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On Equivariant Maps Related to the Space of Pairs of Exceptional Jordan Algebras

by

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Abstract. Let $\mathcal{J}$ be the exceptional Jordan algebra and $V = \mathcal{J} \oplus \mathcal{J}$. We construct an equivariant map from V to $\mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ defined by homogeneous polynomials of degree 8 such that if $x \in V$ is a generic point, then the image of x is the structure constant of the isotope of $\mathcal{J}$ corresponding to x. We also give an alternative way to define the isotope corresponding to a generic point of $\mathcal{J}$ by an equivariant map from $\mathcal{J}$ to the space of trilinear forms.

1. Introduction

Let k be a field of characteristic not equal to 2, 3, k^{sep} the separable closure of k and $\overline{k}$ the algebraic closure of k. Let $\widetilde{\mathbb{O}}$ be the split octonion over k. It is the normed algebra over k obtained by the Cayley–Dickson process (see [2, pp.101–110]). If A is the algebra of 2×2 matrices, $\widetilde{\mathbb{O}}$ is $A(+)$ with the notation of [2]. An *octonion* is, by definition, a normed algebra which is a k-form of $\widetilde{\mathbb{O}}$. Let $\mathbb{O}$ be an octonion. We use the notation $\|x\|$ for the norm of $x \in \mathbb{O}$. If $a \in k$, $\|ax\| = a^2\|x\|$. Also if $x, y \in \mathbb{O}$, then $\|xy\| = \|x\|\|y\|$. For $x, y \in \mathbb{O}$, let

$$Q(x, y) = \frac{1}{2}(\|x + y\| - \|x\| - \|y\|) .$$

This is a non-degenerate symmetric bilinear form such that $Q(x, x) = \|x\|$. Let $W \subset \mathbb{O}$ be the orthogonal complement of $k \cdot 1$ with respect to Q. If $x = x_1 + x_2$ where $x_1 \in k \cdot 1$, $x_2 \in W$, then we define $\overline{x} = x_1 - x_2$ and call it the *conjugate* of x. Note that $\|x\| = x\overline{x}$. For $x \in \mathbb{O}$, we define the trace $\mathrm{tr}(x)$ by $\mathrm{tr}(x) = x + \overline{x}$. It is easy to verify that

$$\mathrm{tr}(xy) = \mathrm{tr}(yx) , \quad 2Q(x, y) = \mathrm{tr}(x\overline{y}) .$$

Let $\mathrm{GL}(n)$ be the group of $n \times n$ invertible matrices over k. If V is a finite dimensional vector space, then we denote the group of invertible linear maps from V to V by $\mathrm{GL}(V)$.

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Let $\mathcal{J}$ be the exceptional Jordan algebra over k. Elements of $\mathcal{J}$ are of the form:

$$X = \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix}, \quad s_i \in k,\ x_i \in \mathbb{O}\ \ (i = 1, 2, 3).$$

The multiplication of $\mathcal{J}$ is defined as follows:

$$X \circ Y = \frac{1}{2}(XY + YX),$$

where the multiplication used on the right-hand side is the multiplication of matrices.

The algebraic groups E_6 and GE_6 are given by

$$E_6 = \{L \in \mathrm{GL}(\mathcal{J}) | {}^{\forall} X \in \mathcal{J},\ \det(LX) = \det(X)\},$$

$$GE_6 = \{L \in \mathrm{GL}(\mathcal{J}) | {}^{\forall} X \in \mathcal{J}, \det(LX) = c(L)\det(X) \text{ for some } c(L) \in \mathrm{GL}(1)\}$$

respectively. Then $c : GE_6 \to \mathrm{GL}(1)$ is a character and there exists an exact sequence

$$0 \to E_6 \hookrightarrow GE_6 \xrightarrow{c} \mathrm{GL}(1) \to 0. \tag{1.1}$$

It is known that E_6 is a smooth connected quasi-simple simply-connected algebraic group of type E_6 (see [5, p.181, Theorem 7.3.2]). The terminology "quasi-simple" means that its inner automorphism group is simple (see [4, p. 136]). The smoothness of the group follows from the fact that the dimension of E_6 as a variety and the dimension of the Lie algebra of E_6 coincide (see the proof of [5, p.181, Theorem 7.3.2]).

Let $H_1 = E_6$, $G_1 = GE_6$, $H = H_1 \times \mathrm{GL}(2)$ and $G = G_1 \times \mathrm{GL}(2)$. Let $V = \mathcal{J} \otimes \mathrm{Aff}^2$. We regard elements of V as the set of $x = x_1v_1 + x_2v_2$ where $x_1, x_2 \in \mathcal{J}$ and v_1, v_2 are variables. The action of $g = (g_1, g_2) \in G$ where $g_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on V is given by

$$g(x_1v_1 + x_2v_2) = g_1(x_1)(av_1 + cv_2) + g_1(x_2)(bv_1 + dv_2). \tag{1.2}$$

For $x = x_1v_1 + x_2v_2 \in V$, we put $F_x(v) = F_x(v_1, v_2) = \det(x)$. Then $F_x(v)$ is a binary cubic form. Let $\Delta(x)$ be the discriminant of $F_x(v)$ as a polynomial of v. Let

$$w = w_1v_1 + w_2v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} v_2 \in \mathcal{J} \otimes \mathrm{Aff}^2. \tag{1.3}$$

Then it is easy to see that $F_w(v) = v_1v_2(v_1 - v_2)$ and $\Delta(w) = 1$. The pair (G, V) is an irreducible regular *prehomogeneous vector space* (see [3, Proposition 3.2] and [6]) and the polynomial $\Delta(x)$ is what we call a *relative invariant polynomial.* We define $V^{\mathrm{ss}} = \{x \in V \mid \Delta(x) \neq 0\}$. Points in V^{ss} are called *semi-stable points.* As we pointed out above, $w \in V_k^{\mathrm{ss}}$. The polynomial $\Delta(x)$ is of degree 12. If we put $\chi(g) = c(g_1)^4 \det(g_2)^6$ for $g = (g_1, g_2) \in G$, then $\Delta(gx) = \chi(g)\Delta(x)$.

A Jordan algebra $\mathcal{M}$ is called an *isotope* of $\mathcal{J}$ if $\mathcal{M} \otimes k^{\mathrm{sep}} \cong \mathcal{J} \otimes k^{\mathrm{sep}}$ and the "determinant" of $\mathcal{M}$ is a constant multiple of that of $\mathcal{J}$. For details, the reader should see [5, pp.154–158]. If $\mathcal{M}_1, \mathcal{M}_2$ are isotopes of $\mathcal{J}$ and $\mathfrak{n}_1 \subset \mathcal{M}_1, \mathfrak{n}_2 \subset \mathcal{M}_2$ are cubic étale subalgebras, then the pairs $(\mathcal{M}_1, \mathfrak{n}_1)$, $(\mathcal{M}_2, \mathfrak{n}_2)$ are defined to be equivalent if there exists a k-isomorphism $\mathcal{M}_1 \to \mathcal{M}_2$ which induces an isomorphism from $\mathfrak{n}_1$ to $\mathfrak{n}_2$. Let $\mathrm{JIC}(k)$ be the set of equivalence classes of pairs $(\mathcal{M}, \mathfrak{n})$ as above.

In [3], to each point in V_k^{ss}, a pair $(\mathcal{M}, \mathfrak{n}) \in \mathrm{JIC}(k)$ was associated. It is proved in [3, Theorem 5.8] that there is a bijective correspondence between the set $G_k \backslash V_k^{ss}$ of rational orbits and $\mathrm{JIC}(k)$. Moreover, an equivariant map $m : V \to \mathcal{J}$ was defined and the Jordan algebra corresponding to $x \in V_k^{ss}$ was explicitly described using the point $m(x)$ (see [3, Section 5]).

Let $\mathfrak{t} \subset \mathcal{J}$ be the subalgebra of diagonal matrices, which is isomorphic to k^3. Let $\mathrm{Aut}(\mathcal{J})$ be the algebraic group of automorphisms of the Jordan algebra $\mathcal{J}$. We define $\mathrm{Aut}(\mathcal{J}, \mathfrak{t})$ to be the subgroup of $\mathrm{Aut}(\mathcal{J})$ consisting of automorphisms L such that $L(\mathfrak{t}) = \mathfrak{t}$. It is proved in ([3, Theorem 3.1, Lemma 4.2]) that $G_w \cong \mathrm{GL}(1) \times \mathrm{Aut}(\mathcal{J}, \mathfrak{t})$. The first Galois cohomology set $\mathrm{H}^1(k, \mathrm{GL}(1) \times \mathrm{Aut}(\mathcal{J}, \mathfrak{t}))$ can be identified with $\mathrm{H}^1(k, \mathrm{Aut}(\mathcal{J}, \mathfrak{t}))$ and there is a natural map $\mathrm{H}^1(k, \mathrm{Aut}(\mathcal{J}, \mathfrak{t})) \to \mathrm{H}^1(k, \mathrm{Aut}(\mathcal{J}))$. One can show by standard argument that elements of $\mathrm{H}^1(k, \mathrm{Aut}(\mathcal{J}))$ correspond bijectively with k-forms of $\mathcal{J}$.

Suppose that $x = g_x w \in V_k^{ss}$ where $g_x \in G_{k^{\mathrm{sep}}}$. Then $h_x : \mathrm{Gal}(k^{\mathrm{sep}}/k) \ni \sigma \mapsto g_x^{-1} g_x^{\sigma} \in G_{w\, k^{\mathrm{sep}}}$ is a 1-cocycle, which defines an element, say c_x of $\mathrm{H}^1(k, G_w)$. This element c_x does not depend on the choice of g_x. Since there is a natural map $\mathrm{H}^1(k, G_w) \cong \mathrm{H}^1(k, \mathrm{Aut}(\mathcal{J}, \mathfrak{t})) \to \mathrm{H}^1(k, \mathrm{Aut}(\mathcal{J}))$ ($\cong$ means bijection), c_x determines a k-form of $\mathcal{J}$. If x corresponds to a pair $(\mathcal{M}, \mathfrak{n})$, then $\mathcal{M}$ is the k-form of $\mathcal{J}$ which is determined by c_x. The underlying vector space of $\mathcal{M}$ is $\mathcal{J}$. To define a Jordan algebra structure on $\mathcal{J}$, it is enough to define the product structure, which is given by an element of $\mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ (we call this element the "structure constant").

If we choose bases of $V, \mathcal{J}$ as k-vector spaces, the map $m : V \to \mathcal{J}$ is given by homogeneous polynomials of degree 4 on V. The structure constant which is associated to elements of $\mathcal{J}$ is, with the denominator multiplied, given by homogeneous polynomials of degree 11 on $\mathcal{J}$. So the structure constant of $\mathcal{M}$ is given by homogeneous polynomials of degree 44 on V.

The first purpose of this paper is to prove the following theorem (see Section 2).

THEOREM 1.4. *There is an equivariant map* $\mathrm{S} : V \ni x \mapsto \mathrm{S}_x \in \mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ *defined by homogeneous polynomials of degree* 8 *on* V *such that if* $x \in V_k^{ss}$ *corresponds to the pair* $(\mathcal{M}, \mathfrak{n}) \in \mathrm{JIC}(k)$, *then* $\Delta(x)^{-1}\mathrm{S}_x$ *is the structure constant of the Jordan algebra* $\mathcal{M}$.

Let $a \in \mathcal{J}$ and $\det(a) \neq 0$. In [5, p. 155], for $X, Y \in \mathcal{J}$, the product structure $X \circ_a Y$ defining the isotope $\mathcal{J}_a$ and the corresponding symmetric bilinear form $Q_a(X, Y)$ are given (see (3.5)). Then $\mathcal{J}^3 \ni (X, Y, Z) \mapsto \mathrm{T}_a(X, Y, Z) = Q_a(X \circ_a Y, Z)$ is a trilinear form on $\mathcal{J}$. To construct T_a first equivariantly to provide an alternative way to define the product structure on $\mathcal{J}_a$ is another purpose of this paper.

We prove the following theorem in Section 3.

THEOREM 1.5. *There is an equivariant map* $\mathrm{T} : \mathcal{J} \ni a \mapsto \mathrm{T}_a \in \mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J}, k)$ *defined by homogeneous polynomials of degree* 6 *such that if for* $X, Y \in \mathcal{J}$, $X \circ_a Y \in \mathcal{J}$ *is the element such* $Q_a(X \circ_a Y, Z) = \det(a)^{-1}\mathrm{T}_a(X, Y, Z)$ *for all* $Z \in \mathcal{J}$, *then this product structure coincides with that of the isotope* $\mathcal{J}_a$.

2. Equivariant map I

We prove Theorem 1.4 in this section. We first define basic notions and then define the desired equivariant map.

We denote the 3×3 diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \alpha_3 \in k^\times$ by diag $(\alpha_1, \alpha_2, \alpha_3)$. $\mathcal{J}^{n\otimes}$ is the tensor product of n copies of $\mathcal{J}$. We define a symmetric trilinear form D on $\mathcal{J}$ by

$$
\begin{aligned}
6D(X, Y, Z) = &\det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X) \\
&+ \det(X) + \det(Y) + \det(Z)
\end{aligned}
$$

for $X, Y, Z \in \mathcal{J}$. Let $\mathrm{Tr}(X)$ be the sum of diagonal entries of $X \in \mathcal{J}$. For $X, Y \in \mathcal{J}$, we define a symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathcal{J}$ by

$$\langle X, Y\rangle = \mathrm{Tr}(X \circ Y)\,.$$

One can verify by direct computation that the symmetric bilinear form $\langle\ ,\ \rangle$ satisfies the following equation:

$$\langle X \circ Y, Z\rangle = \langle X, Y \circ Z\rangle, \quad {}^\forall X, Y, Z \in \mathcal{J}\,. \tag{2.1}$$

For $X, Y \in \mathcal{J}$, the cross product $X \times Y$ is, by definition, the element satisfying the following equation:

$$\langle X \times Y, Z\rangle = 3D(X, Y, Z)\,, \quad {}^\forall Z \in \mathcal{J}\,.$$

Let $e = I_3$. Then the following equations are satisfied (see [5, p.122, Lemma 5.2.1]).

$$ {}^\forall X \in \mathcal{J}, \quad X \circ (X \times X) = \det(X)e\,, \quad e \times e = e\,. \tag{2.2}$$

For any $g \in \mathrm{GL}(\mathcal{J})$, we define $\widetilde{g} \in \mathrm{GL}(\mathcal{J})$ by

$$\langle g(X), \widetilde{g}(Y)\rangle = \langle X, Y\rangle\,, \quad {}^\forall X, Y \in \mathcal{J}\,.$$

The following lemma is proved in [5, p.180, Proposition 7.3.1].

LEMMA 2.3. *The map $g \mapsto \widetilde{g}$ is an automorphism of H_1 with order 2 and for all $X, Y \in \mathcal{J}$,*

$$g(X \times Y) = \widetilde{g}(X) \times \widetilde{g}(Y), \ \ \widetilde{g}(X \times Y) = g(X) \times g(Y)\,.$$

COROLLAY 2.4. If $g \in G_1$ and $X, Y, Z, W \in \mathcal{J}$, then

$$g((X \times Y) \times (Z \times W)) = c(g)^{-1}\,(g(X) \times g(Y)) \times (g(Z) \times g(W))\ .$$

Proof. There exists $t \in \overline{k}$ such that $t^3 = c(g_1)$. Then $t^{-1}g \in H_{1\overline{k}}$. So

$$
\begin{aligned}
g\,((X \times Y) \times (Z \times W)) &= t(t^{-1}g)\,((X \times Y) \times (Z \times W)) \\
&= t\left(t^{-1}g(X) \times t^{-1}g(Y)\right) \times \left(t^{-1}g(Z) \times t^{-1}g(W)\right) \\
&= c(g)^{-1}\,(g(X) \times g(Y)) \times (g(Z) \times g(W))\ .
\end{aligned}
$$

□

We now define an equivariant map

$$\mathrm{S} : V \ni x \mapsto \mathrm{S}_x \in \mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$$

such that $\mathrm{S}_w(X, Y) = X \circ Y$ for all $X, Y \in \mathcal{J}$.

Let

$$X = \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix}, \quad Y = \begin{pmatrix} t_1 & y_3 & \overline{y_2} \\ \overline{y_3} & t_2 & y_1 \\ y_2 & \overline{y_1} & t_3 \end{pmatrix} \tag{2.5}$$

where $s_i, t_i \in k$, $x_i, y_i \in \mathbb{O}$ for $i = 1, 2, 3$. By computation, $X \circ Y$ is the following matrix:

(2.6)

$$\frac{1}{2}\begin{pmatrix} 2s_1t_1 + \mathrm{tr}(x_3\overline{y_3} + x_2\overline{y_2}) & \begin{matrix} s_1y_3 + t_2x_3 + \overline{x_2}\,\overline{y_1} \\ +t_1x_3 + s_2y_3 + \overline{y_2}\,\overline{x_1} \end{matrix} & \begin{matrix} s_1\overline{y_2} + x_3y_1 + t_3\overline{x_2} \\ +t_1\overline{x_2} + y_3x_1 + s_3\overline{y_2} \end{matrix} \\ \begin{matrix} s_1\overline{y_3} + t_2\overline{x_3} + y_1x_2 \\ +t_1\overline{x_3} + s_2\overline{y_3} + x_1y_2 \end{matrix} & 2s_2t_2 + \mathrm{tr}(x_3\overline{y_3} + x_1\overline{y_1}) & \begin{matrix} \overline{x_3}\,\overline{y_2} + s_2y_1 + t_3x_1 \\ +\overline{y_3}\,\overline{x_2} + t_2x_1 + s_3y_1 \end{matrix} \\ \begin{matrix} s_1y_2 + \overline{y_1}\,\overline{x_3} + t_3x_2 \\ +t_1x_2 + \overline{x_1}\,\overline{y_3} + s_3y_2 \end{matrix} & \begin{matrix} y_2x_3 + s_2\overline{y_1} + t_3\overline{x_1} \\ +x_2y_3 + t_2\overline{x_1} + s_3\overline{y_1} \end{matrix} & 2s_3t_3 + \mathrm{tr}(x_2\overline{y_2} + x_1\overline{y_1}) \end{pmatrix}.$$

Note that $\mathrm{tr}(\overline{x_2}y_2) = \mathrm{tr}(y_2\overline{x_2}) = \mathrm{tr}(x_2\overline{y_2})$, etc. In particular, if Y is diagonal, then

$$X \circ Y = \frac{1}{2}\begin{pmatrix} 2s_1t_1 & (t_1+t_2)x_3 & (t_1+t_3)\overline{x_2} \\ (t_1+t_2)\overline{x_3} & 2s_2t_2 & (t_2+t_3)x_1 \\ (t_1+t_3)x_2 & (t_2+t_3)\overline{x_1} & 2s_3t_3 \end{pmatrix}. \tag{2.7}$$

It is known (see [5, p.122]) that

$$\begin{aligned} X \times Y &= X \circ Y - \frac{1}{2}\langle X, e\rangle Y - \frac{1}{2}\langle Y, e\rangle X - \frac{1}{2}\langle X, Y\rangle e + \frac{1}{2}\langle X, e\rangle\langle Y, e\rangle e \\ &= X \circ Y - \frac{1}{2}\mathrm{Tr}(X)Y - \frac{1}{2}\mathrm{Tr}(Y)X - \frac{1}{2}\mathrm{Tr}(X \circ Y)e + \frac{1}{2}\mathrm{Tr}(X)\mathrm{Tr}(Y)e. \end{aligned} \tag{2.8}$$

For $a, b \in \mathcal{J}$ and $i_1, \dots, i_8 = 1, 2$, we define

$$[[i_1, i_2 | i_3, i_4 | i_5, i_6 | i_7, i_8]]_{(a,b)} = v_1 \otimes \cdots \otimes v_8 \tag{2.9}$$

where $v_j = a$ (resp. $v_j = b$) if $i_j = 1$ (resp. $i_j = 2$). For example,

$$[[1, 2|2, 1|2, 1|1, 2]]_{(a,b)} = a \otimes b \otimes b \otimes a \otimes b \otimes a \otimes a \otimes b.$$

We only consider $[[i_1, i_2|i_3, i_4|i_5, i_6|i_7, i_8]]_{(a,b)}$ such that $\{i_{2j-1}, i_{2j}\} = \{1, 2\}$ $(j = 1, \dots, 4)$.

Let

$$t_4(a, b) = \overbrace{(a \otimes b - b \otimes a) \otimes \cdots \otimes (a \otimes b - b \otimes a)}^{4} \in \mathcal{J}^{8\otimes}.$$

Note that $t_4(a, b)$ depends only on $a \wedge b \in \wedge^2 \mathcal{J}$. By expanding all the terms,

$$\begin{aligned} t_4(a, b) = {} & [[1, 2|1, 2|1, 2|1, 2]]_{(a,b)} - [[1, 2|1, 2|1, 2|2, 1]]_{(a,b)} \\ & - [[1, 2|1, 2|2, 1|1, 2]]_{(a,b)} + [[1, 2|1, 2|2, 1|2, 1]]_{(a,b)} \\ & - [[1, 2|2, 1|1, 2|1, 2]]_{(a,b)} + [[1, 2|2, 1|1, 2|2, 1]]_{(a,b)} \\ & + [[1, 2|2, 1|2, 1|1, 2]]_{(a,b)} - [[1, 2|2, 1|2, 1|2, 1]]_{(a,b)} \end{aligned}$$

$$
\begin{aligned}
&- [[2,1|1,2|1,2|1,2]]_{(a,b)} + [[2,1|1,2|1,2|2,1]]_{(a,b)} \\
&+ [[2,1|1,2|2,1|1,2]]_{(a,b)} - [[2,1|1,2|2,1|2,1]]_{(a,b)} \\
&+ [[2,1|2,1|1,2|1,2]]_{(a,b)} - [[2,1|2,1|1,2|2,1]]_{(a,b)} \\
&- [[2,1|2,1|2,1|1,2]]_{(a,b)} + [[2,1|2,1|2,1|2,1]]_{(a,b)}\,.
\end{aligned}
$$

For $v_1, \dots, v_8, X, Y \in \mathcal{J}$, we put

$$\Psi_1(v_1 \otimes \cdots \otimes v_8)(X,Y) = D(v_2,v_5,v_7)D(v_4,v_6,v_8)(v_1 \times v_3) \times (X \times Y)\,,$$

$$\Psi_2(v_1 \otimes \cdots \otimes v_8)(X,Y) = D(v_2,v_5,v_7)D(v_4,v_6,v_8)(D(v_1,v_3,X)Y + D(v_1,v_3,Y)X)\,.$$

Then Ψ_i induces a k-linear map $\mathcal{J}^{8\otimes} \to \mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ for $i = 1, 2$. We define

$$\Phi_{i,(a,b)}(X,Y) = \Psi_i(t_4(a,b))(X,Y) \tag{2.10}$$

for $i = 1, 2$. Then

$$
\begin{array}{ccccccc}
\Phi_i : & V & \to & (\wedge \mathcal{J}^2)^{4\otimes} & \to & \mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J}) \\
& \Downarrow & & \Downarrow & & \Downarrow \\
& (a,b) & \mapsto & t_4(a,b) & \mapsto & \Phi_{i,(a,b)}
\end{array}
$$

is a k-linear map defined by degree 8 polynomials of (a, b).

LEMMA 2.11. *$\Phi_{i,gx}(g_1X, g_1Y) = c(g_1)^3 \det(g_2)^4 g_1(\Phi_{i,x}(X,Y))$ for $i = 1, 2$ and all $g = (g_1, g_2) \in G$, $X, Y \in \mathcal{J}$.*

Proof. If $x = (x_1, x_2)$, then $t_4(x_1, x_2)$ depends only on $x_1 \wedge x_2$. If $g_2 \in \mathrm{GL}(2)$ and $y = (y_1, y_2) = gx$, then $y_1 \wedge y_2 = (\det(g_2))x_1 \wedge x_2$. Since $x_1 \otimes x_2 - x_2 \otimes x_1$ can be identified with $x_1 \wedge x_2$ and it appears in the definition of $t_4(x_1, x_2)$ four times, $t_4(y_1, y_2) = (\det g_2)^4 t_4(x_1, x_2)$. So we may assume that $g_2 = 1$ and only consider the action of G_1.

Suppose that $v_1 \otimes \cdots \otimes v_8$ is a term which appears in the expansion of $t_4(x_1, x_2)$. If $g_1 \in G_1$ and x is replaced by $g_1x = (g_1x_1, g_1x_2)$, then terms which appear in the expansion of $t_4(g_1x_1, g_1x_2)$ are in the form $g_1v_1 \otimes \cdots \otimes g_1v_8$. By Corollary 2.4,

$$
\begin{aligned}
&D(g_1v_2, g_1v_5, g_1v_7)D(g_1v_4, g_1v_6, g_1v_8)(g_1v_1 \times g_1v_3) \times (g_1X \times g_1Y), \\
&\quad = c(g_1)^3 D(v_2,v_5,v_7)D(v_4,v_6,v_8)g_1((v_1 \times v_3) \times (X \times Y)).
\end{aligned}
$$

Also

$$
\begin{aligned}
&D(g_1v_2, g_1v_5, g_1v_7)D(g_1v_4, g_1v_6, g_1v_8)(D(g_1v_1, g_1v_3, g_1X)g_1Y + D(g_1v_1, g_1v_3, g_1Y)g_1X) \\
&\quad = c(g_1)^3 D(v_2,v_5,v_7)D(v_4,v_6,v_8)g_1(D(v_1,v_3,X)Y + D(v_1,v_3,Y)X)\,.
\end{aligned}
$$

Therefore, $\Phi_{i,g_1x}(g_1X, g_1Y) = c(g_1)^3 g_1(\Phi_{i,x}(X,Y))$ for $i = 1, 2$. □

We first evaluate $\Phi_{1,w}$ (see (1.3)).

PROPOSITION 2.12. *For all $X, Y \in \mathcal{J}$, $-18\Phi_{1,w}(X,Y) = X \circ Y - \frac{1}{2}\mathrm{Tr}(Y)X - \frac{1}{2}\mathrm{Tr}(X)Y$.*

Proof. Suppose that $(a, b) = w = (w_1, w_2)$ in the following. If we expand $t_4(a, b)$, then the coefficient of $[[i_1, i_2|i_3, i_4|i_5, i_6|i_7, i_8]]_{(a,b)}$ is 1 (resp. -1) if the number of $j = 1, \dots, 4$ such that $(i_{2j-1}, i_{2j}) = (2, 1)$ is even (resp. odd). We list the sign in $t_4(a, b)$,

$D(v_2, v_5, v_7)D(v_4, v_6, v_8)$ and $v_1 \times v_3$ in the following table assuming that $[[i_1, i_2|i_3, i_4|i_5, i_6|i_7, i_8]]_{(a,b)}$ is in the form (2.9).

(1)	$D(v_2, v_5, v_7)D(v_4, v_6, v_8)$	$v_1 \times v_3$
$+[[1, 2\|1, 2\|1, 2\|1, 2]]$	$D(b, a, a)D(b, b, b)$	$a \times a$
$-[[1, 2\|1, 2\|1, 2\|2, 1]]$	$D(b, a, b)D(b, b, a)$	$a \times a$
$-[[1, 2\|1, 2\|2, 1\|1, 2]]$	$D(b, b, a)D(b, a, b)$	$a \times a$
$+[[1, 2\|1, 2\|2, 1\|2, 1]]$	$D(b, b, b)D(b, a, a)$	$a \times a$
$-[[1, 2\|2, 1\|1, 2\|1, 2]]$	$D(b, a, a)D(a, b, b)$	$a \times b$
$+[[1, 2\|2, 1\|1, 2\|2, 1]]$	$D(b, a, b)D(a, b, a)$	$a \times b$
$+[[1, 2\|2, 1\|2, 1\|1, 2]]$	$D(b, b, a)D(a, a, b)$	$a \times b$
$-[[1, 2\|2, 1\|2, 1\|2, 1]]$	$D(b, b, b)D(a, a, a)$	$a \times b$
$-[[2, 1\|1, 2\|1, 2\|1, 2]]$	$D(a, a, a)D(b, b, b)$	$b \times a$
$+[[2, 1\|1, 2\|1, 2\|2, 1]]$	$D(a, a, b)D(b, b, a)$	$b \times a$
$+[[2, 1\|1, 2\|2, 1\|1, 2]]$	$D(a, b, a)D(b, a, b)$	$b \times a$
$-[[2, 1\|1, 2\|2, 1\|2, 1]]$	$D(a, b, b)D(b, a, a)$	$b \times a$
$+[[2, 1\|2, 1\|1, 2\|1, 2]]$	$D(a, a, a)D(a, b, b)$	$b \times b$
$-[[2, 1\|2, 1\|1, 2\|2, 1]]$	$D(a, a, b)D(a, b, a)$	$b \times b$
$-[[2, 1\|2, 1\|2, 1\|1, 2]]$	$D(a, b, a)D(a, a, b)$	$b \times b$
$+[[2, 1\|2, 1\|2, 1\|2, 1]]$	$D(a, b, b)D(a, a, a)$	$b \times b$

((1) is $[[i_1, i_2|i_3, i_4|i_5, i_6|i_7, i_8]]_{(a,b)}$ with its sign in $t_4(a, b)$.)

Note that if $(a, b) = w$, then

$$D(a, a, a) = D(b, b, b) = 0\,, \quad D(a, a, b) = \frac{1}{3}\,, \quad D(a, b, b) = -\frac{1}{3}\,.$$

So we can ignore terms with the second column including either $D(a, a, a)$ or $D(b, b, b)$. Removing these terms, we obtain the following table.

(1)	$9D(v_2, v_5, v_7)D(v_4, v_6, v_8)$	$v_1 \times v_3$
$-[[1, 2\|1, 2\|1, 2\|2, 1]]$	1	$a \times a$
$-[[1, 2\|1, 2\|2, 1\|1, 2]]$	1	$a \times a$
$-[[1, 2\|2, 1\|1, 2\|1, 2]]$	-1	$a \times b$
$+[[1, 2\|2, 1\|1, 2\|2, 1]]$	-1	$a \times b$
$+[[1, 2\|2, 1\|2, 1\|1, 2]]$	-1	$a \times b$
$+[[2, 1\|1, 2\|1, 2\|2, 1]]$	-1	$b \times a$
$+[[2, 1\|1, 2\|2, 1\|1, 2]]$	-1	$b \times a$
$-[[2, 1\|1, 2\|2, 1\|2, 1]]$	-1	$b \times a$
$-[[2, 1\|2, 1\|1, 2\|2, 1]]$	1	$b \times b$
$-[[2, 1\|2, 1\|2, 1\|1, 2]]$	1	$b \times b$

By the above table,

$$9\Phi_{1,w}(X, Y) = -(2a \times a + a \times b + a \times b + 2b \times b) \times (X \times Y)\,.$$

If we put $c = a + b = \mathrm{diag}(1, 0, -1)$, then

$$9\Phi_{1,w}(X, Y) = -(a \times a + b \times b + c \times c) \times (X \times Y)\,.$$

By (2.7) and (2.8)

$$a \times a = \mathrm{diag}(0, 0, -1),\ \ b \times b = \mathrm{diag}(-1, 0, 0),\ \ c \times c = \mathrm{diag}(0, -1, 0)\,.$$

Therefore, $9\Phi_{1,w}(X, Y) = e \times (X \times Y)$. Since e is the unit element of $\mathcal{J}$, replacing X, Y in (2.8) by $e, X \times Y$, we obtain

$$\begin{aligned} 9\Phi_{1,w}(X, Y) &= X \times Y - \frac{1}{2}\mathrm{Tr}(e)X \times Y - \frac{1}{2}\mathrm{Tr}(X \times Y)e - \frac{1}{2}\mathrm{Tr}(X \times Y)e \\ &\quad + \frac{1}{2}\mathrm{Tr}(e)\mathrm{Tr}(X \times Y)e\,. \\ &= -\frac{1}{2}X \times Y + \frac{1}{2}\mathrm{Tr}(X \times Y)e\,. \end{aligned}$$

By (2.8),

$$\begin{aligned} \mathrm{Tr}(X \times Y) &= \mathrm{Tr}(X \circ Y) - \mathrm{Tr}(X)\mathrm{Tr}(Y) - \frac{3}{2}\mathrm{Tr}(X \circ Y) + \frac{3}{2}\mathrm{Tr}(X)\mathrm{Tr}(Y) \\ &= -\frac{1}{2}\mathrm{Tr}(X \circ Y) + \frac{1}{2}\mathrm{Tr}(X)\mathrm{Tr}(Y)\,. \end{aligned}$$

Therefore, again by (2.8),

$$9\Phi_{1,w}(X, Y) = -\frac{1}{2}X \circ Y + \frac{1}{4}\mathrm{Tr}(Y)X + \frac{1}{4}\mathrm{Tr}(X)Y\,. \tag{2.13}$$

Multiplying -2, we obtain the proposition. □

PROPOSITION 2.14. *For all* $X, Y \in \mathcal{J}$, $3\Phi_{2,w}(X, Y) = \mathrm{Tr}(Y)X + \mathrm{Tr}(X)Y$.

Proof. By very similar comutations as in the case of $\Phi_{1,w}(X, Y)$, we obtain

$$\begin{aligned} 3\Phi_{2,w}(X, Y) &= -3(D(a, a, X) + D(b, b, X) + D(c, c, X))Y \\ &\quad - 3(D(a, a, Y) + D(b, b, Y) + D(c, c, Y))X \\ &= (s_3 + s_1 + s_2)Y + (t_3 + t_1 + t_2)X \\ &= \mathrm{Tr}(Y)X + \mathrm{Tr}(X)Y\,. \end{aligned} \tag{2.15}$$

□

For $(a, b) \in V_k$, we define

$$\mathrm{S}_{(a,b)}(X, Y) = -18\Phi_{1,(a,b)}(X, Y) + \frac{3}{2}\Phi_{2,(a,b)}(X, Y)\,. \tag{2.16}$$

The following proposition follows from Lemma 2.11 and (2.13), (2.15).

PROPOSITION 2.17. (1) *If* $x \in V_k$, $g = (g_1, g_2) \in G_k$ *and* $X, Y \in \mathcal{J}$, *then*

$$\mathrm{S}_{gx}(g_1 X, g_1 Y) = c(g_1)^3 \det(g_2)^4 g_1(\mathrm{S}_x(X, Y))\,.$$

(2) $\mathrm{S}_w(X, Y) = X \circ Y$.

Let $\Delta(x)$ be the relative invariant polynomial of degree 12 and $\Delta(w) = 1$ which we defined in Introduction. For $x \in V_k^{\mathrm{ss}}$, we define a k-algebra structure $\circ_x$ on $\mathcal{J}$ by

$$X \circ_x Y \overset{\mathrm{def}}{=} \Delta(x)^{-1}\mathrm{S}_x(X, Y)\,. \tag{2.18}$$

We denote this k-algebra on the underlying vector space $\mathcal{J}$ by $\mathcal{M}_x$. Proposition 2.17 (2) implies that $\mathcal{M}_w$ is isomorphic to $\mathcal{J}$ (the original Jordan algebra structure).

For $g = (g_1, g_2) \in G$, we define an element of G_1 by

$$\mu_g = c(g_1)\det(g_2)^2 g_1 \in G_1\,. \tag{2.19}$$

Note that the map $g \to \mu_g$ is a homomorphism.

THEOREM 2.20. *If* $x, y \in V_k^{\mathrm{ss}}$, $g \in G_{k^{\mathrm{sep}}}$ *and* $y = gx$, *then* $\mu_g : \mathcal{M}_{x\,k^{\mathrm{sep}}} \to \mathcal{M}_{y\,k^{\mathrm{sep}}}$ *is an isomorphism of* k*-algebras*

Proof. Let $X, Y \in \mathcal{J}_{k^{\mathrm{sep}}} = \mathcal{M}_{x\,k^{\mathrm{sep}}}$. Then

$$\begin{aligned}
&\mu_g(X) \circ_y \mu_g(Y)\\
&\quad = \Delta(gx)^{-1}\mathrm{S}_{gx}(c(g_1)\det(g_2)^2 g_1(X), c(g_1)\det(g_2)^2 g_1(Y))\\
&\quad = \Delta(gx)^{-1}c(g_1)^2\det(g_2)^4\mathrm{S}_{gx}(g_1(X), g_1(Y))\\
&\quad = \Delta(gx)^{-1}c(g_1)^2\det(g_2)^4 c(g_1)^3\det(g_2)^4 g_1(\mathrm{S}_x(X,Y)) \text{ by Proposition 2.17}\\
&\quad = \Delta(gx)^{-1}\Delta(x)c(g_1)^4\det(g_2)^6 c(g_1)\det(g_2)^2 g_1(\Delta(x)^{-1}\mathrm{S}_x(X,Y))\\
&\quad = c(g_1)\det(g_2)^2 g_1(X \circ_x Y) = \mu_g(X \circ_x Y)\,.
\end{aligned}$$

Therefore, μ_g is a homomorphism. Since μ_g is obviously bijective, it is an isomorphism. □

The above theorem implies that if $g_x = (g_1, g_2) \in G_{k^{\mathrm{sep}}}$ and $x = g_x w \in V_k^{\mathrm{ss}}$, then $\mu_g^{-1}(\mathcal{M}_x) \subset \mathcal{J}_{k^{\mathrm{sep}}} = \mathcal{M}_{w\,k^{\mathrm{sep}}}$ is a k-form of $\mathcal{J}$. If $(\mathcal{M}, \mathfrak{n}) \in \mathrm{JIC}(k)$ is the pair corresponding to x (see [3, Section 4,5]), $\mathcal{M}$ was characterized in the same manner (see [3, (4.8)]). Therefore, Theorem 1.4 follows.

3. Equivariant map II

In this section, we prove Theorem 1.5.

Let

$$\begin{aligned}
\mathrm{T}_a(X,Y,Z) &= 27D(a,a,X)D(a,a,Y)D(a,a,Z)\\
&\quad - 24D(a,a,a)D(a\times X, a\times Y, a\times Z)
\end{aligned} \tag{3.1}$$

for $a, X, Y, Z \in \mathcal{J}$. Then the map

$$\mathrm{T} : \mathcal{J} \ni a \mapsto \mathrm{T}_a \in \mathrm{Hom}_k(\mathcal{J}\otimes\mathcal{J}\otimes\mathcal{J}, k)$$

is k-linear.

LEMMA 3.2. *If* $a \in \mathcal{J}$, $g \in G_1$ *and* $X, Y, Z \in \mathcal{J}$, *then*

$$\mathrm{T}_{ga}(gX, gY, gZ) = c(g)^3 g(\mathrm{T}_a(X,Y,Z))\,.$$

Proof. There exists $t \in \overline{k}$ such that $t^3 = c(g)$. We put $g_1 = t^{-1}g$. Then $g = tg_1$ and $g_1 \in H_{1\,\overline{k}}$. So,

$$D(ga\times gX, ga\times gY, ga\times gZ) = t^6 D(g_1a\times g_1X, g_1a\times g_1Y, g_1a\times g_1Z)$$

$$= t^6 D(\widetilde{g}_1(a \times X), \widetilde{g}_1(a \times Y), \widetilde{g}_1(a \times Z))$$
$$= t^6 D(a \times X, a \times Y, a \times Z)$$
$$= c(g)^2 D(a \times X, a \times Y, a \times Z).$$

Since $D(ga, ga, gX) = c(g)D(a, a, X)$, etc., the lemma follows. □

The following proposition plays a crucial role in proving Theorem 1.5.

PROPOSITION 3.3. $\mathrm{T}_e(X, Y, Z) = \mathrm{Tr}((X \circ Y) \circ Z)$ $(= \mathrm{Tr}(X \circ (Y \circ Z)))$ *for all* $X, Y, Z \in \mathcal{J}$.

Proof. Since $D(e, e, e) = 1$,

$$\mathrm{T}_e(X, Y, Z) = 27D(e, e, X)D(e, e, Y)D(e, e, Z) - 24D(e \times X, e \times Y, e \times Z)\,.$$

Let

$$X = \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix}, \quad Y = \begin{pmatrix} t_1 & y_3 & \overline{y_2} \\ \overline{y_3} & t_2 & y_1 \\ y_2 & \overline{y_1} & t_3 \end{pmatrix}, \quad Z = \begin{pmatrix} u_1 & z_3 & \overline{z_2} \\ \overline{z_3} & u_2 & z_1 \\ z_2 & \overline{z_1} & u_3 \end{pmatrix} \in \mathcal{J}\,.$$

Note that

$$\begin{aligned} 6D(X, Y, Z) = &\sum_{\{i,j,k\}=\{1,2,3\}} s_i t_j u_k + \sum_{\{i,j,k\}=\{1,2,3\}} \mathrm{tr}(x_i y_j z_k) \\ &- \sum_i s_i \mathrm{tr}(y_i \overline{z_i}) - \sum_i t_i \mathrm{tr}(x_i \overline{z_i}) - \sum_i u_i \mathrm{tr}(x_i \overline{y_i})\,. \end{aligned}$$

By (2.7) and (2.8),

$$e \times X = -\frac{1}{2} \begin{pmatrix} -(s_2+s_3) & x_3 & \overline{x_2} \\ \overline{x_3} & -(s_1+s_3) & x_1 \\ x_2 & \overline{x_1} & -(s_1+s_2) \end{pmatrix}$$

Therefore,

$$\begin{aligned} &48D(e \times X, e \times Y, e \times Z) \\ &\quad = [(s_2+s_3)(t_1+t_3)(u_1+u_2) + (s_2+s_3)(t_1+t_2)(u_1+u_3) \\ &\qquad + (s_1+s_3)(t_2+t_3)(u_1+u_2) + (s_1+s_3)(t_1+t_2)(u_2+u_3) \\ &\qquad + (s_1+s_2)(t_2+t_3)(u_1+u_3) + (s_1+s_2)(t_1+t_3)(u_2+u_3)] \\ &\qquad - \sum_{\{i,j,k\}=\{1,2,3\}} \mathrm{tr}(x_i y_j z_k) \\ &\qquad - (s_2+s_3)\mathrm{tr}(y_1\overline{z_1}) - (s_1+s_3)\mathrm{tr}(y_2\overline{z_2}) - (s_1+s_2)\mathrm{tr}(y_3\overline{z_3}) \\ &\qquad - (t_2+t_3)\mathrm{tr}(x_1\overline{z_1}) + (t_1+t_3)\mathrm{tr}(x_2\overline{z_2}) + (t_1+t_2)\mathrm{tr}(x_3\overline{z_3}) \\ &\qquad - (u_2+u_3)\mathrm{tr}(x_1\overline{y_1}) + (u_1+u_3)\mathrm{tr}(x_2\overline{y_2}) + (u_1+u_2)\mathrm{tr}(x_3\overline{y_3}) \\ &\quad = 2\sum_{\{i,j,k\}=\{1,2,3\}} s_i t_j u_k + 2(s_1 t_1 u_2 + \cdots) - \sum_{\{i,j,k\}=\{1,2,3\}} \mathrm{tr}(x_i y_j z_k) \\ &\qquad - \sum_{i \neq j} s_i \mathrm{tr}(y_j \overline{z_j}) - \sum_{i \neq j} t_i \mathrm{tr}(x_j \overline{z_j}) - \sum_{i \neq j} u_i \mathrm{tr}(x_j \overline{y_j}). \end{aligned}$$

Since $6D(e,e,X) = 2(s_1+s_2+s_3)$, we have

$$
\begin{aligned}
27D(e,e,X)D(e,e,Y)D(e,e,Z) &= (s_1+s_2+s_3)(t_1+t_2+t_3)(u_1+u_2+u_3) \\
&= \sum_i s_i t_i u_i + (s_1 t_1 u_2 + \cdots) + \sum_{\{i,j,k\}=\{1,2,3\}} s_i t_j u_k \,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{T}_e(X,Y,Z) &= \sum_i s_i t_i u_i + \frac{1}{2}\sum_{\{i,j,k\}=\{1,2,3\}} \mathrm{tr}(x_i y_j z_k) \\
&\quad + \frac{1}{2}\sum_{i\neq j} s_i \mathrm{tr}(y_j \overline{z_j}) + \frac{1}{2}\sum_{i\neq j} t_i \mathrm{tr}(x_j \overline{z_j}) + \frac{1}{2}\sum_{i\neq j} u_i \mathrm{tr}(x_j \overline{y_j}) \,.
\end{aligned} \tag{3.4}
$$

Replacing X, Y in (2.6) by $X\circ Y, Z$ respectively and taking the sum of diagonal entries, we can express $4\mathrm{Tr}((X\circ Y)\circ Z)$ in the following manner:

$$
\begin{aligned}
&2(2s_1t_1 + \mathrm{tr}(x_3\overline{y}_3 + \overline{x}_2 y_2))u_1 \\
&\quad + \mathrm{tr}((s_1 y_3 + x_3 t_2 + \overline{x_2}\,\overline{y_1} + t_1 x_3 + y_3 s_2 + \overline{y_2}\,\overline{x_1})\overline{z_3} \\
&\quad + (s_1\overline{y_2} + x_3 y_1 + \overline{x_2} t_3 + t_1 \overline{x_2} + y_3 x_1 + \overline{y_2} s_3) z_2) \\
&\quad + 2(2s_2 t_2 + \mathrm{tr}(x_1\overline{y_1} + \overline{x_3} y_3))u_2 + \mathrm{tr}((\overline{x_3}\,\overline{y_2} + s_2 y_1 + x_1 t_3 \\
&\quad + \overline{y_3}\,\overline{x_2} + t_2 x_1 + y_1 s_3)\overline{z_1} + (\overline{x_3} t_1 + s_2\overline{y_3} + x_1 y_2 + \overline{y_3} s_1 + t_2\overline{x_3} + y_1 x_2) z_3) \\
&\quad + 2(2s_3 t_3 + \mathrm{tr}(x_2\overline{y_2} + \overline{x_1} y_1))u_3 \\
&\quad + \mathrm{tr}((x_2 t_1 + \overline{x_1}\,\overline{y_3} + s_3 y_2 + y_2 s_1 + \overline{y_1}\,\overline{x_3} + t_3 x_2)\overline{z_2} \\
&\quad + (x_2 y_3 + \overline{x_1} t_2 + s_3 \overline{y_1} + y_2 x_3 + \overline{y_1} t_2 + s_3 \overline{x_1}) z_1) \,.
\end{aligned}
$$

This coincides with 4 times (3.4). □

The construction of the isotope defined for $a \in \mathcal{J}$ ($\det(a) \neq 0$) is given in [5, p.155]. Let

$$
\begin{aligned}
Q_a(X,Y) &= -6\det(a)D(X,Y,a) + 9D(X,a,a)D(Y,a,a)\,, \\
\Phi_a(X,Y) &= 4\det(a)^3 (X\times a)\times(Y\times a) \\
&\quad + \frac{1}{2}(\det(a)^2 Q_a(X,Y) - Q_a(X,a)Q_a(Y,a))a\,.
\end{aligned} \tag{3.5}
$$

Then the product structure and the associated bilinear form of the isotope corresponding to $a \in \mathcal{J}$ is given by

$$
\begin{aligned}
\mathcal{J}^2 \ni (X,Y) &\mapsto X \circ_a Y \overset{\mathrm{def}}{=} \det(a)^{-4}\Phi_a(X,Y) \in \mathcal{J}\,, \\
\mathcal{J}^2 \ni (X,Y) &\mapsto \langle X,Y\rangle_a \overset{\mathrm{def}}{=} \det(a)^{-2} Q_a(X,Y) \in k\,.
\end{aligned}
$$

(see [5, p.147, Proposition 5.6.2], [5, p.153, Proposition 5.8.2], [5, p.155, Proposition 5.9.2]).

Here we provide an alternative way to define the product structure on $\mathcal{J}_a$.

DEFINITION 3.6. Suppose that $a \in \mathcal{J}, \det(a) \neq 0$. For $X, Y \in \mathcal{J}$, we define $X \circ_a Y \in \mathcal{J}$ to be the element such that $Q_a(X \circ_a Y, Z) = \det(a)^{-1}\mathrm{T}_a(X, Y, Z)$ for all $Z \in \mathcal{J}$.

Since Q_a is a non-degenerate bilinear form, the definition of $X \circ_a Y$ is well-defined.

THEOREM 3.7. *If $a \in \mathcal{J}$, $\det(a) \neq 0$, $g_a \in G_{1\,k^{\mathrm{sep}}}$, $a = g_a e \in \mathcal{J}$, then $\mathcal{J} \otimes k^{\mathrm{sep}} \ni X \mapsto gX \in \mathcal{J}_a \otimes k^{\mathrm{sep}}$ induces an isomorphism of k-algebras $\mathcal{J} \otimes k^{\mathrm{sep}} \cong \mathcal{J}_a \otimes k^{\mathrm{sep}}$.*

Proof. Suppose that $X, Y, Z \in \mathcal{J} \otimes k^{\mathrm{sep}}$. Since $\det(a) = c(g)\det(e) = c(g)$,

$$\begin{aligned} Q_a(g(X \circ Y), gZ) &= c(g)^2 Q_e(X \circ Y, Z) = c(g)^2\mathrm{T}_e(X, Y, Z) = c(g)^{-1}\mathrm{T}_a(gX, gY, gZ) \\ &= \det(a)^{-1}\mathrm{T}_a(gX, gY, gZ) = Q_a(gX \circ_a gY, gZ)\,. \end{aligned}$$

Since this holds for all $Z \in \mathcal{J} \otimes k^{\mathrm{sep}}$, $g(X \circ Y) = gX \circ_a gY$. Therefore, g induces an isomorphism of k-algebras $\mathcal{J} \otimes k^{\mathrm{sep}} \cong \mathcal{J}_a \otimes k^{\mathrm{sep}}$. □

In the situation of Theorem 3.7, $\mathcal{J}_a$ can be identified with the k-form $g^{-1}(\mathcal{J}_a) \subset \mathcal{J}_{k^{\mathrm{sep}}}$. Therefore, this $\mathcal{J}_a$ coincides with the isotope $\mathcal{J}_a$ constructed in [5].

Let $m : V \to \mathcal{J}$ be the equivariant map in [3, Section 5]. Since m is defined by homogeneous polynomials of degree 4, $Q_{m(x)}, \mathrm{T}_{m(x)}$ are defined by homogeneous polynomials of degrees 16, 24 respectively. Can we construct lower degree equivariant maps from V to the space of bilinear forms and trilinear forms on $\mathcal{J}$ to give an alternative way to define the product structure on $\mathcal{M}_x$ (see (2.18))? It does not seem so. For example, if there were such a quadratic form Q_x depending on x, the degree must be $16 - 12 = 4$. However, there seems to be only one equivariant map of degree 4 from $\wedge^2\mathcal{J}$ to the space of bilinear forms on $\mathcal{J}$ by the calculation of the software "LiE" ([1]) as follows.

```
> alt_tensor(2,[1,0,0,0,0,0])
     1X[0,0,1,0,0,0]
> sym_tensor(2,[0,0,0,0,0,1])
     1X[0,0,0,0,0,2] +1X[1,0,0,0,0,0]
> sym_tensor(2,[0,0,1,0,0,0])
     1X[0,0,0,0,0,2] +1X[0,0,2,0,0,0]
     +1X[0,1,0,0,1,0] +1X[1,0,0,0,0,0]
     +1X[1,1,0,0,0,0] +1X[2,0,0,0,0,1]
```

We can construct an equivariant map from $\wedge^2\mathcal{J}$ to the space of bilinear forms on $\mathcal{J}$ such that $w_1 \wedge w_2$ corresponds to the bilinear form

$$\mathcal{J}^2 \ni (X, X) \mapsto \sum_{i=1}^{3} s_i^2 - \sum_{i=1}^{3} \|x_i\| \in k$$

where as

$$\mathrm{Tr}(X \circ X) = \sum_{i=1}^{3} s_i^2 + \sum_{i=1}^{3} \|x_i\|\,.$$

Therefore, we cannot obtain $\mathrm{Tr}(X \circ X)$.

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