\, L_0(\Delta_{[MU]}) = 0$ holds. In section 3, we shall show that some pairs in the list $(*)$ do have this property.

We end this section by giving formulas that are needed in the proof of the main theorem. See [4] for details. For a 3-dimensional reflexive polytope Δ, denote by $\Delta^{[1]}$ the set of all edges of Δ, and for an edge $\Gamma \in \Delta^{[1]}$, the dual edge in the polar dual polytope Δ^* is denoted by Γ^*. The number of lattice points on an edge Γ is denoted by $l(\Gamma)$, whilst $l(\Gamma) - 2$ by $l^*(\Gamma)$. We have

$$\mathrm{rank}\, L_0(\Delta) = \sum_{\Gamma\in\Delta^{[1]}} l^*(\Gamma) l^*(\Gamma^*)\,. \tag{1}$$

$$\rho(\Delta) = \sum_{\Gamma\in\Delta^{[1]}} l(\Gamma^*) - 3\,. \tag{2}$$

Note that $\mathrm{rank}\, L_0(\Delta) = \mathrm{rank}\, L_0(\Delta^*)$ by the formula.

3. Main result

The chief aim of this section is to prove the following statements.

THEOREM 3.1. *For pairs (B, B') of singularities, if one takes compactifications F, F' as in Table 1, and polytopes Δ, Δ' as in Table 2, then,*

(i) *Δ and Δ' are reflexive,*
(ii) *Δ^* is isomorphic to Δ' up to lattice isometry of $\mathbb{Z}^3$,*
(iii) *$\Delta_F \subset \Delta$, and $\Delta_{F'} \subset \Delta'$ hold, and*
(iv) $\mathrm{rank}\, L_0(\Delta) = 0$.

Moreover, $\rho(\Delta) + \rho(\Delta') = 20$.

Proof. $Z_{1,0}$ **case.** The defining polynomials of singularities $B = Z_{1,0}$ and $B' = Z_{1,0}$ are the same $f = f' = x^5 y + x y^3 + z^2$.

TABLE 1. Compactifications of singularities

B	F	F'	B'
$Z_{1,0}$	$X^5Y + XY^3 + Z^2 + W^{10}X^2$	$X^5Y + XY^3 + Z^2 + W^{14}$	$Z_{1,0}$
$U_{1,0}$	$X^3Y + Y^2Z + Z^3 + WX^4$	$XZ^3 + X^2Y + Y^3 + W^9$	$U_{1,0}$
$Z_{2,0}$	$X^5Z + XY^3 + Z^2 + W^7Y$	$X^5Y + WY^3 + XZ^2 + W^7$	Q_{17}
$W_{1,0}$	$X^6 + Y^2Z + Z^2 + W^6Z$	$X^6 + Y^2Z + Z^2 + W^{12}$	$W_{1,0}$

TABLE 2. Polytopes that make the pairs polytope dual

B	vertices of Δ	vertices of Δ'	B'
$Z_{1,0}$	$\{(-1,0,1), (-1,0,0), (0,1,-1), (2,3,-1), (2,2,-1), (1,-1,-1), (0,-1,-1)\}$	$\{(0,2,-1), (-1,1,-1), (-1,-1,-1), (5,-1,-1), (4,0,-1), (1,0,0), (-1,-1,1)\}$	$Z_{1,0}$
$U_{1,0}$	$\{(-1,0,2), (0,1,0), (1,2,-1), (1,1,-1), (0,-1,0), (0,-1,-1)\}$	$\{(1,0,-1), (0,-1,-1), (-1,-1,-1), (-1,2,-1), (1,2,-1), (1,0,1), (0,-1,2), (-1,-1,2)\}$	$U_{1,0}$
$Z_{2,0}$	$\{(-1,-1,2), (0,-1,0), (1,-1,0), (1,-1,1), (1,2,-3), (0,0,-1)\}$	$\{(-1,2,-1), (-1,-1,1), (-1,-1,-1), (6,-1,-1), (2,1,-1), (0,-1,1)\}$	Q_{17}
$W_{1,0}$	$\{(-1,0,1), (-1,0,0), (1,2,-1), (2,3,-1), (0,-1,0)\}$	$\{(-1,-1,-1), (5,-1,-1), (1,3,-1), (-1,3,-1), (-1,-1,1)\}$	$W_{1,0}$

Take a compactification of f as $F = W^{10}X^2 + X^5Y + XY^3 + Z^2$ in the weighted projective space $\mathbb{P}(1,2,4,7)$. Note that F is a different compactification from the one in [3].

Take a compactification of f' as $F' = W^{14} + X^5Y + XY^3 + Z^2$ in the weighted projective space $\mathbb{P}(1,2,4,7)$. Note that F' is the same compactification as in [3].

The polytope Δ contains the Newton polytope of F: indeed, by taking a basis $\boldsymbol{e}_1 = (-6,1,1,0)$, $\boldsymbol{e}_2 = (2,1,-1,0)$, $\boldsymbol{e}_3 = (-7,0,0,1)$ for $\mathbb{R}^3$, one can see that monomials $W^{10}X^2$, X^5Y, XY^3, Z^2 are respectively corresponding to vertices

$$(0,1,-1),\ (2,2,-1),\ (1,-1,-1),\ (-1,0,1)\,.$$

The polytope Δ' contains the Newton polytope of F': indeed, by taking a standard basis $\boldsymbol{e}'_1 = (-2,1,0,0)$, $\boldsymbol{e}'_2 = (-4,0,1,0)$, $\boldsymbol{e}'_3 = (-7,0,0,1)$ for $\mathbb{R}^3$, one can see that monomials $W^{14},\ X^5Y,\ XY^3,\ Z^2$ are respectively corresponding to vertices

$$(-1,-1,-1),\ (4,0,-1),\ (0,2,-1),\ (-1,-1,1)\,.$$

The dual polytope Δ'^* of Δ' is a convex hull of vertices

$$(0,0,1),\ (-1,-2,-3),\ (-1,-3,-5),\ (1,-1,-1),\ (1,0,0),\ (0,1,0),\ (-1,-1,-3)$$

that is mapped to isomorphically from Δ by a transformation of $\mathbb{R}^3$ by the matrix

$$M := \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 2 & 4 \end{pmatrix}$$

that is, $(x,y,z)M = (x',y',z')$ for $(x,y,z) \in \Delta$ and $(x',y',z') \in \Delta'$.

Therefore, Δ and Δ' are reflexive and the pair is polytope dual.

By the formula (1), one gets $\operatorname{rank} L_0(\Delta) = \operatorname{rank} L_0(\Delta^*) = 0$ because for all edges in Δ satisfy $l^*(\Gamma)l^*(\Gamma^*) = 0$. In fact, at least either Γ or Γ^* has no lattice points in its interior.

By the formula (2), one can compute that

$$\rho(\Delta) = 17 - 3 = 14, \quad \rho(\Delta^*) = 9 - 3 = 6$$

thus one has

$$\rho(\Delta) + \rho(\Delta^*) = 20\,.$$

$U_{1,0}$ **case.** The defining polynomials of singularities $B = U_{1,0}$ and $B' = U_{1,0}$ are $f = x^3y + y^2z + z^3,\ f' = x'z'^3 + x'^2y' + y'^3$, respectively.

Take a compactification of f as $F = WX^4 + X^3Y + Y^2Z + Z^3$ in the weighted projective space $\mathbb{P}(1,2,3,3)$. Note that F is a different compactification from the one in [3].

Take a compactification of f' as $F' = W'^9 + X'Z'^3 + X'^2Y' + Y'^3$ in the weighted projective space $\mathbb{P}(1,3,3,2)$. Note that F' is the same compactification as in [3].

The polytope Δ contains the Newton polytope of F: indeed, by taking a basis $\boldsymbol{e}_1 = (-5,1,1,0)$, $\boldsymbol{e}_2 = (1,1,-1,0)$, $\boldsymbol{e}_3 = (-3,0,0,1)$ for $\mathbb{R}^3$, one can see that monomials $WX^4,\ X^3Y,\ Y^2Z,\ Z^3$ are respectively corresponding to vertices

$$(1,2,-1),\ (1,1,-1),\ (0,-1,0),\ (-1,0,2)\,.$$

The polytope Δ' contains the Newton polytope of F': indeed, by taking a standard basis $\boldsymbol{e}'_1 = (-3,1,0,0)$, $\boldsymbol{e}'_2 = (-3,0,1,0)$, $\boldsymbol{e}'_3 = (-2,0,0,1)$ for $\mathbb{R}^3$, one can see that monomials $W'^9,\ X'Z'^3,\ X'^2Y',\ Y'^3$ are respectively corresponding to vertices

$$(-1,-1,-1),\ (0,-1,2),\ (1,0,-1),\ (-1,2,-1)\,.$$

The dual polytope Δ'^* of Δ' is a convex hull of vertices

$$(0,0,1),\ (-1,0,0),\ (-1,1,0),\ (0,1,0),\ (1,0,0),\ (0,-1,-1)$$

that is mapped to isomorphically from Δ by a transformation of $\mathbb{R}^3$ by the matrix

$$M = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

that is, $(x, y, z)M = (x', y', z')$ for $(x, y, z) \in \Delta$ and $(x', y', z') \in \Delta'$.

Therefore, Δ and Δ' are reflexive and the pair is polytope dual.

By the formula (1), one gets $\operatorname{rank} L_0(\Delta) = \operatorname{rank} L_0(\Delta^*) = 0$ because for all edges in Δ satisfy $l^*(\Gamma)l^*(\Gamma^*) = 0$. In fact, at least either Γ or Γ^* has no lattice points in its interior.

By the formula (2), one can compute that

$$\rho(\Delta) = 20 - 3 = 17, \quad \rho(\Delta^*) = 6 - 3 = 3$$

thus one has

$$\rho(\Delta) + \rho(\Delta^*) = 20\,.$$

$Z_{2,0}$ and Q_{17} case. The defining polynomials of singularities $B = Z_{2,0}$ and $B' = Q_{17}$ are $f = x^5z + xy^3 + z^2,\ f' = x^5y + y^3 + xz^2$, respectively.

Take a compactification of f as $F = W^7Y + X^5Z + XY^3 + Z^2$ in the weighted projective space $\mathbb{P}(1, 1, 3, 5)$. Note that F is the same compactification as in [3].

Take a compactification of f' as $F' = W^7 + X^5Y + WY^3 + XZ^2$ in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$. Note that F' is the same compactification as in [3].

The polytope Δ contains the Newton polytope of F: indeed, by taking a basis $\boldsymbol{e}_1 = (-3, 3, 0, 0)$, $\boldsymbol{e}_2 = (-8, 0, 1, 1)$, $\boldsymbol{e}_3 = (-6, 1, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials $W^7Y, X^5Z,\ XY^3,\ Z^2$ are respectively corresponding to vertices

$$(0, 0, -1),\ (1, -1, 1),\ (1, 2, -3),\ (-1, -1, 2)\,.$$

The polytope Δ' contains the Newton polytope of F': indeed, by taking a standard basis $\boldsymbol{e}'_1 = (-1, 1, 0, 0)$, $\boldsymbol{e}'_2 = (-2, 0, 1, 0)$, $\boldsymbol{e}'_3 = (-3, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials $W^7,\ X^5Y,\ WY^3,\ XZ^2$ are respectively corresponding to vertices

$$(-1, -1, -1),\ (4, 0, -1),\ (-1, 2, -1),\ (0, -1, 1)\,.$$

The dual polytope Δ'^* of Δ' is a convex hull of vertices

$$(-1, -3, -4),\ (0, -2, -3),\ (0, 1, 0),\ (1, 0, 0),\ (0, 0, 1),\ (-1, -2, -3)$$

that is mapped to isomorphically from Δ by a transformation of $\mathbb{R}^3$ by the matrix

$$M := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

that is, $M(x, y, z) = (x', y', z')$ for $(x, y, z) \in \Delta$ and $(x', y', z') \in \Delta'$.

Therefore, Δ and Δ' are reflexive and the pair is polytope dual.

By the formula (1), one gets $\operatorname{rank} L_0(\Delta) = \operatorname{rank} L_0(\Delta^*) = 0$ because for all edges in Δ satisfy $l^*(\Gamma)l^*(\Gamma^*) = 0$. In fact, at least either Γ or Γ^* has no lattice points in its interior.

By the formula (2), one can compute that

$$\rho(\Delta) = 18 - 3 = 15, \quad \rho(\Delta^*) = 8 - 3 = 5$$

thus one has

$$\rho(\Delta) + \rho(\Delta^*) = 20\,.$$

$W_{1,0}$ **case.** The defining polynomials of singularities $B = B' = W_{1,0}$ are the same $f = f' = x^6 + y^2 z + z^2$.

Take a compactification of f as $F = X^6+Y^2Z+Z^2+W^6Z$ in the weighted projective space $\mathbb{P}(1, 2, 3, 6)$. Note that F is a different compactification from the one in [3].

Take a compactification of f' as $F' = X'^6 + Y'^2Z' + Z'^2 + W'^{12}$ in the weighted projective space $\mathbb{P}(1, 2, 3, 6)$. Note that F' is the same compactification as in [3].

The polytope Δ contains the Newton polytope of F: indeed, by taking a basis $\boldsymbol{e}_1 = (-5, 1, 1, 0)$, $\boldsymbol{e}_2 = (1, 1, -1, 0)$, $\boldsymbol{e}_3 = (-6, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials X^6, Y^2Z, Z^2, W^6Z are respectively corresponding to vertices

$$(2, 3, -1),\ (0, -1, 0),\ (-1, 0, 1),\ (-1, 0, 0)\,.$$

The polytope Δ' contains the Newton polytope of F': indeed, by taking a standard basis $\boldsymbol{e}'_1 = (-2, 1, 0, 0)$, $\boldsymbol{e}'_2 = (-3, 0, 1, 0)$, $\boldsymbol{e}'_3 = (-6, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials X'^6, Y'^2Z', Z'^2, W'^{12} are respectively corresponding to vertices

$$(5, -1, -1),\ (-1, 1, 0),\ (-1, -1, 1),\ (-1, -1, -1)\,.$$

The dual polytope Δ'^* of Δ' is a convex hull of vertices

$$(0, 1, 0),\ (-1, -1, -3),\ (0, -1, -2),\ (1, 0, 0),\ (0, 0, 1)$$

that is mapped to isomorphically from Δ by a transformation of $\mathbb{R}^3$ by the matrix

$$M := \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

that is, $M(x, y, z) = (x', y', z')$ for $(x, y, z) \in \Delta$ and $(x', y', z') \in \Delta'$.

Therefore, Δ and Δ' are reflexive polytopes and the pair is polytope dual.

By the formula (1), one gets $\operatorname{rank} L_0(\Delta) = \operatorname{rank} L_0(\Delta^*) = 0$ because for all edges in Δ satisfy $l^*(\Gamma)l^*(\Gamma^*) = 0$. In fact, at least either Γ or Γ^* has no lattice points in its interior.

By the formula (2), one can compute that

$$\rho(\Delta) = 21 - 3 = 18, \quad \rho(\Delta^*) = 5 - 3 = 2$$

thus one has

$$\rho(\Delta) + \rho(\Delta^*) = 20\,.$$

□

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