立教大学数理物理学研究センター

# 平成27年度活動報告書

平成28年2月


立教大学　数理物理学研究センター

センター長　江口 徹

# 立教大学数理物理学研究センター平成２７年度活動報告

数理物理学研究センターは平成２４年度４月に発足し、立教大学における数理物理学研究の推進とポスドク、院生などの教育、研究の場として活動を行っている.

現在、数理物理学研究センター構成員は

学内　江口徹、筧三郎、小林努、小森靖、
　　　佐藤信哉、神保道夫、原田知広、疋田 泰章、山田裕二
学外：加藤晃史、斉藤義久、立川裕二

である.
センターの今年度の主な活動内容は

１．隔週に開催される数理物理学セミナー
　　15 回開催
２．平成28年１月9日-11日に開催された国際研究集会「Rikkyo MathPhys 2016」

である.

上記１の数理物理学セミナーは数理物理学の最近の様々な進展に関して、専門の研究者を招いて毎回１時間３０分程度の講演を行なうもので、通常のセミナーよりも導入部に時間をかけてより広い分野の聴衆が参加できるようにしている.
２は科研費などからの支援を受けて行った国際研究集会で海外から 2 名、国内から 9 名の研究者が最新の研究成果の発表を行った. 超弦理論や可解模型に関する話題が会議の中心的なテーマで、60 名あまりの参加者があり議論が盛り上がって盛会であった. 多くの院生の方々に会議の運営のために協力して頂いた.

これらのセミナー、講演で用いられたスライドは数理物理学研究センターのホームページ
https://sites.google.com/a/rikkyo.ac.jp/mathphys/

に公開される．

平成２８年２月

立教大学数理物理学センター長

江口　徹（立教大学特任教授）

I. 研究概要

江口は5年程前に、K3曲面の楕円種数をN=4超共形代数の指標で展開すると低い次数の展開係数がMathieu群M24の規約表現の次元の和で表される事を見いだした。この現象はJ関数に関する有名なmonstrous moonshineに良く似ているため広く関心がもたれ、Mathieu moonshineと呼ばれようになった．その後Mathieu moonshineは展開係数の全ての次数でなりたつことが数学的に証明された。

またMathieu moonshineの拡張であるUmbral moonshineが発見され、その他にも何種類かの新らしいmoonshineが発見されたが、これらのmoonshine現象の起源や機構はまだ十分に解明されていない．現在も詳しい研究が続けられている．

江口は今年度、N=4超対称性を持つリュービル理論を調べて理論の中心電荷が$c=6$の場合と$c=6\ k$（$k$は整数)の場合に一種の双対性が有る事を示し、Mathieu moonshineとUmbral moonshineの関係を説明する事を試みた．

II. 発表論文 (2011∼2015年度)

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4. ”Mathieu moonshine and Superconformal algebra”, Arithmetic and Algebraic Geometry 2014, 1月2014年, 東大駒場
5. コロキウム ”Mathieu moonshine and Superconformal algebra”, Simons Center for Geometry and Physics, New York, September 2013
6. ”Superconformal algebra and Mathieu group”, Conference on ”Algebraic geometry, modular forms and applications to physics ”, Edinburgh, November (2012)
7. ”Superconformal algebra and moonshine phenomenon”, Conference on ”Geometry and Physics”, Munich, November (2012)

8. ”Superconformal algebra and moonshine phenomenon”, Conference on ”Strings, branes and Supergravity”, Istanbul, Aug. (2011)

9. ”K3 surface and Mathieu group M24”, Conference on ”Mathieu monnshine”, ETH, Switzerland, July (2011).

10. ”K3 surface and Mathieu group M24”, Conference on ”Three generations of string theory”, IHE, Paris, May (2011)

IV. その他

**加藤 晃史 (かとう あきし)**

I. 研究概要箙 (quiver) とその変異 (mutation) は，クラスター代数とともに，可積分系・低次元トポロジー・表現論・代数幾何学・WKB 解析などさまざまな分野に共通して現れる構造として注目を集めている．特に，箙の変異列 (mutation sequence) から系統的にゲージ理論や 3 次元双曲多様体を構成する方法が提唱され，その不変量を数学的に厳密に解析する手段の開発が必要となった．

加藤は寺嶋郁二氏 (東京工業大学) との共同研究において、与えられた箙変異の列 $\gamma$ (quiver mutation loop = クラスター代数の exchange graph 上のループに相当) に対し、分配 $q$ 級数 $Z(\gamma)$ と呼ばれる母関数を定義した。これは、以下のような著しい性質を持つ。(1) $Z(\gamma)$ は箙変異の列 $\gamma$ の反転操作や巡回シフトのもとで不変であり、圏論的なモノドロミーの不変量と考えられる。(2) 箙変異の列 $\gamma$ の変形に対し、量子ダイログと同様なペンタゴン関係式を満たす。(3) ADE 型ディンキン図形やそのペアから自然に定義される分配 $q$ 級数は、アフィン・リー環に附随する coset 型共形場理論に現れるフェルミ型 (準粒子型) 指標公式に一致し、適当な $q$ ベキ補正のもとで $Z(\gamma)$ は保型形式となる。(4) reddening sequence というクラスの箙変異列 $\gamma$ に対し、分配級数は量子ダイログの積で表され、combinatorial Donaldson-Thomas invariant と一致する。分配 $q$ 級数は組合せ論的データのみから定義され、箙が表す数学的対象の詳細には依らないので、双対性の背後にある共通の性質や量子化の機構を追究する上で役立つと期待される。

現在は分配級数の 3 次元多様体不変量としての性質および、分配級数を不変性を損なわずに切断して有限の多項式にする可能性を追究している。

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3. 加藤晃史 : “圏論と物理学” 数理科学 第 54 巻 2 号 pp. 52-58 (2016 年 2 月号 特集：幾何学における圏論的思考ー幾何学の新展開に迫るー )


III. 口頭発表

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6. 加藤晃史 "Quiver mutation loops and partition $q$-series" Aspects of Integrability in Mathematics and Physics, 大阪市立大学, 2015 年 3 月

## 小林　努（こばやし　つとむ）

I. 研究概要

- スカラー場と計量から構成され、オイラー・ラグランジュ方程式が 2 階となる最も一般的な理論であるホルンデスキ理論、ならびに物理的自由度の数を変えないその拡張理論に関する研究をおこなった。そのような理論にもとづくさまざまな宇宙論とその帰結について議論した。また、そのような理論におけるブラックホール解の安定性解析をおこない、あるクラスのブラックホール解がすべて摂動的に不安定であることを示した。

II. 発表論文

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3. “Galilean Creation of the Inflationary Universe,”
日本物理学会秋季大会 (大阪市立大学, 9 月 27 日, 2015)
4. “Breaking of Vainshtein screening in scalar-tensor theories beyond Horndeski,”
日本物理学会年次大会 (早稲田大学, 3 月 21 日, 2015)

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6. “Exact black hole solutions in shift symmetric scalar-tensor theories,”
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7. “Vainshtein mechanism in the Horndeski theory and beyond,”
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IV. その他

- 特になし

## 小森　靖 (こもり やすし)

I. 研究概要

- 量子多体系と多重ゼータ関数を主な対象として研究を行っている. 2015 年度はリー群に付随する多重ゼータや Euler-Zagier 多重ゼータ関数の拡張として, 超平面配置のゼータ関数の研究を行い (松本耕二氏 (名古屋大) と津村博文氏 (首都大) との共同研究), さらに多重ゼータの負での値の正則化について研究を行った (松本耕二氏 (名古屋大), 古庄英和氏 (名古屋大), 津村博文氏 (首都大) との共同研究).

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斉藤　義久 (さいとう　よしひさ)

I. 研究概要

(1) 量子群の幾何学的表現論；幾何学的な立場から結晶基底の研究をしている。quiver と呼ばれる有限有向グラフから出発し、quiver に付随する代数多様体を考える。その代数多様体の余接バンドルのラクランジアン部分多様体の既約成分全体の集合に結晶構造が定義でき、さらに結晶として量子群の結晶基底と同型になることを証明した。また同様の方法で量子群の既約最高ウエイト表現の結晶基底も幾何学的に構成できることを示した。

(2) 量子群の表現のなす圏の構造；$\mathfrak{sl}_2$ に付随する制限型量子群の有限次元表現の圏のテンソル圏としての構造を調べた。具体的には，任意の直既約表現同士のテンソル積の直既約分解則を完全に決定した。結果として，$\mathfrak{sl}_2$ に付随する制限型量子群の有限次元表現の圏が，テンソル圏としてブレイド圏ではないことを証明した。

(3) 楕円ヘッケ代数の表現論とその応用；楕円ルート系に付随するヘッケ代数を定義し、二重アフィンヘッケ代数との比較を行った。また、楕円ヘッケ代数の表現論を直交多項式の理論に応用し、shifted Jack 多項式の代数的構造を明らかにした。さらに q-KZ 方程式の特殊解との関係も明らかにした。

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IV. その他

- なし．

佐藤　信哉 (さとう のぶや)

I. 研究概要

- フォン・ノイマン環のコホモロジー理論は，純代数的な環や代数の場合と異なり，高次のホッホシルトコホモロジー群は自明となってしまい，環の情報を引き出すことが出来ないことが知られている．フォン・ノイマン環とその部分環を考えて，相対ホッホシルトコホモロジー群を考えても，高次の群は自明となってしまい，やはり環の情報を引き出すことが出来ないことが知られており，フォン・ノイマン環に対する「意味のある」コホモロジー理論を構築することは難しい問題とされている．私はフォン・ノイマン環の部分因子環の研究の観点から，この問題に取り組んでいる．部分因子環は，指数が有限で，深さも有限であるとき，それに付随して2-テンソル圏が存在し部分因子環の完全不変量を与える．2-テンソル圏に対するコホモロジー理論の一般論は現在のところ知られておらず，本年度の私の研究では，そのような 2-テンソル圏に対するコホモロジー理論を構築するための基礎的な考察を行った．また，2015 年に Popa-Shlyakhtenko-Vaes がある種の部分因子環に対するコホモロジー理論を発表した．こちらについては私の研究アプローチとは異なるのだが，興味深い結果であるので，比較して調べている．

神保 道夫 (じんぼう みちお)

I. 研究概要

昨年度に引き続き $\mathfrak{gl}_1$ 量子トロイダル代数のベーテ仮説の研究を行った。ボレル部分代数の有限型表現という概念を導入し、1 有限表現および 2 有限表現上でトレースをとることにより Baxter の TQ 関係式が導かれることを示した。これによりベーテ方程式とフォック表現に付随する転送行列自身の固有値の導出ができた。同様の構成を量子アフィン代数の場合に行うことにより, Frenkel-Hernandez の論文で予想となっていたベーテ方程式の導出が可能になる。これらについては論文を準備中である。(B. Feigin 氏 (Landau 研究所), E. Mukhin 氏 (Indiana 大学), 三輪哲二氏（京都大学）との共同研究)

II. 発表論文 (2011∼2015 年度)

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8. D. Hernandez and M. Jimbo, Asymptotic representations and Drinfeld raional fractions, *Compositio Math.* **148** (2012) 1593–1623
9. M. Jimbo, T. Miwa and F. Smirnov, Fermionic structure in the sine-Gordon model:form factors and null vectors, *Nucl. Phys.* **B852** (2011) 390–440.

III. 口頭発表 (2011∼2015 年度)

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2. Quantum toroidal algebras and Bethe ansatz, 招待講演, workshop “Baxter 2015: Exactly solved models and beyond”, Australian National Univ., 2015 年 7 月 20 日
3. Quantum toroidal $\mathfrak{gl}_1$ and Bethe ansatz, Mathematical Physics Seminar, SEN Saclay, 2015 年 6 月 8 日
4. Quantum toroidal $\mathfrak{gl}_1$ and Bethe ansatz, Mathematical Physics Seminar, Cergy-Pontoise University, 2015 年 6 月 1 日
5. Fermionic basis of local fields in integrable models, “Moshe Flato Lecture Series”, Ben-Gurion Univ., 2015 年 3 月 12 日
6. Fermionic basis in integrable models:profile and prospect, 招待講演, Mathematical Statistical Physics, 京都, 2013 年 7 月 29 日–8 月 3 日

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8. Representations of quantum toroidal algebras, 招待講演, workshop "Recent Advances in Quantum Integrable Systems", Angers, France, 2012 年 9 月 11 日
9. Fermionic basis of local operators in quantum integrable models, 招待講演, International Congress of Mathematical Physics, Aalborg, Denmark, 2012 年 8 月 6 日
10. Fermions acting on local operators in the XXZ model: a review, workshop "Symmetries, Integrable Systems and Representations", 招待講演, Lyon, 2011 年 12 月 15 日

立川 裕二 (たちかわ ゆうじ)

I. 研究概要

2015 年度は 2014 年度に引き続き五次元、六次元の超対称場の理論の研究をすすめた。六次元 $\mathcal{N}=(2,0)$ 超共形場理論を一般のリーマン面にコンパクト化して得られる四次元 $\mathcal{N}=2$ 理論のことを class S 理論と呼ぶが、論文 4, 1 では大森、清水、米倉とともに六次元 $\mathcal{N}=(1,0)$ 超対称共形場理論を $T^2$ にコンパクト化して得られる四次元 $\mathcal{N}=2$ 理論と class S 理論との関係について詳細な考察を行った。また、論文 3 では渡辺とともに A 型 class S 理論の線演算子のみたす代数関係について調べた。Class S 理論については、リーマン面が三つ穴つき球面である際に得られる $T_N$ とよばれる四次元理論が基本になるが、この理論の性質はここ数年来調べられてきたものの、さまざまな著者のいろいろな原著論文に内容が散在していたため、PTEP 誌にレビューを依頼された。それが論文 2 である。

II. 発表論文

2015 年発表の主要なもののみ挙げる。


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2. “Recent advances in SUSY”, Strings 2014 Conference, Princeton & IAS, USA, 2014.

3. “On 2d TQFTs whose values are hyperkähler cones,” String-Math 2011 Conference, University of Pennsylvania, Philadelphia, USA, 2011

4. “4d gauge theories and 2d CFTs,” Lecture at 5th Asian Winter School on Strings, Particles and Cosmology, Jeju Island, Korea, 2011

IV. その他

2016 年の New Horizons in Physics Prize を受賞した。

**西納　武男 (にしのう　たけお)**

I. 研究概要多様体の退化を用いて多様な幾何構造の研究を行った。昨年度から研究していたK3曲面上の有理曲線の存在に関する古くから知られた予想に関する論文を完成させた。

種数が高い正則曲線の研究で重要な, 次数の高い頂点を持つトロピカル曲線に付随する退化した正則曲線の変形に関する詳細な記述を与え, 論文にまとめた。

Université Paris Diderot - Paris 7 の Tony Yue YU と共同で複素トーラス上の代数曲線のトロピカル曲線による記述を研究した。

II. 発表論文

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3. Takeo Nishinou and Tony Yue YU, *Realization of tropical curves in abelian surfaces.* Oberwolfach reports, to appear.
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III. 口頭発表

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10. トロピカル曲線と複素幾何学, 立教大談話会 2014.5.7.

## 原田　知広　(はらだ ともひろ)

I. 研究概要

- 一般相対論の基礎的諸問題とその宇宙物理学および宇宙論への応用に関する研究

II. 発表論文


1. Takafumi Kokubu, Hideki Maeda and Tomohiro Harada, "Does the Gauss-Bonnet term stabilize wormholes?," Class. Quant. Grav. **32** (2015) 23, 235021 [arXiv:1506.08550 [gr-qc]].
2. Takafumi Kokubu and Tomohiro Harada, "Negative tension branes as stable thin shell wormholes," Class. Quant. Grav. **32** (9/2015) 20, 205001 [arXiv:1411.5454 [gr-qc]] (20pp).
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5. B. J. Carr and Tomohiro Harada, "The separate universe problem: 40 years on", Phys. Rev. D **91** (4/2015) 8, 084048 (16pp).
6. Ken-ichi Nakao, Masashi Kimura, Tomohiro Harada, Mandar Patil and Pankaj S. Joshi, "How small can an over-spinning body be in general relativity?," Phys. Rev. D **90** (12/2014) 12, 124079.
7. Tomohiro Harada and Masashi Kimura, "Black holes as particle accelerators: a brief review," Class. Quant. Grav. **31** (11/2014) 243001.
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III. 口頭発表

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2. 原田知広、「Can an over-spinning Kerr geometry be the source of ultra-high energy cosmic rays and neutrinos?」、第 17 回特異点研究会「特異点と時空、および関連する物理」、慶應義塾大学、2016 年 1 月 9 日-11 日
3. Tomohiro Harada, "Primordial black hole formation from cosmological fluctuations", 27 November 2015, Department of Physics, Waseda University, Tokyo, Japan
4. Tomohiro Harada, "Primordial black hole formation from cosmological fluctuations", 20 October 2015, Department of Physics, Nagoya University, Nagoya, Japan
5. Tomohiro Harada, "Primordial black hole formation from cosmological fluctuations",

16 October 2015, Department of Physics, Hokkaido University, Sapporo, Japan

6. 原田知広（立教大理・教授)、Sanjay Jhingan (Jamia Millia Islamia・教授)、「Szekeres 解の宇宙物理学への応用」、日本物理学会 2015 年秋季大会、大阪市立大学、2015 年 9 月 25 日

7. 原田知広（立教大理・准教授)、柳哲文（名大・助教)、中間智弘（東大ビッグバン・D2)、古賀恭敬（立教大院理・M1)、「宇宙論的非線形ゆらぎと原始ブラックホール形成」、日本物理学会 2015 年年次大会、早稲田大学、2015 年 3 月 24 日

8. Tomohiro Harada, "Primordial black hole formation from cosmological fluctuations", the international conference "Hot Topics in General Relativity and Gravitation", 9-15 Aug 2015, Quy Nhon, Vietnam. (invited talk)

9. Tomohiro Harada, "High energy particle collision and collisional Penrose process near a Kerr black hole", the workshop "One Hundred Years of Strong Gravity", 10-12 Jun 2015, Instituto Superior Técnico in Lisbon, Lisbon, Portugal. (invited talk)

10. 原田知広、「宇宙論的長波長解と原始ブラックホール形成」、第 16 回特異点研究会「特異点と時空、および関連する物理」、名古屋大学、2015 年 1 月 10 日-12 日

IV. その他

- なし。

疋田　泰章　　(ひきだ　やすあき)

I. 研究概要

- 高いスピンのゲージ理論と超弦理論との関係性に関する研究を行った。高いスピンのゲージ対称性を破ることで、高いスピンの場が質量を持つようになる。その質量をゲージ/重力対応を利用することで求めた。高いスピンの理論が 3 次元と 4 次元の二つの場合について解析を行った。さらに、超弦理論との関係性について議論した。
- 弦の世界面の理論の研究も行った。2007 年に開発した 3 次元反ド・ジッター空間上の弦を解析する新たな手法を、より広いクラスの理論に応用できるよう拡張した。

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8. Y. Hikida, "Conical defects and $\mathcal{N} = 2$ higher spin holography," JHEP **1308**, 127 (2013).
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III. 口頭発表

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2. "Higher spin $AdS_3$ holography and superstring theory," 研究セミナー, 名古屋大学, 2015 年 11 月.
3. "Higher spin $AdS_3$ holography and superstring theory," 研究セミナー, 大阪大学, 2015 年 10 月.
4. "Higher spin $AdS_3$ holography and superstring theory," 研究セミナー, 京都大学, 2015 年 10 月.
5. "Higgs phenomenon and $\mathcal{N} = 3$ higher spin holography," International Workshop on Strings, Black Holes and Quantum Information, Miyagi, Japan, September, 2015.

6. “$\mathcal{N} = 3$ higher spin holography and Higgs phenomenon,” 研究セミナー, 茨城大学, 2015 年 6 月.

7. “$\mathcal{N} = 3$ higher spin holography and Higgs phenomenon,” 研究セミナー, 東京大学本郷, 2015 年 6 月.

8. “$\mathcal{N} = 3$ higher spin holography and Higgs phenomenon,” 素粒子論研究室コロキウム, 日本大学, 2015 年 5 月.

9. “$\mathcal{N} = 3$ higher spin holography and Higgs phenomenon,” 研究セミナー, KEK, 2015 年 5 月.

10. “Marginal deformations and the Higgs phenomenon in higher spin $AdS_3$ holography,” 研究セミナー, 東京大学駒場, 2015 年 4 月.

IV. その他

- 集中講義, “高いスピンのゲージ理論とその応用,” 京大基研, 2015 年 3 月.

立教大学　数理物理学研究センター　# 数理物理学セミナー

立教大学 数理物理学研究センター　　　　　　　　　　　第１回セミナー

# One-day workshop on isomonodromic deformations

日時：2015 年 4 月 25 日（土）10:00 - 17:30 ←今回は土曜日に行います。
場所：　立教大学池袋キャンパス 4353 教室（4 号館別棟）
※会場がやや分かりにくいのでご注意下さい。

講師１：　川上　拓志　氏［青山学院大学］10:00 - 12:00
題目１：　分岐型線型方程式に付随する 4 次元 Painlevé 型方程式
概要１：　東京大学の坂井氏，中村氏との共同研究において，アクセサリーパラメータを 4 つ持つ線型方程式の特異点を合流させることにより 4 次元Painlevé 型　方程式の退化図式を得た．しかし，Painlevé 方程式の場合にそうであったように，4 次元 Painlevé 型方程式においても，分岐型不確定　特異点を持つ線型方程式を考えなければ，完全な退化図式は得られない．この講演では，線型方程式の不確定特異点の周りの標準形及びその標準形の退化について説明し，それにより退化図式がどのように拡張されるか説明する。

講師２：　中村　あかね　氏［東京大学］13:30 - 17:30
題目２：　４次元パンルヴェ型方程式の自励極限
概要２：　パンルヴェ型方程式の等スペクトラル極限として可積分系が得られるが、これらの系の特徴を捉えるためにスペクトラル曲線のファイブレーションを用いる。８種類の２次元自励パンルヴェ方程式の Hamiltonian $H_{\mathrm{VI}}$～$H_{\mathrm{I}}$ に対するスペクトラル曲線ファイブレーションとして楕円曲面が得られるが、これらの曲面に現れる特異ファイバーの Dynkin 型は $D_4, D_5, D_7, D_8, E_6, E_7, E_8$ である。40 種類の４次元パンルヴェ型方程式の自励極限として得られる可積分系に対して、スペクトラル曲線ファイブレーションを具体的に構成し、現れる特異ファイバーの浪川-上野型について述べる。たとえば自励４次元行列パンルヴェ方程式に対するスペクトラル曲線ファイブレーションの特異ファイバーとして、対応する２次元系の特異ファイバーに楕円曲線が１本付け加わったものが現れる。

立教大学 数理物理学研究センター　　　　　　　　　第 2 回セミナー

# ミレニアム問題と繰りこみ群


## 伊東恵一 氏

立教大学


場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 5 月 13 日（水）16 時 40 分 - 18 時 10 分


## — 概要 —

クレイ研究所のミレニアム問題の (i) ナヴィエ・ストークス方程式の解の存在, (ii) 4 次元 YM 場の構成, は数理物理学の Central Problems であるが, その解法は誰も知らない. これらは高度に非線形であり, 摂動項は各エネルギーレベルが絡み合い急速に増え, 係数が発散する困難に直面する. 現在のところこれを処理できる唯一の方法は繰りこみ群と言われる「マルチ・スケール汎関数積分法」であろう. これは一種の非線形漸化式の形をとるが、まずそれを単純化した「階層近似模型」といわれる Toy Model を解説し, 現在講演者が研究している YM 場の簡略版である, といっても未解決であるが, 2 次元シグマ模型の質量生成の証明について議論する. 時間があれば YM についても触れる予定である.

# Determinantal martingales and determinantal processes


香取眞理 氏

中央大学


場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 5 月 20 日（水）16 時 40 分 - 18 時 10 分


― 概要 ―

Dyson's Brownian motion (BM) model with β=2 (the GUE type) was originally introduced as an eigenvalue process of Hermitian-matrix-valued BM in random matrix theory. In probability theory, it is constructed as Doob's harmonic transform of absorbing BM in a Weyl chamber and realized as a system of noncolliding BMs in one-dimension. In mathematical physics, it provides a typical example of solvable stochastic model in the sense that all spatio-temporal correlation functions can be expressed by determinants and all of them are controlled by a single function called the correlation kernel. The purpose of the present talk is to clarify the direct connection between the harmonic transform and the determinantal solvability by introducing a notion of determinantal martingales. As an application, we discuss the trigonometric- and elliptic-functional extensions and some discretization of Dyson's BM model with β=2. In general, martingales are stochastic processes representing fluctuations. The present work will imply a new point of view in statistical mechanics to study relationship between fluctuations and correlations in nonequilibrium interacting particle systems.

# 多変数 Meixner, Charlier, Krawtchouk 多項式

渋川元樹 氏

大阪大学

場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 6 月 5 日（金）16 時 40 分 - 18 時 10 分
（普段とは曜日が異なります）

— 概要 —

離散型の直交多項式系の代表例である，Meixner, Charlier,Krawtchouk 多項式に関しては q-類似，有限体類似，Hahn, Racah 多項式(一般超幾何多項式)への拡張，多変数類似等の様々な拡張が知られている．本講演では古典的な Meixner 多項式他の復習からはじめ，特に，非負整数上の離散和で直交関係式が定まる Meixner 多項式と，積分で直交関係式が定まる連続型の直交多項式系の代表例である Laguerre 多項式との間の母函数を用いた対応を与える．この事実は一変数の場合においても知られていなかったが，本講演では更にこの対応を対称錐上の調和解析を用いて多変数化する．より具体的には，従来知られていた青本-Gelfand 型の超幾何函数で表示される多変数化とは異なる，一般(Jack)二項係数を用いた Meixner 多項式他の多変数類似を構成し，Faraut-Koranyi らにより導入された多変数 Laguerre 多項式との，母函数を用いた対応を与える．この結果と多変数 Laguerre 多項式に関する既知の結果とを併せることで，多変数 Meixner 多項式他の母函数，直交性，差分方程式といった基本的性質を導出する．また時間があれば，行列式表示や，多変数 Meixner 多項式他に関連するいくつかの予想や展開についても述べたい．

立教大学 数理物理学研究センター　　　　　　　　　　　　　　　第 5 回セミナー

# 量子情報と物理との接点


## 藤井啓祐 氏

京都大学


場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 6 月 17 日（水）16 時 40 分 - 18 時 10 分

## — 概要 —


近年、エンタングルメントなど量子情報分野で用いられている概念を用いて多体量子系の特徴付けが盛んに行われている。また、量子情報を雑音から保護するために開発された量子誤り訂正符号が、トポロジカル秩序を示す量子多体系の可解模型として利用され、その一般的な性質に関する議論が展開されている。一方で、古典スピングラス理論に代表される統計力学の知見に基づいて量子誤り訂正符号の雑音耐性が議論されている。本セミナーでは、これら対応の背景にある量子情報の基本知識を解説し、量子情報と物理分野との接点を探る。関連した最近の研究に関しても紹介する。

# ５次元版（$q$-変形版）AGT 予想とその結晶化

大久保勇輔 氏

名古屋大学

場所：立教大学理学部４号館４階 4408 号室
日時：2015 年７月１日（水）16 時 40 分 - 18 時 10 分
（普段とは会場が異なります）

## ― 概要 ―

AGT 予想とは２次元共形場理論の Virasoro 共形ブロックと４次元 SU(2)ゲージ理論の Nekrasov 公式が一致するという予想である。また、この両者の理論を $q$-変形したもの同士、つまり変形 Virasoro 代数（や Ding-Iohara 代数）と５次元ゲージ理論との間にも対応（５次元版 AGT 予想）があることが知られている。これらの予想には未だ完全な証明は与えられていないが、Macdonald 多項式という直交多項式が５次元版 AGT 予想を解析する１つの重要な道具となっている。例えば共形ブロックを計算する際に、その計算結果を組合せ論的に明示的に表すことができる良い基底が AGT 予想によって示唆されているのだが、５次元版 AGT 予想ではその良い基底が一般化された Macdonald 多項式によって表せることが知られている。

ところで、Macdonald 多項式は Jack 多項式と Hall-Littlewood 多項式という直交多項式を内包している。Jack 多項式への退化と同じ退化を５次元版 AGT 予想に施したものが４次元 AGT 予想であるが、Hall-Littlewood 多項式への退化と同じ退化極限を施した場合をその結晶化と呼ぶ。（変形パラメータ $q$ の $q$=0 極限を見ていることから、量子群の結晶基底に準えて。）この結晶化された場合では物事は簡単化され、様々な予想が証明可能になる。

本講演では AGT 予想の外観から始めて、その $q$ 変形と一般化 Macdonald 多項式について説明し、さらにその結晶化についても紹介する。

立教大学 数理物理学研究センター　　　　　　　　　　第 7 回セミナー

# 可積分性を用いた AdS/CFT の ダイナミカルな理解への挑戦

風間洋一 氏

立教大学

場所：立教大学理学部 4 号館 4 階 4408 号室
日時：2015 年 7 月 15 日（水）16 時 40 分 - 18 時 10 分
（普段とは会場が異なります）

## ― 概要 ―

AdS/CFT が提唱されてからすでに 18 年になろうとしているが、依然としてその強/弱双対性を特徴としたダイナミカルなメカニズムは明らかにされていない。この講演では、AdS/CFT の例の中でも最も基本的である、 $AdS_5 \times S^5$ 時空中の弦理論と 4 次元の $\mathcal{N}$=4 の極大超対称性を持った 4 次元超対称ヤン・ミルズ理論の対応に焦点をあて、ヤン・ミルズ理論のゲージ不変な non-BPS 複合演算子とそれに対応する弦理論の頂点演算子の 3 点関数に対する双対性の検証とその理解の概要を、可積分性を最大限利用した最近の我々の仕事に沿って解説する。

立教大学 数理物理学研究センター 第 8 回セミナー

# エンタングルメントと繰り込み群


西岡辰磨 氏

東京大学


場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 9 月 30 日（水）16 時 40 分 - 18 時 10 分

— 概要 —


繰り込み群は量子場の理論がエネルギーの変化とともにどのように変化するのかを記述する重要な手法である。場の理論の ``有効自由度” はエネルギーの減少と共に単調減少すると期待されているが、そのような ``有効自由度” が任意の場の理論に対して定義できるかどうかは未だ明らかではない。近年、場の理論の真空のもつれを測る指標であるエンタングルメントエントロピーを用いることで 2 次元と 3 次元では繰り込み群の下で単調減少する関数が構成できることが分かってきた。本講演ではこの構成法を紹介と問題点について述べる。

# Kostant-Toda 階層, Totally nonnegative matrix と特異曲線

岩尾慎介 氏

青山学院大学

場所：立教大学理学部 4 号館 4 階 4408 号室
日時：2015 年 10 月 15 日（木）16 時 40 分 - 18 時 10 分
(普段とは会場と曜日が異なります)

## — 概要 —

Totally nonnegative matrix とは，その全ての小行列式が非負となる正方行列のことである．Totally nonnegative matrix 全体のなす集合上には，行列の分解を経由してある種の離散力学系が定義されるが，これは，古典可積分系として知られている Kostant-Toda 階層の特別な場合となっている．本講演では，以下の 2 点について解説する．

1．特異曲線を用いた（非周期的）Kostant-Toda 階層の解の構成．

2．Kostant-Toda 階層の totally nonnegative part の，特異ピカール群を用いた特徴づけ．

本講演の内容は，西山享氏（青山学院大学），小川竜氏（東海大学）との共同研究である．

立教大学 数理物理学研究センター　　　　　　　　　　第10回セミナー

# 情報幾何学的手法を用いたゲージ重力対応の研究


松枝宏明 氏

仙台高専


場所：立教大学理学部4号館4階 4410号室
日時：2015年10月21日（水）16時40分‐18時10分
(普段とは会場が異なります)

— 概要 —


AdS/CFT対応やホログラフィー原理を量子情報の視点から理解しようという試みが盛んに研究されている．笠・高柳公式やエンタングルメントくりこみ群はその有用な例である．一方，情報幾何学の方法も最近注目され始めてきている．とりわけ，量子系の部分密度行列から構成したFisher計量は，量子古典変換を直接表現しており，AdS/CFT対応をストリング理論とは異なる視点から眺める上で非常に有用であると期待される．本講演では量子データから古典的な幾何を構成する一般的方法とその情報理論的意味を紹介する．また空間1次元格子フェルミオン模型を例にとって，量子系のどのようなモデルパラメータが古典幾何の自然な座標に変換されるかを検討する．加えて最近興味が持たれているエンタングルメント熱力学の情報幾何学的再構成やアインシュタイン方程式の情報幾何学的導出についても述べる．

# 楕円 Ding-Iohara-Miki 代数と関連する話題

齋藤 洋介 氏

大阪市大

場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 11 月 11 日（水）16 時 40 分 - 18 時 10 分

― 概要 ―

楕円 Ding-Iohara-Miki 代数は Ruijsenaars 作用素という q-差分作用素の自由場表示を通じて導入された楕円量子群の一種である．本講演では，楕円 Ding-Iohara-Miki 代数が導入される様子や関連する話題，Ding-Iohara-Miki 代数の modular double の表現に関する予想について説明する．

# Strong CP problem and axion on the lattice

北野 龍一郎 氏

KEK

場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 11 月 25 日（水）16 時 40 分 - 18 時 10 分

— 概要 —

In QCD, while the phase in the quark masses are physical through the axial anomaly, it must be extremely small to be consistent with the measurement of the electric dipole moment of the neutron. We review and discuss this mystery, called the strong CP problem, and present an approach from the lattice QCD. Especially, we discuss the temperature dependence of the topological property of QCD which has physical significance in the estimation of the QCD axion in the Universe.

# パンルヴェ方程式と weight 系


## 千葉 逸人 氏

九州大学


場所：立教大学理学部 4 号館 4 階 4407 号室
日時：2015 年 12 月 9 日（水）16 時 40 分 - 18 時 10 分

— 概要 —

微分方程式の weight とは、Newton 図形から定まる自然数の組であり、方程式の不変量である。講演では、weight に付随するトーリック多様体を用いたパンルヴェ方程式の解析法について解説する。また逆に、与えられた weight で適切な条件を満たすものに対し、対応するパンルヴェ方程式を再構築する。特にいくつかの新しい４次元パンルヴェ方程式を紹介する。

# 高いスピンのゲージ理論とその超弦理論への応用

疋田 泰章 氏

立教大学

場所：立教大学理学部 4 号館 4 階 4404 号室
日時：2016 年 1 月 20 日（水）16 時 40 分 - 18 時 10 分

## ― 概要 ―

超弦理論には高いスピンの状態が数多く存在し、それらの状態の理解が弦の理論としての理解に不可欠である。特に高エネルギー極限では、高いスピンのゲージ対称性が現れ、その対称性を利用することで解析可能となると信じられている。最近ゲージ/重力対応を応用することで、高いスピンのゲージ理論と超弦理論との間の具体的な対応関係が議論できるようになってきた。本講演では、これらの発展の解説を行うとともに、私たちの提案について紹介する。

立教大学 数理物理学研究センター　　　　　　　　臨時セミナー

# Direct Proof of Mirror Theorem of Projective Hypersurfaces


## 秦泉寺 雅夫 氏

北海道大学


場所：立教大学理学部 4 号館 2 階 4232 号室
日時：2016 年 2 月 24 日（水）16 時 40 分 - 18 時 10 分
（普段とは会場が異なります）

― 概要 ―


現在においてはミラー定理はギベンタールによる I 関数を用いた定式化と証明が一般的であるが、本講演では講演者による仮想構造定数を用いた、「複素射影空間内の超曲面のグロモフ-ウィッテン不変量に対するミラー定理」の証明を紹介する。（ただし、まだ論文は未発表である。）鍵となるのは、「コンチェビッチによる固定点定理を用いたグロモフ-ウィッテン不変量の計算結果」を留数積分の形に翻訳するテクニックと、講演者によるミラー写像と B 模型の相関関数の展開係数の留数積分表示である。この二つを組み合わせると、ミラー定理は単なる被積分有理関数の組合せ論的恒等式に帰着される。難しい点は、コンチェビッチの計算結果を留数積分表示する際に、対角的寄与というものを取り除く必要がある事であり、この点をどう切り抜けるかについても解説したい。

International Symposium

# RIKKYO MathPhys 2016

January 09-11, 2016

International Symposium

# RIKKYO MathPhys 2016

## January 9 (Sat) - 11 (Mon), 2016

Room M301 and M302, McKim Hall (bldg 15), Rikkyo University

## Invited Speakers :


## Organizer :

Research Center for Mathematical Physics

Tohru Eguchi, Yasuaki Hikida, Michio Jimbo, Saburo Kakei

Rikkyo Universty

## Address :

3-34-1 Nishi-Ikebukuro, Toshima-ku, Tokyo Japan 171-8501

Website: https://sites.google.com/a/rikkyo.ac.jp/rikkyo-mathphys-2016/home

# RIKKYO MathPhys 2016

立教大学

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International Symposium

# RIKKYO MathPhys 2016

January 9 [Sat] - 11 [Mon], 2016

Date:

January 9 (Sat) - 11 (Mon), 2016

Venue:

Rikkyo University, McKim Hall (bldg 15)

Room M302 (except in the morning of January 9)

Room M301 (in the morning of January 9)

Access to Rikkyo University (Ikebukuro Campus)

Campus Map (Ikebukuro Campus)

Invited Speakers:

S. Aoki (YITP)

Y. Hatsuda (Geneva)
T. Imamura (Chiba)
A. Ishibashi (Kinki)
M. Katori (Chuo)
K. Lee (KIAS)
Y. Nakayama (IPMU)
J. Suzuki (Shizuoka)
F. Takahashi (Tohoku)
K. Takasaki (Kinki)
S. Yanagida (RIMS)

Banquet:

The conference banquet will take place on
January 9th (Saturday), 2016,
at “Fuji-dana” on the second floor of Main Dining Hall.
Banquet fee: JPY 3,000

Organizer:

Research Center for Mathematical Physics
Tohru Eguchi, Yasuaki Hikida, Michio Jimbo, Saburo Kakei
Rikkyo Universty

Website Links:

Rikkyo Research Center for Mathematical Physics (In Japanese)
Rikkyo University

# Timetable

| | January 9 (Sat.) | January 10 (Sun.) | January 11 (Mon.) | |
|---|---|---|---|---|
| | | F. Takahashi | Y. Nakayama | 10:00–11:00 |
| 10:30–11:00 | Registration | | | |
| | | Break | Break | 11:00–11:15 |
| 11:00–12:00 | K. Lee * | J. Suzuki | M. Katori | 11:15–12:15 |
| 12:00–14:00 | Lunch | Lunch | Lunch | 12:15–14:00 |
| 14:00–15:00 | S. Yanagida | Y. Hatsuda | S. Aoki | |
| 15:00–15:30 | Tea | Tea | | |
| 15:30–16:30 | A. Ishibashi | T. Imamura | | |
| 16:30–16:40 | Break | | | |
| 16:40–17:40 | K. Takasaki | | | |
| | Banquet (6:00 p.m.) | | | |

Venue: M302, McKim Hall (bldg. 15) (* except in the morning of January 9; M301)

| Invited Speakers: | *Titles:* |
|---|---|
| Kimyeong Lee (KIAS) | *6 dim supersymmetric theories* |
| Shintarou Yanagida (RIMS) | *K-theoretic stable envelopes of instanton moduli spaces and intertwiners between Fock representations of quantum toroidal algebra* |
| Akihiro Ishibashi (Kinki Univ.) | *Instabilities of asymptotically AdS black holes* |
| Kanehisa Takasaki (Kinki Univ.) | *Topological vertex and quantum mirror curves* |
| | |
| Fuminobu Takahashi (Tohoku Univ.) | *Axion cosmology and the Witten effect* |
| Junji Suzuki (Shizuoka Univ.) | *A form factor expansion approach to correlations of the spin 1/2 XXZ chain* |
| Yasuyuki Hatsuda (Univ. Geneva) | *Exact quantization conditions for relativistic integrable systems* |
| Takashi Imamura (Chiba Univ.) | *Determinantal structures in a random polymer model* |
| | |
| Yu Nakayama (Kavli IPMU/Caltech) | *Search for dead-end CFTs* |
| Makoto Katori (Chuo Univ.) | *Determinantal interacting particle systems* |
| Sinya Aoki (YITP) | *Recent developments and challenges in lattice QCD* |

International Symposium

# RIKKYO MathPhys 2016

January 9 (Sat.) - 11 (Mon.), 2016
M301 and M302, McKim Hall (bldg. 15), Rikkyo University

## Titles and Abstracts

January 9 (Saturday)

**Kimyeong Lee (KIAS)**
Title: 6 dim supersymmetric theories
Abstract:
Recent progress in 6d superconformal field theories and little string theories are summarized. In this talk, we discuss in detail their physics by calculating the 2-dim elliptic genus of self-dual strings in these theories. We focus on the enhanced global symmetry of 6d SCFT and T-duality of LST.

**Shintarou Yanagida (RIMS)**
Title: K-theoretic stable envelopes of instanton moduli spaces and intertwiners between Fock representations of quantum toroidal algebra
Abstract:
We give a geometric interpretation of intertwiners between Fock representations of the quantum toroidal gl1 algebra, which were introduced by Awata, B. Feign and Shiraishi. The main ingredient is the analysis of K-theoretic stable envelopes of instanton moduli spaces on the complex plane. As a consequence, we can prove some parts of the conjectures on K-theoretic AGT correspondence proposed in the previous collaboration with Awata, B. Feigin, Hoshino, Kanai and Shiraishi.

**Akihiro Ishibashi (Kinki Univ.)**
Title: Instabilities of asymptotically AdS black holes
Abstract:
Asymptotically AdS black hole with rotation plays an interesting role in the AdS-CFT context. We show that any asymptotically AdS black hole with an ergoregion with respect to the horizon Killing vector field must be linearly unstable. For this purpose we adapt the canonical energy method of Hollands-Wald, which was originally formulated for asymptotically flat case and applied only for the axisymmetric perturbations. However, our analysis for asymptotically AdS black holes requires no restrictions on the gravitational perturbations, apart from the appropriate behavior at AdS infinity. This immediately implies, for example, that Kerr-AdS black holes with rotation speeds above the Hawking-Reall bound are gravitationally unstable. We also briefly discuss possible final states of such unstable AdS black holes.

**Kanehisa Takasaki (Kinki Univ.)**
Title: Topological vertex and quantum mirror curves?
Abstract:
Topological vertex is a diagrammatic method for constructingthe partition functions of topological string theory on non-compact toric Calabi-Yau threefolds. We present a few cases, including the so called "closed topological vertex", where open string amplitudes can be computed explicitly by this method. These expressions of open string amplitudes can be used to derive "quantum mirror curves". This is a joint work with Toshio Nakatsu.

January 10 (Sunday)

**Fuminobu Takahashi (Tohoku Univ.)**
Title: Axion cosmology and the Witten effect
Abstract:
After giving a review on the current status of the axion cosmology, I will discuss how the Witten effect affects the QCD axion dynamics. In the presence of monopoles, the theta-term of an Abelian gauge theory becomes physical, and the monopoles are known to become dyons. This is the Witten effect. I will show that the Witten effect can suppress the axion abundance as well as isocurvature perturbations.

**Junji Suzuki (Shizuoka Univ.)**
Title: A form factor expansion approach to correlations of the spin 1/2 XXZ chain
Abstract:
We study static and dynamical correlation functions of the spin 1/2 XXZ chain in the massive regime, based on a form factor expansion of the correlations. The resultant expression is valid at any finite temperatures and with an arbitrary magnetic field for the static case. We argue that there exist three different expressions in the zero temperature and zero magnetic field limit and demonstrate the consistency among them. This talk is based on the collaboration with M. Dugave, F. Goehmann and K. Kozlowski.

**Yasuyuki Hatsuda (Univ. Geneva)**
Title: Exact quantization conditions for relativistic integrable systems
Abstract:
In this talk, I will report recent progress on a relation between spectral theory and topological string theory. Using mirror symmetry, the topological string is described by an algebraic curve, called a mirror curve. The quantization of such a curve is naturally related to a quantum mechanical operator. I will show that the eigenvalue problem of this operator is exactly solved by a quantization condition, whose building blocks are the refined topological string amplitudes in the so-called Nekrasov-Shatashvili limit. Based on this result, we finally found exact quantization conditions for a wide class of relativistic integrable systems, including the relativistic Toda lattice, associated with toric Calabi-Yau threefolds.

**Takashi Imamura (Chiba Univ.)**
Title: Determinantal structures in a random polymer model
Abstract:
A directed polymer in random media is a most typical object belonging to the Kardar-Parisi-Zhang (KPZ) universality class. In 1+1 dimensional case, we have known some exactly solvable models with nice mathematical structures. In the zero-temperature case, some determinanal structures of the models have been clarified based on the properties of RSK correspondence and Schur function. In the finite temperature case, on the other hand, we have recently been obtaining exact solutions of the polymer free energy distributions using the Macdonald polynomial, dualities etc. However, we have not yet understand very much the determinantal structures behind the models with finite temperature.

In this talk, I will report our recent result on the O'Connell-Yor model, which is a typical exactly solvable model of the random polymer (arXiv:1506.05548). We reveal some determinantal structures of the model and obtain a representation for the moment generating function of the polymer partition in terms of a determinantal measure, which is valid for arbitrary positive temperature. In the zero-temperature limit this measure goes to the probability measure of the eigenvalues for the Gaussian Unitary Ensemble (GUE) in random matrix theory. This is the joint work with Tomohiro Sasamoto.

January 11 (Monday)

**Yu Nakayama (Kavli IPMU / Caltech)**
Title: Search for dead-end CFTs
Abstract:
Can we find CFTs without any relevant deformation? This question is important
(A) in applications to collective phenomena called "self-organized criticality"
(B) in search for candidate CFT duals of the moduli stabilized universe (which we have never found so far)
(C) an obstruction to regularize the CFT S-matrix.
In this talk I will discuss perturbative as well as non-perturbative searches for such CFTs. We also report the up-to-date conformal bootstrap results to derive necessary conditions for the emergent symmetry enhancement that are relevant for the symmetry protected dead-end CFTs that are alleged to be realized in nature.

**Makoto Katori (Chuo Univ.)**
Titile:Determinantal interacting particle systems
Abstract:
Eigenvalue distributions of random matrices provide typical examples of determinantal (fermion) point processes on a line (e.g., the Gaussian unitary ensemble) or on a plane (e.g., the Ginibre ensemble). Corresponding to that Dyson introduced models of interacting Brownian motions as dynamical extensions of random matrix ensembles, the notion of determinantal point processes has been dynamically extended; for a set of observables,

if all spatio-temporal correlation functions are given by determinants specified by a single integral kernel (the correlation kernel), then the process is said to be determinantal [1]. Recently a sufficient condition for determinantal processes was proved by using probability theory concerning harmonic transforms, martingales, and conformal invariance of the complex Brownian motion [2].

In the present talk, I show a variety of examples satisfying the condition, that is, the interacting particle systems having the determinantal martingale representations (DMR). There some systems related with the Chern-Simons theory and with the Kardar-Parisi-Zhang (KPZ) equation are discussed. The solvability of determinantal interacting particle systems is characterized by the entire function which defines the conformal transform of a complex Brownian motion and determines the DMR.

[1] M. Katori : Bessel Processes, Schramm-Loewner Evolution, and the Dyson Model, Springer Briefs in Mathematical Physics, Vol.11, Springer (2016).
[2] M. Katori : Determinantal martingales and noncolliding diffusion processes, Stochastic Process. Appl. 124 (2014) 3724-3768.

**Sinya Aoki (YITP)**
Title: Recent developments and challenges in lattice QCD
Abstract:
After a brief introduction of lattice QCD, I will review the recent developments in lattice QCD, by showing the latest results of numerical simulations. Results include the hadron spectroscopy with the isospin breaking effects,electroweak matrix elements and the QCD equation of states at finite temperature. As a challenge in lattice QCD, I will also report on hadron interactions in QCD such as the nuclear force, from the formulation to some latest results using Japanese flagship K computer.

# 6 dim supersymmetric theories

Kimyeong Lee, KIAS
RYKKO MathPhys 2016

Stefano Bolognesi, Hirotaka Hayasi, Hee-Cheol Kim, Joonho Kim, Seok Kim, Sung-Soo Kim, Eunkyung Koh, Sungjay Lee, Jaemo Park, Masato Taki, Cumrun Vafa, Futoshi Yagi, Ho-Ung Yee

## 5,6 dimensions

- QFT in higher dimension is non-renormalizable.
- Ultra-violet incomplete, Non-Lagrangian
- Some supersymmetric theories can be UV completed.
- 5d superconformal field theories (5d SCFTs)
- 6d superconformal field theories (6d SCFTs)
- 6d little string theories (LSD)
- Interesting implications in lower dimensions

## Goal

- Understand these theories by calculating various exact and non exact quantities.
- Study BPS objects
- Study the elliptic genus of self-dual strings

## Outline

- 6d superconformal field theories
  - 6d SCFTs
  - E-strings elliptic genus
  - $SU(N)_T$-[A,N+8] theories
- 6d little string theories
  - type IIA and type IIB LSTs
- Conclusion

## 6d SCFTs

- N=(2,0) SCFTs: ADE (nonabelian tensor multiplet)
  - $A_N$ on N M5 branes, $D_N$ on N M5+OM5 branes
  - type IIB on $C^2/\Gamma_{ADE}$ singularity
- N=(1,0) SCFTs: quiver with multiple nonabelian YMs + tensor multiplets + hypermultiplets
  - M5 branes on $C^2/\Gamma_{ADE}$ singularity
  - M5 branes near $E_8$ M9 wall
- 6d SCFTs → 5,4,3,2d QFTs

## 6d (2,0) SCFTs

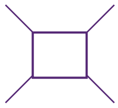
Witten'95, Seiberg Witten '96

- superconformal symmetry: $OSp(2,6|2) \supset O(2,8) \times Sp(2)_R$
  - fields: $B$, $\Phi_I$, $\Psi$
  - selfdual strength $H=dB={}^*H$, purely quantum $\hbar=1$
  - superconformal symmetry: $OSp(2,6|2) \supset O(2,8) \times Sp(2)_R$
- tensor branch: M2 branes connecting M5 branes = selfdual
- We do not know how to write down the theory for nonabelian case.
- $N^3$ degrees of freedom

## 6d (1,0) SCFTs

Seiberg'96,Danielsson et.al.'97

- supercharge Q (1,0), ε-spinor (0,1)
  - (1,0) vector multiplet: gaugino (0,1)
  - (1,0) hypermultiplet: higgsino (1,0)
  - tensor multiplet: (1,0)
- gauge anomaly due to vector and matter 1-loop
- anomaly polynomial



$$\mathrm{Tr}_R F^4 = \alpha_R \mathrm{tr} F^4 + c_R (\mathrm{tr} F^2)^2$$

$$\alpha_R = 0 \;\; for \;\; SU(2), SU(3), G_2, SO(8), F_4, E_6, E_7, E_8$$

$$c_{\mathrm{tot}} = \left[ c_{Ad} - \sum_{R \text{ matter}} c_R \right] \geq 0$$

## Anomalies

Green-Schwarz Mechanism

$$H^2 + \sqrt{c_{\mathrm{tot}}} B \wedge \mathrm{tr} F \wedge F + \sqrt{c_{\mathrm{tot}}} \Phi \mathrm{tr} F^2$$
$$dH = \sqrt{c_{\mathrm{tot}}} \mathrm{tr} F \wedge F$$

$c_{tot}=0$: tensor decouples, and LST

Global Anomaly: SU(2),SU(3),$G_2$ Bershadsky,Vafa'97

$$\pi_6(SU(2)) = \mathbf{Z}_{12}, \;\; \pi_6(SU(3)) = \mathbf{Z}_6, \;\; \pi_6(G_2) = \mathbf{Z}_3$$

$$n_2 - 4 = 0 \bmod 2 \text{ for } SU(2)$$
$$n_3 - n_6 = 0 \bmod 6 \text{ for } SU(3)$$
$$n_7 - 1 = 0 \bmod 3 \text{ for } G_2$$

## Simple gauge group without no quartic

Bershadsky and Vafa '97

F-theory on elliptic CY 3-folds with a non-compact base

LST (Bhardwaj' 15, Bhardwaj et.al. '15) → $D^2=0$

| G | hyper | $D^2$=-8 | $D^2$=-7 | $D^2$=-6 | $D^2$=-5 | $D^2$=-4 | $D^2$=-3 | $D^2$=-2 | $D^2$=-1 | $D^2$=0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $E_7$ | $n_{56/2}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $E_6$ | $n_{27}$ | | | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $F_4$ | $n_{26}$ | | | | 0 | 1 | 2 | 3 | 4 | 5 |
| SO(8) | $n_{8(v,c,s)}$ | | | | | 0 | 1 | 2 | 3 | 4 |
| $G_2$ | $n_7$ | | | | | | 1 | 4 | 7 | 10 |
| SU(3) | $n_3$ | | | | | | 0 | 6 | 12 | 18 |
| SU(2) | $n_2$ | | | | | | | 4 | 10 | 16 |

Higgsable along the column

## 6d SU(n)+ tensor + matter

| R | $\alpha_R$ | $c_R$ |
|---|---|---|
| fund | 1 | 0 |
| asym | n-8 | 3 |
| sym | n+8 | 3 |
| adj | 2n | 6 |

1. vector + 2n fund + tensor
   - two NS6 on n D6: two M5 on $A_{n-1}$ singularity
2. vector + 8 fund+ 1 asym+ tensor
   - O8+8D8 with n D6 connecting 1/2 NS brane on O8 and NS5 brane
3. vector + 16 fund+ 2 asym: n D6 connecting two 1/2 NS5s on a pair of O8+8D8 branes, LST
4. vector+ 1 asym + 1 sym: n D6 connecting two 1/2 NS5s on $O8^-$ and $O8^+$ , LST

## N M5s on ADE singularities

Heckman et.al. '15

Fractionalization of NS5 branes

$A_N$ = N D6 with n NS5 : [SU(N)]-SU(N)-SU(N)…-[SU(N)]

$D_{N+4}$= [SO(2N+8)]-Sp(N)-SO(2N+8)-Sp(N)….Sp(N)-[SO(2N +8)]

$E_6$: +.. (N-1) $E_6$ gauge theories +2 $E_6$ global symmetries:

[$E_6$] -T-SU(3)-T-$E_6$-T -SU(3)-… … -SU(3)-T- [$E_6$]

$E_7$: (N-1) E7 gauge theories+ 2$E_7$ global symmetries..,

[E7]-T-SU(2)-SO(7)-SU(2)-T- E7-T-……… -T-[E7]

$E_8$: (N-1) $E_8$ gauge theories +2$E_8$ global symmetries

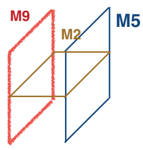
[$E_8$]-T-T-SU(2)-$G_2$-T-$F_4$-T-$G_2$-SU(2)-T-T-E8-T-……-T-T-[E8]

## E8 (1,0) Theory

Joonho Kim, Seok Kim, KL, Jaemo Park, Cumrun Vafa (14)

Hwang, JKim,SKim,Park(14), Haghighat,Lockhart,Vafa(14), Cai,Huang,Sun(14),Haghighat,Klemm,Lockhart,Vafa (14),..

- M2 branes between 2 M5 branes = M-string elliptic genus
- M2 branes between M5 and M9 branes = E-string elliptic genus
- wrap $x^{11}$ to a circle with $E_8$ Wilson line **248 →120+128** with SO(16) symmetry
- D8+O8,NS5,D2,
- D6(un-compactify $S^1$ in IR)

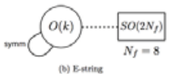



| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| NS5 | • | • | • | • | • | | | | • | |
| D8-O8 | • | • | • | • | • | • | • | • | • | |
| D2 | • | | | | | | | | • | • |
| D6 | • | • | • | • | • | | | | • | • |

$x^0x^9$ 2d SQFT on D2 branes

# UV theory on D2 branes

- The theory on D4 (wrap instead NS5)=Sp(k) theory
- Symmetries $SO(4)_{1234} \times SO(3)_{567}$ =$SU(2)_L \times SU(2)_R \times SU(2)_I$
  $\alpha,\beta,\cdots=1,2 \quad \dot\alpha,\dot\beta,\cdots=1,2 \quad A,B,\cdots=1,2,$
- boundary condition + boundary degrees of freedom
- 2d field content: vector : $O(n)$ antisymmetric $(A_\mu, \lambda_+^{\dot\alpha A})$
  hyper : $O(n)$ symmetric $(\varphi_{\alpha,\dot\beta}, \lambda_-^{\alpha A})$
  Fermi : $O(n) \times SO(16)$ bifundamental $\Psi_l$
- 2d N=(0,4) SUSY $Q^{\dot\alpha A}$ dictates the interaction
- SO(16)→$E_8$ symmetry enhancement in IR




# Elliptic Genus

Gadde and Gukov (13), Benini,Eager,Hori,Tachikawa I,II(13)

- Take (0,2) subset of (0,4) SUSY,
- Define partition function for n-strings

$$Z_n(q,\epsilon_{1,2},m_l) = \mathrm{Tr}_{RR}\left[(-1)q^{H_L}\bar q^{H_R} e^{2\pi i\epsilon_1(J_1+J_2)} e^{2\pi\epsilon_2(J_2+J_I)} \prod_{l=1}^{8} e^{2\pi i m_l F_l}\right]$$

$J_1, J_2, J_I$ are the Cartans of $SU(2)_L \times SU(2)_R \times SU(2)_I$

$F_l$ are the Cartans of $SO(16)$

- All string sum: $Z = \Sigma_{n=0} Z_n$
- Path integral representation of $Z_n$ : O(n) gauge theory

# Holonomy

- gauge zero mode= O(n) flat connection =O(2p) and O(2p+1) cases
- eigenvalues $u_i=u_{1i}+\tau u_{2i}$, of the holonomy $\exp(u_{1i}\sigma_2)$, $\exp(u_{2i}\sigma_2)$

**O(2p)**

$$(ee) : u=(\pm u_1,\cdots,\pm u_p);\quad u=(\pm u_1,\cdots,\pm u_{p-2},0,\frac12,\frac{1+\tau}{2},\frac{\tau}{2})$$
$$(eo) : u=(\pm u_1,\cdots,\pm u_{p-1},0,\frac{\tau}{2});\quad u=(\pm u_1,\cdots,\pm u_{p-1},\frac12,\frac{1+\tau}{2})$$
$$(oe) : u=(\pm u_1,\cdots,\pm u_{p-1},0,\frac12);\quad u=(\pm u_1,\cdots,\pm u_{p-1},0,\frac{\tau}{2},\frac{1+\tau}{2},\frac{\tau}{2})$$
$$(oo) : u=(\pm u_1,\cdots,\pm u_{p-1},0,\frac{1+\tau}{2});\quad u=(\pm u_1,\cdots,\pm u_{p-1},\frac{\tau}{2},\frac12)$$

**O(2p+1)**

$$(ee) : u=(\pm u_1,\cdots,\pm u_p,0);\quad u=(\pm u_1,\cdots,\pm u_{p-1},\frac12,\frac{1+\tau}{2},\frac{\tau}{2})$$
$$(eo) : u=(\pm u_1,\cdots,\pm u_p,\frac{\tau}{2});\quad u=(\pm u_1,\cdots,\pm u_{p-1},\frac12,\frac{1+\tau}{2},0)$$
$$(oe) : u=(\pm u_1,\cdots,\pm u_p,\frac12);\quad u=(\pm u_1,\cdots,\pm u_{p-1},\frac{\tau}{2},\frac{1+\tau}{2},0)$$
$$(oo) : u=(\pm u_1,\cdots,\pm u_p,\frac{1+\tau}{2});\quad u=(\pm u_1,\cdots,\pm u_{p-1},0,\frac{\tau}{2},\frac12)$$

# Determinant

- hyper, fermi, vector

$$Z_{\text{sym. hyper}} = \prod_{\rho\in\text{sym}} \frac{i\eta(q)}{\theta_1(q,\epsilon_1+\rho(u))}\cdot\frac{i\eta(q)}{\theta_1(q,\epsilon_2+\rho(u))}$$
$$Z_{SO(16)\text{ Fermi}} = \prod_{\rho\in\text{fund}}\prod_{l=1}^{8}\frac{\theta_1(q,m_l+\rho(u))}{i\eta(q)}$$
$$Z_{\text{vector}} = \prod_{i=1}^{r}\left(\frac{2\pi\eta^2 du_i}{i}\cdot\frac{\theta_1(\epsilon_1+\epsilon_2)}{i\eta}\right)\cdot\prod_{\alpha\in\text{root}}\frac{\theta_1(\alpha(u))\theta_1(\epsilon_1+\epsilon_2+\alpha(u))}{i^2\eta^2}.$$

$$\sum_a \frac{1}{|W_a|}\cdot\frac{1}{(2\pi i)^r}\oint Z^{(a)}_{\text{1-loop}},\quad Z^{(a)}_{\text{1-loop}} \equiv Z^{(a)}_{\text{vector}} Z^{(a)}_{\text{sym. hyper}} Z^{(a)}_{SO(16)\text{ Fermi}},$$

- Integration: Jeffery-Kirwan Residues

Benini-Eager-Hori-Tachikawa

# Calculations..

- single string: Ganor and Hanany, Klemm, Mayr and Vafa
- two E-strings: Haghighat, Lockhart,Vafa
- 3,4 E-strings, any E-strings…
- 5d *N*=1 YM theory on D4 with $N_f$=8: Hwang, Kim[2],Park
- $E_8$ symmetry is manifest for lower number of strings

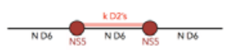
$$Z_1 = \sum_{i=1}^{4}\frac{Z_{1(i)}}{2} = -\frac{\Theta(q,m_l)}{\eta^6\theta_1(\epsilon_1)\theta_1(\epsilon_2)},$$
$$\Theta(\tau,m_l) = \frac12\sum_{n=1}^{4}\prod_{l=1}^{8}\theta_n(\tau,m_l).$$

$$Z_2(\tau,\epsilon_{1,2},m_l) = \frac12 Z_{2(0)} + \frac14\sum_{a=1}^{6} Z_{2(a)}$$

$$Z_{2(1)} = \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^8\theta_1(m_l)\theta_2(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)},\quad Z_{2(2)} = \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^8\theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)}$$
$$Z_{2(3)} = \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^8\theta_1(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)},\quad Z_{2(4)} = \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^8\theta_2(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)}$$
$$Z_{2(5)} = \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^8\theta_1(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)},\quad Z_{2(6)} = \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^8\theta_2(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)}$$

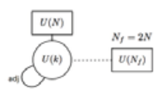
K. Mohri (02), K. Sakai (14)    Cai, Huang, Sun(14)

# SU(N) + 2N N Fund




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| NS5 | • | • | • | • | • | • | – | – | – | – |
| D6 | • | • | • | • | • | • | • | – | – | – |
| D2 | • | • | – | – | – | – | • | – | – | – |

- Selfdual Strings=Instanton Strings
- 1+1 dynamics of ADHM data (0,4) Language

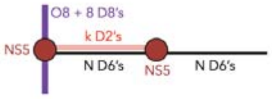



| Field | Type | $U(k)$ | $U(N)$ | $U(N_f)$ |
|---|---|---|---|---|
| $(A_\mu, \lambda^{A\dot\alpha})$ | vector | adj | – | – |
| $(a_{\alpha\dot\beta}, \chi^A_\alpha)$ | hyper | adj | – | – |
| $(q_{\dot\alpha}, \psi^A)$ | hyper | **k** | $\bar{\mathbf{N}}$ | – |
| $(\Xi_l)$ | Fermi | **k** | – | $\bar{\mathbf{N}}_f$ |

- 1+1 anomaly cancellation
- Straightforward calculation

## SU(N) + Anti+ (N+8)Fund

JKim,SKim,KL 1510

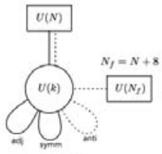



| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| NS5 | • | • | • | • | • | • | – | – | – | – |
| D6 | • | • | • | • | • | • | • | – | – | – |
| O8-D8 | • | • | • | • | • | • | – | • | • | • |
| D2 | • | • | – | – | – | – | • | – | – | – |

- N D6 branes give $Z_N$ ALF space
- SU(0), SU(1),SU(2): Anti-symmetric multiplet is trivial.
- SU(3): Anti-symmetric multiplet is equivalent to fundamental representation.
  - SU(3)_T + 12 fundamental hypermultiplet
- SU(3)+12 can be obtained from Higgsing of G2+ 7 fundamental hyper

Global Symmetry Enhancement for N=3 to SU(12) ?

## String dynamics




(a)

| Field | Type | $U(k)$ | $U(N)$ | $U(N_f)$ | $U(1)_A$ |
|---|---|---|---|---|---|
| $(A_\mu, \lambda^{\dot{\alpha}A})$ | vector | **adj** | – | – | 0 |
| $(a_{\alpha\dot{\beta}}, \chi^A_\alpha)$ | hyper | **adj** | – | – | 0 |
| $(q_{\dot{\alpha}}, \psi^A)$ | hyper | **k** | $\overline{\mathbf{N}}$ | – | 0 |
| $(\Xi_l)$ | Fermi | **k** | – | $\overline{\mathbf{N}}_\mathbf{f}$ | 0 |
| $(\varphi_A, \Phi^{\dot{\alpha}})$ | twisted hyper | **sym** | – | – | +1 |
| $(\Psi_\alpha)$ | Fermi | **anti** | – | – | +1 |
| $(\psi)$ | Fermi | **k** | **N** | – | +1 |

(b)

- SU(0), SU(1),SU(2): Anti-symmetric multiplet is trivial.
- SU(3): Anti-symmetric multiplet is equivalent to fundamental representation.
  - SU(3)_T + 12 fundamental hypermultiplet
- SU(3)+12 can be obtained from Higgsing of G2+ 7 fundamental hyper

## Global symmetry enhancement

- For N=1, single D6 gives $Z_1$ singularity which is trivial in IR.
  - superior to no D6 as the SO(4)=SU(2)xSU(2) symmetry transverse to M5 brane manifests.
  - the flavor symmetry SU(9) should be enhanced to $E_8$
- For N=2, the global symmetry SO(20) in Sp(1) description is manifest
  - the flavor symmetry SU(10) should be enhanced to SO(20)
- For N=3, the global symmetry SU(11)xU(1) should be merged to SU(12)

## Elliptic Genus

Benini,Eager,Hori,Tachikawa 1305,1308

Supercharge $Q \equiv Q_-^{\dot{1}2}$ and $Q^\dagger \equiv Q_-^{\dot{2}1}$

$$Z_k = \mathrm{Tr}_{RR}\left[(-1)^F q^{H_L}\bar{q}^{H_R} e^{2\pi i\epsilon_1(J_1+J_R)} e^{2\pi i\epsilon_2(J_2+J_R)} e^{2\pi i M A}\prod_{l=1}^{N_f} e^{2\pi i m_l F_l}\prod_{i=1}^{N} e^{2\pi i a_i G_i}\right]$$

$$Z_{\text{vector}} = d\phi_1\wedge\cdots\wedge d\phi_k\cdot(2\pi\eta^2)^k\prod_{\alpha\in\text{root}}\frac{\theta_1(\alpha(\phi))}{i\eta}$$

$$Z_{\text{chiral}} = \prod_{\rho\in\text{weight}(R)}\frac{i\eta}{\theta_1(\rho(\phi)+2\epsilon_-J_l+2\epsilon_+(J_r+J_R)+z\cdot F)}$$

$$Z_{\text{Fermi}} = \prod_{\rho\in\text{weight}(R)}\frac{\theta_1(\rho(\phi)+2\epsilon_-J_l+2\epsilon_+(J_r+J_R)+z\cdot F)}{i\eta},$$

$$Z_{1-\text{loop}} = Z_{\text{vector}}\prod_{\text{chiral}} Z_{\text{chiral}}\prod_{\text{Fermi}} Z_{\text{Fermi}}.$$

$$\frac{1}{|W|}\oint Z_{1-\text{loop}} = \frac{1}{|W|}\sum_{\phi_*}\text{JK-Res}_{\phi_*}(\mathbf{Q}_*,\mathfrak{n})\,Z_{1-\text{loop}}$$

$$\text{JK-Res}_{\phi_*}(\mathbf{Q}_*,\mathfrak{n})\frac{d\phi_1\wedge\cdots\wedge d\phi_r}{Q_{j_1}(\phi-\phi_*)\cdots Q_{j_r}(\phi-\phi_*)} = \begin{cases}|\det(Q_{j_1},\cdots,Q_{j_r})|^{-1} & \text{if } \mathfrak{n}\in\text{Cone}(Q_{j_1},\cdots,Q_{j_r})\\ 0 & \text{otherwise}\end{cases}$$

$\mathbf{Q}_* = (Q_1,\cdots,Q_r)$ is a set of $r\geq k$ charge vectors,

## Single string

$$\oint d\phi\,\frac{\eta^3\theta_1(2\epsilon_+)}{i\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\cdot\prod_{i=1}^{N}\frac{\eta\theta_1(\phi+a_i+M)}{\theta_1(\epsilon_+\pm(\phi-a_i))}\cdot\frac{\eta^2}{\theta_1(-\epsilon_+\pm(2\phi+M))}\cdot\prod_{l=1}^{N+8}\frac{\theta_1(\phi-m_l)}{\eta}$$

with n>0, we choose the poles of positive charge Q

$$\epsilon_+ +\phi - a_j = 0 \quad (j=1,\cdots,N), \qquad -\epsilon_+ + 2\phi + M = 0,$$

- $\phi = a_j - \epsilon_+ \quad (j=1,\cdots,N)$

$$-\sum_{j=1}^{N}\frac{\eta^{-6}\prod_{l=1}^{N+8}\theta_1(a_j-\epsilon_+-m_l)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(2a_j-3\epsilon_++M)}\cdot\prod_{i\neq j}\frac{\theta_1(a_i+a_j-\epsilon_++M)}{\theta_1(a_j-a_i)\theta_1(2\epsilon_+-(a_j-a_i))}$$

- $\phi = \frac{\epsilon_+-M}{2}+\ell_I$ for $\ell = \{0,\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}\}\quad (I=1,2,3,4)$

$$-\frac{1}{2}\frac{\eta^{-6}}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\left[\frac{\prod_{l=1}^{N+8}\theta_1(\frac{\epsilon_+-M}{2}-m_l)}{\prod_{i=1}^{N}\theta_1(\frac{3\epsilon_+-M}{2}-a_i)}+(-1)^N\sum_{I=2}^{4}\frac{\prod_{l=1}^{N+8}\theta_I(\frac{\epsilon_+-M}{2}-m_l)}{\prod_{i=1}^{N}\theta_I(\frac{3\epsilon_+-M}{2}-a_i)}\right]$$

## Two strings

$$\oint\frac{d\phi_{1,2}}{2}\,\frac{-\eta^6\theta_1(2\epsilon_+)^2}{\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2}\prod_{i\neq j}\frac{\theta_1(\phi_{ij})\theta_1(\phi_{ij}+2\epsilon_+)}{\theta_1(\phi_{ij}+\epsilon_1)\theta_1(\phi_{ij}+\epsilon_2)}\prod_{l=1}^{N+8}\frac{\theta_1(\phi_{1,2}-m_l)}{\eta^2}\prod_{i=1}^{N}\frac{\eta^2\theta_1(\phi_{1,2}+a_i+M)}{\theta_1(\epsilon_+\pm(\phi_{1,2}-a_i))}$$
$$\times\frac{\eta^4\theta_1(\epsilon_-\pm(\phi_1+\phi_2+M))}{\theta_1(-\epsilon_+\pm(\phi_1+\phi_2+M))\theta_1(-\epsilon_+\pm(2\phi_{1,2}+M))}.$$

We adopt the concise notations such as $\phi_{ij}\equiv\phi_i-\phi_j$, $a_{mn}\equiv a_m-a_n$, $\theta_I(\phi_{i,j}+b)\equiv\theta_I(\phi_i+b)\theta_I(\phi_j+b)$, $\theta_I(a_{m,n}+b)\equiv\theta_I(a_m+b)\theta_I(a_n+b)$, $\theta_{I,J}(b)\equiv\theta_I(b)\theta_J(b)$. The Weyl group $W\subset U(2)$ is $\mathbb{Z}_2$. After picking an auxiliary vector $\mathfrak{n}$ to be $(+1,+1)$, we collect all contributing residues given as follows.

Poles

$(\phi_1,\phi_2) = (a_m-\epsilon_+, a_n-\epsilon_+)$ for $m\neq n$.

$(\phi_1,\phi_2) = (\frac{\epsilon_+-M}{2}+\ell_I, a_m-\epsilon_+)$ and $(\phi_1,\phi_2) = (a_m-\epsilon_+, \frac{\epsilon_+-M}{2}+\ell_I)$

more…..

# Single String for N=2

2d O(k) for 6d Sp(1)

$$-\frac{\eta^2}{\theta_1(\epsilon_{1,2})}\sum_{I=1}^{4}\frac{\eta^2}{\theta_I(\epsilon_+\pm a)}\prod_{l=1}^{10}\frac{\theta_I(m_l)}{\eta}$$

2d U(k) for 6d SU(2)

$$-\frac{\eta^{-6}}{\theta_1(\epsilon_{1,2})}\left[\frac{\prod_{l=1}^{10}\theta_1(a-\epsilon_+-m_l)}{\theta_1(2a-3\epsilon_++M)}\frac{\theta_1(-\epsilon_++M)}{\theta_1(2a)\theta_1(2\epsilon_+-2a)}+(\pm a\to\mp a)\right]-\frac{\eta^{-6}}{\theta_1(\epsilon_{1,2})}\sum_{I=1}^{4}\frac{\prod_{l=1}^{10}\theta_I(\frac{\epsilon_+-M}{2}-m_l)}{2\theta_I(\frac{3\epsilon_+-M}{2}\pm a)}$$

Expand in q power

$$t=e^{2\pi i\epsilon_+},\ u=e^{2\pi i\epsilon_-},\ y_i=e^{2\pi i\hat{m}_i},\ \overline{y}=e^{2\pi i\overline{m}},\ Y=e^{2\pi iM},\ w_i=e^{2\pi i\hat{a}_i},\ \overline{w}=e^{2\pi i\overline{a}}.$$

$$\frac{t}{(1-tu)(1-tu^{-1})}\Big[q^{-1/2}+\frac{q^{1/2}\cdot t^2}{(1-t^2w_1^2)(1-t^2w_1^{-2})}\Big(-\chi_{\mathbf{512}}^{SO(20)}\chi_{1/2}^{SU(2)}(w_1)+\chi_{\mathbf{512}}^{SO(20)}\chi_{1/2}^{SU(2)}(t)$$
$$+\chi_{\mathbf{20}}^{SO(20)}\chi_{1/2}^{SU(2)}(t)\chi_{3/2}^{SU(2)}(w_1)-\chi_{\mathbf{20}}^{SO(20)}\chi_{3/2}^{SU(2)}(t)\chi_{1/2}^{SU(2)}(w_1)-\chi_{\mathbf{190}}^{SO(20)}\chi_{1}^{SU(2)}(w_1)+\chi_{1/2}^{SU(2)}(t)\chi_{1/2}^{SU(2)}(u)$$
$$+\chi_{3/2}^{SU(2)}(t)\chi_{1/2}^{SU(2)}(u)-\chi_{1/2}^{SU(2)}(t)\chi_{1/2}^{SU(2)}(u)\chi_{1}^{SU(2)}(w_1)+\chi_{2}^{SU(2)}(t)\chi_{1}^{SU(2)}(w_1)-\chi_{1}^{SU(2)}(t)\chi_{2}^{SU(2)}(w_1)$$
$$+\chi_{\mathbf{190}}^{SO(20)}\chi_{1}^{SU(2)}(t)\Big)+\mathcal{O}(q^{3/2})\Big].$$

$q^{-1/2}$: zero point energy

# Single String for N=3

$$\mathbf{12}\longrightarrow\mathbf{1}_{-11}+\mathbf{11}_{+1}$$
$$\overline{\mathbf{12}}\longrightarrow\mathbf{1}_{+11}+\overline{\mathbf{11}}_{-1}$$
$$\mathbf{143}\longrightarrow\mathbf{1}_0+\mathbf{11}_{12}+\overline{\mathbf{11}}_{-12}+\mathbf{120}_0,$$

Expand in q power

$$\frac{t^2}{(1-tu)^2(1-tu^{-1})^2}\Big[q^{-1}\cdot\frac{t\,\chi_{1/2}^{SU(2)}(t)}{(1+tu^{-1})(1+tu)}+q^0\cdot\Big(t^{-2}\chi_{\mathbf{8}}^{SU(3)}+t^{-1}(\chi_{1/2}^{SU(2)}(u)-\chi_{\mathbf{3}}^{SU(3)}\chi_{\overline{\mathbf{12}}}^{SU(12)}$$
$$+\chi_{\overline{\mathbf{3}}}^{SU(3)}\chi_{\mathbf{12}}^{SU(12)})+\chi_{\mathbf{143}}^{SU(12)}+1+\chi_{\mathbf{8}}^{SU(3)}+\mathcal{O}(t^1)\Big)+\mathcal{O}(q^1)\Big].\qquad(3.26)$$

UV and IR description of self-dual strings

UV does not need to respect IR symmetry

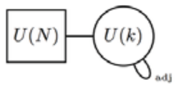
# Little String Theories

- Low energy dynamics of NS5 branes + fundamental strings in the limit where gravity decouples: fix $l_s$ and take $g_s=0$
  - type IIA, compactify one of $R^5$ transverse to 6d (2,0) SCFT
  - type IIB, S-dual of D5-D1 system and decouple gravity
    - UV completion of 6d N=2 SYM theory (ADE) $\frac{4\pi^2}{g^2_{YM,NS5}}=\frac{1}{2\pi\alpha'}=T_{F1}$
- Two theories on $R^{1+4}\times S^1$ with momentum p and winding w are T-dual to each other with exchange of p and w.
- elliptic genus of instanton strings and M-strings are needed to show this.

# type IIB (1,1) LST

| | **x⁰** | x¹ | x² | x³ | x⁴ | **x⁵** | x⁶ | x⁷ | x⁸ | x⁹ |
|---|---|---|---|---|---|---|---|---|---|---|
| N NS5 | x | x | x | x | x | x | | | | |
| k F1 | x | | | | | x | | | | |
| Wilson | | | | | | x | | | | |

- 6d N= (1,1) Super Yang-Mills, Instanton Strings
- Torus: $x^0$-$x^5$,
- U(N) Wilson line: $e^{2\pi iR_{IIB}A_5}=\mathrm{diag}(e^{2\pi i\alpha_1},e^{2\pi i\alpha_2}\cdots,e^{2\pi i\alpha_N})$
- **KK momentum: fractional ones** $P_5-A_5=\frac{n-\alpha_i+\alpha_j}{R_{IIB}}$
- Elliptic genus
- KK-mode contribution + LST contribution




$$Z_{\rm IIB}(\alpha_i,\epsilon_\pm,m;q,w)=Z_{\rm KK}^{\rm IIB}(\alpha_i,\epsilon_\pm,m;q)Z_{\rm string}^{\rm IIB}(\alpha_i,\epsilon_\pm,m;q,w).$$

Aharony-Berkooz'99,J.Kim,S.Kim,KL'15

# type IIB (1,1) LST

- 6d SYM $Z^{\rm IIB}_{\rm 6d\,SYM}(\alpha_i,\epsilon_\pm,m;q)=PE\left[I_+\sum_{i>j}^{N}e^{2\pi i(\alpha_i-\alpha_j)}+I_+\left(N+\sum_{i\neq j}^{N}e^{2\pi i(\alpha_i-\alpha_j)}\right)\frac{q}{1-q}\right],$
- $I_+(\epsilon_\pm,m)=\frac{\sinh\frac{2\pi i(m+\epsilon_+)}{2}\sinh\frac{2\pi i(m-\epsilon_+)}{2}}{\sinh\frac{2\pi\epsilon_1}{2}\sinh\frac{2\pi\epsilon_2}{2}}.$
- Instanton string: k-instantons in U(N) theory

$$Z_k(\alpha_i,\epsilon_\pm,m;q)=\sum_{Y:\sum_i|Y_i|=k}\prod_{i,j=1}^{N}\prod_{s\in Y_i}\frac{\theta_1(q;E_{ij}+m-\epsilon_-)\,\theta_1(q;E_{ij}-m-\epsilon_-)}{\theta_1(q;E_{ij}-\epsilon_1)\,\theta_1(q;E_{ij}+\epsilon_2)},$$

where

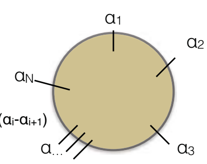
$$E_{ij}=\alpha_i-\alpha_j-\epsilon_1h_i(s)+\epsilon_2v_j(s).$$

'$s$' denotes a box in the Young diagram $Y_i$. $h_i(s)$ is the distance from the box '$s$' to the edge on the right side of $Y_i$ that one reaches by moving horizontally. $v_j(s)$ is the distance from '$s$' to the edge on the bottom side of $Y_j$ that one reaches by moving vertically. The elliptic genus of the

# type IIA (2,0) LST

| | **x⁰** | x¹ | x² | x³ | x⁴ | **x⁵** | x⁶ | x⁷ | x⁸ | x⁹ | **x¹⁰** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| M5 | x | x | x | x | x | x | | | | | $2\pi\alpha_iR_M$ |
| M2 | x | | | | | x | | | | | $2\pi R_M(\alpha_i,\alpha_i$ |

- Torus: $x^0$-$x^5$, M-circle $x^{10}$
- Position of M5 branes : $2\pi\alpha_iR_M$
- tension of M2 branes connecting two M5 Branes i and i+1th ~$(\alpha_i-\alpha_{i+1})$
- T-duality between type IIB and type IIA
  - type IIB winding(instanton string)=type IIA KK (instanton)




$$m_{\rm IIB\ winding}=\frac{2\pi R_{\rm IIB}}{2\pi\alpha'}=\frac{R_{\rm IIB}}{\alpha'}\xrightarrow{\rm T-dual}\frac{1}{R_{\rm IIA}}=m_{\rm IIA\ KK}.$$

  - type IIB fractional KK = type IIA fractional winding

$$m^{\rm frac}_{\rm IIB\ KK}=\frac{\alpha_{i,i+1}}{R_{\rm IIB}}\xrightarrow{\rm T-dual}(\alpha_{i,i+1})\frac{R_{\rm IIA}}{\alpha'}=\alpha_{i,i+1}(2\pi R_{\rm IIA})T_{\rm F1}$$
$$=2\pi\alpha_{i,i+1}R_M(2\pi R_{\rm IIA})T_{\rm M2}=m^{\rm frac}_{\rm IIA\ winding},$$

# type IIA (2,0) LST

| | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ | $x^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| N NS5 | x | x | x | x | x | x | | | | $2\pi a_i R_M$ |
| $n_i$ D2 | x | | | | | x | | | | $2\pi R_M(a_i, a_{i+1})$ |
| 1 D6 | x | x | x | x | x | x | | | | x |

- IIA elliptic genus $Z_{\rm IIA}(\alpha_i,\epsilon_\pm,m;q',w') = {\rm Tr}\Big[(-1)^F w'^{W} q'^{H_L}\bar{q}'^{H_R} e^{2\pi i\alpha_i\Pi_i} e^{2\pi i\epsilon_-(2J_{1L})} e^{2\pi i m(2J_{2L})} e^{2\pi i\epsilon_+(2J_{1R}+2J_{2R})}\Big]$
- Split to KK and string contributions:

$$Z_{\rm IIA}(\alpha_i,\epsilon_\pm,m;q',w') = Z^{\rm IIA}_{\rm pert}(\epsilon_\pm,m;q')Z^{\rm IIA}_{\rm string}(\alpha_i,\epsilon_\pm,m;q',w')\ .$$

- KK mode:

$$Z^{\rm IIA}_{\rm pert}(\epsilon_\pm,m;q') = \left(Z^{\rm IIA}_{\text{rank 1 pert}}(\epsilon_\pm,m;q')\right)^N = PE\left[NI_-(\epsilon_\pm,m)\frac{q'}{1-q'}\right]$$

$$I_\pm(\epsilon_\pm,m) = \frac{\sinh\frac{2\pi i(m+\epsilon_\pm)}{2}\sinh\frac{2\pi i(m-\epsilon_\pm)}{2}}{\sinh\frac{2\pi i\epsilon_1}{2}\sinh\frac{2\pi i\epsilon_2}{2}}\ .$$

- String contribution (D2 branes connecting NS5 branes..) $SU(2)_D$ x U(1) R-symmetry

$$Z^{\rm IIA}_{\rm string}(\alpha_i,\epsilon_\pm,m;q',w') = \sum_{n_i=0}^\infty e^{2\pi i\sum_{i=1}^N n_i\alpha_{i,i+1}} Z^{(n_1,\cdots,n_N)}_{\rm string}(\epsilon_\pm,m;q')\ . \qquad (3.12)$$

Haghighat,Kozcaz,Lockhart,Vafa:hep-th/1310.1185

where

$$Z^{(n_1,\cdots,n_N)}_{\rm string}(\epsilon_\pm,m;q') = \sum_{(Y_1,\cdots,Y_N)\,|Y_i|=n_i}\prod_{i=1}^N\prod_{(a,b)\in Y_i}\frac{\theta_1(q';E^{(a,b)}_{i,i+1}-m+\epsilon_-)\theta_1(q';E^{(a,b)}_{i,i-1}+m+\epsilon_-)}{\theta_1(q';E^{(a,b)}_{i,i}+\epsilon_1)\theta_1(q';E^{(a,b)}_{i,i}-\epsilon_2)}\ , \qquad (3.13)$$

and

$$E^{(a,b)}_{ij} = (Y_{i,a}-b)\epsilon_1 - (Y^T_{j,b}-a)\epsilon_2\ ,\quad e^{-2\pi i(\alpha_{i,i+1})} = e^{-2\pi i m_i}w' \qquad (3.14)$$

$Y_{a,i}$ is the length of the $a$ th row of the Young diagram $Y_i$. $Y^T_{a,i}$ is the length of the $a$ th column of the Young diagram $Y_i$.

# T-duality

- type IIA : Extra -bound state at the center of Taub-NUT (Without NS5, there is still a bound state of fundamental string at D6.
- The correct result for type IIA $\hat{Z}_{\rm IIA}(\alpha_i,\epsilon_{1,2},m,q',w') = \dfrac{Z_{\rm IIA}(\alpha_i,\epsilon_{1,2},m,q',w')}{Z_{\rm extra}(w')}$.

$$Z_{\rm extra}(w') \equiv \prod_{n=1}^\infty\frac{1}{1-(w')^n} \sim \eta(w')^{-1}.$$

- T-duality

$$\hat{Z}_{\rm IIA}(\alpha_i,\epsilon_\pm,m;q',w')|_{q'\to w,w'\to q} = Z_{\rm IIB}(\alpha_i,\epsilon_\pm,m;q,w)\ ,$$

# T-duality : Rank 1

- type IIB $Z^{\rm IIB}_{U(1)\,\rm pert}(\epsilon_\pm,m;q) = PE\left[I_-(\epsilon_\pm,m)\dfrac{q}{1-q}\right]\ .$

$$Z^{\rm IIB}_{U(1)\,\rm string}(\epsilon_\pm,m;q,w) = \sum_{k=0}^\infty w^k Z_k(\epsilon_\pm,m;q)$$

$$I_k = \sum_{Y:|Y|=k}\prod_{s\in Y}\frac{\theta_1(q;E(s)+m-\epsilon_-)\,\theta_1(q;E(s)-m-\epsilon_-)}{\theta_1(q;E(s)-\epsilon_1)\,\theta_1(q;E(s)+\epsilon_2)}\ .$$

$$E(s) = -\epsilon_1 h(s) + \epsilon_2 v(s)\ .$$

- Hecke transformation

$$Z_{U(1)\,\rm string}(\epsilon_\pm,m;q,w) = \exp\left[\sum_{n=1}^\infty\frac{1}{n}w^n\sum_{\substack{ad=n\\ a,d\in\mathbb{Z}}}\sum_{b(\text{mod }d)}Z_{1-\rm inst}\left(a\epsilon_\pm,am;\frac{a\tau+b}{d}\right)\right]$$

- the same functional form $Z_{extra}Z^{IIB}_{pert}Z^{IIB}_{U(1)inst}(\epsilon_\pm,m;q,w) = Z^{IIA,N=1}_{bulk}Z^{IIA,N=1}_{string}(\epsilon_\pm,m;q',w')$
- T-duality (q,w)↔(w',q') requires the symmetric function of (q,w)

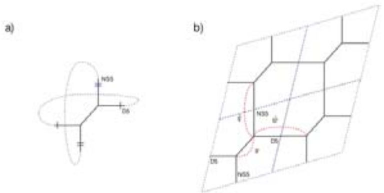
# T-duality : Rank 1

$$Z_{\rm IIA\ N=1}(\epsilon_\pm,m;q',w') = PE\Big[I_-(\epsilon_\pm,m)z_{\rm sp}(\epsilon_\pm,m)\Big]$$

$$\begin{aligned}
z_{\rm sp}(\epsilon_\pm,m;q',w') &= (q'+w') + (q'^2+w'^2) + (q'w')\left[tu+\frac{t}{u}+\frac{1}{tu}+\frac{u}{t}-uy-\frac{y}{u}-\frac{u}{y}-\frac{1}{uy}\right]\\
&+ q'^3 + w'^3 + (q'^2w'+q'w'^2)\Big[t^2u^2+\frac{t^2}{u^2}+\frac{u^2}{t^2}+\frac{1}{t^2u^2}+t^2+\frac{1}{t^2}-tu^2y-\frac{ty}{u^2}-\frac{tu^2}{y}-\frac{t}{u^2y}\\
&-\frac{y}{tu^2}-\frac{u^2}{ty}-\frac{1}{tu^2y}-\frac{u^2y}{t}+tu+\frac{t}{u}+\frac{1}{tu}+\frac{u}{t}-2ty-\frac{2t}{y}-\frac{2}{ty}-\frac{2y}{t}+2u^2+\frac{2}{u^2}-uy-\frac{y}{u}\\
&-\frac{u}{y}-\frac{1}{uy}+y^2+\frac{1}{y^2}+4\Big] + (q'^4+w'^4) + (q'^3w'+q'w'^3)\Big[t^3u^3+\frac{t^3}{u^3}+\frac{u^3}{t^3}+\frac{1}{t^3u^3}+t^3u+\frac{t^3}{u}\\
&+\frac{u}{t^3}+\frac{1}{t^3u}-t^2u^3y-\frac{t^2y}{u^3}-\frac{t^2u^3}{y}-\frac{t^2}{u^3y}-\frac{u^3y}{t^2}-\frac{y}{t^2u^3}-\frac{u^3}{t^2y}-\frac{1}{t^2u^3y}+t^2u^2+\frac{t^2}{u^2}+\frac{u^2}{t^2}\\
&+\frac{1}{t^2u^2}-2t^2uy-\frac{2t^2y}{u}-\frac{2t^2u}{y}-\frac{2t^2}{uy}-\frac{2uy}{t^2}-\frac{2y}{t^2u}-\frac{2u}{t^2y}-\frac{2}{t^2uy}+2t^2+\frac{2}{t^2}+2tu^3+\frac{2t}{u^3}\\
&+\frac{2}{tu^3}+\frac{2u^3}{t}-2tu^2y-\frac{2ty}{u^2}-\frac{2tu^2}{y}-\frac{2t}{u^2y}-\frac{2y}{tu^2}-\frac{2u^2}{ty}-\frac{2}{tu^2y}-\frac{2u^2y}{t}+tuy^2+\frac{ty^2}{u}+\frac{tu}{y^2}\\
&+\frac{t}{uy^2}+\frac{y^2}{tu}+\frac{u}{ty^2}+\frac{1}{tuy^2}+\frac{uy^2}{t}+6tu+\frac{6t}{u}+\frac{6}{tu}+\frac{6u}{t}-4ty-\frac{4t}{y}-\frac{4}{ty}-\frac{4y}{t}-u^3y-\frac{y}{u^3}\\
&-\frac{u^3}{y}-\frac{1}{u^3y}+u^2y^2+\frac{y^2}{u^2}+\frac{u^2}{y^2}+\frac{1}{u^2y^2}+4u^2+\frac{4}{u^2}-5uy-\frac{5y}{u}-\frac{5u}{y}-\frac{5}{uy}+2y^2+\frac{2}{y^2}+8\Big]
\end{aligned}$$

$t = e^{2\pi i\epsilon_+}, u = e^{2\pi i\epsilon_-}, y = e^{2\pi im}$

# Triality : Rank 1




$\hat{q} = qy^{-1}, \hat{w} = wy^{-1}, y = e^{2\pi im}$

Triality: exchange of $\hat{q},\hat{w},\hat{y}$ Hollowood,Iqbal,Vafa, 0310272

$$\hat{Z}_{\rm N=1}(\epsilon_\pm;\hat{q},\hat{w},y) = PE\left[I_{\rm com}(\epsilon_\pm)y\right]Z_{\rm extra}(q)Z_{U(1)\,\rm IIB}(\epsilon_\pm,m;\hat{q}y,\hat{w}y)\ .$$

$I_{\rm com}(\epsilon_\pm)$ is given as follows,

$$I_{\rm com}(\epsilon_\pm) = \frac{1}{2\sinh\frac{2\pi i\epsilon_1}{2}2\sinh\frac{2\pi i\epsilon_2}{2}} = \frac{t}{(1-tu)(1-tu^{-1})}.$$

# Triality : Rank 1

We write down the index, $\hat{Z}_{\rm N=1}$, as follows,

$$\hat{Z}_{\rm N=1}(\epsilon_\pm;\hat{q},\hat{w},y) = PE\Big[I_{\rm com}\hat{z}_{\rm sp}(\epsilon_\pm;\hat{q},\hat{w},y)\Big], \qquad (4.21)$$

where $I_{\rm com}$ is given by eq.(4.20). $\hat{z}_{\rm sp}$ can be obtained as follows,

$$\begin{aligned}
\hat{z}_{\rm sp}(\epsilon_\pm;\hat{q},\hat{w},y) &= \hat{q}+\hat{w}+y-(u+u^{-1})(\hat{q}\hat{w}+\hat{q}y+\hat{w}y)+\frac{(1+u^2)(t+u+t^2u+tu^2)}{tu^2}\hat{q}\hat{w}y\\
&+(\hat{q}^2\hat{w}+\hat{q}\hat{w}^2+\hat{q}^2y+\hat{q}y^2+\hat{w}^2y+\hat{w}y^2)-(u+u^{-1})(\hat{q}^2\hat{w}^2+\hat{q}^2y^2+\hat{w}^2y^2)\\
&-\frac{(u^2+1)\left(t^2(u^2+1)+2tu+u^2+1\right)}{tu^2}\hat{q}\hat{w}y(\hat{q}+\hat{w}+y)\\
&+(\hat{q}^3\hat{w}^2+\hat{q}^2\hat{w}^3+\hat{q}^3y^2+\hat{q}^2y^3+\hat{w}^3y^2+\hat{w}^2y^3)+\frac{(1+u^2)(t+u+t^2u+tu^2)}{tu^2}\hat{q}\hat{w}y(\hat{q}^2+\hat{w}^2+y^2)\\
&+\frac{t^4(u^5+u^3+u)+t^3(u^6+4u^4+4u^2+1)}{t^2u^3}\hat{q}\hat{w}y(\hat{q}\hat{w}+\hat{q}y+\hat{w}y)\\
&+\frac{t^2(3u^4+7u^2+3)u+t(u^6+4u^4+4u^2+1)+u^5+u^3+u}{t^2u^3}\hat{q}\hat{w}y(\hat{q}\hat{w}+\hat{q}y+\hat{w}y)\\
&-(u+u^{-1})(\hat{q}^3\hat{w}^3+\hat{q}^3y^3+\hat{w}^3y^3)\\
&-\frac{(u^2+1)\left(t^4(u^4+u^2+1)+3t^3(u^3+u)\right)}{t^2u^3}\hat{q}\hat{w}y(\hat{q}^2\hat{w}+\hat{q}\hat{w}^2+\hat{q}^2y+\hat{q}y^2+\hat{w}^2y+\hat{w}y^2)\\
&-\frac{(u^2+1)\left(2t^2(u^4+3u^2+1)+3t(u^3+u)+u^4+u^2+1\right)}{t^2u^3}\hat{q}\hat{w}y(\hat{q}^2\hat{w}+\hat{q}\hat{w}^2+\text{(cyclic)})
\end{aligned}$$

## T-duality : Rank 2

$$v_1 = e^{2\pi i\alpha_{12}},\ \text{and}\ v_2 \equiv q v_1^{-1} \qquad q' = w,\ q' = w.$$

$$Z_{\mathrm{IIB\,U(2)}}(\alpha_i, \epsilon_\pm, m; w, v_i) = PE\left[ I_{\mathrm{com}}(t,u) \sum_{i,j,k=0}^{\infty} F_{ijk}^{\mathrm{IIB\,U(2)}}(t,u,y) w^i v_1^j v_2^k \right].$$

$$Z_{\mathrm{IIA\,N=2}}(\alpha_i, \epsilon_\pm, m; w, v_i) = PE\left[ I_{\mathrm{com}}(t,u) \sum_{i,j,k=0}^{\infty} F_{ijk}^{\mathrm{IIA\,N=2}}(t,u,y) w^i v_1^j v_2^k \right].$$

Weyl $F_{ijk}^{\mathrm{U(2)\,IIB}}(t,u,y) = F_{ikj}^{\mathrm{U(2)\,IIB}}(t,u,y)$

T-duality $F_{ijk}^{\mathrm{IIA\,N=2}} = F_{ijk}^{\mathrm{IIB\,U(2)}} \equiv F_{ijk}^{\mathrm{N=2}}.$

## T-duality : Rank 2

$$\begin{aligned}
F_{211}^{\mathrm{N=2}} = &-ty^4 - uy^4 - \frac{y^4}{t} - \frac{y^4}{u} + 2t^2y^3 + 2u^2y^3 + 6tuy^3 + \frac{6uy^3}{t} + \frac{2y^3}{t^2} + \frac{6ty^3}{u} + \frac{6y^3}{tu} + \frac{2y^3}{u^2} \\
&+ 14y^3 - t^3y^2 - u^3y^2 - 9tu^2y^2 - 33ty^2 - 9t^2uy^2 - 33uy^2 - \frac{9u^2y^2}{t} - \frac{33y^2}{t} - \frac{9uy^2}{t^2} - \frac{y^2}{t^3} - \frac{9t^2y^2}{u} \\
&- \frac{33y^2}{u} - \frac{9y^2}{t^2u} - \frac{9ty^2}{u^2} - \frac{9y^2}{tu^2} - \frac{y^2}{u^3} + 4tu^3y + \frac{4u^3y}{t} + 28t^2y + 10t^2u^2y + \frac{10u^2y}{t^2} + 28u^2y + 4t^3uy \\
&+ 52tuy + \frac{52uy}{t} + \frac{4uy}{t^3} + \frac{28y}{t^2} + \frac{4t^3y}{u} + \frac{52ty}{u} + \frac{52y}{tu} + \frac{4y}{t^3u} + \frac{10t^2y}{u^2} + \frac{28y}{u^2} + \frac{10y}{t^2u^2} + \frac{4ty}{u^3} + \frac{4y}{tu^3} \\
&+ 90y - 8t^3 - 3t^2u^3 - 8u^3 - 3t^3u^2 - 29tu^2 - 86t - 29t^2u - 86u - \frac{29u^2}{t} - \frac{86}{t} - \frac{3u^3}{t^2} - \frac{29u}{t^2} \\
&- \frac{3u^2}{t^3} - \frac{8}{t^3} - \frac{29t^2}{u} - \frac{86}{u} - \frac{29}{t^2u} - \frac{3t^3}{u^2} - \frac{29t}{u^2} - \frac{29}{tu^2} - \frac{3}{t^3u^2} - \frac{3t^2}{u^3} - \frac{8}{u^3} - \frac{3}{t^2u^3} + \frac{4u^3}{ty} + \frac{52u}{ty} \\
&+ \frac{10u^2}{t^2y} + \frac{28}{t^2y} + \frac{4u}{t^3y} + \frac{4t^3}{uy} + \frac{52t}{uy} + \frac{52}{tuy} + \frac{4}{t^3uy} + \frac{10t^2}{u^2y} + \frac{28}{u^2y} + \frac{10}{t^2u^2y} + \frac{4t}{u^3y} + \frac{4}{tu^3y} - \frac{t^3}{y^2} \\
&- \frac{u^3}{y^2} - \frac{9tu^2}{y^2} - \frac{33t}{y^2} - \frac{9t^2u}{y^2} - \frac{33u}{y^2} - \frac{9u^2}{ty^2} - \frac{33}{ty^2} - \frac{9u}{t^2y^2} - \frac{1}{t^3y^2} - \frac{9t^2}{uy^2} - \frac{33}{uy^2} - \frac{9}{t^2uy^2} - \frac{9t}{u^2y^2} \\
&- \frac{9}{tu^2y^2} - \frac{1}{u^3y^2} + \frac{2t^2}{y^3} + \frac{2u^2}{y^3} + \frac{6tu}{y^3} + \frac{14}{y^3} + \frac{6u}{ty^3} + \frac{2}{t^2y^3} + \frac{6t}{uy^3} + \frac{6}{tuy^3} + \frac{2}{u^2y^3} - \frac{t}{y^4} - \frac{u}{y^4} - \frac{1}{ty^4} \\
&- \frac{1}{uy^4} + \frac{4tu^3}{y} + \frac{28t^2}{y} + \frac{10t^2u^2}{y} + \frac{28u^2}{y} + \frac{4t^3u}{y} + \frac{52tu}{y} + \frac{90}{y}.
\end{aligned}$$

## Conclusion

- A lot more to be learned about 6d SCFTs and LST
- Elliptic genus for selfdual strings is a powerful tool.
- Not all the UV description of these strings are known. It remains as a challenge.
- More approaches to 6d theories: bootstrap, gravity dual….
- Relation to lower dimensional physics (5,4,3,2,1,0)

International Symposium RIKKYO MathPhys 2016

# Instabilities of asymptotically AdS black holes

Akihiro Ishibashi
at Rikkyo U. on 9 Jan. 2016

based on 1512.02644
w/ S.R. Green, S. Hollands, R.M. Wald

## Anti-de Sitter (AdS) spacetimes

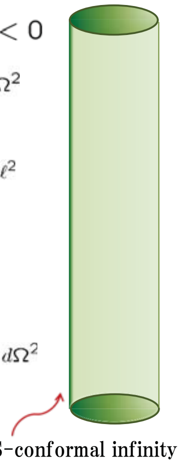
- A solution to Einstein's equations w/ $\Lambda < 0$

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- A hyperbolic space

$$-(X^0)^2-(X^1)^2+(X^2)^2+(X^3)^2+\cdots+(X^{D+1})^2 = -\ell^2$$

- Non-globally hyperbolic
  ⇒ need boundary conditions for defining dynamics
- AdS conformal infinity
  - is a lower-dim. spacetime $ds^2 = -dt^2 + d\Omega^2$ where dual-field-theory resides




## AdS spacetimes as playgrounds

**Physics**

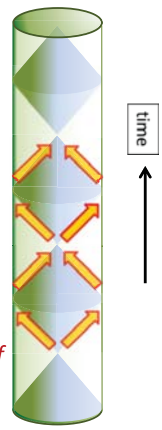
- Ground state of supergravity
- Bulk spacetime for RS brane-world model
- Near-Horizon-Geometry of Extremal Black Holes
- AdS-CFT correspondence and its applications
- Against Dark-Energy/ Inflationary Cosmology
- Closed system

**Mathematics**

- Maximally symmetric space
- Lorentzian hyperbolic space
- Allow various topology etc…

## Dynamics in AdS

- Waves can (typically) reach AdS-infinity, bounce-off and return into the bulk within finite (coordinate) time
- AdS is like a confined box, whose conformal boundary acts just like a mirror.
- In AdS (under reflection boundary condition), No energy dissipation.
- Physical mechanism responsible for the *stability of Minkowski* spacetime is the *dissipation by dispersion*; the energy of perturbations radiates away to infinity. This is not the case for AdS




## Is AdS stable?

Positive-Energy Theorem:
If the matter satisfies certain energy condition for all regular, asymptotically AdS initial data

$$E \geq 0$$

And only for exact AdS spacetime $E = 0$

⇓

AdS is a *ground state*

⇓

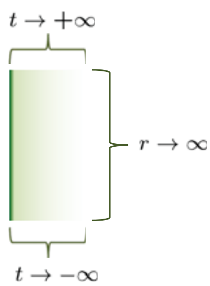
AdS should be stable(?)

## As a mathematical problem

- AdS is non-globally hyperbolic
  *Is initial-boundary value problem well-posed?*
  yes! (Friedrich 95, Al-Wald 04 )
- AdS is rigid! (M. Anderson 06)

Under AdS boundary conditions that asymptotic timelike-future, past, and spatial infinity be exact AdS, the inside must also be exact AdS

Any finite excitation might eventually explore all configurations consistent with the conserved quantities—including black holes



- Based on the linear perturbation analysis

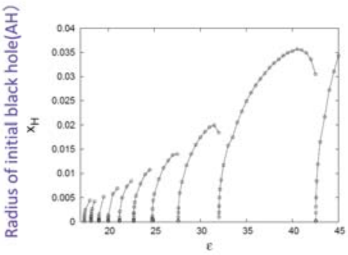
  Conjecture: Pure AdS is dynamically unstable (Dafermos-Holzegel 06)

  Conjecture: All asymptotically AdS spacetimes are dynamically unstable (Holzegel-Smulevici)

- AdS boundary acts like a confining box, hence AdS is like a closed Universe for the fields inside. It should be singular according to Hawking-Penrose's singularity theorem (Dias-Horowitz-Santos)

## *Turbulent* instability: Numerical results

The energy cascades from low frequency to high frequency




Initial small perturbations grow by repeating bounce-off by AdS infinity

Black hole forms even starting from arbitrarily small initial perturbations

Initial data amplitude and a sequence of critical amplitude

$x_H(\varepsilon) \sim (\varepsilon - \varepsilon_n)^\gamma$

Spherically symmetric, mass-less scalar field

Bizon, Rostworowski 2011 Jalmuzna, Rostworowski, Bizon 2011

Vacuum, gravitational waves Dias, Horowitz, Santos 2011

## Various instabilities in asymptotically AdS spacetimes

- AdS itself:
  - Fields w/ Mass below BF-bound
  - Choice of boundary conditions below unitarity-bound
  - Weakly turbulent instability
- Charged AdS-black hole
  - e.g.. scalar-hair condensation in holographic superconductor
- Rotating AdS black hole
  - Superradiant instability

## Purpose

Increasing evidence of superradiant instabilities of *rapidly* rotating (i.e., above *Hawking-Reall bound*) AdS black holes.

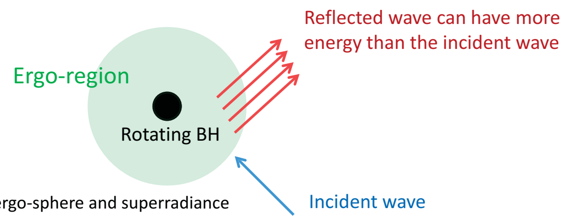
We *prove* that any such rapidly rotating AdS black hole is gravitationally *unstable*.

## Superradiant scattering by rotating BH

Stationary black hole has a "stationary" Killing field $T = \dfrac{\partial}{\partial t}$ which is *timelike* near infinity

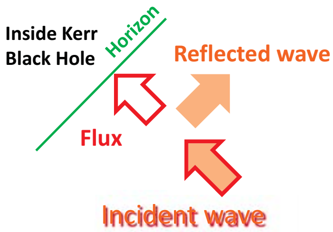
When black hole is rotating with angular velocity $\Omega_H$, there is an Ergosphere(−region) around the hole, in which $T^a$ becomes *spacelike*.

Reflected wave by ergosphere can be amplified.




Top view of ergo-sphere and superradiance

## Spacetime diagram of BH scattering




Conservation law:

$$E_I = E_R + \mathit{Flux}$$

If $\mathit{Flux} < 0$ ➡ $E_R > E_I$

Reflected wave gets amplified

Zel'dovich 72 Starobinsky73

e.g. Scalar field of modes : $\phi = e^{-i\omega t + im\varphi} f(r,\theta)$

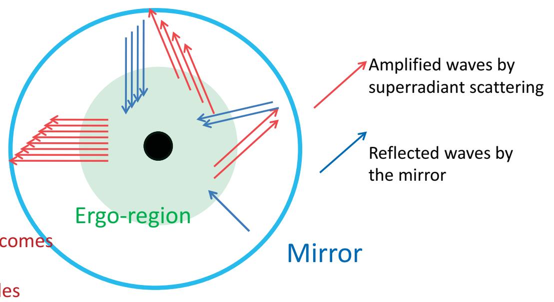
$\mathit{Flux} = \omega(\omega - m\Omega_H)\int_H |f|^2$ $\Omega_H$: Horizon angular velocity

$\mathit{Flux} < 0$ for $0 < \omega < m\Omega_H$ Superradiant modes

## Superradiant instability

If the rotating BH is surrounded by a reflecting mirror …

Amplitude of the wave will grow unboundedly
due to repeated superradiant scattering

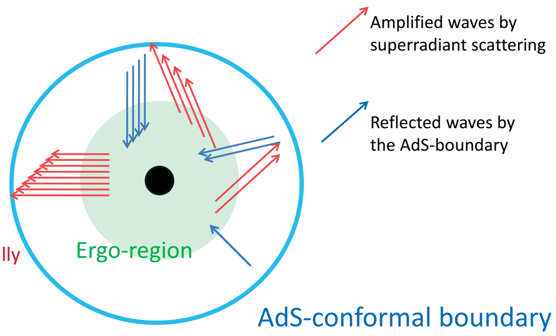



## Effective Mirrors

- Boundary conditions by hand: Black hole bomb
  Press-Teukolsky Nature 238,211 (1972)
  Cardoso-Dias-Lemos-Yoshida PRD70,044039
- Inner edge of accretion disk
  Van Putten Science 284, 115 (1999)
  Aguirre, APJ 529 L9 (2000)
- Massive bosonic fields
  Detweiler (1980)
  See e.g., Pani-Cardoso-Gualtieri-Berti-AI (2012)
- Kaluza-Klein mass for Gravitational waves / Kerr-brane
  Pani-Gualtieri-Cardoso -AI (2015)
- AdS Black hole : AdS curvature
  Hawking-Reall 99
  Cardoso-Dias 04 Cardoso-Dias-Yoshida 06
  Kodama et al 07 Uchikata-Yoshida-Futamase 09

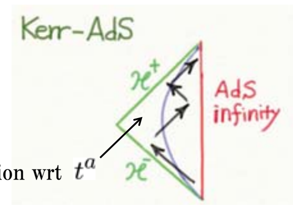
## Superradiant instability of Kerr-AdS

- Amplified scattered waves bounce back and forth between the Black Hole and AdS-infinity




## Instability of rotating AdS black holes

- Rotating AdS BH $\Rightarrow$ Superradiant instability




Ergoregion wrt $t^a$

Hawking-Reall 99, Cardoso-Dias 04
Cardoso-Dias-Yoshida 06,
Kodama 07 Murata-Soda 08
Kunduri-Lucietti-Reall 06
Uchikata-Yoshida-Futamase 09
Kodama-Konoplya-Zhidenko 09
Cardoso-Dias-Hartnett-Lehner-Santos 14
… etc.
See e.g review Brito-Cardoso-Pani 15

- Hawking−Reall bound

Slow−rotation $\Omega_H \le 1/\ell$ ➡ Horizon Killing vector field $K^a = T^a + \Omega_H \phi^a$
$\Rightarrow$ causal everywhere outside the horizon

➡ $E = -\int dS^a K^b T_{ab} \ge 0$ Stable wrt test scalar fields

## Stationary-rotating black hole in AdS

$T^a$ : Stationary Killing field (Asymptotic time-translation)

$\phi^a$ : Rotational Killing field

$K^a = T^a + \Omega_H \phi^a$: Horizon Killing field
(tangent to the horizon generators)

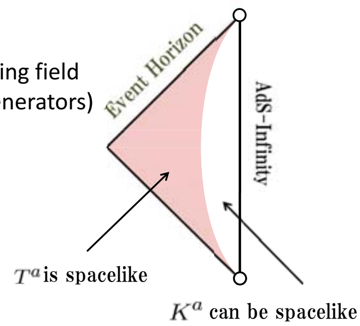
At Event Horizon:

$T^a$ : becomes *spacelike*
$K^a$ : *null* (by definition)

At AdS Infinity:

$T^a$ : *timelike* (by definition)
$K^a$ : becomes either *timelike, null,* or *spacelike*

Slow-rotation (timelike, null) Rapid-rotation (spacelike)

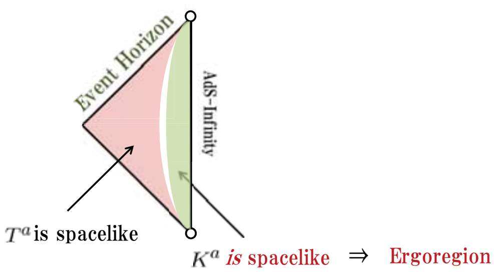



**Def.** Ergoregion of asymptotically AdS black hole

A region where the horizon Killing vector field is spacelike

Slow−rotation → No ergoregion wrt $K^a = T^a + \Omega_H \phi^a$

Rapid−rotation $\Omega_H > 1/\ell$ there exists an ergoregion near AdS infinity



**Def.** An ergoregion of asymptotically AdS black hole

A region where the horizon Killing vector field is spacelike

Slow-rotation → No ergoregion wrt $K^a = T^a + \Omega_H \phi^a$

Rapid-rotation $\Omega_H > 1/\ell$ there exists an ergoregion near AdS infinity

**Theorem** Green-Hollands-AI-Wald

Any asymptotically globally AdS black hole with Killing horizon is unstable if it admits an ergoregion with respect to the horizon Killing field $K^a$

## Stability analysis

- The standard approach:
  Search for unstable modes by solving the linearized Einstein equations.

  -- feasible for Kerr-AdS BH, as we have decoupled master equations
  Teukolsky equations for Kerr-AdS (but difficult in practice)

  -- hopeless in higher dimensions
  or in more complicated background w/ matter fields

- Our approach : based on canonical energy method
  -- need only to solve the linearized constraint equations for initial data
  Analysis is greatly simplified
  -- is applicable for many interesting cases
  e.g. higher dimensions,
  more complicated background w/ matter fields

## Sketch of proof.

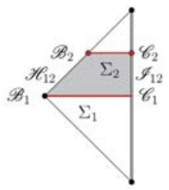
Symplectic form for gravitational perturbations $\gamma_{ab}$

$$W(\Sigma;\gamma_1,\gamma_2) \equiv \int_\Sigma \star w(g;\gamma_1,\gamma_2)$$

$$w^a = \frac{1}{16\pi} P^{abcdef}(\gamma_{2bc}\nabla_d \gamma_{1ef} - \gamma_{1bc}\nabla_d \gamma_{2ef})$$

The Canonical energy of the initial data for perturbations

$$\mathcal{E}_K(\gamma) = W_\Sigma(g;\gamma,\pounds_K\gamma)$$




The canonical energy has the following properties:

1. Gauge-invariant
2. Monotonically decreasing $\mathcal{E}_{\Sigma_2} \leq \mathcal{E}_{\Sigma_1}$ if the flux at boundary is positive

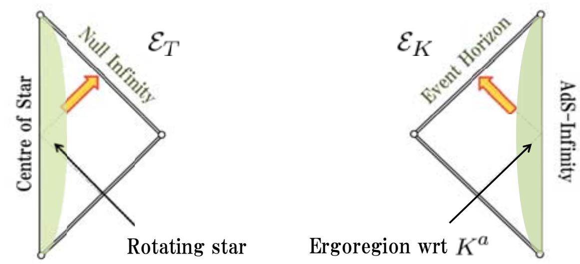
$$\frac{1}{4\pi}\int_{\mathscr{H}_{12}} (K^c\nabla_c u)\delta\sigma_{ab}\delta\sigma^{ab} \geq 0$$

$\sigma_{ab}$ : shear along $\mathscr{H}_{12}$

## c.f. Canonical energy and stability analysis

Positive Flux at Null Infinity
wrt Stationary Killing field $T^a$

Positive Flux at Event horizon
wrt Horizon Killing field $K^a$




Instability of rotating relativistic stars
Friedman 78

Superradiant instability of AdS black holes

## Sketch of Proof

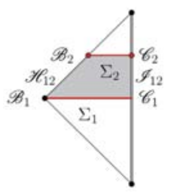
Symplectic form for gravitational perturbations

$$W(\Sigma;\gamma_1,\gamma_2) \equiv \int_\Sigma \star w(g;\gamma_1,\gamma_2)$$

$$w^a = \frac{1}{16\pi} P^{abcdef}(\gamma_{2bc}\nabla_d \gamma_{1ef} - \gamma_{1bc}\nabla_d \gamma_{2ef})$$

The Canonical energy of the initial data for perturbations

$$\mathcal{E}_K(\gamma) = W_\Sigma(g;\gamma,\pounds_K\gamma)$$




The canonical energy has the following properties:

1. Gauge-invariant
2. Monotonically decreasing $\mathcal{E}_{\Sigma_2} \leq \mathcal{E}_{\Sigma_1}$ if the flux at boundary is positive

- wish to show the existence of $\gamma_{ab}$ with which $\mathcal{E}_K(\gamma) < 0$ in ergoregion

## Expression of canonical energy

- Initial data $(\delta q_{ab}, \delta p^{ab})$ for gravitational perturbation $\gamma_{ab}$
- Lapse and shift $(N, N^a)$ for horizon Killing field $K^a = Nn^a + N^a$
- Assume $(\delta q_{ab}, \delta p^{ab})$ have a support in a compact subset $U$ in the ergo-region where we choose $N = 0$

The resulting expression

$$\mathcal{E}_K(\delta q_{ab}, \delta p^{ab}) = -\frac{1}{16\pi}\int_\Sigma N^a\left(-2\delta p^{bc}D_a\delta q_{bc} + 4\delta p^{cb}D_b\delta q_{ac} + 2\delta q_{ac}D_b\delta p^{cb} - 2p^{cb}\delta q_{ad}D_b\delta q_c{}^d + p^{cb}\delta q_{ad}D^d\delta q_{cb}\right).$$

WKB method ⇒ expansion with respect to $1/\omega \ll 1$

$$\gamma_{ab} = \exp(i\omega\chi)\left[\gamma^{(0)}_{ab} + \frac{1}{\omega}\gamma^{(1)}_{ab} + \dots\right]$$

Eikonal equation : $\nabla^a\chi\nabla_a\chi = 0$

In ergoregion : $K^a\nabla_a\chi > 0$

Constraint equations :

$$\delta q_{ab} = \left(\sum_{n>0} Q^{(n)}_{ab}(i\omega)^{-n}\right)\exp(i\omega\chi)$$

$$\delta p_{ab} = \left(\sum_{n\geq 0} P^{(n)}_{ab}(i\omega)^{-n+1}\right)\exp(i\omega\chi)$$

$$\begin{pmatrix} -D^a\chi(D_a\chi)Q^{(n)c}_{c} + D^a\chi(D^b\chi)Q^{(n)}_{ab} \\ P^{(n)}_{ab}D^b\chi \end{pmatrix} = C^{(n)}$$

$C^{(n)}$: depends on the lower order WKB approximations $(Q^{(m)}_{ab}, P^{(m)}_{ab})$

WKB method ⇒ expansion with respect to $1/\omega \ll 1$

$$\gamma_{ab} = \exp(i\omega\chi)\left[\gamma^{(0)}_{ab} + \frac{1}{\omega}\gamma^{(1)}_{ab} + \dots\right]$$

Eikonal equation : $\nabla^a\chi\nabla_a\chi = 0$

In ergoregion : $K^a\nabla_a\chi > 0$

The canonical energy can take the form in the ergoregion

$$\mathcal{E} = -\frac{\omega^2}{8\pi}\int (K^a\nabla_a\chi)\cdot\|\gamma\|^2\cdot\sin^2(\omega\chi) + O(\omega)$$

$\mathcal{E}_K < 0$ As large negative as one wants in ergoregion, where $K^a\nabla_a\chi > 0$

WKB method ⇒ expansion with respect to $1/\omega \ll 1$

$$\gamma_{ab} = \exp(i\omega\chi)\left[\gamma^{(0)}_{ab} + \frac{1}{\omega}\gamma^{(1)}_{ab} + \dots\right]$$

Eikonal equation : $\nabla^a\chi\nabla_a\chi = 0$

In ergoregion : $K^a\nabla_a\chi > 0$

The canonical energy can take the form in the ergoregion

$$\mathcal{E} = -\frac{\omega^2}{8\pi}\int (K^a\nabla_a\chi)\cdot\|\gamma\|^2\cdot\sin^2(\omega\chi) + O(\omega)$$

WKB solution does not satisfy the linearized constraints, but the failure is as small as one wants.
By applying the Corvino-Schoen method, we can correct the initial data for the perturbations so that it satisfies the constraints. □

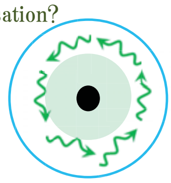
## Summary

We have shown that any asymptotically globally AdS black hole with an ergoregion is unstable.

What is the final end point?

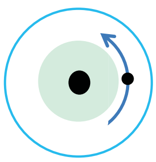
## What is the final state of unstable AdS BHs?

Gravitational wave condensation?



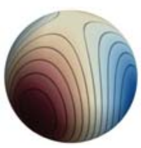
Gravitational waves cannot dissipate due to AdS boundary

Black moons? Dias-Santos-Skenderis



Black-resonators (a time-periodic sol. )? Dias-Santos-Way 15

They allow only a single, helical Killing field w/ axial-symmetry broken...



## Summary

We have shown that any asymptotically globally AdS black hole with an ergoregion is unstable.

We require only that the black hole have a single Killing vector field normal to the horizon and no restrictions on gravitational perturbations (not necessary to be axisymmetric as in the asymptotically flat case).

Black moons and Black resonators, which admit only a single helical Killing field, are also unstable.

# Topological vertex
# and quantum mirror curves

Kanehisa Takasaki, Kinki University

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1. Topological vertex
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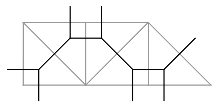
Based on

1. K. T., arXiv:1301.4548 [math-ph]
2. K. T. and T. Nakatsu, arXiv:1507.07053, J. Phys. A: Math. Gen. 49 (2016), 025201 (28pp)

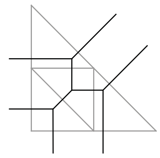


## 1. Topological vertex

### Web diagrams of non-compact toric Calabi-Yau threefolds



Strip geometry



Closed topological vertex

### Topological vertex

"Topological vertex" (Aganagic, Klemm, Mariño and Vafa 2003) is a diagrammatic method to construct the partition functions (or amplitudes) of topological string theory on non-compact toric Calabi-Yau threefolds.

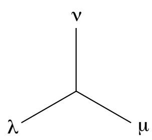




### Vertex weight

$$C_{\lambda\mu\nu} = q^{\kappa(\mu)/2} s_{{}^t\nu}(q^{-\rho}) \sum_{\eta\in\mathcal{P}} s_{{}^t\lambda/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-{}^t\nu-\rho})$$

- $\lambda = (\lambda_i)_{i=1}^\infty$, $\mu = (\mu_i)_{i=1}^\infty$, $\nu = (\nu_i)_{i=1}^\infty$ are partitions representing Young diagrams of arbitrary shapes. ${}^t\nu$ denotes the conjugate partition of $\nu$.




- $\kappa(\mu)$ is the second Casimir invariant

$$\kappa(\mu) = \sum_{i=1}^\infty \mu_i(\mu_i - 2i + 1) = 2 \sum_{(i,j)\in\mu} (j - i)$$





### Vertex weight (cont'd)

$$C_{\lambda\mu\nu} = q^{\kappa(\mu)/2} s_{{}^t\nu}(q^{-\rho}) \sum_{\eta\in\mathcal{P}} s_{{}^t\lambda/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-{}^t\nu-\rho})$$

- $s_{{}^t\nu}(q^{-\rho})$, $s_{{}^t\lambda/\eta}(q^{-\nu-\rho})$ and $s_{\mu/\eta}(q^{-{}^t\nu-\rho})$ are special values of the Schur/skew Schur functions $s_{{}^t\nu}(\boldsymbol{x})$, $s_{{}^t\lambda/\eta}(\boldsymbol{x})$, $s_{\mu/\eta}(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, \cdots)$, at

$$q^{-\rho} = (q^{i-1/2})_{i=1}^\infty, \quad q^{-\nu-\rho} = (q^{-\nu_i+i-1/2})_{i=1}^\infty,$$
$$q^{-{}^t\nu-\rho} = (q^{-{}^t\nu_i+i-1/2})_{i=1}^\infty$$

- Vertex weights are glued together along the internal lines. Edge weights are assigned to those lines.

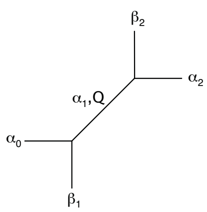




### Examples (local $\mathbb{P}^1$ geometry)

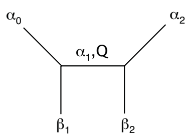
1. Open string amplitudes of $X = \mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{CP}^1$ (resolved conifold):

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = \sum_{\alpha_1\in\mathcal{P}} C_{\alpha_1\alpha_0\beta_1}(-Q)^{|\alpha_1|} C_{{}^t\alpha_1\alpha_2\beta_2}$$




2. Open string amplitudes of $X = \mathcal{O}\oplus\mathcal{O}(-2) \to \mathbb{CP}^1$:

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = \sum_{\alpha_1\in\mathcal{P}} C_{\alpha_1\alpha_0\beta_1}(-Q)^{|\alpha_1|} \times (-1)^{|\alpha_1|} q^{-\kappa(\alpha_1)/2} C_{\alpha_2{}^t\alpha_1\beta_2}$$




(A framing factor is inserted in this case.)





### Building blocks of operator formalism

- Charge-zero sector of fermionic Fock space spanned by the ground states $\langle 0|$, $|0\rangle$ and the excited states

$$\langle\lambda| = \langle -\infty| \cdots \psi^*_{\lambda_i - i+1} \cdots \psi^*_{\lambda_2-1}\psi^*_{\lambda_1},$$
$$|\lambda\rangle = \psi_{-\lambda_1}\psi_{-\lambda_2+1} \cdots \psi_{-\lambda_i+i-1} \cdots |-\infty\rangle$$

- Fermion bilinears

$$L_0 = \sum_{n\in\mathbb{Z}} n{:}\psi_{-n}\psi^*_n{:}, \quad K = \sum_{n\in\mathbb{Z}} (n-1/2)^2{:}\psi_{-n}\psi^*_n{:},$$
$$J_m = \sum_{n\in\mathbb{Z}} {:}\psi_{m-n}\psi^*_n{:}, \quad m \in \mathbb{Z},$$
$$V^{(k)}_m = q^{-km/2} \sum_{n\in\mathbb{Z}} q^{kn}{:}\psi_{m-n}\psi^*_n{:}, \quad k, m \in \mathbb{Z}.$$

- Vertex operators

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^\infty \frac{z^k}{k} J_{\pm k}\right), \quad \Gamma'_\pm(z) = \exp\left(-\sum_{k=1}^\infty \frac{(-z)^k}{k} J_{\pm k}\right)$$

and the multi-variable extensions

$$\Gamma_\pm(\boldsymbol{x}) = \prod_{i\ge 1} \Gamma_\pm(x_i), \quad \Gamma'_\pm(\boldsymbol{x}) = \prod_{i\ge 1} \Gamma'_\pm(x_i)$$

Fermionic expression of vertex weight

$$\begin{aligned} C_{\lambda\mu\nu} &= s_{{}^t\nu}(q^{-\rho})\langle {}^t\lambda|\Gamma_-(q^{-\nu-\rho})\Gamma_+(q^{-{}^t\nu-\rho})q^{K/2}|\mu\rangle \\ &= s_{{}^t\nu}(q^{-\rho})\langle \lambda|\Gamma'_-(q^{-\nu-\rho})\Gamma'_+(q^{-{}^t\nu-\rho})q^{-K/2}|{}^t\mu\rangle \end{aligned}$$

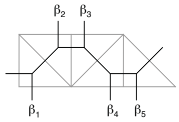
(Okounkov, Reshetikhin and Vafa 2003)



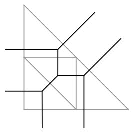


Cases where amplitudes are computed explicitly

1. Strip geometry
Iqbal and Kashani-Poor 2004 (by topological vertex)




2. Closed topological vertex
Bryan and Karp 2003, Karp, Liu and Mariño 2005 (by algebraic geometry and topological vertex)
Sułkowski 2006 (by crystal model and topological vertex)



Our goal: To derive a $q$-difference equation that may be interpreted as a quantization of the mirror curve.

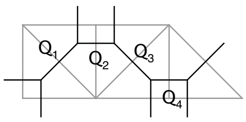


## 2. Strip geometry

General setup

(a) The toric diagram is a triangulation of a strip (trapezoid of height 1).

(b) The web diagram has $N$ vertices, $N-1$ internal lines and $N+2$ external lines (or "legs").




Example of strip geometry ($N=5$)

(c) The internal lines are assigned with Kähler parameters $Q_1, \ldots, Q_{N-1}$.

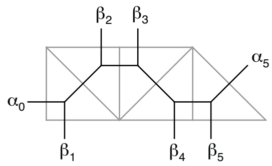




General setup (cont'd)

(d) The external lines are assigned with partitions $\alpha_0, \beta_1, \ldots, \beta_N, \alpha_N$.

(e) The sign (or type) $\sigma_n$ of the $n$-th vertical leg are defined as




$$\sigma_n = \begin{cases} +1 & \text{if the leg points} \uparrow \\ -1 & \text{if the leg points} \downarrow \end{cases}$$

$\sigma_2 = \sigma_3 = +1,$
$\sigma_1 = \sigma_4 = \sigma_5 = -1$

Let $Z^{\alpha_0\alpha_N}_{\beta_1\ldots\beta_N}$ denote the open string amplitude constructed by topological vertex.





Infinite-product formula of amplitude

If $\alpha_0 = \alpha_N = \emptyset$, the amplitude can be computed explicitly with the aid of the Cauchy identities for skew Schur functions (Iqbal and Kashani-Poor 2004):

$$\begin{aligned} Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N} &= s_{{}^t\beta_1}(q^{-\rho})\cdots s_{{}^t\beta_N}(q^{-\rho}) \\ &\quad \times \prod_{1\le m<n\le N}\prod_{i,j=1}^\infty (1 - Q_{mn}q^{-{}^t\beta_i^{(m)} - \beta_j^{(n)} + i + j - 1})^{-\sigma_m\sigma_n} \end{aligned}$$

where

$$\beta^{(n)} = \begin{cases} \beta_n & \text{if } \sigma_n = +1, \\ {}^t\beta_n & \text{if } \sigma_n = -1, \end{cases} \qquad Q_{mn} = Q_m Q_{m+1}\cdots Q_{n-1}$$





Fermionic formula of amplitude

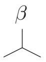
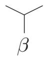
The infinite-product formula holds only for $\alpha_0 = \alpha_N = \emptyset$. For general cases, the following fermionic expression is available:

$$\begin{aligned} Z^{\alpha_0\alpha_N}_{\beta_1\cdots\beta_N} &= q^{(1-\sigma_1)\kappa(\alpha_0)/4} q^{(1+\sigma_N)\kappa(\alpha_N)/4} s_{{}^t\beta_1}(q^{-\rho})\cdots s_{{}^t\beta_N}(q^{-\rho}) \\ &\quad \times \langle {}^t\alpha_0|\Gamma_-^{\sigma_1}(q^{-\beta^{(1)}-\rho})\Gamma_+^{\sigma_1}(q^{-{}^t\beta^{(1)}-\rho})(\sigma_1 Q_1 \sigma_2)^{L_0}\cdots \\ &\quad \times \Gamma_-^{\sigma_{N-1}}(q^{-\beta^{(N-1)}-\rho})\Gamma_+^{\sigma_{N-1}}(q^{-{}^t\beta^{(N-1)}-\rho})(\sigma_{N-1}Q_{N-1}\sigma_N)^{L_0} \\ &\quad \times \Gamma_-^{\sigma_N}(q^{-\beta^{(N)}-\rho})\Gamma_+^{\sigma_N}(q^{-{}^t\beta^{(N)}-\rho})|\alpha_N\rangle \end{aligned}$$

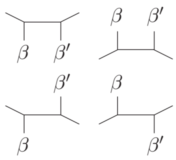
where $\Gamma^\sigma_\pm$ denote $\Gamma_\pm$ if $\sigma = +1$ and $\Gamma'_\pm$ if $\sigma = -1$ (Eguchi and Kanno 2003, Bryan and Young 2008 for special cases; Nagao 2009 and Sułkowski 2009 for general cases)

## Fermionic formula of amplitude (cont'd)

### Vertex operators

 $\longrightarrow \quad \Gamma_-(q^{\beta+\rho})\Gamma_+(q^{{}^t\beta+\rho})$  $\longrightarrow \quad \Gamma'_-(q^{{}^t\beta+\rho})\Gamma'_+(q^{\beta+\rho})$

### Propagators



$\longrightarrow \quad Q^{L_0}$

$\longrightarrow \quad (-Q)^{L_0}$





## Wave functions

Wave functions are defined as

$$\Psi_n(x) = \sum_{k=0}^{\infty} \frac{Z_{n,(1^k)}}{Z_{n,\emptyset}} x^k, \quad \tilde{\Psi}_n(x) = \sum_{k=0}^{\infty} \frac{Z_{n,(k)}}{Z_{n,\emptyset}} x^k$$

for $n = 0, 1, \ldots, N, N+1$, where

$$Z_{0,\lambda} = Z^{\lambda\emptyset}_{\emptyset\cdots\emptyset},\ Z_{n,\lambda} = Z^{\emptyset\emptyset}_{\cdots\emptyset\lambda\emptyset\cdots}\ (1 \le n \le N),\ Z_{N+1,\lambda} = Z^{\emptyset\lambda}_{\emptyset\cdots\emptyset}$$

Remark: $Z_{n,\lambda}$'s are the coefficients of Schur function expansion of a KP tau function $\tau_n$. In this sense, these wave functions are Baker-Akhiezer functions (at the initial time $\boldsymbol{t} = \boldsymbol{0}$) that correspond to free fermion fields $\psi(-x), \psi^*(x)$.





## $\Psi_n$ and $\tilde{\Psi}_n$ for $n = 1, \ldots, N$ are $q$-hypergeometric series

$$\Psi_n(x) = 1 + \sum_{k=1}^{\infty} \frac{C_n(1)C_n(q)\cdots C_n(q^{k-1})}{B_n(1)B_n(q)\cdots B_n(q^{k-1})(1-q)\cdots(1-q^k)} q^{k/2}x^k$$

where $B_n(y)$ and $C_n(y)$ are Laurent polynomials in $y$:

$$B_n(y) = \prod_{m<n,\,\sigma_m\sigma_n>0} (1 - Q_{mn}y^{\sigma_n}) \times \prod_{m>n,\,\sigma_m\sigma_n>0} (1 - Q_{nm}y^{-\sigma_n}),$$

$$C_n(y) = \prod_{m<n,\,\sigma_m\sigma_n<0} (1 - Q_{mn}y^{\sigma_n}) \times \prod_{m>n,\,\sigma_m\sigma_n<0} (1 - Q_{nm}y^{-\sigma_n})$$

Remark: When $N = 1$, $\Psi$ reduces to a quantum dilog:

$$\Psi(x) = 1 + \sum_{k=1}^{\infty} \frac{q^{k/2}x^k}{(1-q)\cdots(1-q^k)} = \prod_{i=1}^{\infty}(1 - q^{i-1/2}x)^{-1}$$





## $\Psi_1, \ldots, \Psi_N$ satisfy $q$-difference equation

$$\Psi_n(x) - \Psi_n(qx) = q^{1/2}x\frac{C_n(q^{x\partial_x})}{B_n(q^{x\partial_x})}\Psi_n(x)$$

or, equivalently,

$$B_n(q^{-1}q^{x\partial_x})(1 - q^{x\partial_x})\Psi_n(x) = q^{1/2}xC_n(q^{x\partial_x})\Psi_n(x)$$

Remark: 1. For the resolved conifold, these equations are derived by Kashani-Poor 2006, Hyun and Yi 2006, Gukov and Sułkowski 2011 and Zhou 2012.

2. These equations are also studied in the context of the AGT correspondence (Kozçaz, Pasquetti and Wyllard 2010, Awata and Kanno 2010, Taki 2010) and the vortex partition function (Dimofte, Gukov and Hollands 2010, Bonelli, Tanzini and Zhao 2011).





## Classical limit

As $q \to 1$ ($\hat{y} = q^{x\partial_x} \to y$), the $q$-difference equation

$$B_n(q^{-1}q^{x\partial_x})(1 - q^{x\partial_x})\Psi_n(x) = q^{1/2}xC_n(q^{x\partial_x})\Psi_n(x)$$

reduces to the algebraic equation

$$B_n(y)(1-y) = xC_n(y)$$

of the mirror curve. In this sense, the $q$-difference equation may be thought of as a quantization of the mirror curve.

Remark: The Newton polygon of $B_n(y)(1-y) - xC_n(y)$ can be mapped to the outline of the toric diagram by an $SL(2,\mathbb{Z})$ transformation.





## How the equations for different $n$'s are related

$$x = (1-y)\frac{B_n(y)}{C_n(y)} \overset{(\heartsuit)}{\longrightarrow} \tilde{x} = (1-\tilde{y})\frac{B_{n+1}(\tilde{y})}{C_{n+1}(\tilde{y})}$$

$$(\heartsuit) \quad y = Q_n^{\sigma_n}\tilde{y}^{\sigma_n\sigma_{n+1}},\ x = g(y,\tilde{y})\tilde{x}^{\sigma_n\sigma_{n+1}}$$

where

$$g(y,\tilde{y}) = \begin{cases} -\tilde{y}^{-1} & \text{if } \sigma_n = +1,\ \sigma_{n+1} = +1, \\ 1 & \text{if } \sigma_n = +1,\ \sigma_{n+1} = -1, \\ y\tilde{y} & \text{if } \sigma_n = -1,\ \sigma_{n+1} = +1, \\ -y & \text{if } \sigma_n = -1,\ \sigma_{n+1} = -1 \end{cases}$$

Birational map $(\heartsuit)$ preserves the symplectic structure:

$$d\log x \wedge d\log y = d\log\tilde{x} \wedge d\log\tilde{y}$$

### What about $\Psi_0$ and $\Psi_{N+1}$ ?

They are related to quantum dilogs. If $\sigma_1 = +1$, $\Psi_0(x)$ is a product of quantum dilogs and satisfies the equation

$$\Psi_0(qx) = (1 - q^{1/2}x)^{-1} \prod_{n=2}^{N} (1 - Q_{1,n-1}q^{1/2}x)^{-\sigma_n} \Psi_0(x).$$

If $\sigma_1 = -1$, the factor $q^{\kappa(\alpha_0)/2}$ shows up in the amplitude, and $\Psi_0(x)$ satisfies the equation

$$\Psi_0(qx) = (1 + q^{1/2}xq^{x\partial_x}) \prod_{n=2}^{N} (1 + Q_{1,n-1}q^{1/2}xq^{x\partial_x})^{-\sigma_n} \Psi_0(x)$$

Similar results hold for $\Psi_{N+1}(x)$ as well.





### What about $\Psi_0$ and $\Psi_{N+1}$ ? (cont'd)

The classical limit of the $q$-difference equation is the algebraic equation

$$y = (1 - x)^{-1} \prod_{n=2}^{N} (1 - Q_{1,n-1}x)^{-\sigma_n}$$

for $\sigma = +1$ and

$$y = (1 + xy) \prod_{n=2}^{N} (1 + Q_{1,n-1}xy)^{-\sigma_n}$$

for $\sigma = -1$. This equation can be transformed to the equation $\tilde{x} = (1 - \tilde{y})B_1(\tilde{y})/C_1(\tilde{y})$ associated with $\Psi_1(x)$ by a birational symplectic map $(x, y) \mapsto (\tilde{x}, \tilde{y})$:

$$d\log y \wedge d\log x = d\log\tilde{x} \wedge d\log\tilde{y}$$

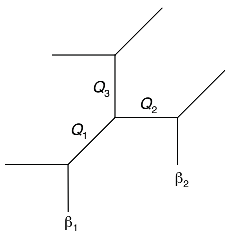


## 3. Closed topological vertex

### Setup

(a) Three internal lines are assigned with Kähler parameters $Q_1, Q_2, Q_3$

(b) Two vertical external lines are assigned with partitions $\beta_1, \beta_2$. All other external lines are given $\emptyset$.



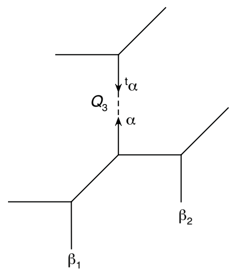

Let $Z^{\text{ctv}}_{\beta_1\beta_2}$ denote the open string amplitude in this setting.





### Method of computation of $Z^{\text{ctv}}_{\beta_1\beta_2}$

- Reconstruct the amplitude $Z^{\text{ctv}}_{\beta_1\beta_2}$ by gluing a single vertex $C_{{}^t\alpha\emptyset\emptyset}$ to the amplitude of a strip geometry.
- Borrow tools from our previous study of the melting crystal models (5D $U(1)$ instanton sum and its variants) (Nakatsu and K.T, since 2007) to compute the sum over $\alpha \in \mathcal{P}$.




Gluing a vertex (top) to a strip geometry (bottom)





### Result of computation of $Z^{\text{ctv}}_{\beta_1\beta_2}$

$$\begin{aligned}Z^{\text{ctv}}_{\beta_1\beta_2} = q^{\kappa(\beta_2)/2} &\prod_{i,j=1}^{\infty} (1 - Q_1Q_2q^{-\beta_{1i}-{}^t\beta_{2j}+i+j-1})^{-1} \\ &\times \langle{}^t\beta_1|\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})(-Q_1)^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})(-Q_3)^{L_0} \\ &\times \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})(-Q_2)^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})|{}^t\beta_2\rangle.\end{aligned}$$

Remark: 1. $Y_{\beta_1\beta_2} = \langle{}^t\beta_1|\cdots|{}^t\beta_2\rangle$ happens to be the open string amplitude of yet another strip geometry (triple-$\mathbb{P}^1$ geometry). This coincidence is a key to derive a $q$-difference equation.
2. Letting $\beta_1 = \beta_2 = \emptyset$, this expression reduces to the known result of the closed string amplitudes (Bryan and Karp 2003, Karp, Liu and Mariño 2005).





### Wave functions

Wave functions are defined as

$$\Psi(x) = \sum_{k=0}^{\infty} \frac{Z^{\text{ctv}}_{(1^k)\emptyset}}{Z^{\text{ctv}}_{\emptyset\emptyset}} x^k, \quad \tilde{\Psi}(x) = \sum_{k=0}^{\infty} \frac{Z^{\text{ctv}}_{(k)\emptyset}}{Z^{\text{ctv}}_{\emptyset\emptyset}} x^k$$

along with the auxiliary wave functions

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{Y_{(1^k)\emptyset}}{Y_{\emptyset\emptyset}} x^k, \quad \tilde{\Phi}(x) = \sum_{k=0}^{\infty} \frac{Y_{(k)\emptyset}}{Y_{\emptyset\emptyset}} x^k$$

obtained from the main part $Y_{\beta_1\beta_2} = \langle{}^t\beta_1|\cdots|{}^t\beta_2\rangle$ of $Z^{\text{ctv}}_{\beta_1\beta_2}$.

## Wave functions (cont'd)

- The coefficients of $\Psi(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\Phi(x) = \sum_{k=0}^{\infty} b_k x^k$, $a_0 = b_0 = 1$, are related as

$$a_k = b_k \prod_{i=1}^{k} (1 - Q_1 Q_2 q^{i-1})^{-1}$$

- $\Phi(x)$ is build from quantum dilogs, and satisfies the $q$-difference equation

$$\Phi(qx) = \frac{(1-q^{1/2}x)(1-Q_1Q_3q^{1/2}x)}{(1-Q_1q^{1/2}x)(1-Q_1Q_2Q_3q^{1/2}x)}\Phi(x).$$





## Transforming $q$-difference equation

The $q$-difference equation

$$(1-Q_1q^{1/2}x)(1-Q_1Q_2Q_3q^{1/2}x)\Phi(qx)$$
$$= (1-q^{1/2}x)(1-Q_1Q_3q^{1/2}x)\Phi(x)$$

for $\Phi(x)$ is transformed to the $q$-difference equation

$$(1 - Q_1Q_2q^{-2}q^{x\partial_x} - Q_1q^{1/2}x)(1 - Q_1Q_2q^{-1}q^{x\partial_x} - Q_1Q_2Q_3q^{1/2}x)\Psi(qx)$$
$$= (1 - Q_1Q_2q^{-2}q^{x\partial_x} - q^{1/2}x)(1 - Q_1Q_2q^{-1}q^{x\partial_x} - Q_1Q_3q^{1/2}x)\Psi(x)$$

for $\Psi(x)$. Let us rewrite this equation as

$$H(x, q^{x\partial_x})\Psi(x) = 0$$

and examine the $q$-difference operator $H(x, q^{x\partial_x})$.





## Reducing $q$-difference equation to final form

The operator $H(x, q^{x\partial_x})$ can be factorized as

$$H(x, q^{x\partial_x}) = (1 - Q_1Q_2q^{-2}q^{x\partial_x})K(x, q^{x\partial_x})$$

where

$$K(x, q^{x\partial_x}) = (1 - Q_1Q_2q^{-1}q^{x\partial_x})(1 - q^{x\partial_x}) - (1 + Q_1Q_3)q^{1/2}x$$
$$+ Q_1(1 + Q_2Q_3)q^{1/2}xq^{x\partial_x} + Q_1Q_3qx^2.$$

Since the prefactor $1 - Q_1Q_2q^{-2}q^{x\partial_x}$ is invertible on the space of power series of $x$, the equation $H(x, q^{x\partial_x})\Psi(x) = 0$ reduces to

$$K(x, q^{x\partial_x})\Psi(x) = 0$$

This is the final form of our quantum mirror curve.





## Classical limit

As $q \to 1$, the $q$-difference operator

$$K(x, q^{x\partial_x}) = (1 - Q_1Q_2q^{-1}q^{x\partial_x})(1 - q^{x\partial_x}) - (1 + Q_1Q_3)q^{1/2}x$$
$$+ Q_1(1 + Q_2Q_3)q^{1/2}xq^{x\partial_x} + Q_1Q_3qx^2$$

turns into the polynomial

$$K_{\rm cl}(x, y) = (1 - Q_1Q_2y)(1 - y) - (1 + Q_1Q_3)x$$
$$+ Q_1(1 + Q_2Q_3)xy + Q_1Q_3x^2$$
$$= ax^2 + bxy + cy^2 + dx + ey + f$$

of degree two as expected.





## What about wave functions obtained from $\beta_2 = (k), (1^k)$ ?

The $q$-difference equations become slightly more complicated because of the framing factor $q^{\kappa(\beta_2)/2}$.

## What about putting $\beta_1$ and $\beta_2$ on other legs ?

The open string amplitude can be expressed in a similar form. However $q^{K/2}$'s remain in the operator product, and they prevent us from doing explicit computation. A similar difficulty takes place when one attempts to prolong the branches of the closed topological vertex.



## Conclusion

We have derived a quantization $K(x, q^{x\partial_x})\Psi(x) = 0$ of the mirror curve $K_{\rm cl}(x, y) = 0$ in the following special cases by explicit computation of $\Psi(x)$.

1. Strip geometry  $\Psi(x)$ is a $q$-hypergeometric series or related to quantum dilogarithmic functions:

$$K(x, q^{x\partial_x}) = B_n(q^{-1}q^{x\partial_x})(1 - q^{x\partial_x}) - q^{1/2}xC_n(q^{x\partial_x}),$$

$$K(x, q^{x\partial_x}) = q^{x\partial_x} - A(x), \text{ or } K(x, q^{x\partial_x}) = q^{x\partial_x} - A(xq^{x\partial_x})$$

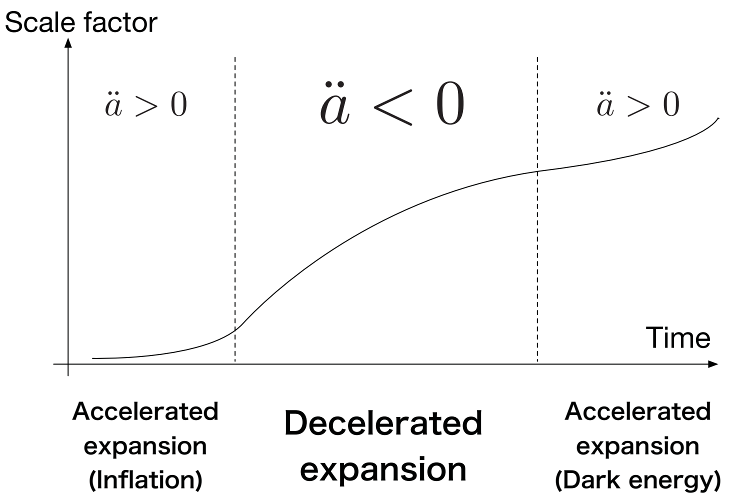
2. Closed topological vertex  $\Psi(x)$ is obtained from the dilogarithmic wave function $\Phi(x)$ of a triple-$\mathbb{P}^3$ geometry by a simple transformaiton. The final form $K(x, q^{x\partial_x}) = Ax^2 + Bxq^{x\partial_x} + Cq^{2x\partial_x} + Dx + Eq^{x\partial_x} + F$ of $K$ is derived in an intricate way.

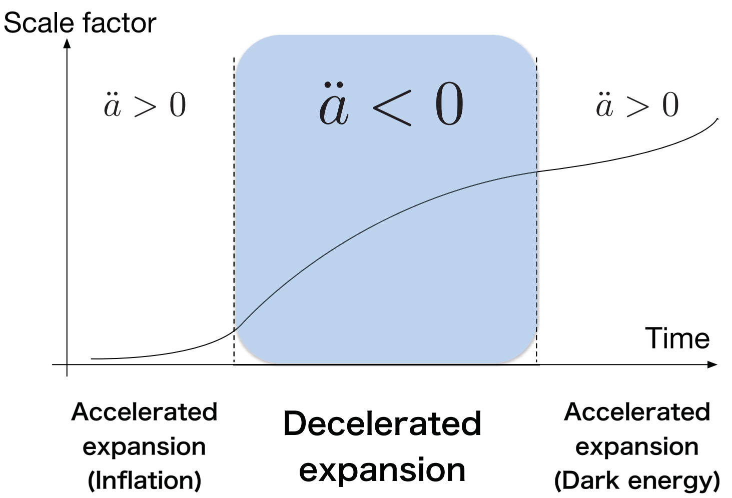
# Axion Cosmology and the Witten effect

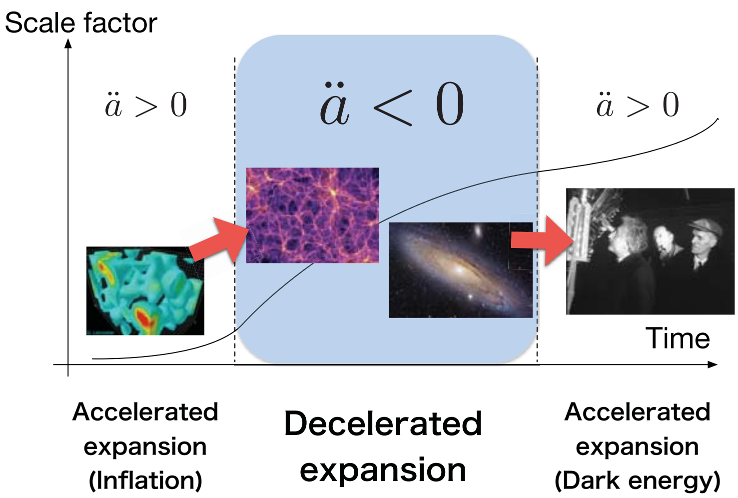
10th Jan. 2016
RIKKYO MathPhys 2016

Fuminobu Takahashi
(Tohoku)

Kawasaki, FT, Yamada, 1511.05030, to appear in Phys. Lett. B.



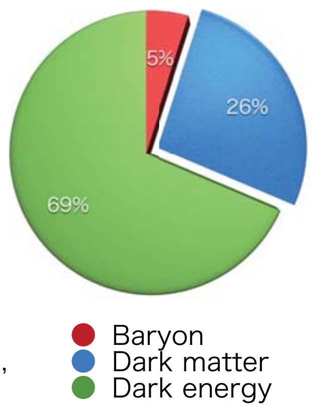

## Dark Matter

The presence of DM has been firmly established.

$$\Omega_{\rm DM} \simeq 0.26$$

DM may be made of an as-yet-undiscovered particle.

e.g. WIMPs, sterile neutrinos, axions, etc.




## What we know about DM

- ✓ DM gravitates.
- ✓ DM is electrically neutral.
- ✓ DM has only (very) weak interactions with the SM particles.
  - DM may have self-interactions.
- ✓ DM is non-relativistic.
- ✓ DM is long-lived.

$$\tau_{\rm DM} \gtrsim 20 t_0$$

Enqvist et al, 1505.05511

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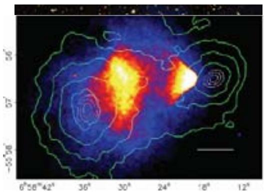


# Why long-lived?

**1) Symmetry**

- e.g. R-parity (LSP), topological charge (monopoles)

**2) Light mass**

- e.g. modulus lifetime: $\tau \simeq \frac{M_p^2}{m_{\rm DM}^3}$

**3) Very weak interactions**

- e.g. Hidden sector, gravity sector

The light mass and/or weak int. may be due to symmetry (e.g. chiral, shift sym, SUSY), extra dim or compositeness.

These are not exclusive: e.g. axion [2&3], gravitino [1,2,3]

# The Strong CP Problem

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^{a\mu\nu}\tilde{G}^a_{\mu\nu}$$

Experimental bound from neutron electric dipole moment reads

$$|\theta| < 10^{-10}$$

Why is $\theta$ so small is the strong CP problem.

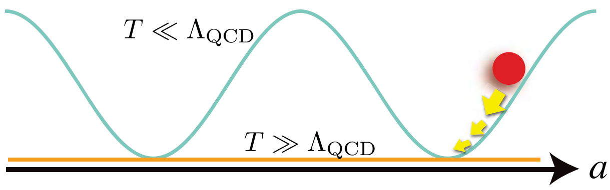
cf. More precisely, the physical combination is given by

$$\bar{\theta} \equiv \theta - \arg\det(M_u M_d)$$

which makes the problem even more puzzling.

A plausible solution is the Peccei-Quinn mechanism where the CP phase is promoted to a dynamical variable: Peccei, Quinn \`77, Weinberg \`78, Wilczek \`78

$$\mathcal{L}_\theta = \left(\theta + \frac{a}{f_a}\right) \frac{g_s^2}{32\pi^2} G^{a\mu\nu}\tilde{G}^a_{\mu\nu}$$



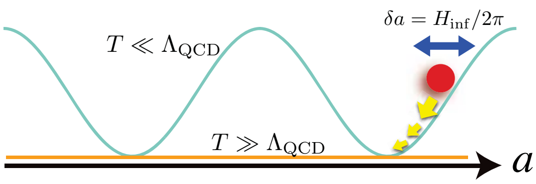

The QCD axion dynamically cancels the strong CP phase.

Accordingly, the axion DM is produced as coherent oscillations [misalignment mechanism].

$$\Omega_a h^2 = 0.18\, \theta_i^2 \left(\frac{f_a}{10^{12}\ {\rm GeV}}\right)^{1.19} \left(\frac{\Lambda_{\rm QCD}}{400\ {\rm MeV}}\right)$$

If the axion exists during inflation, it acquires isocurvature fluctuations.

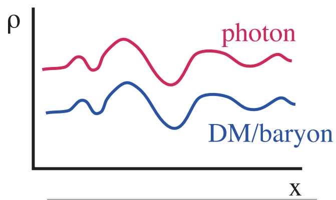
$$\frac{\delta\Omega_a}{\Omega_a} = 2\frac{\delta\theta_i}{\theta_i}$$




# Axion isocurvature perturbations

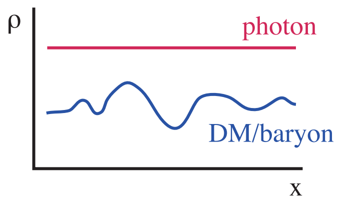
Fluctuation of time

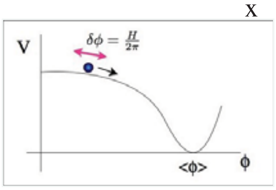
Adiabatic perturbation

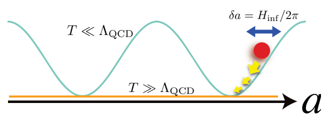



Fluctuations btw different components

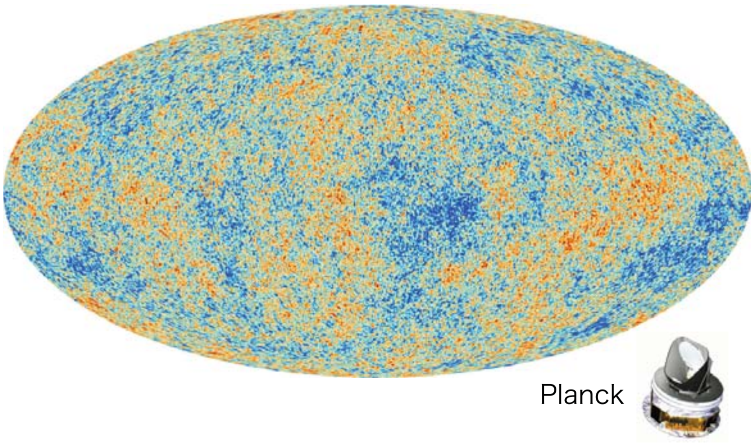
Isocurvature perturbation



## CMB angular power spectrum

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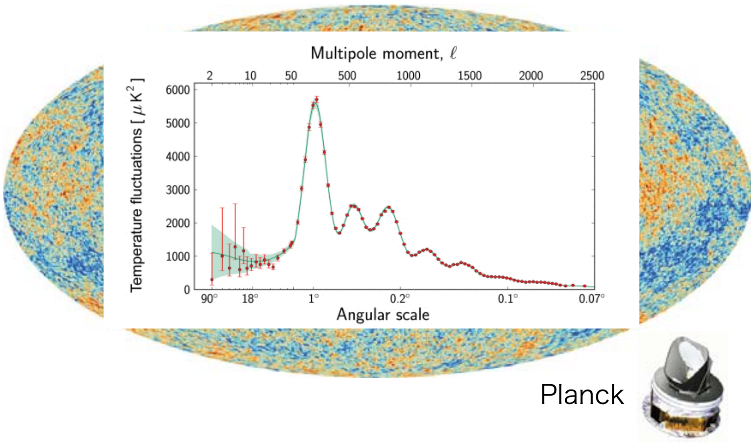


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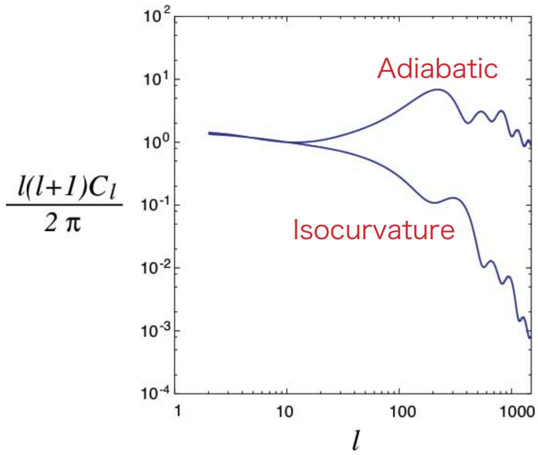
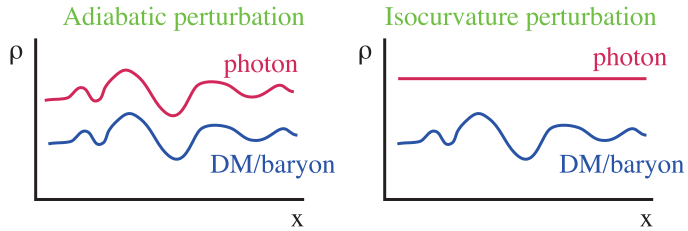



(Taken from Kawasaki's slide)

## Axion isocurvature perturbations

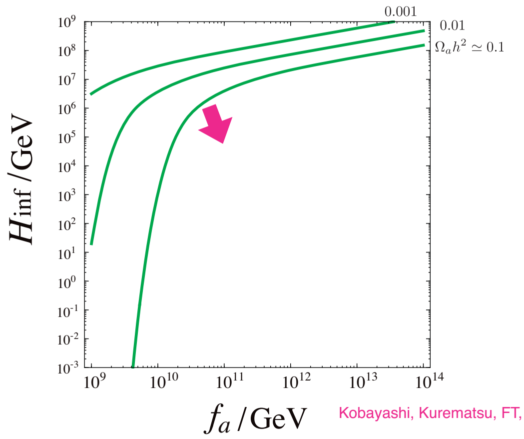



$$S = 2\frac{\Omega_a}{\Omega_{\rm CDM}}\frac{\delta\theta_i}{\theta_i} = \frac{\Omega_a}{\Omega_{\rm CDM}}\frac{H_{\rm inf}}{\pi\theta_i f_a}$$

$$\beta_{\rm iso} = \frac{\mathcal{P}_S}{\mathcal{P}_{\mathcal{R}} + \mathcal{P}_S} < 0.038 \quad (95\%\,{\rm CL})$$

Planck 2015
(Planck TT, TE, EE + lowP)

## Isocurvature constraint on $H_{inf}$




Kobayashi, Kurematsu, FT, 1304.0922

Axion DM is in severe tension w/ many inflation models!

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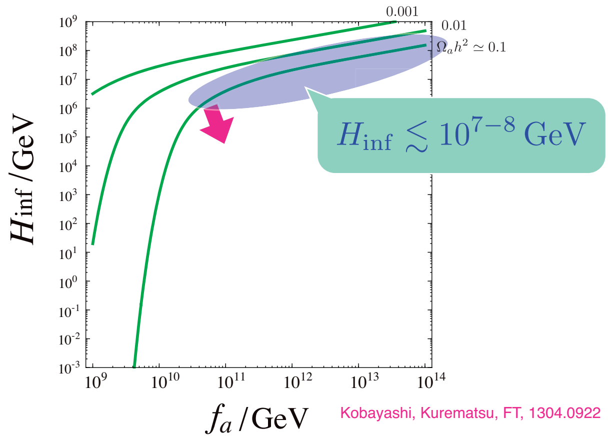


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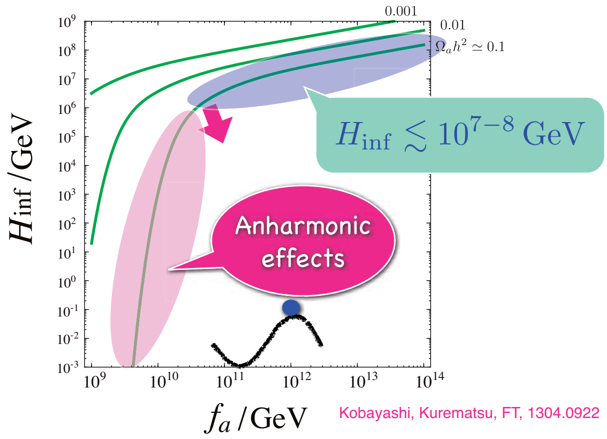


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**Axion DM is in severe tension w/ many inflation models!**

## Solutions to isocuvature problem

- Restoration of Peccei-Quinn symmetry during inflation. Linde and Lyth `90 Lyth and Stewart `92

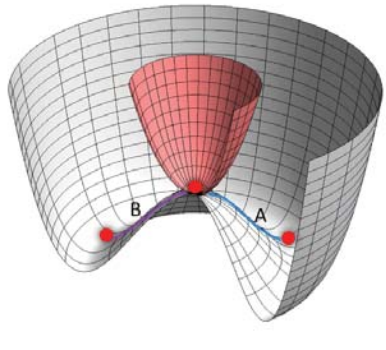


Figure taken from M. Kawasaki's slide

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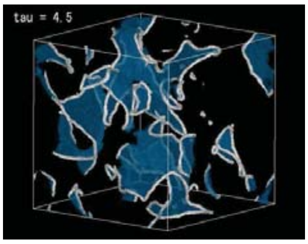
Hiramatsu, Kawasaki, Saikawa and Sekiguchi, 1202.5851,1207.3166

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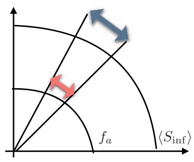
Hiramatsu, Kawasaki, Saikawa and Sekiguchi, 1202.5851,1207.3166

- (Super-)Planckian saxion field value during inflation. (Saxion could be the inflaton) Linde and Lyth `90 Linde, `91




Axion: phase component
Saxion: radial component

$$S = \frac{f_a + \sigma}{\sqrt{2}} e^{ia/f_a}$$

## Solutions to isocuvature problem

- Restoration of Peccei-Quinn symmetry during inflation.
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Hiramatsu, Kawasaki, Saikawa and Sekiguchi, 1202.5851,

- Super-Planckian saxion field value during inflation. (Saxion could be the inflaton) Linde and Lyth `90 Linde, `91
- Heavy axions during inflation $m_a^2 \gtrsim H_{\rm inf}^2$
  - Stronger QCD during inflation cf. Dvali, `95, Jeong, FT 1304.8131 Choi et al, 1505.00306
  - Extra explicit PQ breaking Dine, Anisimov hep-ph/0405256 Higaki, Jeong, FT, 1403.4186, Barr and J.E.Kim, 1407.4311

## Witten effect

Witten `79

Let us first consider the $\theta$-term of hidden U(1):

This $\theta$ is not the strong CP phase.

$$\mathcal{L}_\theta = \frac{\theta e^2}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{\theta e^2}{8\pi^2} \mathbf{E}\cdot\mathbf{B}$$

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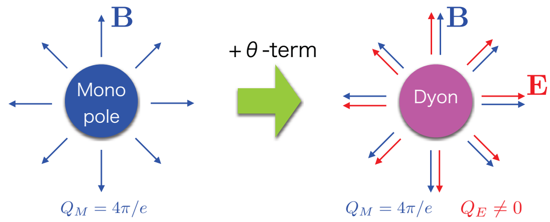
The term, however, has an interesting effect in the **monopole background**.




# Witten effect

Witten `79

Consider a static EM field and a monopole sitting at the origin:

due to Coleman

$$\mathbf{E} = -\mathrm{grad}\phi = -\nabla A^0 \qquad \mathbf{B} = \nabla\times\mathbf{A} + \frac{Q_M}{4\pi}\frac{\hat{r}}{r^2}$$

Monopole

Then, the Lagrangian is

$$L_\theta = \int d^3r\left(-\frac{\theta e^2}{8\pi^2}\right)\mathbf{E}\cdot\mathbf{B}$$
$$= \int d^3r\left(-\frac{\theta e^2}{8\pi^2}\right)(-\nabla A^0)\cdot\left(\nabla\times\mathbf{A} + \frac{Q_M}{4\pi}\frac{\hat{r}}{r^2}\right)$$
$$= -\int d^3r A^0\left(\frac{\theta e^2 Q_M}{8\pi^2}\right)\nabla\left(\frac{\hat{r}}{4\pi r^2}\right)$$
$$= -\int d^3r A^0\left(\frac{\theta e^2 Q_M}{8\pi^2}\delta^{(3)}(\mathbf{r})\right)$$

Electric charge at the origin

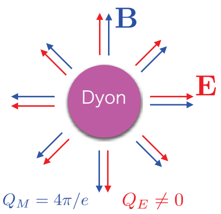
# Witten effect

Witten `79

A monopole acquires an electric charge, and becomes a dyon if $\theta \neq 0$.

$$Q_E = ne + \frac{e\theta}{2\pi} \qquad n = 0, \pm1, \pm2, \cdots$$

A monopole magnetic charge $Q_M = 4\pi/e$ is used.




The mass of dyons is heavier than the monopole mass:

$$M_D = M_M + \frac{1}{2}\frac{m_W^2}{M_M}\left(n - \frac{\theta}{2\pi}\right)^2$$
$$\simeq v\sqrt{Q_M^2 + Q_E^2}$$

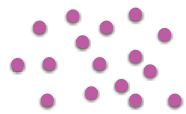
The non-zero value of $\theta$ costs more energy as it induces an electric field around the monopole.

# Axion coupled to hidden monopoles

Kawasaki, FT, Yamada, 1511.05030

Consider the axion coupling to U(1)$_H$:

$$\mathcal{L}_\theta = \frac{e^2 a}{32\pi^2 f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{e^2 a}{8\pi^2 f_a}\mathbf{E}\cdot\mathbf{B}$$



Then, the QCD axion acquires an extra mass about $a = 0$ by the Witten effect in the presence of hidden monopoles:

$$m_{a,M}^2 \simeq \frac{\alpha}{16\pi^2}\frac{n_M}{r_c f_a^2}$$

cf. Fischler and Preskill, `83

$n_M$ :(anti-)monopole number density $\qquad \alpha = \frac{e^2}{4\pi}$

$r_c \simeq \frac{1}{ev} = \frac{1}{m_W}$ : monopole radius

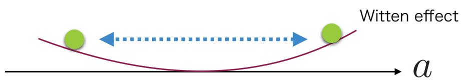
- The extra axion mass naturally decreases as the monopole density is diluted by the cosmic expansion.

Consistent w/ the neutron EDM constraint.

# Evolution of axion

- The axion starts to oscillate when $m_{a,M}^2 \simeq H^2$.

$$T_{\rm osc,1} \simeq 65\,{\rm GeV}\alpha^2\left(\frac{\Omega_M h^2}{0.12}\right)\left(\frac{f_a}{10^{12}\,{\rm GeV}}\right)^{-2}.$$

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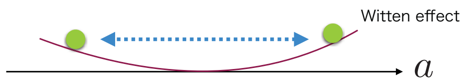
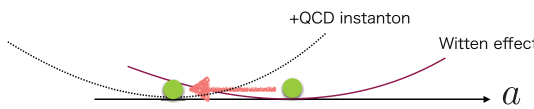



- No extra axion coherent oscillations are induced during the QCD phase transition if $m_{a,M}/H \gtrsim \mathcal{O}(10)$.

$$f_a \lesssim 10^{12}\,{\rm GeV}\alpha^{1.1}\left(\frac{\Omega_M h^2}{0.12}\right)^{0.55}.$$

"Adiabatic suppression mechanism"
Linde `96.



# Axion abundance

Kawasaki, FT, Yamada, 1511.05030

The final axion abundance is given by

$$\Omega_a h^2 \simeq 3\times 10^{-4}\alpha^{-2}\theta_i^2\left(\frac{0.12}{\Omega_M h^2}\right)\left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^3$$

**The axion abundance is suppressed, and it is inversely proportional to the monopole abundance!**

**The Witten effect also solves the axion domain wall problem** when the PQ symmetry spontaneously broken after inflation.

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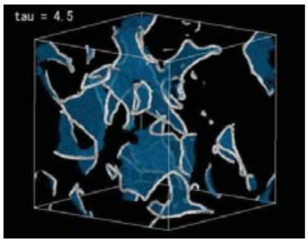
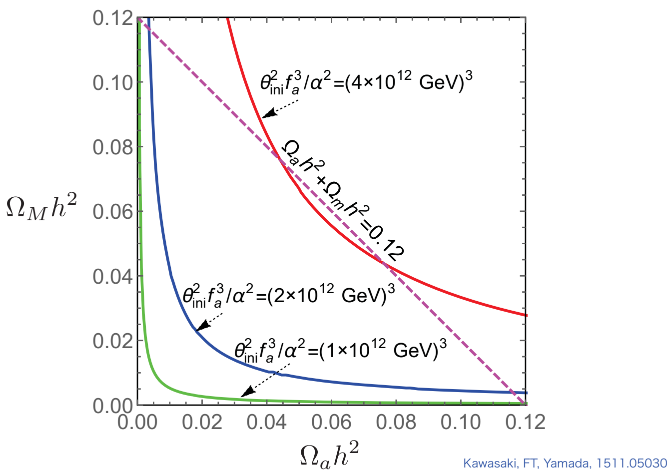



The string-wall network collapses when the axion starts to oscillate due to the Witten effect, and the axion field is set to be the hidden U(1) CP conserving minimum.

# QCD axion and monopole DM

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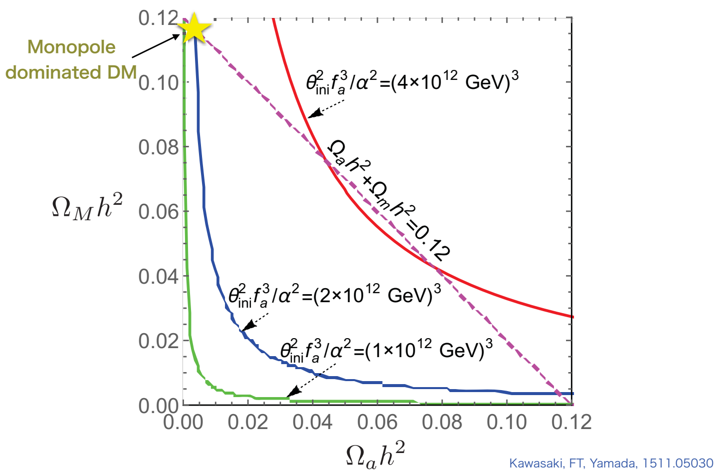
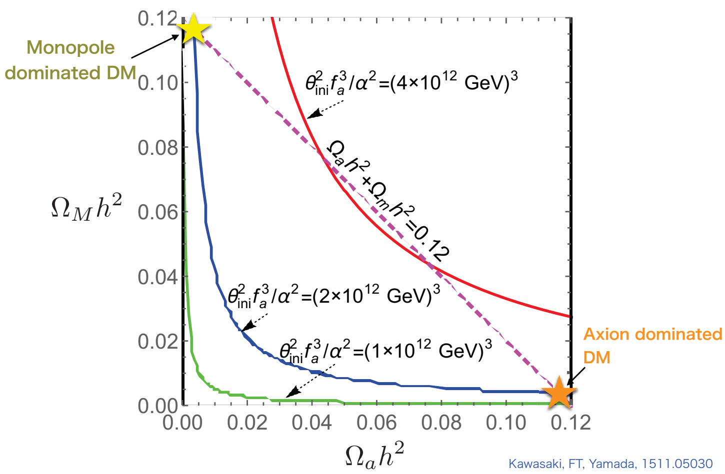



# QCD axion and monopole DM

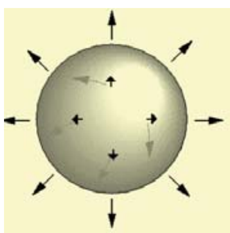



# 't Hooft-Polyakov monopole

Consider SU(2)H→U(1)H by an adjoint Higgs $\phi^a$

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\phi^a D_\mu\phi^a - \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right)^2 \qquad a=1,2,3$$

Vacuum mfd. = $S^2$



The scalar field values at spatial infinity map out $S^2$ in the vacuum manifold = $S^2$ : $\pi_2(S^2) = \mathbf{Z}$

Monopole w/ $Q_M = \frac{4\pi}{e}$ $M_M \sim \frac{4\pi v}{e}$

Monopoles are stable due to the topological charge. Once produced, they contribute to DM.

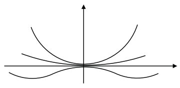
In addition there are massive gauge bosons $W^\pm$ of mass, $m_W = ev$ and a massless hidden photon.

# Monopole DM

Murayama, Shu 0905.1720
,Baek, Ko, Park 1311.1035, Khoze, Ro 1406.2291

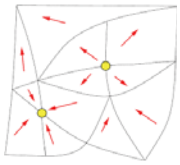
The Kibble-Zurek mechanism

Relaxation time scale: $\tau = \tau_0 \left|\frac{T-T_c}{T_c}\right|^{-\mu}$



Correlation length: $\zeta = \zeta_0 \left|\frac{T-T_c}{T_c}\right|^{-\nu}$ $\quad \mu \simeq \nu \sim 0.5 - 0.7$

$$Y_M \simeq 10^{-2}\left(\frac{30T_c}{\sqrt{8\pi}M_p}\right)^{3\nu/(1+\nu)}$$



Monopole-antimonopole annihilation

Motion of monopoles is determined by the competition between the diffusion and the monopole-antimonopole attraction.

$$Y_M \simeq \frac{B}{\pi g^2 x}\sqrt{\frac{45}{8g_*}}\frac{M_M}{M_p} \qquad x = \min\left[B^2/\alpha^2, \alpha\right]$$

$$B = 6\zeta(3)/\pi^2$$

# Monopole DM

The monopoles with mass $M_M = (1-10)\,\mathrm{PeV}$ account for the observed DM.
Murayama, Shu 0905.1720

In the minimal case of the 't Hooft-Polyakov monopole w/o other charged particles, the massive gauge bosons give a larger contribution.
Baek, Ko, Park 1311.1035, Khoze, Ro 1406.2291

e.g. The observed DM is realized with $v \simeq 10^2\,\mathrm{TeV}, \alpha = \mathcal{O}(0.1)$ for which the monopole fraction is O(10)%.

N.B. The massive gauge boson abundance can be reduced by adding matter fields.

# Discussion

- The monopoles (as well as W') have self-interactions which may solve the small-scale tension of CDM.
Baek, Ko, Park 1311.1035, Khoze, Ro 1406.2291

# Discussion


- The monopoles (as well as W') have self-interactions which may solve the small-scale tension of CDM.
Baek, Ko, Park 1311.1035, Khoze, Ro 1406.2291


- The monopole DM may also have the SM electromagnetic mini-charges via the kinetic mixing:

$$\frac{\phi_a}{M}F_a^{\mu\nu}F_{\mu\nu}^{(\mathrm{EM})} \supset \frac{v}{M}F_3^{\mu\nu}F_{\mu\nu}^{(\mathrm{EM})} \qquad \langle\phi_a\rangle = (0,0,v)$$

For $v \sim 10^2\,\mathrm{TeV}, \quad M \sim M_p$ the kinetic mixing is $\sim 10^{-13}$ and it is consistent with observations. cf. Jaeckel and Ringwald, 1002.0329

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- If $U(1)_H$ is broken by another Higgs, monopoles and W' become unstable and decay into the SM particles via the portal to the SM Higgs. The QCD axion abundance can be suppressed for even larger $f_a$ .
Kawasaki, FT, Yamada, 1511.05030

# Summary

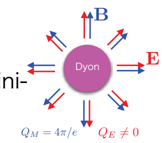
✓The QCD axion solves the strong CP problem, and it is a good DM candidate.

✓However, the QCD axion DM has a tension with many high- and intermediate-scale inflation models.

✓The Witten effect suppresses the QCD axion abundance in the presence of hidden monopoles, thereby relaxing the isocurvature bound.

✓The axion domain wall problem is also solved.

✓(Hidden) monopole DM is self-interacting and mini-charged DM.

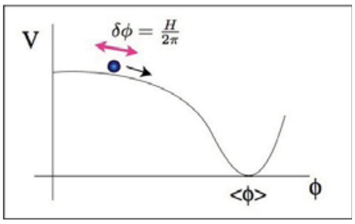


Back-ups

# Scalar mode

$$ds^2 = -(1+\Phi)dt^2 + a(t)^2(1+2\Psi)d\mathbf{x}^2$$

$\Phi$ : gravitational potential    $\Psi$ : curvature perturbations




Inflaton's quantum fluctuations induce **fluctuations in time and volume.**

$$\Phi \sim \frac{\delta\rho}{\rho} \sim H\delta t \sim H_{\rm inf}\frac{\delta\phi}{\dot{\phi}} \sim \left|\frac{V^{3/2}}{V' M_P^3}\right|$$

Super-horizon modes do not evolve.

RIKKYO MathPhys 2016

# Exact Quantization Conditions for Relativistic Integrable Systems

Yasuyuki Hatsuda
(University of Geneva)

Based on arXiv:1410.3382, 1511.02860, 1512.03061

Collaborators: S. Franco, A. Grassi & M. Marino

[Conceptual overlap with Takasaki-san's talk]

# Main Results

**Topological string theory can be used to solve a class of relativistic integrable systems!**

1. Starting with an algebraic curve describing a mirror Calabi-Yau, one can define a quantum mechanical operator. We conjectured exact forms of a **spectral determinant** and a **quantization condition** for this operator by using topological strings. Grassi-YH-Marino, 1410.3382

Mirror curve → Quantum operator

Spectral determinant → Quantization condition

2. Using a known relation between topological strings and integrables systems, we proposed **exact quantization conditions for the relativistic Toda lattice** YH-Marino, 1511.02860

Mirror curve ≡ Spectral curve

Quantum mirror curve ≡ Baxter equation

3. This conjecture was extended to a wide class of integrable systems, called **cluster integrable systems** by Goncharov and Kenyon

Toric CYs

Goncharov-Kenyon '11

Integrable systems ← solve — Topological strings

Franco-YH-Marino, 1512.03061

In all cases, the exact quantization conditions **universally** have a beautiful **S-dual** structure!

Schematically...

$$\sum_m C_{km}\left(\frac{\partial}{\partial t_m}F^{\rm NS}(\vec{t};\hbar)+\frac{\partial}{\partial t_m^{\rm D}}F^{\rm NS}(\vec{t}^{\rm D};\hbar^{\rm D})\right)=2\pi\left(n_k+\frac{1}{2}\right)$$

Free energy in NS limit

$$t_m^{\rm D}=\frac{2\pi t_m}{\hbar},\qquad \hbar^{\rm D}=\frac{4\pi^2}{\hbar}$$

In the special limit (4d limit), our results reduce to quantization conditions by Nekrasov and Shatashvili

# Plan of Talk

1. Topological Strings and Quantum Mechanical Systems
2. Exact Quantization Conditions for Relativistic Integrable Systems

# 1. Topological Strings and Quantum Mechanical Systems

# Mirror Symmetry

- **Mirror symmetry relates two different Calabi-Yau manifolds**
- **Two different topological strings on these CYs are equivalent**
- **This duality is very important both conceptually and practically**

**A-model on X $\equiv$ B-model on Y**

**Kaehler structure    Complex structure**

- **B-model on mirror CY is described by an algebraic curve (a mirror curve)**

$$\mathcal{O}_X(\mathrm{e}^x, \mathrm{e}^p) = H$$

- **These curves are sufficient to construct the all-genus topological string partition function by topological recursion a la Eynard and Orantin ("Remodeling formalism")** Bouchard et al. '07

# Examples of Mirror Curves

| $X$ | $\mathcal{O}_X(x,p)$ |
|---|---|
| local $\mathbb{P}^2$ | $\mathrm{e}^x + \mathrm{e}^p + \mathrm{e}^{-x-p}$ |
| local $\mathbb{F}_0$ | $\mathrm{e}^x + m\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-p}$ |
| local $\mathbb{F}_1$ | $\mathrm{e}^x + m\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-x-p}$ |
| local $\mathbb{F}_2$ | $\mathrm{e}^x + m\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-2x-p}$ |
| local $\mathcal{B}_2$ | $m_2\mathrm{e}^x + m_1\mathrm{e}^p + \mathrm{e}^{-x} + \mathrm{e}^{-p} + \mathrm{e}^{x+p}$ |
| local $\mathcal{B}_3$ | $m_1\mathrm{e}^{-x} + \mathrm{e}^x + m_2\mathrm{e}^{-p} + \mathrm{e}^p + m_3\mathrm{e}^{x+p} + \mathrm{e}^{-x-p}$ |

# Quantizing Mirror Curves

- **Here we see another aspect of the mirror curves**
- **Let us consider a quantization of a given mirror curve**
- **The mirror curve naturally leads to a quantum mechanical operator**

$$\hat{H} = \hat{\mathcal{O}}_X(\mathrm{e}^{\hat{x}}, \mathrm{e}^{\hat{p}}) \qquad [\hat{x}, \hat{p}] = \mathrm{i}\hbar$$

- **Claim: The inverse operator of $\hat{\mathcal{O}}_X$ is a trace class operator, and has an infinite number of discrete eigenvalues** Grassi-YH-Marino '14 Kashaev-Marino '15

$$\hat{\rho}_X = \hat{\mathcal{O}}_X^{-1} \qquad \hat{\rho}_X|\psi_n\rangle = \lambda_n|\psi_n\rangle$$

| $X$ | $\mathcal{O}_X(x,p)$ |
|---|---|
| local $\mathbb{P}^2$ | $\mathrm{e}^x + \mathrm{e}^p + \mathrm{e}^{-x-p}$ |
| local $\mathbb{F}_0$ | $\mathrm{e}^x + m\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-p}$ |
| local $\mathbb{F}_1$ | $\mathrm{e}^x + m\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-x-p}$ |
| local $\mathbb{F}_2$ | $\mathrm{e}^x + m\mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-2x-p}$ |
| local $\mathcal{B}_2$ | $m_2\mathrm{e}^x + m_1\mathrm{e}^p + \mathrm{e}^{-x} + \mathrm{e}^{-p} + \mathrm{e}^{x+p}$ |
| local $\mathcal{B}_3$ | $m_1\mathrm{e}^{-x} + \mathrm{e}^x + m_2\mathrm{e}^{-p} + \mathrm{e}^p + m_3\mathrm{e}^{x+p} + \mathrm{e}^{-x-p}$ |

- **This naturally leads to a quantum eigenvalue problem**

$$\hat{\rho}_X|\psi_n\rangle = \lambda_n(\hbar)|\psi_n\rangle$$

- **Remarkably, this problem can be solved for any Planck constant by using topological string results!**
- **Our strategy is to construct the exact spectral determinant first, and then read off its zeros**

## Spectral Determinant

- **Definition**

$$\Xi_X(\kappa) := \mathrm{Det}(1+\kappa\hat{\rho}_X) = \prod_{n=0}^{\infty}(1+\kappa\lambda_n)$$

- **It has zeros at** $\kappa = -\lambda_n^{-1}$
- **If the spectral determinant is known, by construction, the spectral problem is solved**

---

- **Our conjecture: For a given (genus one) mirror curve, one can explicitly construct the spectral determinant** Grassi-YH-Marino '14

$$\Xi_X(\kappa;\hbar) = \mathrm{e}^{J_X(\mu;\hbar)}\Theta_X(\mu;\hbar) \qquad \kappa = \mathrm{e}^{\mu}$$

$$\Theta_X(\mu;\hbar) = \sum_{n\neq 0}\exp[J_X(\mu+2\pi \mathrm{i} n;\hbar)]$$

- **The building block $J_X$ is related to the refined topological string free energy on X**
- **Higher genus generalization**

Codesido-Grassi-Marino '15

---

- **Explicit form of $J_X$**

$$J_X(\mu;\hbar) = J_X^{\mathrm{WKB}}(\mu;\hbar) + J_X^{\mathrm{np}}(\mu;\hbar)$$

$$J_X^{\mathrm{WKB}}(\mu;\hbar) = \frac{t}{2\pi}\frac{\partial F^{\mathrm{NS}}(t;\hbar)}{\partial t} + \frac{\hbar^2}{2\pi}\frac{\partial}{\partial \hbar}\left(\frac{F^{\mathrm{NS}}(t;\hbar)}{\hbar}\right)$$

$$J_X^{\mathrm{np}}(\mu;\hbar) = F^{\mathrm{GV}}\left(\frac{2\pi t}{\hbar} + \pi \mathrm{i} B_X; \frac{4\pi^2}{\hbar}\right)$$

$$F^{\mathrm{GV}}(t;g_s) := \lim_{\epsilon_2\to -\epsilon_1} F^{\mathrm{refined}}(t;\epsilon_1 = g_s, \epsilon_2)$$

$$F^{\mathrm{NS}}(t;\hbar) := \lim_{\epsilon_2\to 0} \epsilon_2 F^{\mathrm{refined}}(t;\epsilon_1 = \hbar, \epsilon_2)$$

---

## Remarks

- **The WKB part is visible in the semi-classical limit, but the NP part is invisible**

$$\text{WKB part} \sim \mathrm{e}^{-t}$$

$$\text{NP part} \sim \mathrm{e}^{-\frac{2\pi t}{\hbar}}$$

- **The modulus $t$ is related to $\mu$ in a non-trivial way (quantum corrected mirror map)**

Aganagic et al. '11

$$t = \Pi_A(z;\hbar), \qquad z = \mathrm{e}^{-r\mu}$$

---

## Further Remarks

- **A quantum mirror curve leads to a difference equation**
- **This difference equation takes the same form as the Baxter equation**

$$\mathcal{O}_{\mathbb{F}_0} = \mathrm{e}^{x} + m\mathrm{e}^{-x} + \mathrm{e}^{p} + \mathrm{e}^{-p} = H$$

$$\psi(x+\mathrm{i}\hbar) + \psi(x-\mathrm{i}\hbar) = (H - \mathrm{e}^{x} - m\mathrm{e}^{-x})\psi(x)$$

- **If the square integrability of the wave function is required, only the discrete spectra of H are allowed**

---

## Quantization Condition

- **The spectral determinant has information on all the eigenvalues**

$$\Xi_X(\kappa) = \prod_{n=0}^{\infty}(1+\kappa\lambda_n)$$

- **What we have to do is to read off its zeros**
- **The vanishing condition leads to an exact quantization condition, which is valid for arbitrary Planck constant**

- **We explicitly wrote down vanishing condition** Grassi-YH-Marino '14
- **Then, Wang, Zhang and Huang rewrote it as a much more beautiful form**
- **The final result (schematically)** Wang et al. '15

$$\frac{\partial}{\partial t}F^{\mathrm{NS}}(t;\hbar) + \frac{\partial}{\partial t^{\mathrm{D}}}F^{\mathrm{NS}}(t^{\mathrm{D}};\hbar^{\mathrm{D}}) = 2\pi\left(n+\frac{1}{2}\right)$$

$$t = \Pi_A(H;\hbar)$$ **(Quantum mirror map)**

- **Invariant under**

$$t \leftrightarrow t^{\mathrm{D}} = \frac{2\pi t}{\hbar}, \quad \hbar \leftrightarrow \hbar^{\mathrm{D}} = \frac{4\pi^2}{\hbar}$$ **S-duality!**

# A Test

- **Our conjecture is quite heuristic, and thus we have to check it carefully**
- **Compute the spectra in two ways, independently**
- **Solve the quantization condition**
- **Diagonalize the Hamiltonian numerically by harmonic oscillator basis**
- **Compare them**

- **Here we consider a simple example, local P2**

$$\hat{\rho}_{\mathbb{P}^2} = (\mathrm{e}^{\hat{x}} + \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{x}-\hat{p}})^{-1} \qquad |\psi_n\rangle = \sum_m a_{nm}|m\rangle$$

- **Result for $\hbar = 4\pi$**

| Degree | $E_0 = -\log\lambda_0$ |
|---|---|
| 1 | 3.7776432527296085597046797 |
| 3 | 3.7777062505593500784461494 |
| 6 | 3.7777062585822008247337693 |
| 8 | 3.7777062585822069972270331 |
| 10 | 3.7777062585822069986877030 |
| 11 | 3.7777062585822069986880502 |
| 12 | 3.7777062585822069986880709 |
| Numerical value | 3.7777062585822069986880709 |

**We have checked for various hbar**

## 2. Exact Quantization Conditions for Relativistic Integrable Systems

# Generalization

- **Can we extend the previous result to more interesting eigenvalue problems?**
- **The answer is YES!**
- **We can solve eigenvalue problems for known integrable systems**
- **A familiar example is the relativistic Toda lattice**
- **Our result can be further generalized to wider (less familiar) integrable systems**

## Toda Lattice and Topological Strings

- **There is a well-known relation between the Toda lattice and the 4d N=2 supersymmetric gauge theory (Seiberg-Witten theory)** Gorsky et al. '95

**Spectral curve of Toda lattice**
**≡ Seiberg-Witten curve**

- **This relation is generalized to the relativistic system and the 5d theory/ topological string theory** Nekrasov '96

# Relativistic Toda Lattice

- A relativistic generalization of the Toda lattice — Ruijsenaars '90
- It is still integrable
- In the non-relativistic limit, it reduces to the standard Toda lattice
- This system is related to the 5d SU(N) pure supersymmetric gauge theory

# Integrability of RTL

- Lax operator

$$L_n(z;R) = \begin{pmatrix} z - z^{-1}\mathrm{e}^{Rp_n} & R\mathrm{e}^{-q_n} \\ -R\mathrm{e}^{q_n+Rp_n} & 0 \end{pmatrix}$$

$$[q_n, p_m] = \mathrm{i}\hbar\delta_{nm}$$

- Transfer matrix generates mutually commuting Hamiltonians

$$2T(z;R) = \mathrm{Tr}[L_N(z;R)\cdots L_1(z;R)]$$

$$[T(z;R), T(w;R)] = 0$$

$$2T(z;R) = \sum_{k=0}^{N}(-1)^k z^{N-2k} H_k(q_1,p_1;\dots;q_N,p_N)$$

- Our goal is to diagonalize N-1 Hamiltonians
- They have discrete eigenvalues
- Correspondig geometry in the topological string

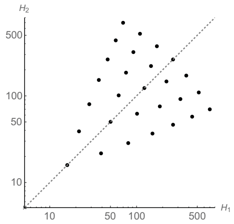
Relativistic Toda

↕

5d pure SU(N) SYM

↕

Topological string on $A_{N-1}$ geometry




Spectra for N=3

- Our quantization condition is easily extended to generic CYs
- The main difference is that in general there are many moduli of CYs
- Roughly speaking, these moduli are related to eigenvalues of commuting Hamiltonians in non-trivial ways (quantum mirror map)

- Our quantization conditions (schematically) — YH-Marino, 1511.02860

$$\sum_m C_{km}\left(\frac{\partial}{\partial t_m}F^{\mathrm{NS}}(\vec{t};\hbar) + \frac{\partial}{\partial t_m^{\mathrm{D}}}F^{\mathrm{NS}}(\vec{t}^{\mathrm{D}};\hbar^{\mathrm{D}})\right) = 2\pi\left(n_k + \frac{1}{2}\right)$$

$$t_m = \Pi_{A_m}(\vec{H};\hbar)$$ (Quantum mirror map)

- We need the NS free energy
- The refined free energy on $A_{N-1}$ geometry was computed by Taki-san, using the refined topological vertex — Taki '07
- We used his result, and took the NS limit

# Result for N=2

- The Hamiltonian

$$H(q_1,p_1;q_2,p_2) = \sum_{n=1}^{2}(1 + q^{-1/2}R^2\mathrm{e}^{q_n - q_{n+1}})\mathrm{e}^{Rp_n}$$

- In the COM frame, this reduces to

$$H = R^2(\mathrm{e}^{\xi} + \mathrm{e}^{-\xi}) + \mathrm{e}^{R\mu} + \mathrm{e}^{-R\mu}$$

- This is equivalent to the Hamiltonian for local $F_0$
- Thus this has already been solved

# Result for N=3

- **There are two commuting Hamiltonians**

$$H_1 = \sum_{n=1}^{3} (1 + q^{-1/2} R^2 e^{q_n - q_{n+1}}) e^{R p_n}$$

$$H_2 = \sum_{n=1}^{3} (1 + q^{-1/2} R^2 e^{q_{n-1} - q_n}) e^{-R p_n}$$

$$[H_1, H_2] = 0$$

- **Eigenvalues are labeled by two quantum numbers ($n_1$,$n_2$)**

### Eigenvalues (0,0) for $\hbar = 2\pi$

| Order | $H_1 = H_2$ |
|---|---|
| 1 | 39.1104429532554969 |
| 5 | 39.1678325080157194 |
| 15 | 39.1678190762795935 |
| 23 | 39.1678190762768699 |
| Numerical value | 39.1678190762768699 |

### Eigenvalues (1,0) for $\hbar = 2\pi$

| Order | $H_1$ | $H_2$ |
|---|---|---|
| 1 | 61.7259698869968 | 152.405034932001 |
| 6 | 61.9664326975787 | 152.359676263718 |
| 12 | 61.9664190066106 | 152.359672000995 |
| 18 | 61.9664190064911 | 152.359672001068 |
| Numerical value | 61.9664190064911 | 152.359672001068 |

### Eigenvalues (0,0) for various hbar

| Order | $\hbar = \pi$ | $\hbar = 3\pi$ |
|---|---|---|
| 1 | 15.7049125387 | 93.5756026639 |
| 5 | 15.8136736201 | 93.5700657722 |
| 10 | 15.8137840616 | 93.5700660274 |
| 15 | 15.8137841054 | 93.5700660274 |
| Numerical value | 15.8137841054 | 93.5700660274 |

| Order | $\hbar = 3$ | $\hbar = 10$ |
|---|---|---|
| 1 | 15.0088290209 | 109.495054032 |
| 5 | 15.1620665172 | 109.443122441 |
| 10 | 15.1622789237 | 109.442994726 |
| 15 | 15.1622789846 | 109.442994727 |
| Numerical value | 15.1622789846 | 109.442994727 |

**Our quantization conditions correctly reproduce the true eigenvalues!**

# Comments

- **For a given non-compact toric CY threefold, one can construct mutually commuting Hamiltonians systematically (cluster integrable systems)** Goncharov-Kenyon '11
- **Our quantization conditions indeed diagonalize these Hamiltonians!**
- **We have explicitly checked it for resolved C^3/Z_5 and C^3/Z_6 geometries**

Franco-YH-Marino '15

# Summary

**Topological string theory can be used to solve a class of relativistic integrable systems!**

- **We have checked this fact in some explicit examples, the relativistic Toda lattice etc, at the very quantitative level.**
- **It would be interesting to check it further for more complicated examples**

# Open Problems

- **How to construct eigenfunctions? Open topological strings? 5d AGT?**

[Takasaki-san's talk]

- **What is the origin of the S-dual structure in the quantization conditions? Modular double?**

**Modular double in relativistic Toda lattice**

$$q = e^{\pi i b^2}, \quad \tilde{q} = e^{\pi i b^{-2}}$$ Karchev et al. '01

## Determinantal structures in a random polymer model

Takashi Imamura

Department of mathematics and informatics, Chiba university

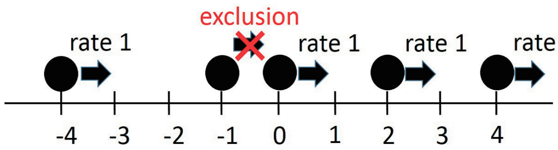
Joint work with Tomohiro Sasamoto

Ref: arXiv:1506.05548

10th January 2016



## Totally asymmetric simple exclusion process (TASEP)




- Rules:
  - Each particle hop to the right neighboring site with rate 1 if the site is empty.
  - The particle cannot hop if the right neighboring is occupied (exclusion interaction).
- Applications: Nonequilibrium statistical physics (ionic conductor, traffic flow, surface growth)
- Integrable structures:
  - Stationary state (1990$\sim$): matrix product method (Derrida et al)
  - Dynamics (2000$\sim$): random matrices(Johansson, Spohn)
  - Integrable probability (2010$\sim$): Sasamoto-Spohn, Amir-Corwin-Quastel Borodin-Corwin



## A basic question: the tagged particle problem

- $X(s)$ : the position of a particular particle (called the tagged particle) at time $s$. (we set $X(0)=0$)
- Law of $X(s)$ for large $s$. (e.g. average, variance, distributions, etc.)
- Simple random walk (without exclusion):

$$X(s)\sim s, \text{for large } s: \text{ law of large number,}$$

$$\lim_{s\to\infty}\text{Prob}\left(\frac{X(s)-s}{s^{\frac{1}{2}}}<u\right)=\int_{-\infty}^{u}\frac{e^{-y^2/2}}{\sqrt{2\pi}}: \text{ central limit theorem}$$

These are known to be universal.

- TASEP: Limiting behavior when both $s$ and $N$ (the number of particles) are large.
  Does the exponent of fluctuation deviate from $1/2$?
  Does the distribution become non Gaussian?



## Random matrix theory: GUE Dyson's Brownian motion

- $N\times N$ Hermitian matrix, $a_{ii}\sim B_{ii}(t)$, $a_{ij}\sim B_{ij}(t)/\sqrt{2}$, $b_{ij}\sim\tilde{B}_{ij}(t)/\sqrt{2}$
  $B_{ij}(t)$, $\tilde{B}_{ij}(t)$: 1d standard Brownian motions

$$\begin{pmatrix} a_{11} & a_{12}+\sqrt{-1}b_{12} & \cdots & a_{1N}+\sqrt{-1}b_{1N} \\ a_{12}-\sqrt{-1}b_{12} & a_{22} & \cdots & a_{2N}+\sqrt{-1}b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1N}-\sqrt{-1}b_{1N} & a_{2N}-\sqrt{-1}b_{2N} & \cdots & a_{NN} \end{pmatrix}$$

- The probability density function (pdf) of eigenvalues at $t$: $x_1>x_2>\cdots>x_N$ is given by level repulsion

$$P_{\text{GUE}}(x_1,\cdots,x_N;t)=\prod_{j=1}^{N}\frac{e^{-x_j^2/2t}}{j!\sqrt{2\pi t}}\cdot\prod_{1\le j<k\le N}(x_k-x_j)^2$$

- Product of two (Vandermonde) determinant$\sim$ $N$ free Fermions.



## Gaussian Unitary Ensemble (GUE)

- $P_{\text{GUE}}(x_1,\cdots,x_N;t)=\prod_{j=1}^{N}\frac{e^{-x_j^2/2}}{j!\sqrt{2\pi}}\cdot\prod_{1\le j<k\le N}(x_k-x_j)^2$
- By the random matrix technique (Tracy-Widom(1998)), we readily obtain the Fredholm determinant representation.

$$\text{Prob}\left(x_1(t)<s\right)=\det\left(1-K_{\text{GUE}}\right)_{L^2((s,\infty))} \quad \text{Fredholm determinant}$$

$$:=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\prod_{j=1}^{k}\int_s^{\infty}dx_j\cdot\det\left(K_{\text{GUE}}(x_l,x_m;t)\right)_{l,m=1}^{k},$$

$$K_{\text{GUE}}(x_1,x_2;t)=\frac{e^{-x_1^2/2}}{\sqrt{2\pi t}}\sum_{j=0}^{N-1}\frac{H_j(x_1/\sqrt{2t})H_j(x_2/\sqrt{2t})}{2^k k!},$$

$H_k(x)$ : $k$th Hermite polynomial

- Determinantal point process (free Fermion)

$$\det\left(K_{\text{GUE}}(x_l,x_m)\right)_{l,m=1}^{k}: \ k\text{-point correlation function of the eigenvalues}$$



## GUE Tracy-Widom distribution

- (scaled) largest eigenvalue distribution of GUE

$$F_2(s)=\lim_{N\to\infty}\text{Prob}\left((x_1-\sqrt{2N})\sqrt{2}N^{1/6}\le s\right)$$

- Fredholm determinant: Forrester(1993)

$$F_2(s)=\det(1-P_sKP_s):=\sum_{k=0}^{\infty}\frac{1}{k!}\prod_{l=1}^{k}\left(\int_s^{\infty}dx_l\right)\det\left(K(x_m,x_n)\right)_{m,n=1}^{k}$$

$$\text{Airy kernel} K(x,y)=\int_0^{\infty}d\lambda \text{Ai}(x+\lambda)\text{Ai}(y+\lambda),$$

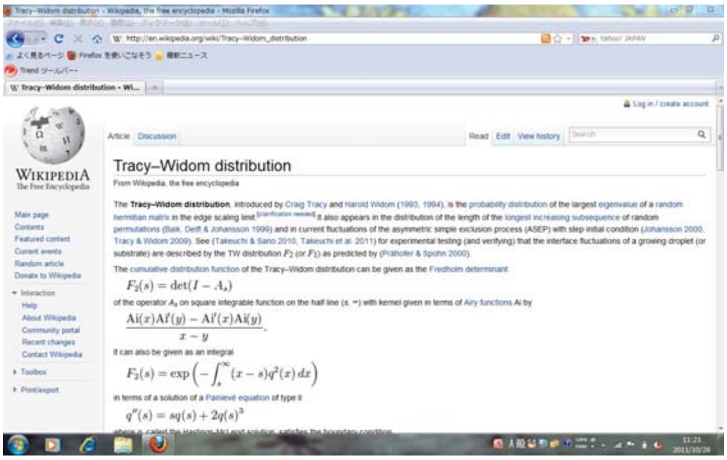
$P_s$ : Projection onto$(s,\infty)$

- Painlevé II: Tracy-Widom(1994)

$$F_2(x)=\exp\left(-\int_s^{\infty}dx(x-s)q(x)^2\right)$$

$$\frac{\partial^2}{\partial x^2}q(x)=2q(x)^2+xq(x),\ q(x)\sim\text{Ai}(x),\ x\to\infty$$

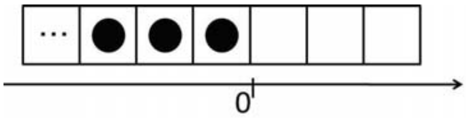
## Tracy-Widom distribution






## Tagged particle for TASEP

- Step initial condition




- $X_N(t)$: Position of the $N$th particle at $t$

K. Johansson(2000)

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{X_N(t)-AN}{DN^{1/3}} < s\right) = F_2(s)$$

with $t = N$

1/3: Kardar-Parisi-Zhang universality class
$F_2(s)$ : GUE Tracy-Widom distribution

- This relation implies a connection between TASEP and random matrix theory.

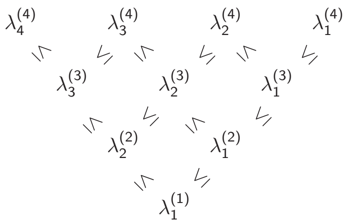


## Unified theory

- To unify the TASEP ($N$ particles) and the GUE (rank $N$), we introduce new stochastic process on the $N(N+1)/2$ dimensional Gelfand-Tsetlin cone $\mathrm{GT}_N$.

$$\mathrm{GT}_N := \{\{\lambda_i^{(j)}\}_{1\le i\le j\le N} \in \mathbb{Z}^{N(N+1)/2} | x_{\ell+1}^{(m+1)} \le x_\ell^{(m)} \le x_\ell^{(m+1)} \text{ with } 1 \le \ell \le m \le N-1\}.$$

- Each $\lambda_i^{(j)}$ moves to $\lambda_i^{(j)}+1$ with rate 1 (Poisson random walk) with exclusion and push interaction





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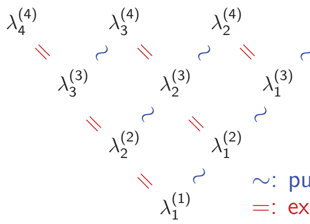

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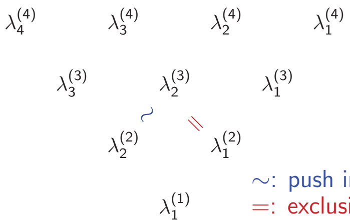

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## Joint distribution and the Schur functions

- Borodin-Ferrari (2008) showed that the probability density function of the positions at time $s$ is described as

$$P\left(\{\lambda_i^{(j)}\}_{1\le i\le j\le N}; s\right) = \frac{1}{e^{Ns}} s_{\lambda^{(1)}}(1) s_{\lambda^{(2)}/\lambda^{(1)}}(1)\cdots s_{\lambda^{(N)}/\lambda^{(N-1)}}(1)\cdot s_{\lambda^{(N)}}(\rho_s)$$

where $s_\lambda(x)$ is the Schur function and $\rho_t$ means the Plancherel specialization.

- Jacobi-Trudi formula:

$$s_{\lambda/\mu}(a_1,\cdots,a_m) = \det\left(h_{\lambda_i-\mu_j+j-i}(a_1,\cdots,a_m)\right)$$

where $h_k(a_1,\cdots,a_m)$: $k$th complete symmetric polynomial

$$h_k(a_1,\cdots,a_m) = \sum_{\substack{k_1,\cdots,k_m=0\\ k_1+\cdots+k_m=k}}^{k} a_1^{k_1}\cdots a_m^{k_m}, \quad h_k(\rho_s) = \frac{s^k}{k!}$$

## Two marginals

- The left marginal: TASEP
- The top marginal: the discretized GUE.

$$P\left(\{\lambda_i^{(N)}\}_{i=1,2,\cdots,N}\right)$$
$$= \frac{1}{e^{Ns}} s_{\lambda^{(N)}}(\overbrace{1,\cdots,1}^{N \text{ times}}) s_{\lambda^{(N)}}(\rho_s)$$
$$= \frac{1}{e^{Ns}} \prod_{j=1}^{N} \frac{s^{\lambda_j^{(N)}} e^{-s}}{\lambda_j^{(N)}!} \cdot \prod_{1<i<j<N} (\lambda_i^{(N)} - \lambda_j^{(N)})^2$$
$$P_{\rm GUE}(x;t) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2}}{j!\sqrt{2\pi}} \cdot \prod_{1\le j<k\le N} (x_k - x_j)^2$$

- $$\lambda_N^{(N)} = \text{the position of the } N\text{th particle of TASEP} = \text{the smallest particle of the discrete GUE}$$

$$\begin{array}{ccccccccc} \lambda_3^{(3)} & & & & \lambda_2^{(3)} & & & & \lambda_1^{(3)} \\ & = & & \sim & & = & & \sim & \\ & & \lambda_2^{(2)} & & & & \lambda_1^{(2)} & & \\ & & & = & & \sim & & & \\ & & & & \lambda_1^{(1)} & & & & \end{array}$$

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## Macdonald processes

- Borodin-Corwin (2011) introduced a $q$-deformation of Borodin-Ferrari (2008) on the interacting random walk on $\mathrm{GT}_N$.

$$P\left(\{\lambda_i^{(j)}\}_{1\le i\le j\le N}\right) = \frac{1}{e^{Nt}} s_{\lambda^{(1)}}(1) s_{\lambda^{(2)}/\lambda^{(1)}}(1)\cdots s_{\lambda^{(N)}/\lambda^{(N-1)}}(1)\cdot s_{\lambda^{(N)}}(\rho_t)$$
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where $P_{\lambda/\mu}$ and $Q_\lambda$ are the $q$-Whittaker functions.
- The $q$-Whittaker functions are the Macdonald symmetric functions (two parameters $(q,t)$ generalizations of the Schur function) with $t=0$. When $q=0$, they become the Schur function.

Rates of each $\lambda_i^{(j)}$ depends on $q$ and the configurations.

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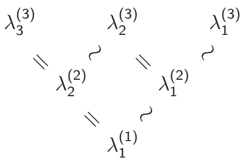
  The exclusion process with jump rate $1 - q^{\lambda_{j-1}^{(j-1)} - \lambda_j^{(j)}}$.
- The top marginal: the $q$-Whittaker measure

$$\frac{1}{e^{Ns}} P_{\lambda^{(N)}}(\overbrace{1, \cdots, 1}^{N \text{ times}}) Q_{\lambda^{(N)}}(\rho_s)$$

- Not a product of two deteminants

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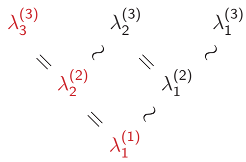


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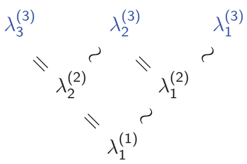


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## Fredholm determinant

- $\lambda_N^{(N)}$ =the position of the $N$th particle of $q$-TASEP
  =the smallest particle in the $q$-Whittaker measure
  Not a product of two determinants
- Macdonald polynomials do not inherit the determinantal structure of the Schur polynomial.
- Nevertheless Borodin-Corwin (2011) found a Fredholm determinant representation for the $q$-monent generating function

$$\left\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \right\rangle = \sum_{n=0}^{\infty} \frac{\zeta^k}{(q;q)_k} \left\langle q^{k\lambda_N^{(N)}} \right\rangle$$

where $\zeta \in \mathbb{C}$, $\langle \cdot \rangle$ represents the expectation value with respect to the $q$-Whittaker measure and

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$$
$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1})$$



## Fredholm determinant

Borodin-Corwin (2011)

$$\left\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \right\rangle \left( = \sum_{n=0}^{\infty} \frac{\zeta^k}{(q;q)_k} \left\langle q^{k\lambda_N} \right\rangle \right) = \det\left(1 + K_\zeta\right)_{L^2(C_0)}.$$

- Fredholm determinant ($C_0$: unit circle around the origin)

$$\det\left(1 + K_\zeta\right)_{L^2(C_0)} = \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{j=1}^{k} \int_{C_0} \frac{dz_j}{z_j} \cdot \det\left(K_\zeta(z_l, z_m)\right)_{l,m=1}^{k}$$

- Borodin-Corwin (2011) used some properties of the Macdonald difference operator.
- Now many stochastic processes which has the Fredholm determinant representaions have been discovered. (Integrable probability)
- These stochastic processes are not a determinantal point process (except the Schur case $q = 0$).
  We want to reveal determinantal structures lying behind the nondeterminantal point processes.



## A Brownian analogue of the $q$-TASEP

- We take a $q \to 1$ limit

$$q = e^{-\epsilon}, \; s = t\epsilon^{-2}, \; \lambda_j^{(k)} = t\epsilon^{-2} - (k+1-2j)\epsilon^{-1}\log\epsilon + y_{k,j}\epsilon^{-1}, \; \epsilon \to 0$$

- The left marginal: Let $Z_j = e^{y_{j,j}}$. Then we have the stochastic differential equations.

$$dZ_j = dZ_{j-1} + Z_j dB_j, \; j = 1, \cdots, N$$

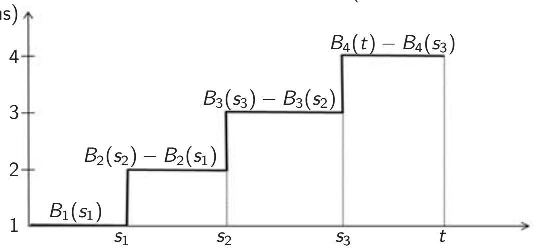
where $Z_0 = 0$, $B_j$, $j = 1, \cdots, N$: independent 1d standard Brownian motions.
They can be solved easily. ($B_j(x,t) = B_j(t) - B_j(s)$, $j = 1, \cdots, N$.)

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{B_1(s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t)} ds_1 \cdots ds_{N-1}$$

- The top martinal: for the q-Whittaker measure $\to$ Whittaker measure
- $Z_N(t)$: The partition function of the O'Connell-Yor random directed polymer model

## The O'Connell-Yor polymer

- It is introduced by O'Connell-Yor(2001). A typical model of the directed polymer in random environment in two dimension (one discrete + one continuous)




- The polymer partition function ($B_j(x,t) = B_j(t) - B_j(s)$, $j = 1, \cdots, N$)

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta(B_1(s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t))} ds_1 \cdots ds_{N-1}$$

$\beta = 1/k_B T$ : inverse temperature



## Motivation

- O'Connell (2010), Borodin-Corwin(2011):

$$\mathrm{Prob}\left(-F_N(t) \le \tilde{s}\right) = \int_{(-\infty,s]} dx_1 \int_{\mathbb{R}^{N-1}} \prod_{j=2}^{N} dx_j \cdot m(x_1, \cdots, x_N; t)$$

$$F_N(t) = \frac{\log Z_N}{\beta}, \quad \tilde{s} = s - \log \beta^2(N-1)/\beta$$

- $m(x_1, \cdots, x_N; t)$: pdf of the Whittaker measure
  It is expressed as the Whittaker functions.
  Not a product of two determinants
  Katori (2012): a probabilistic interpretation.
- Nevertheless, Borodin-Corwin (2011) found out the Fredholm representation

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \det\left(1+L\right)_{L^2(C_0)}$$

- Why does the determinatal structure arise in non-determinantal processes?



## Main result (arXiv:1506.05548)

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t),$$

$f_F(x) = 1/(e^{\beta x}+1)$ : the Fermi distribution function

$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \cdot \prod_{1\le j<k\le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)_{j,k=1}^{N},$$

$$\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - w^2 t/2} \frac{(iw)^k}{\Gamma\left(1+iw/\beta\right)^N}.$$

- $W(x;t)dx$: a signed measure
- A product of the two determinants
  Random matrix techniques can be applied.



## A Fredholm determinant (arXiv:1506.05548)

$$\mathbb{E}\left(\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right) = \det\left(1 - \bar{f}_u K\right)_{L^2(\mathbb{R})}, \quad \bar{f}_u(x) = f_F(x-u) - 1$$

$$K(x, y; t) = \sum_{j=0}^{N-1} \phi_k(x; t)\psi_k(y; t)$$

$$\phi(x; t) = \frac{1}{2\pi i} \oint dv e^{vx - vt^2/2} \frac{\Gamma(v/\beta+1)^N}{v^{k+1}}$$

$$\psi_k(x; t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dw e^{-wx+w^2 t/2} \frac{w^k}{\Gamma\left(1+w/\beta\right)^N}$$

- Biorthogonal relation $\int_{-\infty}^{\infty} \phi_j(x; t)\psi_k(x; t) = \delta_{j,k}$
- We can show the equivalence between our representation and Borodin-Corwin's.

$$\mathbb{E}\left(\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right) = \det\left(1+L\right)_{L^2(C_0)}$$



## The zero-temperature limit $\beta \to \infty$

Now we consider the zero-temperature limit($\beta \to \infty$) of our relations

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t),$$

$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \cdot \prod_{1\le j<k\le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)_{j,k=1}^{N}$$

- In the zero-temperature limit, they go to the result by Baryshnikov(2001), Gravner-Tracy-Widom(2001), Warren(2007).

$$\mathrm{Prob}(-f_N(t) < s) = \int_{-\infty}^{s} dx_1 \cdots dx_N P_{\mathrm{GUE}}(x_1, \cdots, x_N; t).$$

$$f_N(t) = \max_{0=s_0<s_1<\cdots<s_N=t} \sum_{i=1}^{N} \left(B_i(s_i) - B_i(s_{i-1})\right)$$

$$P_{\mathrm{GUE}}(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2t}}{j! t^{j-1}\sqrt{2\pi t}} \cdot \prod_{1\le j<k\le N} (x_k - x_j)^2$$



## The zero-temperature limit $\beta \to \infty$

Fredholm determinant representation

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \det\left(1 - \bar{f}_u K\right)_{L^2(\mathbb{R})},$$

$$K(x, y; t) = \sum_{j=0}^{N-1} \phi_k(x; t)\psi_k(y; t).$$

- In the zero-temperature limit, it goes to

$$\mathrm{Prob}(f_N(t) < s) = \det\left(1 - 1_{(s,\infty)} K_{\mathrm{GUE}}\right)_{L^2(\mathbb{R})}$$

$$f_N(t) = \max_{0=s_0<s_1<\cdots<s_N=t} \sum_{i=1}^{N} \left(B_i(s_i) - B_i(s_{i-1})\right)$$

$$K_{\mathrm{GUE}}(x_1, x_2; t) = \frac{e^{-x_2^2/2t}}{\sqrt{2\pi t}} \sum_{k=0}^{N-1} \frac{H_k(x_1/\sqrt{2t})H_k(x_2/\sqrt{2t})}{2^k k!}.$$

- $\lim_{\beta\to\infty} (2t)^{\frac{k+1}{2}} \pi^{\frac{1}{2}} e^{\frac{x^2}{2t}} \psi_k(x; t) = \lim_{\beta\to\infty} k! \left(\frac{t}{2}\right)^{-\frac{k}{2}} \phi_k(x; t) = H_k\left(\frac{x}{\sqrt{2t}}\right)$

## The scaling limit to the KPZ equation

- The KPZ scaling limit: $F_N(t) \to h(T,X)$

$$\frac{\partial}{\partial t}h(T,X) = \frac{1}{2}\frac{\partial^2}{\partial x^2}h(T,X) + \frac{1}{2}\left(\frac{\partial^2}{\partial x}h(T,X)\right)^2 + \eta(T,X)$$

- By the saddle point analysis, we find in the KPZ scaling, $(\gamma_T = (T/2)^{1/3})$

$$\lim_{N\to\infty}\psi_k(x;t)/C(N) = \lim_{N\to\infty}C(N)\phi_k(x;t) = \frac{\mathrm{Ai}(\xi-\lambda)}{\gamma_T}$$

which leads to the pointwise convergence to the KPZ kernel.

$$K(x_i,x_j) \to \mathcal{K}_{\mathrm{KPZ}}(\xi_i,\xi_j)$$
$$\mathcal{K}_{\mathrm{KPZ}}(\xi_i,\xi_j) = \int_0^\infty \frac{e^{\gamma_T(\xi_i-s)}}{e^{\gamma_T(\xi_i-s)}+1}\, d\lambda \mathrm{Ai}(\xi_i+\lambda)\mathrm{Ai}(\xi_j+\lambda)$$

- This implies that our relation recovers the relation in the KPZ equation. (Sasamoto-Spohn(2010),Amir-Corwin-Quastel(2010))

$$\mathbb{E}_{\mathrm{KPZ}}\left(e^{-e^{\gamma_T(h(X,T)-s)}}\right) = \det(1-\mathcal{K}_{\mathrm{KPZ}})$$



## O'Connell's representation revisted

- O'Connell (2010): A determinantal representation

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{(i\mathbb{R}-\epsilon)^N}\prod_{j=1}^N \frac{d\lambda_j}{\beta}e^{-u\lambda_j+\lambda_j^2 t/2}\Gamma\left(-\frac{\lambda_j}{\beta}\right)^N \cdot s_N\left(\frac{\lambda}{\beta}\right)$$
$$s_N(\lambda) = \frac{1}{(2\pi i)^N N!}\prod_{i<j}\frac{\sin\pi(\lambda_i-\lambda_j)}{\pi}\prod_{i>j}(\lambda_i-\lambda_j) : \text{Sklyanin measure}$$

- We rewrite the above expression and obtained the following.
**Proposition** (arXiv:1506.05548)

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{-\infty}^{\infty}\prod_{\ell=1}^N dt_\ell f_F(t_\ell-u)\cdot\det\left(F_{jk}(t_j;t)\right)_{j,k=1}^N$$
$$F_{jk}(x;t) = \int_{i\mathbb{R}-\epsilon}\frac{d\lambda}{2\pi i}\frac{e^{-\lambda x+\lambda^2 t/2}}{\Gamma\left(\frac{\lambda}{\beta}+1\right)^N}\left(\frac{\pi}{\beta}\cot\frac{\pi\lambda}{\beta}\right)^{j-1}\lambda^{k-1}.$$



## Brownian particle systems with reflection interactions

- In the zero-temperature limit, we have

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{-\infty}^{\infty}\prod_{\ell=1}^N dt_\ell f_F(t_\ell-u)\cdot\det\left(F_{jk}(t_j;t)\right)_{j,k=1}^N$$
$$\xrightarrow[\beta\to\infty]{} \mathrm{Prob}\left(f_N(t)\le u\right) = \int_{-\infty}^{u}\prod_{\ell=1}^N dx_\ell\cdot\det\left(\mathcal{F}_{k-j}(x_j;t)\right)_{i,j=1}^N$$
$$\mathcal{F}_n(x,t) = \int_{i\mathbb{R}-\epsilon}\frac{d\lambda}{2\pi i}\frac{e^{-\lambda x+\lambda^2 t/2}}{x^n},\quad n\in\mathbb{Z}.$$

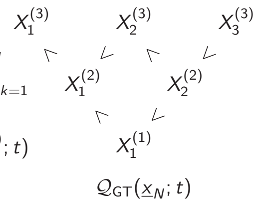
- Warren(2005) found that $\det(\mathcal{F}_{k-j}(x_j;t))$ describes the transition density of the $N$-Brownian particle system where $i+1$th particle is reflected from $i$th and $f_N(t)\sim X_N(t)$.




## Warren's approach

- Warren (2005): The zero-temperature case

$$\mathrm{Prob}\left(f_N(t)\le u\right)$$
$$= \int_{-\infty}^{u}\prod_{\ell=1}^N dx_1^{(\ell)}\cdot\det\left(\mathcal{F}_{k-j}(x_1^{(j)};t)\right)_{j,k=1}^N$$
$$= \int_{-\infty}^{u}\prod_{\ell=1}^N dx_\ell^{(N)}\cdot P_{\mathrm{GUE}}(x_1^{(N)},\cdots,x_N^{(N)};t)$$




- Warren (2005) introduced the relected Brownian motions on the Gelfand-Tsetlin cone whose transition density $\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t)$ is expressed as

$$\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t) = \prod_{1\le i<j\le N}\left(x_i^{(N)}-x_j^{(N)}\right)\cdot\prod_{k=1}^N\frac{\exp\left(-x_k^{(N)2}/2t\right)}{t^{k-1}\sqrt{2\pi t}}\cdot 1_{\mathrm{GT}}(\underline{x}_N).$$

where $1_{\mathrm{GT}}(\underline{x}_k)$ represents the indicator function on GT cone.



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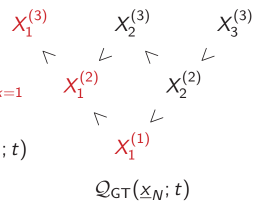


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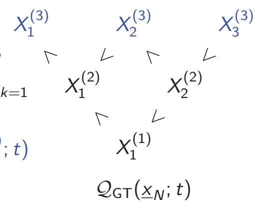


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where $1_{\mathrm{GT}}(\underline{x}_k)$ represents the indicator function on GT cone.



## Warren's approach

- Warren (2005) obtained the following relations

**Proposition (Warren(2005))**

$$\int_{\mathbb{R}^{N(N-1)/2}}dA_1\mathcal{Q}_{GT}(\underline{x}_N;t)=\det\left(\mathcal{F}_{k-j}(x_1^{(j)};t)\right)_{j,k=1}^{N}\prod_{j=1}^{N-1}1_{>0}(x_1^{(j+1)}-x_1^{(j)}),$$

$$\int_{\mathbb{R}^{N(N-1)/2}}dA_2\mathcal{Q}_{GT}(\underline{x}_N;t)=N!P_{\mathrm{GUE}}(x_1^{(N)},\cdots,x_N^{(N)};t)\prod_{j=1}^{N-1}1_{>0}(x_j^{(N)}-x_{j+1}^{(N)}),$$

where the function $1_{>0}(x)$ is the step function and $dA_1$ and $dA_2$ are defined as

$$dA_1=\prod_{2\le i\le j\le N}dx_i^{(j)},\quad dA_2=\prod_{1\le i\le j\le N-1}dx_i^{(j)}.$$

- From this relations, the equality

$$\int_{-\infty}^{u}\prod_{\ell=1}^{N}dx_1^{(\ell)}\cdot\det\left(\mathcal{F}_{k-j}(x_1^{(j)};t)\right)_{j,k=1}^{N}=\int_{-\infty}^{u}\prod_{\ell=1}^{N}dx_\ell^{(N)}\cdot P_{\mathrm{GUE}}(x_1^{(N)},\cdots,x_N^{(N)};t)$$

is established



## Warren's approach

- Warren (2005): The zero-temperature case

$$\mathrm{Prob}\left(f_N(t)\le u\right)$$

$$=\int_{-\infty}^{u}\prod_{\ell=1}^{N}dx_1^{(\ell)}\cdot\det\left(\mathcal{F}_{k-j}(x_1^{(j)};t)\right)_{j,k=1}^{N}$$

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- General $\beta$

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right)=\int_{-\infty}^{\infty}\prod_{\ell=1}^{N}dt_\ell f(t_\ell-u)\cdot\det\left(F_{jk}(t_j;t)\right)_{j,k=1}^{N}$$

$$\stackrel{?}{=}\int_{-\infty}^{\infty}\prod_{j=1}^{N}dx_j f(x_j-u)\cdot W(x_1,\cdots,x_N;t)$$

Is there a similar structure to $\mathcal{Q}_{\mathrm{GT}}(\underline{x}_N;t)$?



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T. Imamura 10th January 2016 28 / 30

## A determinantal weight on $\mathbb{R}^{N(N+1)/2}$

- We define a measure $R_u(\underline{x}_N; t)d\underline{x}_N$ by

$$R_u(\underline{x}_N; t) = \prod_{1\le i\le j\le N} f_i(x_i^{(j)} - x_{i-1}^{(j-1)}) \cdot \det\left(F_{1i}(x_j^{(N)}; t)\right)_{i,j=1}^{N},$$

$$x_0^{(j-1)} = u, \; f_i(x) = \begin{cases} f_F(x) = 1/(e^{\beta x}+1), & i = 1, \text{Fermi} \\ f_B(x) = 1/(e^{\beta x}-1), & i \ge 2, \text{Bose} \end{cases}$$

- **Theorem** (arXiv: 1506.05548)

$$\int_{\mathbb{R}^{\frac{N(N+1)}{2}}} d\underline{x}_N R_u(\underline{x}_N; t)$$
$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_1^{(j)} f_F\left(x_1^{(j)} - u\right) \cdot \det\left(F_{jk}(x_1^j; t)\right)_{j,k=1}^{N}$$
$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j^{(N)} f_F\left(x_j^{(N)} - u\right) \cdot W\left(x_1^{(N)}, \cdots, x_N^{(N)}; t\right),$$



## Summary

- We have considered the O'Connell-Yor random directed polymer

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta(B_1(s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t))} ds_1 \cdots ds_{N-1}$$

- We have obtained the relation

$$\mathbb{E}\left[\exp\left(-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}\right)\right] = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t).$$

To get it, we introduced the determinantal weight on $\mathbb{R}^{\frac{N(N+1)}{2}}$

$$R_u(\underline{x}_N; t) = \prod_{1\le i\le j\le N} f_i(x_i^{(j)} - x_{i-1}^{(j-1)}) \cdot \det\left(F_{1i}(x_j^{(N)}; t)\right)_{i,j=1}^{N}$$

This is a finite-temperature generalization of the pdf of the interacting BMs on GT cone (Warren 2005).

# Search for dead-end CFTs

Yu Nakayama (Kavli IPMU)

**arXiv:1501.02280 + a little bit**

## The rules of the game are as follows:

- We look for conformal field theories (CFTs) without any relevant scalar deformations. We name them dead-end CFTs.
- We do not ask what will happen after introducing relevant deformations (if any).
- We do not impose any continuous global symmetries or discrete global/gauge symmetries.
- We assume that dead-end CFTs are unitary, causal, and have finite energy-momentum tensors.
- Deformations must be physical. In gauge theories, they must be BRST invariant.

Let's play!

## Physical motivations

- Self-organized criticality
- String Landscape
- IR regularization in CFT S-matrix

## Physical motivation

You don't have to tune anything in dead-end CFTs

- Self-organized criticality
  - Non-trivial end points of RG-flows
- Herbertsmithite $ZnCu_3(OH)_6Cl_2$
  - Some quantum spin liquid
  - Actually symmetry protected self-organized criticality
  - Described by conformal QED3 and forbid mass by symmetries

- I'm talking about more strict situations

## Physical motivation

They say string theory predicts $10^{500}$ vacua

- However, what we really want is the universe without any moduli (or light scalar degrees of freedom)
- Suppose our universe has tiny negative CC (should be possible from Landscape Philosophy to solve CC problems)
- Then AdS/CFT says we have CFT without any relevant deformations (after all Higgs is the lightest scalar)
- But are there any known such CFTs?
- As we will discuss, no single such CFT is known in d=3 (or d=4 bulk, i.e. in landscape of our universe)

## Physical motivation

In dead-end CFTs, IR singularity is unavoidable

- Consdider IR regularization in CFT S-matrix
- Recently, there are lot use of concepts of S-matrix in CFTs (Hofman,Maldacena,Zhiboedov, Komargodski…)
  - Conformal collider
  - Proof of a-theorem or scale → conformal invariance
  - Higher spin constraints
- S-matrix has causality, analyticity, unitarity properties that we can use
- Strictly speaking, S-matrix does not exist in CFT, but they assume that, after IR regularization, S-matrix makes sense, and CFT will govern the regularized S-matrix
- Most CFTs seem OK with relevant deformations, but what happens in dead-end CFTs?

## Free dead end CFTs

Can we find any dead-end CFTs?

- Free theory examples in d=4
  - U(1) free Maxwell theory $\int d^4x \frac{1}{e^2} F_{\mu\nu}F^{\mu\nu}$
  - Gauge coupling is exactly marginal (or redundant)
  - No other examples (e.g. Weyl fermions allow Majorana mass)
- Free theory examples in d=3?
  - U(1) free Maxwell theory $\int d^3x \frac{1}{e^2} F_{\mu\nu}F^{\mu\nu}$
    - Scale but not conformal...
  - Chern-Simons theory $\int d^3x k\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}$
    - Trivial local correlators
  - No other examples (e.g. fermions allow real mass)

## Interacting dead end CFTs

Are there any non-trivial examples?

- Yes. From (infinitely) many chiral gauge theories in d=4
- One example: SO(10) gauge theory with $N_f$ (chiral) fermions in spinor (16) rep

$$S = \int d^4x \frac{1}{g^2} \mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + \sum_i^{N_f} \bar{\Psi}_i D \Psi_i$$

- Theta parameter is redundant
- Classically there are no relevant deformations (no mass term in particular)
- Quantum mechanically, gauge coupling runs
  - $N_f \leq 22$ : asymptotic free
  - $N_f \geq N_f^*$ : possibly conformal
  - $N_f < N_f^*$ : possibly confine without mass gap, but we don't know what happens precisely...

## Weakly coupled example 1

Try perturbative analysis of fixed points

- SO(10) gauge theories with $N_f$ chiral fermions in 16 rep
- No difficulty in computing beta function (at least up to 3-loops: c.f. standard model of particle physics)

$$\beta_2 = \frac{g^3}{4\pi^2}\left(-\frac{11}{3}C_g + N_f \frac{3}{2} T_f\right) + \frac{g^5}{4\pi^4}\left(-\frac{34}{3}C_g^2 + N_f\left(\frac{10}{3}C_g + 2C_f\right)T_f\right)$$

- For SO(10) with spinor (16) rep, we have

$$C_g = 8\,, \quad T_f = 2\,, \quad C_f = \frac{45}{8}$$

- We find zero in $10 \leq N_f < 22$
- For $N_f = 21$, the first irrelevant deformation has dimension $\Delta = 4 + \beta'(g)|_{g=g*} = 4.0041002$

## Weakly coupled example 2

To see the validity of perturbations...

- 3-loop beta function can be computed

$$\beta_3 = \frac{g^3}{4\pi^2}\left(-\frac{11}{3}C_g + N_f \frac{3}{2} T_f\right) + \frac{g^5}{4\pi^4}\left(-\frac{34}{3}C_g^2 + N_f\left(\frac{10}{3}C_g + 2C_f\right)T_f\right)$$
$$+ \frac{g^7}{4\pi^6}\left(-\frac{2857}{54}C_g^3 + N_f\left(\frac{1415}{54}C_g^2 + \frac{205}{18}C_g C_f - C_f^2\right)T_f - N_f^2\left(\frac{79}{54}C_g + \frac{11}{9}C_f\right)T_f^2\right)$$

- We find zero in $8 \leq N_f < 22$
- For $N_f = 21$, the first irrelevant deformation has dimension $\Delta = 4 + \beta'(g)|_{g=g*} = 4.00409023$
  - Perturbation theory seems trusted
- Perturbative existence of chiral CFTs without any relevant deformations (dead-end CFT)

## Non-perturbative completion?

But I don't believe in chiral gauge theories!

- Local F-theory construction
- E6 7-branes on Hirzebruch surface F1 with the U(1) flux

$$L = af + b\sigma$$

- Gauge group SO(10) with $n_{16} - n_{\overline{16}} = 3(2a+b)$
- For example, take $a = 4, b = -1$ to get $N_f = 21$
- Lattice construction
- Domain wall fermions + mirror fermion decoupling
  - Kitaev-Wen mechanism: four fermi interactions to gap them
  - Grabowska-Kaplan: Gradient-flow for 5d gauge fields

## 3d example?

So far, so good in d=4 cases, but...

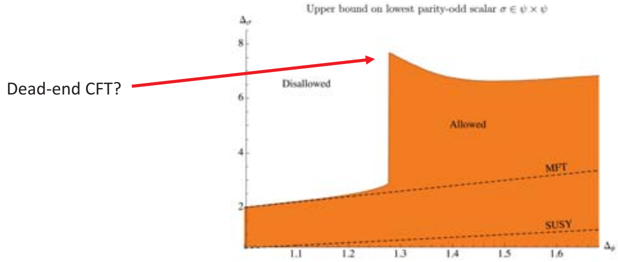
- Possible in d=3?
- This is much more difficult
  - No chiral fermions: we cannot forbid real mass term (fermion must be in real rep in d=3)

$$S = \int d^3x \bar{\psi} D\psi + m\bar{\psi}\psi$$

- Indeed, we can easily argue that there exist no dead-end CFTs within the weakly coupled Lagrangian descriptions in d=3 (except for CS theories)
- Disaster in Landscape dual for our real world??

## Bootstrap example in 3d?

- Non-perturbative search for dead end CFTs in d=3 is urgent (otherwise, string landscape would not make sense)
- There is a promising result from conformal bootstrap with fermions in d=3 (Iliesiuet al)




- At this kink (jump) we have consistent 4pt functions without any relevant scalar operators!

## Symmetry protected Dead-end CFTs

Special thanks to T. Ohtsuki for collaborations on conformal bootstrap

## The rules of the game are as follows:

- We look for conformal field theories (CFTs) without any relevant scalar deformations. We name them dead-end CFTs.
- We do not ask what will happen after introducing relevant deformations (if any).
- ~~We do not impose any continuous global symmetries or discrete global/gauge symmetries.~~
- One may forbid the deformation by global symmetries
- We assume that dead-end CFTs are unitary, causal, and have finite energy-momentum tensors.

Let's play!

## Many examples of SP dead-end CFTs

What about symmetry protected dead-end CFTs?

- Many examples (even vector-like)
- Take Banks-Zaks multi-flavored QCD
- All relevant deformations can be forbidden by chiral symmetries

$$\frac{1}{g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \quad \bar{\Psi}_I\Psi_J \ , \bar{\Psi}_I\gamma_5\Psi_J$$

- In d=3, take multi-flavored QED (c.f. Herbertsmithite)
- All relevant deformations may be forbidden by flavor + CP symmetry (coming from lattice symmetry)

- Relevance to String Landscape is not obvious…

## Necessary conditions for symmetry protected dead-end CFT

- Under which conditions, there is no relevant operators that are singlet under the global symmetries
- Under which conditions, one can impose the global symmetry constraint only from discrete symmetries

= Under which condition, we have an emergent global symmetry out of discrete symmetries

- This question is important in lattice realization

## Was Herbertsmithite symmetry protected dead-end CFT?

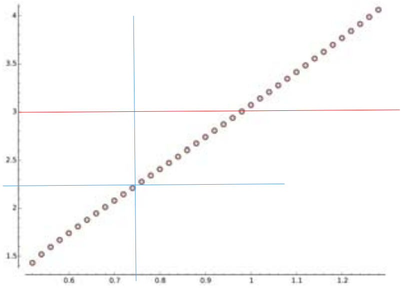
- The model: d=3 QED
- Experimentalists measured(?) dimensions of certain disorder $\Delta_{\bar{\psi}\psi} = 0.75$
- This operator itself can be forbidden by time-reversal symmetry in d=3 (Recall fermion real mass breaks T in d=3)
- The question is if there are any relevant singlet operators?

$$\bar{\psi}\psi \times \bar{\psi}\psi = 1 + (\bar{\psi}\psi)^2 + \cdots$$

- What is the dimension of four-fermi operator?

## Was Herbertsmithite symmetry protected dead-end CFT

- Dimensions of disorder $\Delta_{\bar{\psi}\psi} = 0.75$
- If this number were correct, bootstrap tells symmetry unprotected operator with dimension < 2.3
- It cannot be symmetry protected dead-end CFTs

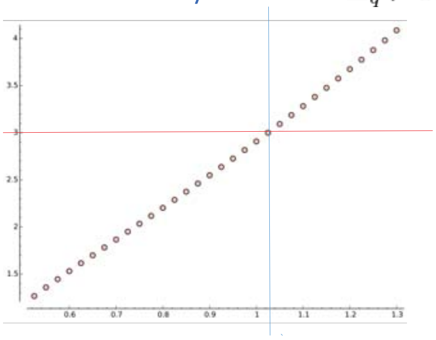



## Necessary conditions for emergent U(1) global symmetries

- Constraint on emergent symmetry in nature
- Suppose we have Z2 symmetry (say strictly realized on lattice), when do we have the IR emergent U(1) global symmetry? (In order to get symmetry protected dead-end CFTs…)
- Applications to controversial physics
- Application 1: On which lattice (square: Z4, honeycomb: Z3, rectangular Z2), does Neel-VBS quantum phase transition show the second order phase transition?
- Application 2: Order of QCD chiral phase transition. Is SU(2)x SU(2) x Z8 effective symmetry (c.f. Aoki et al) enhanced to SU(2) x SU(2) x U(1) in the IR?

## Necessary conditions for emergent U(1) global symmetries

- In order for Z2 symmetry to be enhanced to U(1), charge 2 operator must be irrelevant
- This gives the necessary condition $\Delta_q > 1.05$



## Two applications

- In order for Z2 symmetry to be enhanced to U(1), charge 2 operator must be irrelevant
- This gives the necessary condition $\Delta_q > 1.05$
- In Neel-VBS transition, in simulations by Kawamura et al, it was measured $\Delta_q = 0.865$ for N = 4, so the theory cannot show 2nd order phase transitions on rectangular lattice (unlike some claims that they do)
- In SU(2)xSU(2)x U(1)a symmetric CFT, $\Delta_q = \Delta_{m_{\eta'}} = 0.8$ thus, SU(2) x SU(2) x Z8 cannot be enhanced to SU(2) x SU(2) x U(1)a without fine-tuning (c.f. QCD)

## Summary

- You're very welcome to look for more examples of dead-end CFTs
- Any explicit examples in d=3 is a breakthrough
- It seems trivial to construct such in effective AdS dual. Why does it have to be so hard? Is it related to swampland in the landscape?
- Symmetry protected dead-end CFTs are physically realized
- Many condensed matter papers are inconsistent with conformal bootstrap (Don't trust under 30 lattice size)

# Backups

## How?

Still I want a lattice construction…

- Not completely settled, but some proposals are available
- Starting point is domain wall fermions with left + right 16 fermions at boundaries of 1+4 dim

$$L = \frac{1}{g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{16}D_L\Psi_{16} + \bar{\psi}_{\overline{16}}D_R\psi_{\overline{16}} + \cdots$$

- How to get rid of mirror fermions?
- Kitaev-Wen: add (strong) four-fermi interactions (+suitable UV completion) at boundary (mass without mass term)

$$\delta L = \lambda(\bar{\psi}_{\overline{16}}C_a\psi_{\overline{16}})(\bar{\psi}_{\overline{16}}C_a\psi_{\overline{16}})$$

- Claimed to make mirror fermion massive without breaking SO(10) gauge symmetry

## How?

Still I want another lattice construction…

- Starting point is domain wall fermions with left + right 16 fermions at boundaries of 1+4 dim
- How to get rid of mirror fermions?
- Grabowska-Kaplan: make 5d gauge field satisfy gradient flow. Mirror fermions decouple at high energy

$$\partial_5 A_\nu = \partial^\mu F_{\mu\nu}$$

$$L = \frac{1}{g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{16}D_L\Psi_{16} + \bar{\psi}_{\overline{16}}D_R\psi_{\overline{16}} + \cdots$$

$$D_L = \partial_L + A_L \ , \quad D_R = \partial_R + e^{-\Box}A_R$$

- Zero mode decouples iff anomaly is cancelled, (there remains a subtle topological effect)
- But we need to demonstrate they really work. So far, all the attempts that should work have never worked in chiral lattice gauge theories…

## When?

Supersymmetric dead-end CFTs?

- I don't have any particular motivation, but supersymmetrization of the game may be interesting
  - Mario Brothers → Super Mario Brothers
- Assuming energy-momentum tensor SUSY multiplet is physical
- In d=4, SCA tells N=4 SUSY dead-end CFT does not exist
- This is also true for N=2 SUSY dead-end CFT
- SCA does not tell if N=1 SUSY dead-end CFT exists or does not, but there are no known examples (it would have no flavor symmetries)

## When?

Supersymmetry protected dead-end CFTs?

- Slightly different questions have been studied
- Assuming all the SUSY preserving operators are in physical multiplet
- In d=4, SCA tells all N=4 theories SUSY protected dead-end (so are N=(2,0) in d=6)
- In d=4, SCA tells all N=2 theories without flavor symmetry are SUSY protected dead-end (so are (1,0) in d=6: c.f. yet to be published work by Cordova et al)
- There are some examples with N=1 SUSY protected dead-end CFTs (e.g. chiral SUSY gauge theories)
- However, there may exist deformations whose superconformal multiplets are not physical (e.g. Fayet Illiopolous terms in d=4), so the classification by Cordova et al may not be complete

# Determinantal Interacting Particle Systems

Makoto Katori (Chuo Univ., Tokyo)

International Symposium

**RIKKYO MathPhys 2016**

January 9-11, 2016

Rikkyo University

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## 1. Introduction:

### Interacting Brownian Particles in One-Dimension

**Setting**

$$\boldsymbol{X}(t) = \begin{cases} (X_1(t), X_2(t), \dots, X_N(t)) & N\text{-particle system} \\ (X_1(t), X_2(t), \cdots) & \text{infinite particle system} \end{cases}$$
$$= (X_j)_{j \in \mathbb{I}}, \qquad \mathbb{I} = \text{finite or infinite index set.}$$

$$X_j(t) \in S = \begin{cases} \mathbb{R} \\ \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\} \\ \mathrm{S}^1(r) = [0, 2\pi r). \end{cases} \quad \text{for } j \in \mathbb{I}$$

$$t \in \mathcal{T} = \begin{cases} [0, \infty) & \text{temporally homogeneous} \\ [0, t_*) & \text{temporally inhomogeneous}, 0 < t_* < \infty. \end{cases}$$

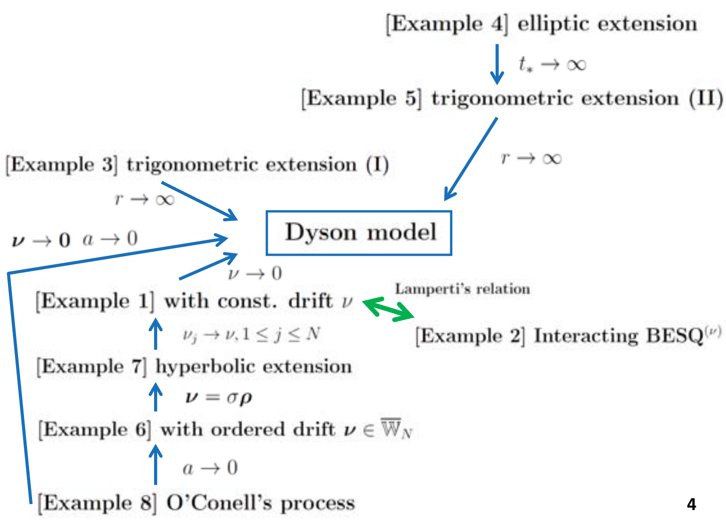
2

**Stochastic Differential Equations (SDEs)**

$$dX_j(t) = \alpha(t, X_j(t)) dB_j(t) - \frac{\partial}{\partial x} \Phi(t, X_j(t)) dt - \frac{1}{2} \frac{\partial}{\partial x_j} \Psi(t, \boldsymbol{X}(t)) dt,$$
$$j \in \mathbb{I}, \quad t \in \mathcal{T}.$$

- $B_j(t), j \in \mathbb{I}$ : independent 1 dim. standard Brownian motions
- $\alpha(t,x)$ : (time-dependent, random) time change
- $\Phi(t,x)$ : single-particle potential
- $\Psi(t, \boldsymbol{x})$ : interaction potential, $\boldsymbol{x} = (x_j)_{j \in \mathbb{I}}$,
  - $\Psi(t, \boldsymbol{x}) = \sum_{k<\ell} \psi(t, x_k, x_\ell)$ : two-body interaction
  - or, **not** given by two-body interaction

3




4

$$dX_j(t) = \alpha(t, X_j(t)) dB_j(t) - \frac{\partial}{\partial x} \Phi(t, X_j(t)) dt - \frac{1}{2} \frac{\partial}{\partial x_j} \Psi(t, \boldsymbol{X}(t)) dt$$

## Examples

[Example 1] Dyson's BM model with constant drift and parameter $\beta > 0$

$$dX_j(t) = dB_j(t) \quad + \quad \frac{\beta}{2} \sum_{\substack{k \in \mathbb{I}, \\ k \neq j}} \frac{1}{X_j(t) - X_k(t)} dt,$$
$$j \in \mathbb{I}, \quad t \in [0, \infty), \quad S = \mathbb{R}.$$

$\alpha(t,x) = 1$,
$\Phi(t,x) = -\nu x, \quad \nu = \text{const.} \in \mathbb{R}$,
$\Psi(t, \boldsymbol{x}) = \sum_{k<\ell} \psi(x_\ell - x_k), \quad \psi(x) = -\beta \ln x.$

(Related Systems) When $\nu = 0$,
- $\beta = 1$ Gaussian orthogonal ensemble (GOE)
- $\beta = 2$ Gaussian unitary ensemble (GUE)
- $\beta = 4$ Gaussian symplectic enemble (GSE)

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$$dX_j(t) = \alpha(t, X_j(t)) dB_j(t) - \frac{\partial}{\partial x} \Phi(t, X_j(t)) dt - \frac{1}{2} \frac{\partial}{\partial x_j} \Psi(t, \boldsymbol{X}(t)) dt$$

## Examples

[Example 1] Dyson's BM model with constant drift and parameter $\beta > 0$

$$dX_j(t) = dB_j(t) + \nu dt + \frac{\beta}{2} \sum_{\substack{k \in \mathbb{I}, \\ k \neq j}} \frac{1}{X_j(t) - X_k(t)} dt,$$
$$j \in \mathbb{I}, \quad t \in [0, \infty), \quad S = \mathbb{R}.$$

$\alpha(t,x) = 1$,
$\Phi(t,x) = -\nu x, \quad \nu = \text{const.} \in \mathbb{R}$,
$\Psi(t, \boldsymbol{x}) = \sum_{k<\ell} \psi(x_\ell - x_k), \quad \psi(x) = -\beta \ln x.$

(Related Systems) When $\nu = 0$,
- $\beta = 1$ Gaussian orthogonal ensemble (GOE)
- $\beta = 2$ Gaussian unitary ensemble (GUE)
- $\beta = 4$ Gaussian symplectic enemble (GSE)

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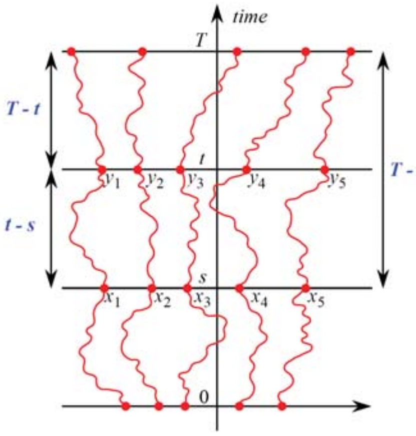
When $\beta = 2$,
Dyson's BM model is realized as the Noncolliding BM.

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# Example 2

[Example 2] **Interacting squared Bessel processes with parameters $\nu > -1, \beta > 0$**

$$dX_j(t) = 2\sqrt{X_j(t)}dB_j(t) + \beta(\nu+1)dt + 2\beta X_j(t) \sum_{\substack{k\in\mathbb{I},\\ k\neq j}} \frac{1}{X_j(t) - X_k(t)}dt,$$

$$j \in \mathbb{I}, \quad t \in [0,\infty), \quad S = \mathbb{R}_+.$$

**(Related Systems)**
Wishart process (Laguerre process)
$\beta = 1, 2, 4, \quad \nu \in \mathbb{N}_0 \equiv \{0, 1, 2, 3, \dots\}$ chiral GOE/GUE/GSE
$\beta = 2, \nu = \frac{1}{2}$ class C
$\beta = 2, \nu = -\frac{1}{2}$ class D
} Altland-Zirnbauer's nonstandard random matrix theory

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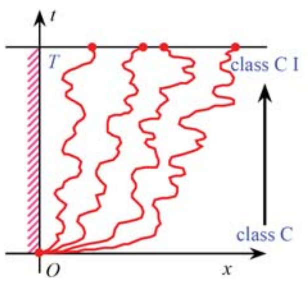
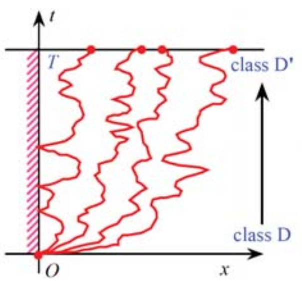
# $\beta$=2 case




Noncolliding particles with an absorbing wall at the origin

Noncolliding particles with a reflecting wall at the origin

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$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$

# Example 3

[Example 3] **trigonometric extension (I) of Dyson's BM model with parameter $\beta > 0$**

$$dX_j(t) = dB_j(t) + \frac{\beta}{4r}\sum_{\substack{k\in\mathbb{I},\\ k\neq j}} \cot\left(\frac{X_j(t) - X_k(t)}{2r}\right)dt,$$

$$j \in \mathbb{I}, \quad t \in [0,\infty), \quad S = \mathrm{S}^1(r) = [0, 2\pi r).$$

$\alpha(t,x) = 1,$
$\Phi(t,x) = 0,$
$\Psi(t,\boldsymbol{x}) = \sum_{k<\ell}\psi(x_\ell - x_k), \quad \psi(x) = -\beta \ln\left[\sin\left(\frac{x}{2r}\right)\right]$

**(Related Systems)**
$\beta = 1, 2, 4$ Gaussian circular ensembles
If $r \to \infty$, **[Example 3]** $\to$ **[Example 1]** with $\nu = 0$ (Dyson's BM model)

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$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$


# Example 4

[Example 4] **elliptic extension of Dyson's BM model with parameter $\beta > 0$**
(K: Probab. Theory Relat. Fields **162** (2015) 637)

$$dX_j(t) = dB_j(t) - \frac{1}{2}\sum_{\substack{1\le k\le N,\\ k\neq j}}\frac{\partial}{\partial x_j}\psi(t, x_j - x_k)dt - \frac{1}{2}\frac{\partial}{\partial x_j}\psi\left(t, \sum_{k=1}^{N} x_k - \kappa_N\right)dt,$$

$$j \in \{1, 2, \dots, N\}, \quad t \in [0, t_*), \quad S = \mathrm{S}^1(r) = [0, 2\pi r).$$

$\alpha(t,x) = 1,$
$\Phi(t,x) = 0,$
$$\Psi(t,\boldsymbol{x}) = \sum_{1\le k<\ell\le N}\psi(t, x_\ell - x_k) + \psi\left(t, \sum_{k=1}^{N} x_k - \kappa_N\right)$$
$$\psi(t,x) = -\beta\ln\left[\vartheta_1\left(\frac{x}{2\pi r}; \frac{iN(t_* - t)}{2\pi r^2}\right)\right],$$

$$\kappa_N = \begin{cases} \pi r(N-1), & \text{if } N \text{ is even} \\ \pi r(N-2), & \text{if } N \text{ is odd,} \end{cases}$$

$$\vartheta_1(v;\tau) = i\sum_{n\in\mathbb{Z}}(-1)^n q^{(n-1/2)^2} z^{2n-1},$$
$$z = e^{\pi i v}, q = e^{\pi i \tau}$$
Jacobi's theta function

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$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$

# Example 4

[Example 4] **elliptic extension of Dyson's BM model with parameter $\beta > 0$**
(K: Probab. Theory Relat. Fields **162** (2015) 637)

$$dX_j(t) = dB_j(t) - \frac{1}{2}\sum_{\substack{1\le k\le N,\\ k\neq j}}\frac{\partial}{\partial x_j}\psi(t, x_j - x_k)dt - \frac{1}{2}\frac{\partial}{\partial x_j}\psi\left(t, \sum_{k=1}^{N} x_k - \kappa_N\right)dt,$$

$$j \in \{1, 2, \dots, N\}, \quad t \in [0, t_*), \quad S = \mathrm{S}^1(r) = [0, 2\pi r).$$


two-body interaction

mean-field-type potential


$\alpha(t,x) = 1,$
$\Phi(t,x) = 0,$
$$\Psi(t,\boldsymbol{x}) = \sum_{1\le k<\ell\le N}\psi(t, x_\ell - x_k) + \psi\left(t, \sum_{k=1}^{N} x_k - \kappa_N\right)$$
$$\psi(t,x) = -\beta\ln\left[\vartheta_1\left(\frac{x}{2\pi r}; \frac{iN(t_* - t)}{2\pi r^2}\right)\right],$$

$$\kappa_N = \begin{cases} \pi r(N-1), & \text{if } N \text{ is even} \\ \pi r(N-2), & \text{if } N \text{ is odd,} \end{cases}$$

$$\vartheta_1(v;\tau) = i\sum_{n\in\mathbb{Z}}(-1)^n q^{(n-1/2)^2} z^{2n-1},$$
$$z = e^{\pi i v}, q = e^{\pi i \tau}$$
Jacobi's theta function


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$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$

## Example 5

[Example 5] **trigonometric extension (II) of Dyson's BM model with parameter** $\beta > 0$

$t_* \to \infty$ (temporally homogeneous limit) of the elliptic Dyson model

$$dX_j(t) = dB_j(t) + \frac{\beta}{4r}\sum_{\substack{k \in \mathbb{I}, \\ k \neq j}} \cot\left(\frac{X_j(t) - X_k(t)}{2r}\right) dt - \frac{\beta}{4r}\tan\left(\frac{1}{2r}\sum_{k=1}^{N} X_k(t)\right) dt,$$

$$j \in \mathbb{I}, \quad t \in [0, \infty), \quad S = \mathrm{S}^1(r) = [0, 2\pi r).$$

$$\alpha(t,x) = 1, \quad \Phi(t,x) = 0,$$

$$\Psi(t, \boldsymbol{x}) = \sum_{k<\ell} \psi^{(2)}(x_\ell - x_k) + \psi^{(\mathrm{mf})}\left(\sum_{k=1}^{N} x_k\right),$$

$$\psi^{(2)}(x) = -\beta \ln\left[\sin\left(\frac{x}{2r}\right)\right], \quad \psi^{(\mathrm{mf})}(x) = -\beta \ln\left[\cos\left(\frac{x}{2r}\right)\right]$$

(Related Systems)

If we take the further limit $r \to \infty$,

[Example 5] $\to$ [Example 1] with $\nu = 0$ (Dyson's BM model)

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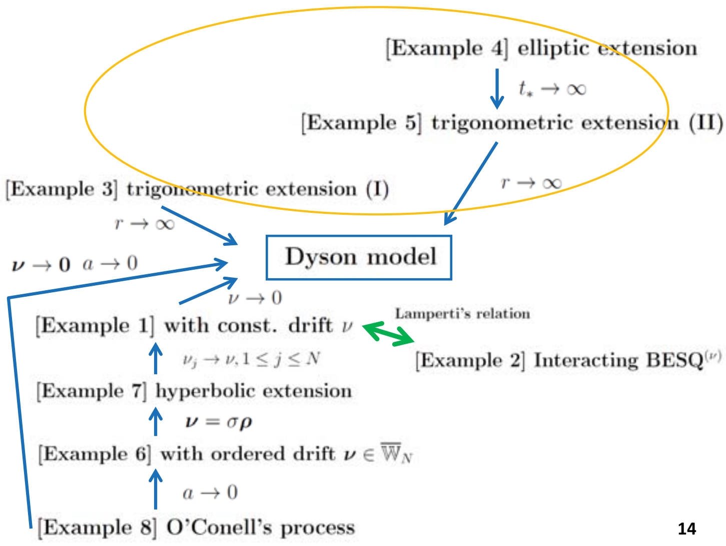


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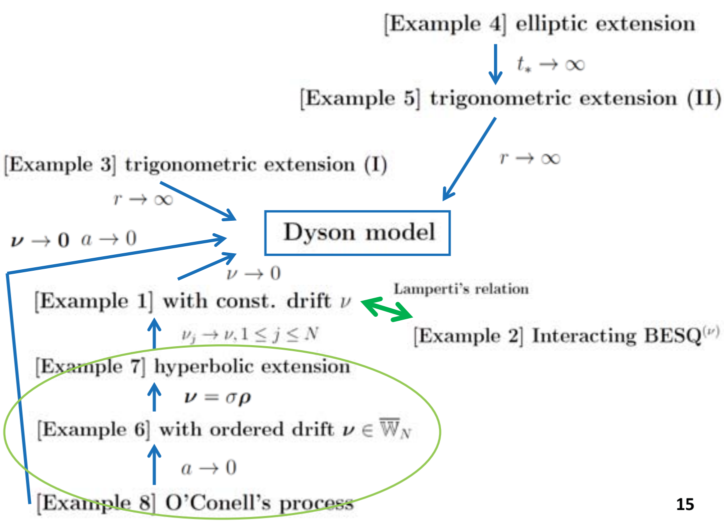


15

$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$

## Example 6

[Example 6] **Noncolliding BM with ordered drift**

$\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_N) \in \overline{\mathbb{W}}_N \equiv \{\boldsymbol{\nu} \in \mathbb{R}^N : \nu_1 \leq \nu_2 \leq \cdots \leq \nu_N\}$

$$dX_j(t) = dB_j(t) - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(\boldsymbol{X}(t))$$

$$j \in \{1, 2, \dots, N\}, \quad t \in [0, \infty), \quad S = \mathbb{R}.$$

$$\alpha(t,x) = 1,$$

$$\Phi(t,x) = 0,$$

$$\Psi(t, \boldsymbol{x}) = -2\ln\left(\det_{1 \leq k, \ell \leq N}[e^{\nu_k x_\ell}]\right).$$

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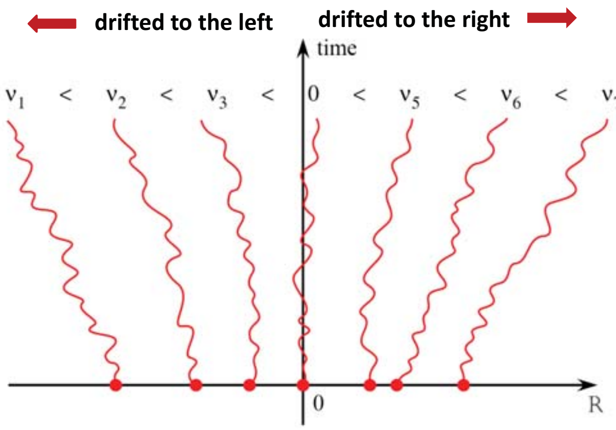
### Noncolliding BM with ordered drift-coefficients

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$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$

## Example 7

[Example 7] **hyperbolic extension of Dyson's BM model**

A special case of **[Example 6]** with $\nu_j = \sigma\rho_j \equiv \sigma\left(j - \frac{N+1}{2}\right), j \in \{1, 2, \dots, N\}$

$$dX_j(t) = dB_j(t) + \frac{\sigma}{2}\sum_{\substack{k \in \mathbb{I}, \\ k \neq j}} \coth\left[\frac{\sigma}{2}(X_j(t) - X_k(t))\right] dt,$$

$$j \in \{1, 2, \dots, N\}, \quad t \in [0, \infty), \quad S = \mathbb{R}.$$

$$\alpha(t,x) = 1, \quad \Phi(t,x) = 0,$$

$$\Psi(t, \boldsymbol{x}) = \sum_{k<\ell} \psi(x_\ell - x_k), \quad \psi(x) = -2\ln\left[\sinh\left(\frac{\sigma x}{2}\right)\right]$$

(Related Systems)

Chern-Simon theory $(\mathrm{S}^3, \mathrm{U}(N))$

Stieltjes-Wigert determinantal point process (log-normal potential)

(see Takahashi-K: J. Math. Phys. **53** (2012) 103305, **55** (2014) 093302)

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$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt$$

## Example 8

[Example 8] O'Connell's process (the Whittaker process) with parameters $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_N) \in \overline{\mathbb{W}}_N \equiv \{\boldsymbol{\nu} \in \mathbb{R}^N : \nu_1 \le \nu_2 \le \cdots \le \nu_N\}$

$$dX_j(t) = dB_j(t) - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(\boldsymbol{X}(t))$$

$$j \in \{1, 2, \dots, N\}, \quad t \in [0, \infty), \quad S = \mathbb{R}.$$

$\alpha(t,x) = 1, \quad \Phi(t,x) = 0,$

$\Psi(t, \boldsymbol{x}) = -2 \ln \psi_{\nu}^{(N)}(\boldsymbol{x}/a) \quad$ with $a > 0$,

$$\psi_{\nu}^{(N)}(\boldsymbol{x}) = \int_{\Gamma_N(\boldsymbol{x})} \exp\left[\sum_{j=1}^{N} \nu_j \left(\sum_{k=1}^{j} T_{j,k} - \sum_{k=1}^{j-1} T_{j-1,k}\right) - \sum_{j=1}^{N}\sum_{k=1}^{j}\{e^{-(T_{j,k}-T_{j+1,k})} + e^{-(T_{j+1,k+1}-T_{j,k})}\}\right] d\boldsymbol{T},$$

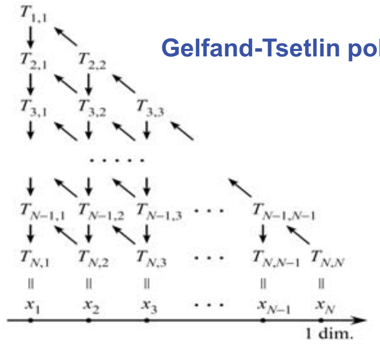
where the integral is performed over the space of all real lower triangular array with size $N$,

$$\boldsymbol{T} = (T_{j,k} : 1 \le k \le j \le N) \quad \text{conditioned } T_{N,k} = x_k, 1 \le k \le N.$$

(Givental's integral representation for the class-one Whittaker function)

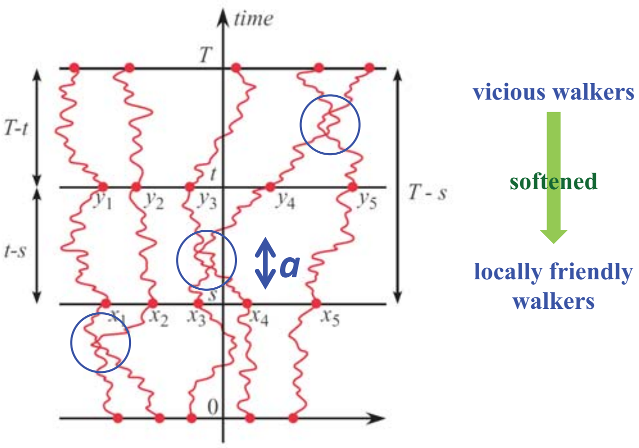


$$\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) = \int_{\Gamma_N(\boldsymbol{x})} \exp\left[\sum_{k=1}^{N} \nu_k \left(\sum_{j=1}^{k} T_{k,j} - \sum_{j=1}^{k-1} T_{k-1,j}\right) - \sum_{k=1}^{N-1}\sum_{j=1}^{k}\left\{e^{-(T_{k,j}-T_{k+1,j})} + e^{-(T_{k+1,j+1}-T_{k,j})}\right\}\right] d\boldsymbol{T}.$$




Gelfand-Tsetlin polytope



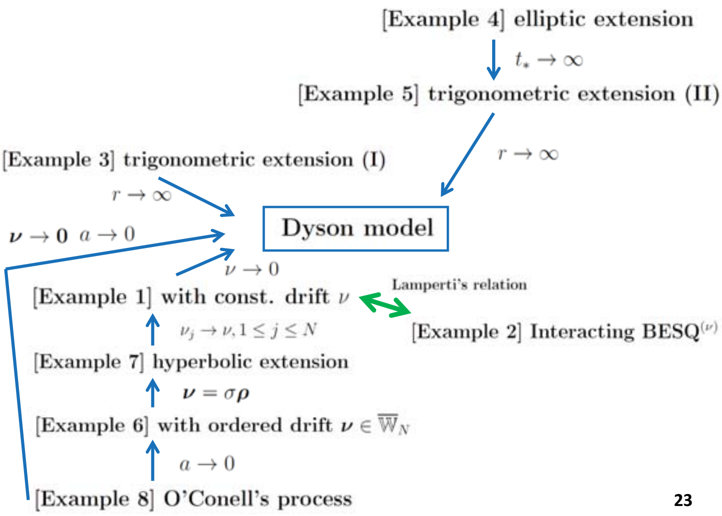





(Related Systems)

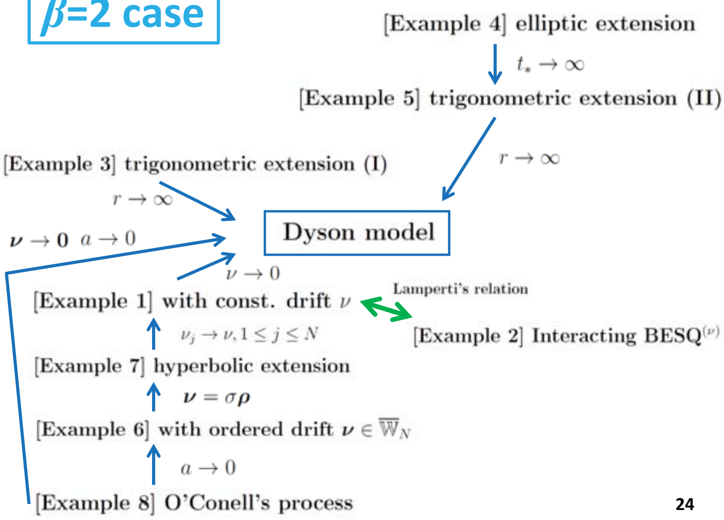
- [Example 6] (Noncolliding BM with $\boldsymbol{\nu} \in \overline{\mathbb{W}}_N$) is a 'tropical analogue' of [Example 8] in the sense that
$$\lim_{a \to 0} a^{N(N-1)/2}\psi_{a\nu}^{(N)}(\boldsymbol{x}/a) = \frac{\det_{1 \le j,k \le N}[e^{\nu_j x_k}]}{\prod_{1 \le j < k \le N}(x_k - x_j)}.$$
- O'Connell's process → $q$-Whittaker process → Macdonald process (Borodin-Corwin)
- $q$-Whittaker process is related to the Sasamoto-Spohn solution of the 1D KPZ equation.

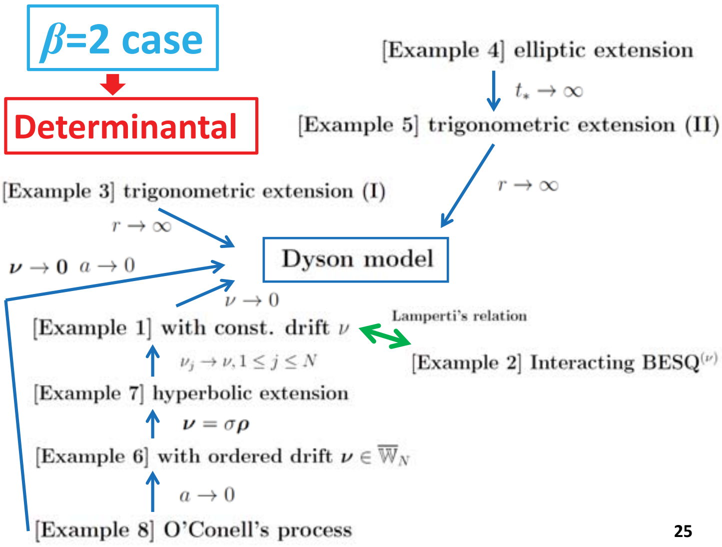







## $\beta$=2 case



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## 2. Preliminaries

- **unlabeled configurations**

SDEs

$$dX_j(t) = \alpha(t, X_j(t))dB_j(t) - \frac{\partial}{\partial x}\Phi(t, X_j(t))dt - \frac{1}{2}\frac{\partial}{\partial x_j}\Psi(t, \boldsymbol{X}(t))dt,$$
$$j \in \mathbb{I}, \quad t \in \mathcal{T}$$

labeled configuration $\boldsymbol{X}(t) = (X_j(t))_{j \in \mathbb{I}}$

unlabeled configuration $\Xi(t, \cdot) = \sum_{j \in \mathbb{I}} \delta_{X_j(t)}(\cdot), \quad t \in \mathcal{T}$

$\Xi(t, \cdot) \in \mathfrak{M}$ : the space of nonnegative integer-valued Radon measures
$\delta_x(\cdot)$ : point mass (delta measure) on $x$
For any compact subset $K \subset S$,

$$\Xi(t, K) \equiv \int_K \Xi(t, dx) = \int_K \sum_{j \in \mathbb{I}} \delta_{X_j(t)}(dx) = \sharp\{j : X_j(t) \in K\}.$$

(the total number of particles in $K$ at time $t$)

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- **probability measure** $\mathbb{P}_\xi$ with the initial configuration $\xi(\cdot) = \Xi(0, \cdot)$
**expectation** $\mathbb{E}_\xi$
- The process is denoted by $(\Xi, \mathbb{P}_\xi)$.
- **spatio-temporal correlation function** $\rho_\xi$

$\mathrm{C_c}$ = the set of all continuous real-valued functions with compact supports on $S$
For an arbitrary number $M \in \mathbb{N} \equiv \{1, 2, 3, \dots\}$,
and for an arbitrary sequence of times $0 \le t_1 < t_2 < \cdots < t_M < \infty$,
$\chi_{t_m} \in \mathrm{C_c}, 1 \le m \le M$ : test functions
$N_m \in \mathbb{N}$ : the number of particles observed at time $t_m, 1 \le m \le M$

symmetrization : $g_{t_m}\left(\sum_{j \in \mathbb{I}} \delta_{x_j}\right) = \sum_{j_1 < j_2 < \cdots < j_{N_m}} \prod_{\ell=1}^{N_m} \chi_{t_m}(x_{j_\ell})$

$$\mathbb{E}_\xi\left[\prod_{m=1}^{M} g_{t_m}(\Xi(t_m))\right] = \prod_{m=1}^{M} \frac{1}{N_m!} \int_{S^{N_m}} \prod_{j=1}^{N_m} \left\{dx_j^{(m)} \chi_{t_m}(x_j^{(m)})\right\} \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; t_2, \boldsymbol{x}_{N_2}^{(2)}; \cdots ; t_M, \boldsymbol{x}_{N_M}^{(M)}\right),$$
$$\boldsymbol{x}_{N_m}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \dots, x_{N_m}^{(m)}), \quad 1 \le m \le M.$$

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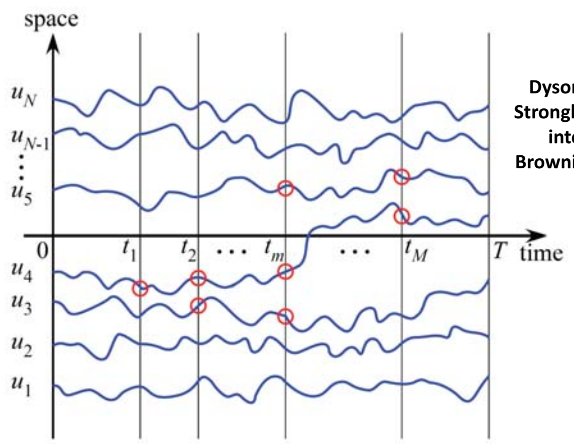


**Dyson's model: Strongly repulsive-interacting Brownian motions**

**We observe particles at points ◯ on the spatio-temporal plane.**

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- **generating function of** $\rho$**'s**

For $M \in \mathbb{N}, 0 \le t_1 < t_2 < \cdots < t_M < \infty, \chi_{t_m} \in \mathrm{C_c}, 1 \le m \le M$

$$G_\xi[\chi_{t_1}, \cdots \chi_{t_M}] \equiv \sum_{\substack{N_m \ge 0, \\ 1 \le m \le M}} \prod_{m=1}^{M} \frac{1}{N_m!} \int_{S^{N_m}} \prod_{j=1}^{N_m} \left\{dx_j^{(m)} \chi_{t_m}(x_j^{(m)})\right\} \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \cdots ; t_M, \boldsymbol{x}_{N_M}^{(M)}\right)$$
$$= \mathbb{E}_\xi\left[\exp\left\{\sum_{m=1}^{M} \int_S f_{t_m}(x) \Xi(t_m, dx)\right\}\right],$$
$$\chi_{t_m}(\cdot) = e^{f_{t_m}(\cdot)} - 1, \quad 1 \le m \le M.$$

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- **generating function of** $\rho$**'s**

For $M \in \mathbb{N}, 0 \le t_1 < t_2 < \cdots < t_M < \infty, \chi_{t_m} \in \mathrm{C_c}, 1 \le m \le M$

$$G_\xi[\chi_{t_1}, \cdots \chi_{t_M}] \equiv \sum_{\substack{N_m \ge 0, \\ 1 \le m \le M}} \prod_{m=1}^{M} \frac{1}{N_m!} \int_{S^{N_m}} \prod_{j=1}^{N_m} \left\{dx_j^{(m)} \chi_{t_m}(x_j^{(m)})\right\} \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \cdots ; t_M, \boldsymbol{x}_{N_M}^{(M)}\right)$$
$$= \mathbb{E}_\xi\left[\exp\left\{\sum_{m=1}^{M} \int_S f_{t_m}(x) \Xi(t_m, dx)\right\}\right],$$
$$\chi_{t_m}(\cdot) = e^{f_{t_m}(\cdot)} - 1, \quad 1 \le m \le M.$$

This is a Laplace transform of $\mathbb{P}_\xi$ (multi-time joint probability distribution).

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- **definition of (space-time) Fredholm determinant**
  Given an integral kernel $\mathbf{K}(s, x; t, y), (s, x), (t, y) \in \mathcal{T} \times S$, and given $M \in \mathbb{N}, 0 \le t_1 < t_2 < \cdots < t_M < \infty$,

$$\mathop{\mathrm{Det}}_{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2,\\ (x,y)\in S^2}} \Big[\delta_{st}\delta_x(\{y\}) + \mathbf{K}(s, x; t, y)\chi_t(y)\Big]$$
$$= \sum_{\substack{N_m \ge 0,\\ 1\le m\le M}} \prod_{m=1}^{M} \frac{1}{N_m!} \int_{S^{N_m}} \prod_{j=1}^{N_m} \Big\{dx_j^{(m)}\chi_{t_m}(x_j^{(m)})\Big\} \det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \Big[\mathbf{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\Big].$$



- **definition of determinantal process**
  If any generating function of $\rho_\xi$ is given by a Fredholm determinant, the process $(\Xi, \mathbb{P}_\xi)$ is said to be **determinantal**.
- In this case, all spatio-temporal correlation functions are given by determinants as

$$\rho_\xi\Big(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\Big) = \det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \Big[\mathbb{K}_\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})\Big],$$

  $0 \le t_1 < \cdots < t_M < \infty, 1 \le N_m \le N, \boldsymbol{x}_{N_m}^{(m)} \in S^{N_m}, 1 \le m \le M \in \mathbb{N}.$
- The RHS is a determinant of the matrix with size $\sum_{m=1}^{M} N_m$
- Here the integral kernel, $\mathbb{K}_\xi : (\mathcal{T} \times S)^2 \mapsto \mathbb{R}$, is a function of initial configuration $\xi$ and is called the **(spatio-temporal) correlation kernel**.



- For any single time $t \in \mathcal{T}$, the spatial correlation functions are given in the form,

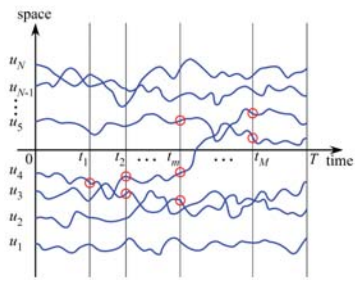
$$\rho_\xi^t(\boldsymbol{x}_{N'}) = \det_{1\le j,k\le N'} \Big[\mathbb{K}_\xi^t(x_j, x_k)\Big], \quad \text{for any } 1 \le N' < N \text{ (or } 1 \le N' < \infty),$$

  where
$$\mathbb{K}_\xi^t(x_j, x_k) = \mathbb{K}_\xi(t, x_j; t, x_k).$$

  Such an statistical ensemble of points on $S$ is called the **determinantal point process** (or **fermion point proccess**).
- In this sense, the notion of **determinantal process** is a **dynamical extension of determinantal point process**.



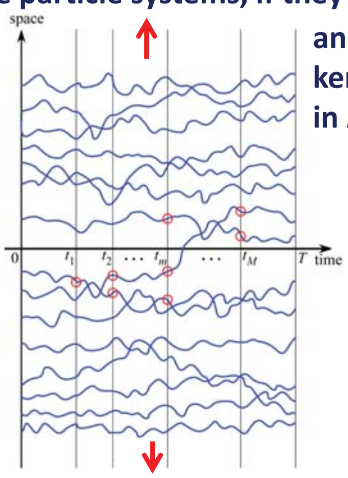
## Extension to Infinite Particle Systems




***N* increases**



**Correlation functions can be defined also for infinite particle systems, if they are determinantal,**




**and if the correlation kernel has a limit in *N* → infinity.**

***N* increases to infinity**



## 3. Family of Determinantal Processes Specified by Martingale Functions

The process $(\Xi, \mathbb{P}_\xi)$ is determinantal with the correlation kernel $\mathbb{K}_\xi$.

$^\exists(Y(t))_{t\in\mathcal{T}}$ : 1-dim. diffusion process with transition probability density
$$p(t, y|s, x), \quad 0 < s < t, \quad s, t \in \mathcal{T}, \quad x, y \in S$$

$^\exists$ a set of **martingale functions** depending on the initial configuration $\xi$
$$\Big\{\mathcal{M}_\xi^v(s, x|t, y) : \xi_*(\{v\}) \ne 0, (s, x), (t, y) \in \mathcal{T} \times S\Big\},$$
where $\xi_* \equiv \operatorname{supp} \xi$.

The process $(\Xi, \mathbb{P}_\xi)$ is determinantal with the correlation kernel $\mathbb{K}_\xi$.

$^\exists (Y(t))_{t \in \mathcal{T}}$ : 1-dim. diffusion process with transition probability density
$p(t, y|s, x), \quad 0 < s < t, \quad s, t \in \mathcal{T}, \quad x, y \in S$

$^\exists$ a set of **martingale functions** depending on the initial configuration $\xi$
$$\left\{ \mathcal{M}_\xi^v(s, x|t, y) : \xi_*(\{v\}) \neq 0, (s, x), (t, y) \in \mathcal{T} \times S \right\},$$
where $\xi_* \equiv \mathrm{supp}\, \xi$.

$$\mathbb{K}_\xi(s, x; t, y) \; = \; \int_S \xi_*(dv) p(s, x|0, v) \mathcal{M}_\xi^v(s, x|t, y) - \mathbf{1}_{(s>t)} p(s, x|t, y),$$
$$\mathbf{1}_{(\omega)} = \begin{cases} 1, & \text{if } \omega \text{ is satisfied}, \\ 0, & \text{otherwise}. \end{cases}$$

If $\xi_* \equiv \mathrm{supp}\, \xi = \xi$ (that is, there is no multiple point in $\xi$), then
$$\mathcal{M}_\xi^v(s, x|t, y) = \mathcal{M}_\xi^v(t, y), \quad \xi(\{v\}) \neq 0.$$

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$(Y(t))_{t \in \mathcal{T}}$

**natural filtration** generated by $Y$, $\mathcal{F}_t \equiv \sigma\Big(Y(s) : s \leq t\Big), t \in \mathcal{T}$.
(the smallest $\sigma$-field generated by $Y(s), s \leq t$)

**definition of martingale functions** $\{\mathcal{M}_\xi^v\}_v$
For any $(s, x) \in \mathcal{T} \times S$, $\mathcal{M}_\xi^v(s, x|t, Y(t)), v \in \xi_*$ are continuous-martingales,
$$\mathrm{E}\Big[\mathcal{M}_\xi^v(s, x|t, Y(t)) \Big| \mathcal{F}_u\Big] = \mathcal{M}_\xi^v(s, x|u, Y(u)) \quad \text{a.s.} \quad \text{for all } 0 \leq u \leq t,$$
where $\mathrm{E}[\cdot]$ denotes the expectation with respect to $Y(t), t \in \mathcal{T}$.

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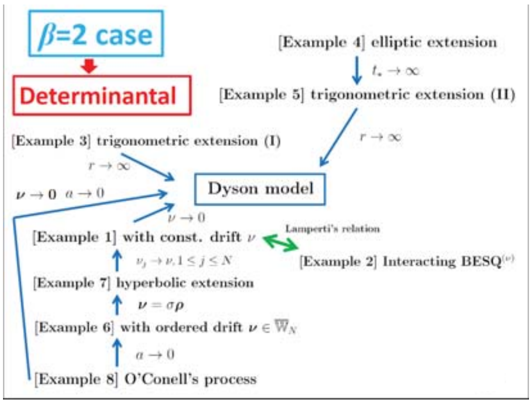


**Claim: Determinantal processes realized in the $\beta$=2 case in the above Examples are all in this family.**

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# Martingale is a betting strategy

① **You bet \$100, the banker bets \$100.** → **You win a bet, you get \$200.**

You lose a bet. You continue the game.

**You stop the game, then your gain is −\$100+\$200=\$100**

② **You double your bet. You bet \$200, the banker bets \$200.** → **You win a bet, you get \$400.**

You lose a bet. You continue the game.

**You stop the game, then your gain is −\$(100+200)+\$400=\$100**

③ **You double your bet again. You bet \$400, the banker bets \$400.**

**You stop the game when you win: Your gain is always+\$100**

- Let $B(t), t \geq 0$ be a **one-dimensional standard Brownian motion (BM)**. Its expectation and conditional expectation are denoted by $\mathrm{E}[\cdots]$ and $\mathrm{E}[\cdots|C]$.
- The **filtration** is the smallest $\sigma$-field (the collection of events which is closed with respect to 'summation' $\cup$) generated by the BM up to time $t$,
$$\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t).$$
- If a process $f(t, B(t)), t \geq 0$ satisfies the following, it is called the continuous-time **martingale**;
$$\mathrm{E}[B(t)|\mathcal{F}_s] = B(s) \quad \text{a.s.} \quad \text{for all } 0 \leq s \leq t.$$
- Martingale is the property of stochastic processes such that **the expectation is conserved in time**.
- Martingale is a stochastic process representing a **fluctuation**.

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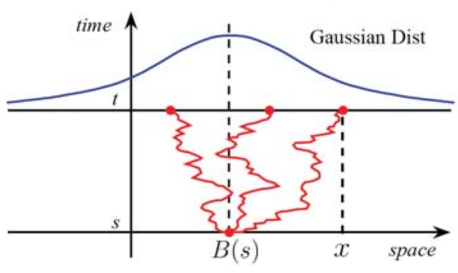
BM is a martingale;
$$\mathrm{E}[B(t)|\mathcal{F}_s] = B(s) \quad \text{a.s.} \quad \text{for all } 0 \leq s \leq t.$$

*Proof.* Let $p(t, y|x)$ be the transition probability density of BM.
$$\mathrm{E}[B(t)|\mathcal{F}_s] \; = \; \int_{-\infty}^{\infty} x p(t-s, x|B(s)) dx$$
$$= \; \int_{-\infty}^{\infty} x \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}} dx = B(s).$$




42

$B(t)$ is a martingale, but $B(t)^2$ is not.
For $0 \le s < t$,

$$\begin{aligned}
\mathrm{E}[B(t)^2|\mathcal{F}_s] &= \int_{-\infty}^{\infty} x^2 p(t-s, x|B(s))dx \\
&= \int_{-\infty}^{\infty} x^2 \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dx \\
&= \int_{-\infty}^{\infty} \{(x-B(s))^2 + 2B(s)(x-B(s)) + B(s)^2\} \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dx \\
&= (t-s) + 0 + B(s)^2 = B(s)^2 + (t-s) \neq B(s)^2.
\end{aligned}$$

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$B(t)$ is a martingale, but $B(t)^2$ is not.
For $0 \le s < t$,

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\mathrm{E}[B(t)^2|\mathcal{F}_s] &= \int_{-\infty}^{\infty} x^2 p(t-s, x|B(s))dx \\
&= \int_{-\infty}^{\infty} x^2 \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dx \\
&= \int_{-\infty}^{\infty} \{(x-B(s))^2 + 2B(s)(x-B(s)) + B(s)^2\} \frac{e^{-(x-B(s))^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dx \\
&= (t-s) + 0 + B(s)^2 = B(s)^2 + (t-s) \neq B(s)^2.
\end{aligned}$$

$$\mathrm{E}[B(t)^2 - t|\mathcal{F}_s] = B(s)^2 - s \quad \text{a.s.} \quad \text{for all } 0 \le s \le t.$$

$m_2(t, B(t)) \equiv B(t)^2 - t$ is a martingale.

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- Let $B(t)$ and $\widetilde{B}(t)$ are independent BMs. The **complex BM** is defined by

$$Z(t) \equiv B(t) + i\widetilde{B}(t), \quad i = \sqrt{-1}, \quad t \ge 0.$$

Let $\mathbf{E}[\cdots] \equiv \mathrm{E}[\widetilde{\mathrm{E}}[\cdots]]$.

- $Z(t)$ is a martingale, since both of the real and imaginary parts are martingales;

$$\begin{aligned}
\mathbf{E}[Z(t)|\mathcal{F}_s] &= \mathbf{E}[B(t) + i\widetilde{B}(t)|\mathcal{F}_s] = \mathrm{E}[B(t)|\mathcal{F}_s] + i\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] = B(s) + i\widetilde{B}(s) \\
&= Z(s) \quad \text{a.s.} \quad \text{for all } 0 \le s \le t.
\end{aligned}$$

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Consider $Z(t)^2 = (B(t) + i\widetilde{B}(t))^2 = B(t)^2 + 2iB(t)\widetilde{B}(t) - \widetilde{B}(t)^2$.

$$\begin{aligned}
\mathbf{E}[Z(t)^2|\mathcal{F}_s] &= \mathrm{E}[B(t)^2|\mathcal{F}_s] + 2i\mathrm{E}[B(t)|\mathcal{F}_s]\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] - \widetilde{\mathrm{E}}[\widetilde{B}(t)^2|\mathcal{F}_s] \\
&= \mathrm{E}[(B(t)^2 - t)|\mathcal{F}_s] + 2i\mathrm{E}[B(t)|\mathcal{F}_s]\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] - \widetilde{\mathrm{E}}[(\widetilde{B}(t)^2 - t)|\mathcal{F}_s] \\
&= (B(s)^2 - s) + 2iB(s)\widetilde{B}(s) - (\widetilde{B}(s)^2 - s) \\
&= B(s)^2 + 2iB(s)\widetilde{B}(s) - \widetilde{B}(s)^2 = Z(s)^2.
\end{aligned}$$

$Z(t)^2$ is also a martingale.

Note that, if $\widetilde{B}(t)$ starts at $0 \Rightarrow \widetilde{\mathrm{E}}[\widetilde{B}(t)] = 0, \widetilde{\mathrm{E}}[\widetilde{B}(t)^2] = t$,

$$\begin{aligned}
\widetilde{\mathrm{E}}[Z(t)^2] &= \widetilde{\mathrm{E}}[B(t)^2 + 2iB(t)\widetilde{B}(t) - \widetilde{B}(t)^2] \\
&= B(t)^2 + 2iB(t)\widetilde{\mathrm{E}}[\widetilde{B}(t)] - \widetilde{\mathrm{E}}[\widetilde{B}(t)^2] \\
&= B(t)^2 + 0 - t \\
&= B(t)^2 - t = m_2(t, B(t)).
\end{aligned}$$

46

Consider $Z(t)^2 = (B(t) + i\widetilde{B}(t))^2 = B(t)^2 + 2iB(t)\widetilde{B}(t) - \widetilde{B}(t)^2$.

$$\begin{aligned}
\mathbf{E}[Z(t)^2|\mathcal{F}_s] &= \mathrm{E}[B(t)^2|\mathcal{F}_s] + 2i\mathrm{E}[B(t)|\mathcal{F}_s]\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] - \widetilde{\mathrm{E}}[\widetilde{B}(t)^2|\mathcal{F}_s] \\
&= \mathrm{E}[(B(t)^2 - t)|\mathcal{F}_s] + 2i\mathrm{E}[B(t)|\mathcal{F}_s]\widetilde{\mathrm{E}}[\widetilde{B}(t)|\mathcal{F}_s] - \widetilde{\mathrm{E}}[(\widetilde{B}(t)^2 - t)|\mathcal{F}_s] \\
&= (B(s)^2 - s) + 2iB(s)\widetilde{B}(s) - (\widetilde{B}(s)^2 - s) \\
&= B(s)^2 + 2iB(s)\widetilde{B}(s) - \widetilde{B}(s)^2 = Z(s)^2.
\end{aligned}$$

$Z(t)^2$ is also a martingale.

Note that, if $\widetilde{B}(t)$ starts at $0 \Rightarrow \widetilde{\mathrm{E}}[\widetilde{B}(t)] = 0, \widetilde{\mathrm{E}}[\widetilde{B}(t)^2] = t$,

$$\begin{aligned}
\widetilde{\mathrm{E}}[Z(t)^2] &= \widetilde{\mathrm{E}}[B(t)^2 + 2iB(t)\widetilde{B}(t) - \widetilde{B}(t)^2] \\
&= B(t)^2 + 2iB(t)\widetilde{\mathrm{E}}[\widetilde{B}(t)] - \widetilde{\mathrm{E}}[\widetilde{B}(t)^2] \\
&= B(t)^2 + 0 - t \\
&= B(t)^2 - t = m_2(t, B(t)).
\end{aligned}$$

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- $B(t)^3$ is not a martingale, but

$$m_3(t, B(t)) \equiv B(t)^3 - 3tB(t) = \widetilde{\mathrm{E}}[Z(t)^3]$$

is a martingale.

- For $n \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$

$$m_n(t, B(t)) \equiv \left(\frac{t}{2}\right)^{n/2} H_n\left(\frac{B(t)}{\sqrt{2t}}\right) = \widetilde{\mathrm{E}}[Z(t)^n]$$

are martingales, where

$$H_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{j!(n-2j)!}(2x)^{n-2j}, \quad n \in \mathbb{N}_0, \quad \textbf{Hermite polynomials}.$$

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## Integral representation of `martingale-polynomials'

$$m_n(t, B(t)) = \widetilde{\mathrm{E}}[Z(t)^n]$$

$$\begin{aligned}
m_n(t,x) &= \widetilde{\mathrm{E}}[(x+i\widetilde{B}(t))^n] \\
&= \int_{-\infty}^{\infty} (x+iy)^n p(t,y|0)dy \\
&= \int_{-\infty}^{\infty} (x+iy)^n \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} dy \\
(x+iy = iw \ &\Leftrightarrow \ y = w + ix) \\
&= \int_{-\infty-ix}^{\infty-ix} (iw)^n \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}} dw \\
&= \int_{-\infty}^{\infty} (iw)^n \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}} dw \\
&\equiv \mathcal{I}[W^n|(t,x)].
\end{aligned}$$

$$\mathcal{I}[f(W)|(t,x)] \equiv \int_{-\infty}^{\infty} f(iw) q(t,w|x) dw$$

$$\text{with} \quad q(t,w|x) = \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}}.$$

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$$\mathcal{I}[f(W)|(t,x)] \equiv \int_{-\infty}^{\infty} f(iw) q(t,w|x) dw$$

$$\text{with} \quad q(t,w|x) = \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}}.$$

(i) $\mathcal{I}[W^n|(t,x)] = m_n(t,x), n \in \mathbb{N}_0$.

(ii) If $f$ is a polynomial function, then $\mathcal{I}[f(W)|(t,B(t))]$ is a martingale.

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$$\mathcal{I}[f(W)|(t,x)] \equiv \int_{-\infty}^{\infty} f(iw) q(t,w|x) dw$$

$$\text{with} \quad q(t,w|x) = \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}}.$$

(i) $\mathcal{I}[W^n|(t,x)] = m_n(t,x), n \in \mathbb{N}_0$.

(ii) If $f$ is a polynomial function, then $\mathcal{I}[f(W)|(t,B(t))]$ is a martingale.

⬇

(iii) If $f$ is an **entire function**, then $\mathcal{I}[f(W)|(t,B(t))]$ is a martingale.

A Consequence of **Conformal Invariance of Complex BM**

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## 4. Examples

**[Example 1] Dyson's BM model with constant drift and parameter** $\beta = 2$

$Y(t) = B(t), t \in [0,\infty)$: 1-dim. standard BM

$$p(t,y|s,x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)}, \quad 0 < s < t, \quad x, y \in S = \mathbb{R}.$$

- $\xi(\mathbb{R}) = N < \infty$, $\operatorname{supp} \xi = \xi$ (no multiple point in $\xi$): $\xi(\cdot) = \sum_{k=1}^{N} \delta_{u_k}(\cdot)$

**entire functions:** $\Phi_\xi^{u_j}(z) = \prod_{1 \le k \le N, \, k \ne j} \frac{z - u_k}{u_j - u_k}, \quad 1 \le j \le N.$

**martingale functions:** $\mathcal{M}_\xi^{u_j}(t,y) = \mathcal{I}\left[e^{\nu(W-u_j)} \Phi_\xi^{u_j}(W) \Big| (t,y)\right], \quad 1 \le j \le N.$

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- $\xi(\cdot) = N\delta_0(\cdot)$, $\quad \xi^* = \operatorname{supp} \xi = \delta_0$ (all $N$ particles are at the origin 0), $\nu = 0$

**an entire function:**

$$\begin{aligned}
\Phi^0_{N\delta_0}((s,x);z) &= \frac{1}{2\pi i} \oint_{C(\delta_0)} d\zeta \frac{p(s,x|0,\zeta)}{p(s,x|0,0)} \frac{1}{z-\zeta} \Phi_\xi^{N\delta_0}(z) \\
&= \sum_{\ell=0}^{N-1} \left(\frac{z}{\sqrt{2s}}\right)^\ell \frac{1}{\ell!} H_\ell\left(\frac{x}{\sqrt{2s}}\right).
\end{aligned}$$

**a martingale:**

$$\begin{aligned}
\mathcal{M}^0_{N\delta_0}(s,x|t,B(t)) &= \mathcal{I}\left[\Phi^0_{N\delta_0}((s,x);W) \Big| (t,B(t))\right] \\
&= \sqrt{\pi} e^{x^2/4s + B(t)^2/4t} \sum_{n=0}^{N-1} \left(\frac{t}{s}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2s}}\right) \varphi_n\left(\frac{B(t)}{\sqrt{2t}}\right), \quad t \in [0,\infty),
\end{aligned}$$

$$\varphi_n(x) = \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(x) e^{-x^2/2}, \quad n \in \mathbb{N}_0 \quad \text{(Hermite orthogonal functions)}.$$

⬇

$$\mathbb{K}_\xi(s,x;t,y) = \int_S \xi_*(dv) p(s,x|0,v) \mathcal{M}_\xi^v(s,x|t,y) - \mathbf{1}_{(s>t)} p(s,x|t,y)$$

**extended Hermite kernel**

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- $\xi(\cdot) = \sum_{j \in \mathbb{Z}} \delta_j(\cdot) \equiv \xi^{\mathbb{Z}}(\cdot)$ : every point of $\mathbb{Z}$ is occupied by one particle (an infinite particle system), $\quad \nu = 0$

**an infinite series of entire functions:**

(the Weierstrass canonical product with genus 0)

$$\begin{aligned}
\Phi^\ell_{\xi^{\mathbb{Z}}}(z) &= \prod_{n \in \mathbb{Z}, \, n \ne \ell} \frac{z-n}{\ell - n} \\
&= \frac{\sin\{\pi(z-\ell)\}}{\pi(z-\ell)} = \frac{1}{2\pi} \int_{|k| \le \pi} dk \, e^{ik(z-\ell)}, \quad \ell \in \mathbb{Z}.
\end{aligned}$$

**an infinite series of martingale functions :**

$$\begin{aligned}
\mathcal{M}^\ell_{\xi^{\mathbb{Z}}}(t,y) &= \mathcal{I}\left[\Phi^\ell_{\xi^{\mathbb{Z}}}(W) \Big| (t,y)\right] \\
&= \frac{1}{2\pi} \int_{|k| \le \pi} dk \, e^{k^2 t/2 + ik(y-\ell)}, \quad \ell \in \mathbb{Z}.
\end{aligned}$$

$$\mathbb{K}_\xi(s,x;t,y) = \int_S \xi_*(dv) p(s,x|0,v) \mathcal{M}_\xi^v(s,x|t,y) - \mathbf{1}_{(s>t)} p(s,x|t,y)$$

⬇

$$\mathbb{K}_{\xi^{\mathbb{Z}}}(s,x;t,y) = \frac{1}{2\pi} \int_{|k| \le \pi} dk \, e^{k^2(t-s)/2 + ik(y-z)} \vartheta_3(x - iks, 2\pi i s) - \mathbf{1}_{(s>t)} p(s,x|t,y).$$

**(nonequilibrium sine process)**

$$\vartheta_3(z,\tau) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i z \ell + \pi i \tau \ell^2}.$$

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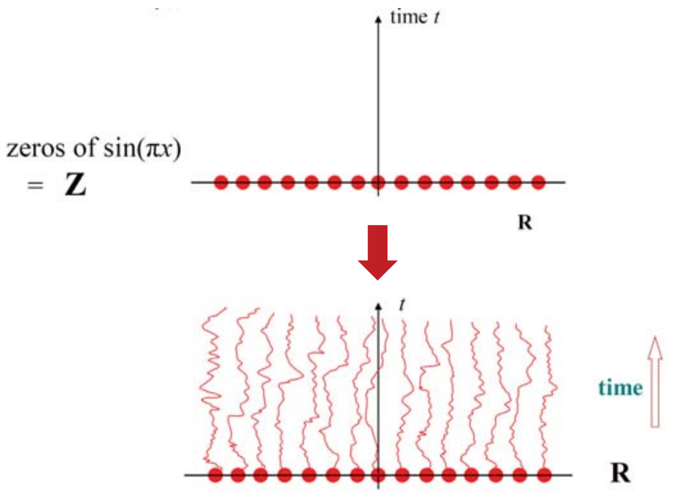


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$$\mathbb{K}_{\xi^{\mathbb{Z}}}(s,x;t,y) = \frac{1}{2\pi}\int_{|k|\le\pi} dk\, e^{k^2(t-s)/2+ik(y-x)}\vartheta_3(x-iks,2\pi is) - \mathbf{1}_{(s>t)}p(s,x|t,y).$$

**(nonequilibrium sine process)**

$$\vartheta_3(z,\tau) = \sum_{\ell\in\mathbb{Z}} e^{2\pi iz\ell+\pi i\tau\ell^2}.$$

**relaxation phenomenon** $\quad s\to\infty, t\to\infty, t-s=\text{fixed}.$

**extended sine kernel (equilibrium sine process)**

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- $\xi(\cdot) = \sum_{n\in\mathbb{N}} \delta_{a_n}(\cdot) \equiv \xi^{\mathcal{A}}(\cdot), \quad \nu = 0, \quad$ where

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty dk\, \cos\left(\frac{k^3}{3}+kx\right),$$

$$\mathcal{A} = \mathrm{Ai}^{-1}(0) = \left\{a_n : n\in\mathbb{N}, \mathrm{Ai}(a_n)=0, 0>a_1>a_2>\cdots\right\}.$$

(zeros of Airy functions)

**an infinite series of entire functions:**

(the Weierstrass canonical product with genus 1)

$$\widehat{\Psi}^{a_\ell}_{\xi^{\mathcal{A}}}(z) = \exp\left[\left(d_1 + \sum_{n=1}^\infty \frac{1}{a_n}\right)(z-a_\ell)\right]\Phi^{a_\ell}_{\xi^{\mathcal{A}}}(z), \qquad \ell\in\mathbb{N},$$

$$\Phi^{a_\ell}_{\xi^{\mathcal{A}}}(z) = \prod_{\substack{n\in\mathbb{N},\\ n\ne\ell}} \frac{a_\ell - a_n}{\ell - a_n}, \quad d_1 = \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} = -\frac{3^{1/3}\Gamma(2/3)}{\Gamma(1/3)}.$$

**an infinite series of martingale functions:**

$$\mathcal{M}^{a_\ell}_{\xi^{\mathcal{A}}}(t,y) = \mathcal{I}\left[\widehat{\Psi}^{a_\ell}_{\xi^{\mathcal{A}}}(W)\middle|(t,y-t^2/4)\right]$$

$$= \frac{e^{-ty/2+t^3/24}}{\mathrm{Ai}'(a_\ell)^2}\int_0^\infty du\, e^{-ut/2}\mathrm{Ai}(a_\ell+u)\mathrm{Ai}(y+u), \quad \ell\in\mathbb{N}.$$

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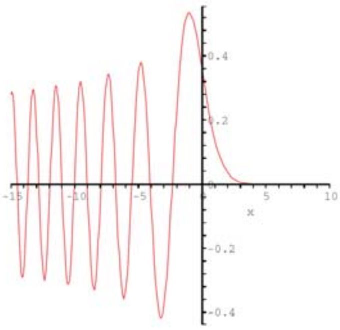
**Airy function**

$\mathrm{Ai}(x)$

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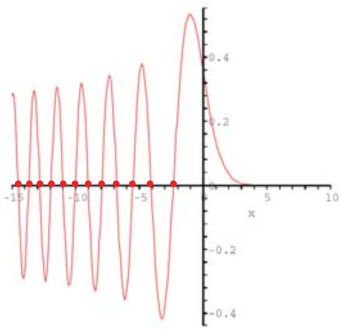
**Airy function**

$\mathrm{Ai}(x)$

**Airy zeros**

$a_\ell < 0, \ell\in\mathbf{N}$

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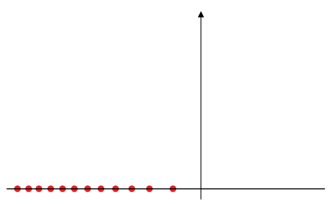
**Airy zeros**



$a_\ell < 0, \ell\in\mathbf{N}$

**confinement in the negative region R-**
**at time $t=0$**

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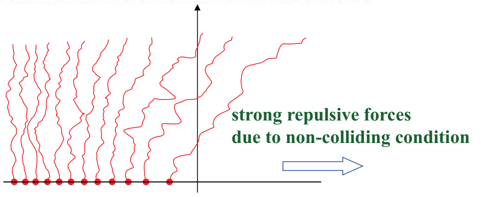
- Zeros of Airy function $\mathrm{Ai}(z)$ are all located on the negative part of the real axis $\mathbb{R}_-$ = confinement of particles in $\mathbb{R}_-$ at $t=0$
- In order to properly control the effect of such initial confinement of particles in the negative region of $\mathbb{R}_-$, we put the drift term to each Brownian motion.




confinement in the negative region **R-**
at time $t = 0$

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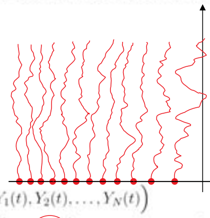
- Zeros of Airy function $\mathrm{Ai}(z)$ are all located on the negative part of the real axis $\mathbb{R}_-$ = confinement of particles in $\mathbb{R}_-$ at $t=0$
- In order to properly control the effect of such initial confinement of particles in the negative region of $\mathbb{R}_-$, we put the drift term to each Brownian motion.




$$\boldsymbol{Y}(t) = \Big(Y_1(t), Y_2(t), \dots, Y_N(t)\Big)$$

$$dY_j(t) = dB_j(t) + \left(\frac{t}{2} + D_{A_N}\right) dt + \sum_{\substack{1 \le k \le N \\ k \ne j}} \frac{dt}{Y_j(t) - Y_k(t)}$$

$$D_{A_N} = d_1 + \sum_{\ell=1}^{N} \frac{1}{a_\ell} \simeq -\left(\frac{12}{\pi^2}\right)^{1/3} N^{1/3} \to -\infty \quad \text{as} \quad N \to \infty$$

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$$\mathbb{K}_\xi(s,x;t,y) = \int_S \xi_*(dv) p(s,x|0,v) \mathcal{M}_\xi^v(s,x|t,y) - \mathbf{1}_{(s>t)} p(s,x|t,y)$$

$$\mathbb{K}_{\xi^{\mathcal{A}}}(s,x;t,y) = \int_0^\infty du \int_{\mathbb{R}} dw\, e^{-ut/2+ws/2} \mathrm{Ai}(y+u)\mathrm{Ai}(x+w) \times \sum_{\ell \in \mathbb{N}} \frac{\mathrm{Ai}(u+a_\ell)\mathrm{Ai}(w+a_\ell)}{\mathrm{Ai}'(a_\ell)^2} - \mathbf{1}_{(s>t)} p_{\mathrm{Ai}}(s,x|t,y),$$

$$p_{\mathrm{Ai}}(s,x|t,y) = e^{-(sx-ty)/2+(s-t)^2/24} p\Big(s, x - s^2/4 \Big| t, y - t^2/4\Big).$$

(nonequilibrium Airy process)

relaxation phenomenon $\qquad s \to \infty, t \to \infty, t-s = \text{fixed}.$

extended Airy kernel (equilibrium Airy process)

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[Example 2] **Interacting squared Bessel process with parameters $\nu > -1$ and $\beta = 2$**

$$Y(t), t \in [0,\infty) = \mathrm{BESQ}^{(\nu)}: \; Y(t) = Y(0) + \int_0^t 2\sqrt{Y(s)}dB(s) + 2(\nu+1)t, \quad t \ge 0,$$

$$p^{(\nu)}(t,y|s,x) = \left(\frac{y}{x}\right)^{\nu/2} \frac{e^{-(x+y)/2(t-s)}}{2(t-s)} I_\nu\left(\frac{\sqrt{xy}}{t-s}\right), \quad 0 < s < t, \quad x, y \in \mathbb{R}_+.$$

- $\xi(\mathbb{R}_+) = N < \infty$, $\mathrm{supp}\,\xi = \xi$ (no multiple point in $\xi$): $\xi(\cdot) = \sum_{k=1}^N \delta_{u_k}(\cdot)$

entire functions: $\Phi_\xi^{u_j}(z) = \prod_{\substack{1 \le k \le N, \\ k \ne j}} \frac{z-u_k}{u_j-u_k}, \quad 1 \le j \le N.$

martingale functions:

$$\mathcal{M}_\xi^{u_j}(t,y) = \widetilde{\mathcal{I}}\Big[\Phi_\xi^{u_j}(W)\Big|(t,y)\Big] \equiv \int_0^\infty \Phi_\xi^{u_j}(-w) \left(\frac{w}{x}\right)^{\nu/2} \frac{e^{(x-w)/2t}}{2t} J_\nu\left(\frac{\sqrt{xw}}{t}\right), \quad 1 \le j \le N.$$

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- $\xi(\cdot) = N\delta_0(\cdot)$ (all $N$ particles are at the origin 0)

$$\mathcal{M}_{N\delta_0}^0(s,x|t,Y(t)) = \Gamma(\nu+1)\left(\frac{x}{2s}\right)^{-\nu/2}\left(\frac{Y(t)}{2t}\right)^{-\nu/2} \times e^{x/4s+Y(t)/4t} \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \left(\frac{t}{s}\right)^n \varphi_n^{(\nu)}\left(\frac{x}{2s}\right) \varphi_n^{(\nu)}\left(\frac{Y(t)}{2t}\right), \quad t \in [0,\infty),$$

$$\varphi_n^{(\nu)}(x) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}} x^{\nu/2} L_n^{(\nu)} e^{-x/2}, \quad n \in \mathbb{N}_0 \quad \text{(Laguerre orthogonal functions).}$$

$$\mathbb{K}_\xi(s,x;t,y) = \int_S \xi_*(dv) p(s,x|0,v) \mathcal{M}_\xi^v(s,x|t,y) - \mathbf{1}_{(s>t)} p(s,x|t,y)$$

extended Laguerre kernel

- $\xi = \sum_{j \in \mathbb{N}} \delta_{u_j}$, $\{u_j\}_{j \in \mathbb{N}} = \{\text{squares of zeros of } J_\nu(z)\}$

$$\mathbb{K}_\xi(s,x;t,y) = \int_S \xi_*(dv) p(s,x|0,v) \mathcal{M}_\xi^v(s,x|t,y) - \mathbf{1}_{(s>t)} p(s,x|t,y)$$

nonequilibrium Bessel process

relaxation phenomenon

extended Bessel kernel (equilibrium Bessel process)

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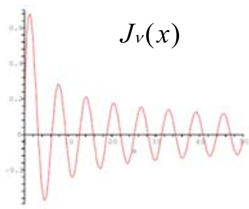
SDEs for finite particle approximation

$$\boldsymbol{X}^{(\nu)}(t) = \Big(X_1^{(\nu)}(t), X_2^{(\nu)}(t), \dots, X_N^{(\nu)}(t)\Big) \in \mathbb{W}_N^{\mathrm{C}} \quad \text{Weyl chamber of type } C_N, \quad t \in [0,\infty)$$

$$dX_j^{(\nu)}(t) = 2\sqrt{X_j^{(\nu)}} dB_j(t) + 2\left\{N + \nu + \sum_{\substack{1 \le k \le N \\ k \ne j}} \frac{X_j^{(\nu)}(t) + X_k^{(\nu)}(t)}{X_j^{(\nu)}(t) - X_k^{(\nu)}(t)}\right\} dt, \quad 1 \le j \le N, \quad t \in [0,\infty),$$

where $B_j(t)$ are independent one-dimensional standard BMs

$J_\nu(x)$



$j_{\nu,\ell} \Rightarrow j_{\nu,\ell}^{\,2}$

$\ell \in \mathbf{N}$

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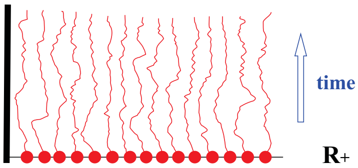
SDEs for finite particle approximation

$$\boldsymbol{X}^{(\nu)}(t) = \left(X_1^{(\nu)}(t), X_2^{(\nu)}(t), \dots, X_N^{(\nu)}(t)\right) \in \mathbb{W}_N^{\mathrm{C}} \quad \text{Weyl chamber of type } C_N, \quad t \in [0,\infty)$$

$$dX_j^{(\nu)}(t) = 2\sqrt{X_j^{(\nu)}}dB_j(t) + 2\left\{N + \nu + \sum_{\substack{1 \le k \le N \\ k \ne j}} \frac{X_j^{(\nu)}(t) + X_k^{(\nu)}(t)}{X_j^{(\nu)}(t) - X_k^{(\nu)}(t)}\right\}dt, \quad 1 \le j \le N, \quad t \in [0,\infty),$$

where $B_j(t)$ are independent one-dimensional standard BMs

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[Example 3] trigonometric extension (I)

$Y(t), t \ge 0$: Markov process on $S = \mathrm{S}^1(r) = [0, 2\pi r)$,

$$p_N^r(t,y|s,x) = \begin{cases} \displaystyle\sum_{\ell \in \mathbb{Z}} p(t, y + 2\pi r\ell | s, x), & \text{if } N \text{ is odd}, \\ \displaystyle\sum_{\ell \in \mathbb{Z}} (-1)^\ell p(t, y + 2\pi r\ell | s, x), & \text{if } N \text{ is even}, \end{cases}$$

$$p(t,y|s,x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)},$$

- $\xi(\mathrm{S}^1(r)) = N < \infty$, $\operatorname{supp}\xi = \xi$ (no multiple point in $\xi$): $\xi(\cdot) = \sum_{k=1}^N \delta_{u_k}(\cdot)$

**entire functions:** $\Phi_\xi^{u_j}(z) = \displaystyle\prod_{\substack{1 \le k \le N, \\ k \ne j}} \frac{\sin((z-u_k)/2r)}{\sin((u_j - u_k)/2r)}, \quad 1 \le j \le N.$

**martingale functions:** $\mathcal{M}_\xi^{u_j}(t,y) = \mathcal{I}\left[\Phi_\xi^{u_j}(W)\Big|(t,y)\right], \quad 1 \le j \le N.$

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[Example 4] elliptic extension

$Y(t), t \ge 0$: Markov process on $S = \mathrm{S}^1(r) = [0, 2\pi r)$,

$$p_N^r(t,y|s,x) = \begin{cases} \displaystyle\sum_{\ell \in \mathbb{Z}} p(t, y + 2\pi r\ell | s, x), & \text{if } N \text{ is even}, \\ \displaystyle\sum_{\ell \in \mathbb{Z}} (-1)^\ell p(t, y + 2\pi r\ell | s, x), & \text{if } N \text{ is odd}, \end{cases}$$

$$p(t,y|s,x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)},$$

- $\xi(\mathrm{S}^1(r)) = N < \infty$, $\operatorname{supp}\xi = \xi$ (no multiple point in $\xi$): $\xi(\cdot) = \sum_{k=1}^N \delta_{u_k}(\cdot)$

**entire functions:**

$$\Phi_\xi^{u_j}(z) = \frac{\vartheta_1((\overline{u} + z - u_j)/2\pi r, N\tau)}{\vartheta_1(\overline{u}/2\pi r, N\tau)} \prod_{\substack{1 \le k \le N, \\ k \ne j}} \frac{\vartheta_1((z-u_k)/2\pi r, N\tau)}{\vartheta_1((u_j - u_k)/2\pi r, N\tau)},$$

$$1 \le j \le N.$$

$$\overline{u} = \sum_{\ell=1}^N u_\ell - \kappa_N, \quad \tau = \frac{it_*}{2\pi r},$$

**martingale functions:** $\mathcal{M}_\xi^{u_j}(t,y) = \mathcal{I}\left[\Phi_\xi^{u_j}(W)\Big|(t,y)\right], \quad 1 \le j \le N.$ 69

[Example 7] hyperbolic extension

$Y(t) = B(t), t \in [0, \infty)$: 1-dim. standard BM

$$p(t,y|s,x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)}, \quad 0 < s < t, \quad x, y \in S = \mathbb{R}.$$

- $\xi(\mathbb{R}) = N < \infty$, $\operatorname{supp}\xi = \xi$ (no multiple point in $\xi$): $\xi(\cdot) = \sum_{k=1}^N \delta_{u_k}(\cdot)$

**entire functions:** $\Phi_\xi^{u_j}(z) = \displaystyle\prod_{\substack{1 \le k \le N, \\ k \ne j}} \frac{\sinh(\sigma(z-u_k)/2)}{\sinh(\sigma(u_j - u_k)/2)}, \ 1 \le j \le N.$

**martingale functions:** $\mathcal{M}_\xi^{u_j}(t,y) = \mathcal{I}\left[\Phi_\xi^{u_j}(W)\Big|(t,y)\right], \quad 1 \le j \le N.$

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- An infinite particle limit: assume $N$'s are odd:

ordered drift coefficients : $\nu_j = \sigma\left(j - \dfrac{N+1}{2}\right) \to \sigma\mathbb{Z}, \quad \text{as } N \to \infty,$

initial configuration : $u_j = a\left(j - \dfrac{N+1}{2}\right) \to a\mathbb{Z}, \quad \text{as } N \to \infty.$

**infinite series of entire functions:**

$$\Phi_{a\mathbb{Z}}^{a\ell}(z) = \frac{\sigma a}{2\pi} \exp\left[\frac{\sigma}{2a}\{z^2 - (a\ell)^2\}\right] e^{\pi^2/2\sigma a} \left(\prod_{n=1}^\infty (1 - e^{-4n\pi^2/\sigma a})\right)^{-3}$$

$$\times \frac{\vartheta_1((z - a\ell)/a, 2\pi i/\sigma a)}{2\sinh[\sigma(z - a\ell)/2]}, \quad \ell \in \mathbb{Z}.$$

$$\sigma \to 0, a = 1$$

$$\Phi_{\mathbb{Z}}^\ell(z) = \frac{\sin[\pi(z-\ell)]}{\pi(z-\ell)}, \quad \ell \in \mathbb{Z}.$$

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[Example 8] O'Connell's process

O'Connell's process (the Whittaker process) is not determinantal.
But if we observe the function

$$\Theta^a(x) = \exp(-e^{-x/a}), \quad a > 0,$$

then we have the Fredholm determinantal expression for the rightmost particle $X_1^a(t)$,

$$\mathbb{E}_{-\infty\rho}^{a,a\nu}\left[\Theta^a(X_1^a(t) - h)\right] = \operatorname*{Det}_{(x,y)\in\mathbb{R}^2}\left[\delta_x(\{y\}) - \mathbf{K}^t(x,y)\mathbf{1}_{(y<h)}\right], \quad h \in \mathbb{R}.$$

The correlation kernel $\mathbf{K}^t$ obtained by Borodin and Corwin is rewritten as the following,

$$\mathbf{K}^t(x,y) = \frac{1}{t}\mathcal{K}^{1/t}\left(\frac{x}{t}, \frac{y}{t}\right), \quad \xi(\cdot) = \sum_{j=1}^N \delta_{\nu_j}(\cdot),$$

$$\mathcal{K}^t(x,y) = \int_{\mathbb{R}} \xi(d\nu) p(t,x|0,\nu) \mathcal{M}_\xi^\nu(t,y),$$

$$\mathcal{M}_\xi^\nu(t,y) = \mathcal{I}\left[\Upsilon_\xi^\nu(W)|(t,y)\right],$$

$$\Upsilon_\xi^\nu(z) = \Gamma(1 - a(\nu - z)) \prod_{\substack{1 \le k \le N, \\ \nu_k \ne \nu}} \frac{\Gamma(a(\nu_k - \nu))}{\Gamma(a(\nu_k - z))}.$$

$$\lim_{a\to 0} \mathbb{E}_{-\infty\rho}^{a,a\nu}\left[\Theta^a(X_1^a(t) - h)\right] = \mathbb{P}_{N\delta_0}^{\nu}\left[X_1(t) > h\right], \quad h \in \mathbb{R}.$$

RHS: Dyson model started from $\xi = N\delta_0$ with the ordered drift $\boldsymbol{\nu} \in \overline{\mathbb{W}}_N$

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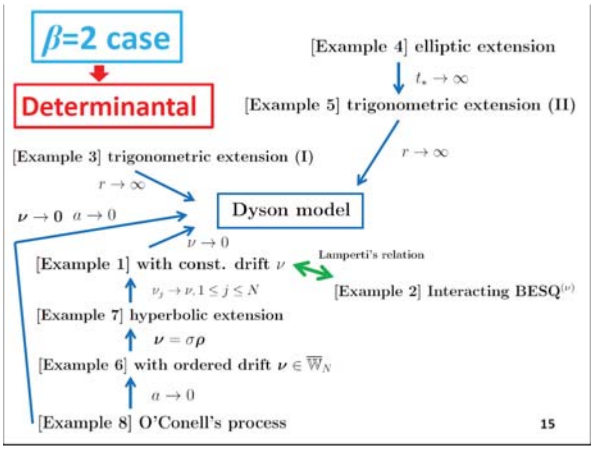
# 5. Concluding Remarks

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- Except for O'Connell's process, the proper entire functions $\{\Phi_\xi^u(z)\}_u$ and the corresponding martingale functions $\{\mathcal{M}_\xi^u(s, x|t, y)\}_u$ are directly obtained from the SDEs.

SDEs $\Longrightarrow$ harmonic transform expression using the process $Y(t), t \in \mathcal{T}$
$\Longrightarrow$ determinantal martingale representation using determinantal identities

See K: Stochastic Process. Appl. **124** (2014) 3724.

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- determinantal identities

**Krattenthaler's identity**

$$\frac{\prod_{1 \leq j < k \leq N} (z_k - z_j)}{\prod_{1 \leq j < k \leq N} (u_k - u_j)} = \det_{1 \leq j,k \leq N} \left[ \Phi_\xi^{u_j}(z_k) \right], \quad \Phi_\xi^u(z) = \prod_{\substack{1 \leq \ell \leq N, \\ u_\ell \neq u}} \frac{z - u_\ell}{u - u_\ell},$$

$$\frac{\prod_{1 \leq j < k \leq N} \sin((z_k - z_j)/2r)}{\prod_{1 \leq j < k \leq N} \sin((u_k - u_j)/2r)} = \det_{1 \leq j,k \leq N} \left[ \widetilde{\Phi}_\xi^{u_j}(z_k) \right], \quad \widetilde{\Phi}_\xi^u(z) = \prod_{\substack{1 \leq \ell \leq N, \\ u_\ell \neq u}} \frac{\sin((z - u_\ell)/2r)}{\sin((u - u_\ell)/2r)}.$$

**Rosengren-Schlosser's identity**

$$\frac{\vartheta_1(\overline{z}_\delta/2\pi r, \tau) \prod_{1 \leq j < k \leq N} \vartheta_1((z_k - z_j)/2\pi r, \tau)}{\vartheta_1(\overline{u}_\delta/2\pi r, \tau) \prod_{1 \leq j < k \leq N} \vartheta_1((u_k - u_j)/2\pi r, \tau)} = \det_{1 \leq j,k \leq N} \left[ \widehat{\Phi}_\xi^{u_j}(z_k) \right],$$

$$\widehat{\Phi}_\xi^u(z) = \frac{\vartheta_1((\overline{u}_\delta + z - u)/2\pi r, \tau)}{\vartheta_1(\overline{u}_\delta/2\pi r, \tau)} \prod_{\substack{1 \leq \ell \leq N, \\ u_\ell \neq u}} \frac{\vartheta_1((z - u_\ell)/2\pi r, \tau)}{\vartheta_1((u - u_\ell)/2\pi r, \tau)},$$

$$\overline{z}_\delta = \sum_{j=1}^N x_j + \delta, \quad \overline{u}_\delta = \sum_{j=1}^N u_j + \delta.$$

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$$\mathrm{E}_{\boldsymbol{u}} \left[ \prod_{m=1}^M \prod_{\ell \in \mathbb{I}} \{1 + \chi_{t_m}(B_\ell(t_m))\} \det_{1 \leq j,k \leq N} \left[ \mathcal{M}_\xi^{u_j}(t, B_k)) \right] \right]$$
$$= \mathop{\mathrm{Det}}_{\substack{(s,t) \in \{t_1, \dots, t_M\}^2, \\ (x,y) \in S}} \left[ \delta_{st} \delta_x(\{y\}) + \mathbb{K}_\xi(s, x; t, y) \chi_t(y) \right],$$
$$0 < t_1 < \cdots < t_M < \infty, \quad \xi = \sum_{j \in \mathbb{I}} \delta_{u_j}.$$

$$\mathbb{K}_\xi(s, x; t, y) = \sum_{j \in \mathbb{I}} p(s, x|0, u_j) \mathcal{M}_\xi^{u_j}(s, x|t, y) - \mathbf{1}_{(s > t)} p(s, x|t, y).$$

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- The origin of the function

$$\Upsilon_\xi^\nu(z) = \Gamma(1 - a(\nu - z)) \prod_{\substack{1 \leq k \leq N, \\ \nu_\ell \neq \nu}} \frac{\Gamma(a(\nu_\ell - \nu))}{\Gamma(a(\nu_\ell - z))}$$

is missing.

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**A Lecture Note entitled**
***`Bessel Processes, Schramm-Loewner Evolution, and the Dyson Model'***
**will be published as**
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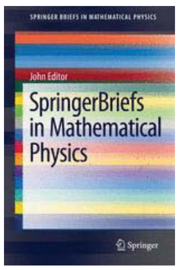


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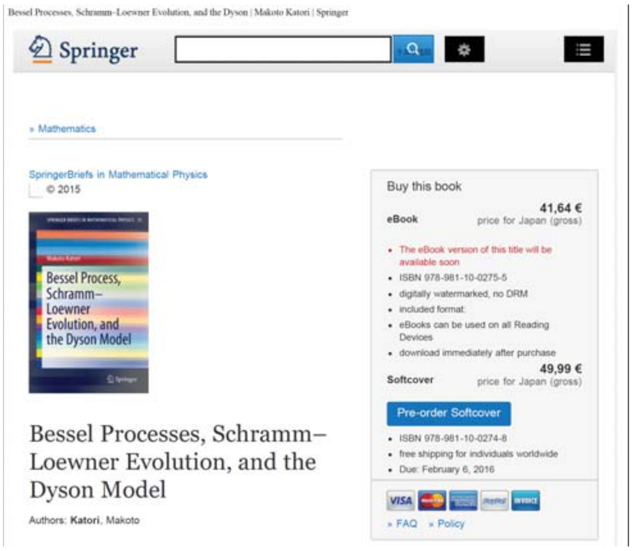

Bessel Processes, Schramm–Loewner Evolution, and the Dyson | Makoto Katori | Springer

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Bessel Process, Schramm–Loewner Evolution, and the Dyson Model

# Bessel Processes, Schramm–Loewner Evolution, and the Dyson Model

Authors: **Katori**, Makoto

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# Recent developments and challenges in lattice QCD

Sinya AOKI
Yukawa Institute for Theoretical Physics, Kyoto University

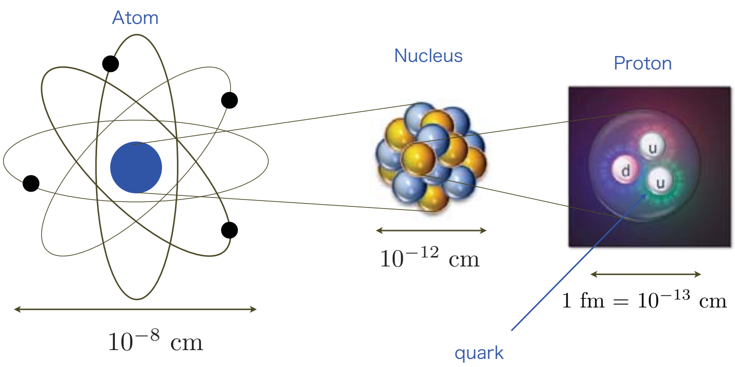
International Symposium “RIKKYO MathPhys 2016”
January 9-11, 2016, Rikkyo University, Tokyo

16年1月11日月曜日

# 1. Introduction
# About lattice QCD

16年1月11日月曜日

## Quarks




Hadrons are made of more fundamental objects, named “quarks”.

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1973: Kobayashi and Maskawa predicted existences of 6 types(“flavor”) of quarks.

Kobayashi

Maskawa,
7th director of YITP

2008 Nobel prize


charge 2e/3

charge -e/3

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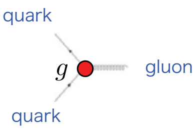
## QCD (Quantum ChromoDynamics)

QCD: theory for dynamics of quarks    cf. QED (Quantum Electrodynamics)

$$\mathcal{L} = \bar q(x)\gamma^\mu\{\partial_\mu + igA_\mu(x)\}q(x) + \frac{1}{4}\{F^a_{\mu\nu}(x)\}^2$$

gluon    quark



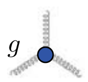

$a = 1 \sim 8$

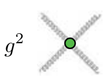
$\bar q A_\mu q = \bar q^A\, T^a_{AB} A^a_\mu\, q^B$    quarks-gluon interaction

$A, B = 1, 2, 3$ (color)    (electrons-photon in QED)

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A_\mu - g f^{abc} A^b_\mu A^c_\nu$    gluon field strength

$(F^a_{\mu\nu})^2$ → self-interaction (absent in QED)




$g$ : unique coupling constant in QCD
universal for all flavors

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## Some Properties of QCD

Asymptotic freedom    effective coupling becomes weaker at shorter distances

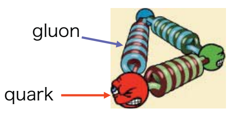
Gross    Politzer    Wilczek

2004 Nobel prize

Quark confinement    effective coupling becomes stronger at longer distances

→ no isolated quark can be observed

structure of nucleon




quark confinement

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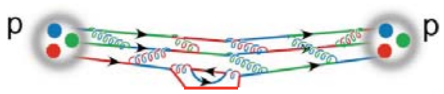
# Difficulties of QCD

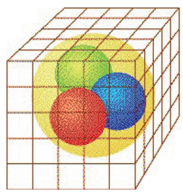



"Free" proton = 3 quarks interacting with each others by exchanging a lot of gluons, so that they move coherently.

Clearly, perturbation theory does not work !

We need a non-perturbative method.

Lattice QCD





# Lattice Field Theories

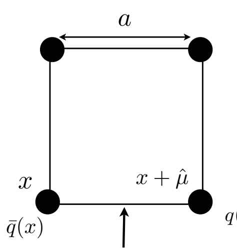
Definition of Quantum Field Theories

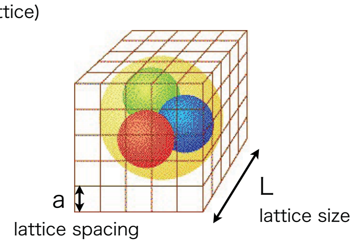
1. Continuum (quantum) Field theories
   - Perturbative expansion: needed to define the theory
   - Divergences ⇨ Regularization/Renormalization
   - Gauge volume ⇨ Gauge fixing
   - Path-integral quantization, canonical quantization
2. Lattice (quantum) field theories
   - does not rely on perturbation theory
   - lattice spacing $a$ ⇨ regularization
   - continuum limit ( $a \to 0$ ) has to be taken (renormalization)
   - (mostly) Path-integral in Euclidean space
   - Strong or weak coupling expansions, Monte Carlo method



# Lattice QCD

define QCD on a discrete space-time (lattice)

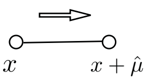


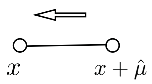

$$U_\mu(x) = e^{igaA_\mu(x)} = 1 + igaA_\mu(x) + \frac{\{igaA_\mu(x)\}^2}{2!} + \cdots \in \mathrm{SU}(3)$$

SU(3) matrix

gluon (lives on link) infinite numbers of gluons (non-perturbative) !




$U_\mu(x)$




$U_{-\mu}(x+\hat\mu) \equiv U_\mu(x)^\dagger$

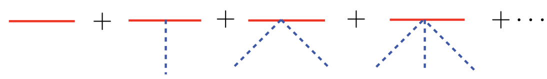


continuum QCD lattice QCD

$$\bar q(x)\gamma^\mu\{\partial_\mu + igA_\mu(x)\}q(x) \xleftarrow{a\to 0} \bar q(x)\gamma^\mu \frac{U_\mu(x)q(x+\hat\mu) - U_{-\mu}(x)q(x-\hat\mu)}{2a}$$

quarks(covariant derivative)

$\bar q(x)U_\mu(x)q(x+\hat\mu) =$



quark interacts with many gluons in a very short distance !

quark action

$$S_F = \sum_{x,\mu} \bar q(x)\gamma^\mu \frac{U_\mu(x)q(x+\hat\mu) - U_{-\mu}(x)q(x-\hat\mu)}{2a} + m\sum_x \bar q(x)q(x)$$

gauge invariant

gauge transformation

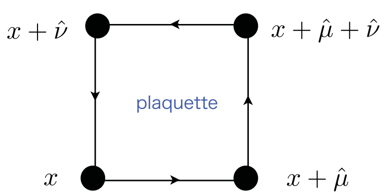
$$q(x) \to \Omega(x)q(x) \qquad \bar q(x) \to \bar q(x)\Omega(x)^\dagger \qquad U_\mu(x) \to \Omega(x)U_\mu(x)\Omega(x+\hat\mu)^\dagger$$



continuum QCD lattice QCD

$$\frac{g^2}{2}\mathrm{tr}\, F_{\mu\nu}(x)^2 \xleftarrow{a\to 0} \mathrm{tr}\, U_\mu(x)U_\mu(x+\hat\mu)U_\mu(x+\hat\nu)^\dagger U_\nu(x)^\dagger$$

gluons




gluon action

$$S_G = \frac{1}{g^2}\sum_x \sum_{\mu\neq\nu} \mathrm{tr}\, U_\mu(x)U_\mu(x+\hat\mu)U_\mu(x+\hat\nu)^\dagger U_\nu(x)^\dagger$$

gauge invariant



# Path integral

continuum QCD

$$\langle \mathcal{O}(A_\mu, q, \bar q)\rangle = \frac{1}{Z}\int \mathcal{D}A_\mu \mathcal{D}q\mathcal{D}\bar q\, \mathcal{O}(A_\mu, q, \bar q)e^{-S_0 - S_{\rm int}}$$
$$= \frac{1}{Z}\int \mathcal{D}A_\mu \mathcal{D}q\mathcal{D}\bar q\, \mathcal{O}(A_\mu, q, \bar q)\sum_{n=0}^\infty \frac{(-S_{\rm int})^n}{n!}e^{-S_0}$$

perturbative expansion

lattice QCD

$$\langle \mathcal{O}(U_\mu, q, \bar q)\rangle = \frac{1}{Z}\int \mathcal{D}U_\mu \mathcal{D}q\mathcal{D}\bar q\, \mathcal{O}(U_\mu, q, \bar q)e^{-S_F - S_G}$$

calculate without perturbative expansion

important properties

$$\int \mathcal{D}U_\mu(x)\, U_\mu(x) = 0$$

gluon is random

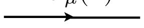
$$\int \mathcal{D}U_\mu(x)\, U_\mu(x)U_\mu(x)^\dagger = \mathbf{1}_{3\times 3}$$

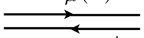
$$\int \mathcal{D}U_\mu(x)\, \det U_\mu(x) = 1$$

## Strong coupling expansion

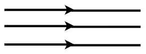
$S_G = O\left(\frac{1}{g^2}\right) \to 0 \qquad g^2 \to \infty$ strong coupling limit

quark path-integral

quark $U_\mu(x)$  $= 0$ by U integral — quark confinement

meson $U_\mu(x)$  $U_\mu(x)^\dagger$ $\neq 0$

after U integral

baryon $U_\mu(x)^3$  $\neq 0$

meson and baryon can propagate !

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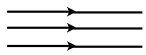
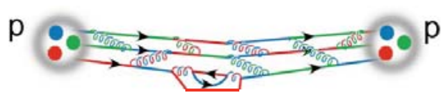
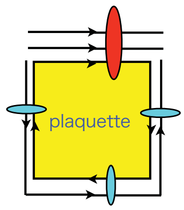
$U_\mu(x)^3$  $\simeq$ 


in terms of perturbation theory

If $\frac{1}{g^2}$ is small but non-zero




$= O\left(\frac{1}{g^2}\right)$ strong coupling expansion

3 quarks can propagate separately but still coherently, as a free baryon.

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## Monte-Carlo simulations

After integral over quarks

$$\langle \mathcal{O}(q,\bar{q},U)\rangle = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}U\ \exp[\bar{q}\,D(U)\,q + S_G(U)]\mathcal{O}(q,\bar{q},U)$$

$$= \int \mathcal{D}U\ \underline{\det D(U) e^{S_G(U)}}\hat{\mathcal{O}}(U)$$

probability of U $\equiv P(U)$ (as long as $P(U) \geq 0$)

Importance sampling according to P(U) "Monte-Carlo simulations"

calculate complicated QCD processes by computer simulations

uses of super-computers are required.

Yet calculations are not so easy.
Recently hadron masses have been accurately calculated. (free hadrons)

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# 2. Hadron spectra

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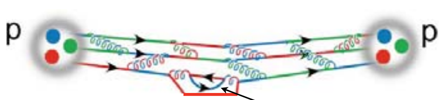
## Hadron mass calculations




creation/annihilation of quark-antiquark pair
"effect of det D(U)"

set $\det D(U) = 1$ : quenched approximation

2-pt correlation function

$$\langle 0|p(0,\vec{0})\frac{1}{V}\sum_{\vec{x}}\bar{p}(t,\vec{x})|0\rangle = \langle 0|p(0,\vec{0})\sum_n |n\rangle\langle n|\frac{1}{V}\sum_{\vec{x}}\bar{p}(t,\vec{x})|0\rangle$$

$$= \sum_n |\langle 0|p(0,\vec{0})|n\rangle|^2 e^{-m_n t} = C_0 e^{-m_0 t} + C_1 e^{-m_1 t} + \cdots$$

extract the ground-state hadron mass $m_0$ from the large $t$ behavior

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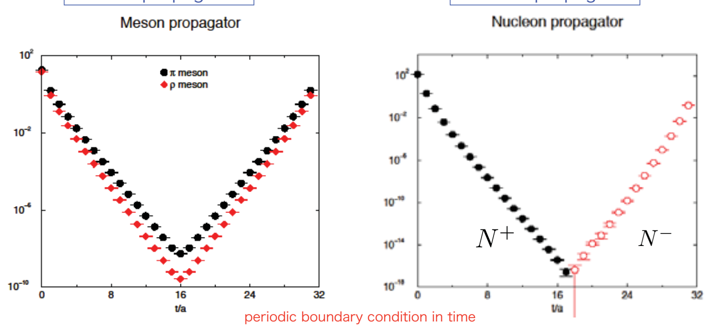
Meson propagator — Nucleon propagator




periodic boundary condition in time

pion is lighter than rho.

Nucleon is lighter than its negative-parity state.

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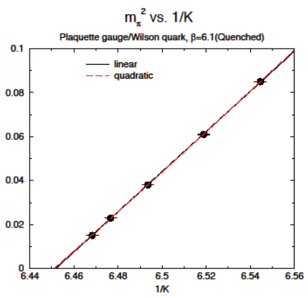
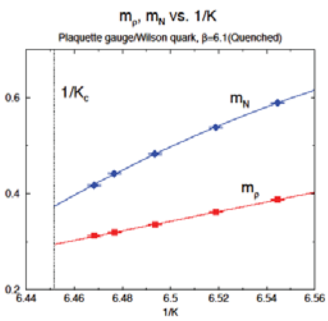
## Chiral extrapolation

It is difficult to make quark mass as small as the “experimental” value in numerical simulations. Extrapolations from heavier quark masses are usually made.

Pion mass




Other hadrons



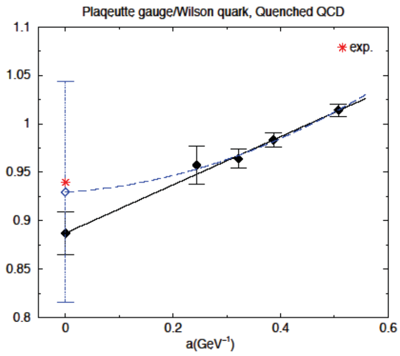

quark mass $\quad 2m_q a = \frac{1}{K} - \frac{1}{K_c}$



## Continuum extrapolation

$a \to 0$ limit should be taken.

Nucleon mass




lattice spacing

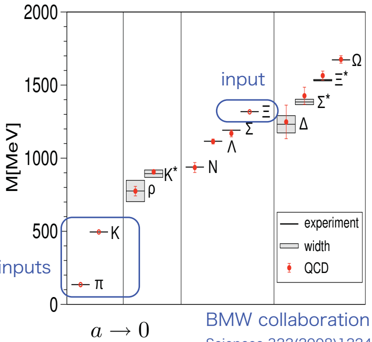
continuum extrapolation by fit

$$m_N(a) = m_N(0) + C_1 a$$
$$m_N(a) = m_N + C_1 a^2 + C_2 a^2$$



## The state of arts for hadron masses

2+1 flavor QCD $\quad m_u = m_d \geq m_s$




$a \to 0$

BMW collaboration
Sciences 322(2008)1224

an agreement between lattice QCD and experiments is good.

Baryon

| Quarks | Octet($\frac{1}{2}$) | Decouplet($\frac{3}{2}$) |
|---|---|---|
| $uuu$ | | $\Delta^{++}$ |
| $uud$ | $N$ $\quad p$ | $\Delta$ $\quad \Delta^+$ |
| $udd$ | $n$ | $\Delta^-$ |
| $ddd$ | | $\Delta^0$ |
| $uus$ | $\Sigma$ $\quad \Sigma^+$ | $\Sigma^*$ $\quad (\Sigma^*)^+$ |
| $uds$ | $\Lambda$ $\quad \Sigma^0, \Lambda^0$ | $(\Sigma^*)^0$ |
| $dds$ | $\Sigma^-$ | $(\Sigma^*)^-$ |
| $uss$ | $\Xi$ $\quad \Xi^0$ | $\Xi^*$ $\quad (\Xi^*)^0$ |
| $dss$ | $\Xi^-$ | $(\Xi^*)^-$ |
| $sss$ | | $\Omega$ $\quad \Omega$ |

Meson

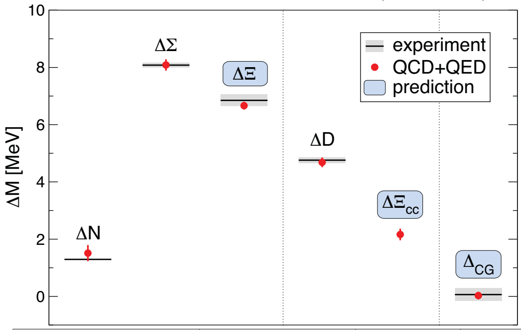
| Quarks | PesudoScala(0) | Vector(1) |
|---|---|---|
| $\bar{u}u - \bar{d}d$ | $\pi$ $\quad \pi^0$ | $\rho$ $\quad \rho^0$ |
| $\bar{d}u, \bar{u}d$ | $\pi^\pm$ | $\rho^\pm$ |
| $\bar{u}u + \bar{d}d$ | $\eta$ | $\omega$ |
| $\bar{s}d, \bar{d}s$ | $K$ $\quad K^0, \bar{K}^0$ | $K^*$ $\quad (K^*)^0, (\bar{K}^*)^0$ |
| $\bar{s}u, \bar{u}s$ | $K^\pm$ | $(K^*)^\pm$ |
| $\bar{s}s$ | $\eta_s$ | $\phi$ |



## Further improvement

Iso-spin breaking effects $\quad m_u \neq m_d$ effect
QED effect ( $\alpha_u = -2\alpha_d$ )

Borsanyi et al.
Science 347(2015) 1452-1455
(arXiv:1406.4088[hep-lat])




1+1+1+1 flavor QCD (u,d,s,c)
+ non-compact QED

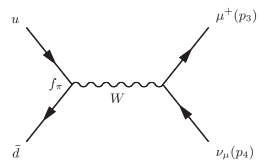
| | mass splitting [MeV] | QCD [MeV] | QED [MeV] |
|---|---|---|---|
| $\Delta N = n - p$ | 1.51(16)(23) | 2.52(17)(24) | -1.00(07)(14) |
| $\Delta\Sigma = \Sigma^- - \Sigma^+$ | 8.09(16)(11) | 8.09(16)(11) | 0 |
| $\Delta\Xi = \Xi^- - \Xi^0$ | 6.66(11)(09) | 5.53(17)(17) | 1.14(16)(09) |
| $\Delta D = D^\pm - D^0$ | 4.68(10)(13) | 2.54(08)(10) | 2.14(11)(07) |
| $\Delta\Xi_{cc} = \Xi_{cc}^{++} - \Xi_{cc}^{+}$ | 2.16(11)(17) | -2.53(11)(06) | 4.69(10)(17) |
| $\Delta_{CG} = \Delta N - \Delta\Sigma + \Delta\Xi$ | 0.00(11)(06) | -0.00(13)(05) | 0.00(06)(02) |



# 3. Weak matrix elements



## 3-1. Decay constants for PS mesons




$\langle 0|\bar{d}\gamma_\mu\gamma_5 u|\pi^+(p)\rangle = ip_\mu f_{\pi^+}, \quad \langle 0|\bar{s}\gamma_\mu\gamma_5 u|K^+(p)\rangle = ip_\mu f_{K^+}.$

axial-vector current $\quad A_\mu^{\pi^+}$

A-P correlation function

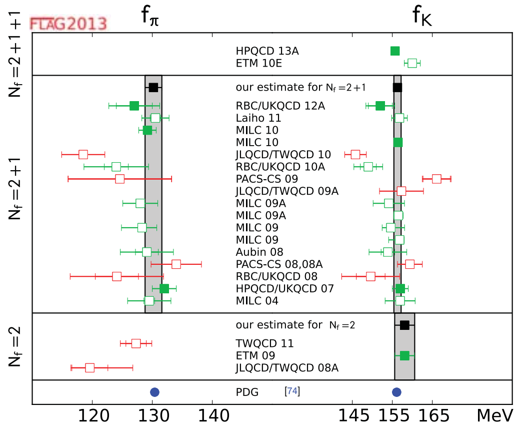
$$\langle 0|A_0^{\pi^+}(0,\vec{0})\frac{1}{V}\sum_{\vec{x}} P^{\pi^-}(t,\vec{x})|0\rangle = \langle 0|A_0^{\pi^+}(0,\vec{0})|\pi^+(\vec{0})\rangle\langle\pi^+(\vec{0})|\frac{1}{V}\sum_{\vec{x}} P^{\pi^-}(t,\vec{x})|0\rangle + \cdots$$
$$= \langle 0|A_0^{\pi^+}|\pi^+(\vec{0})\rangle\langle\pi^+(\vec{0})|P^{\pi^-}|0\rangle e^{-m_\pi t} + \cdots$$

P-P correlation function

$$\langle 0|P(0,\vec{0})\frac{1}{V}\sum_{\vec{x}} P^{\pi^-}(t,\vec{x})|0\rangle = \langle 0|P^{\pi^+}|\pi^+(\vec{0})\rangle\langle\pi^+(\vec{0})|P^{\pi^-}|0\rangle e^{-m_\pi t} + \cdots$$

$$\langle 0|A_0^{\pi^+}|\pi^+(\vec{0})\rangle = m_\pi f_{\pi^+}$$

## The latest lattice results




Particle Data Group

Lattice

$f_\pi = 130.2\ (1.4)$ MeV $(N_f = 2+1)$,
$f_K = 156.3\ (0.9)$ MeV $(N_f = 2+1)$,
$f_K = 158.1\ (2.5)$ MeV $(N_f = 2)$.

$f_\pi^{(\mathrm{PDG})} = 130.41\ (0.20)$ MeV,
$f_K^{(PDG)} = 156.1\ (0.8)$ MeV,

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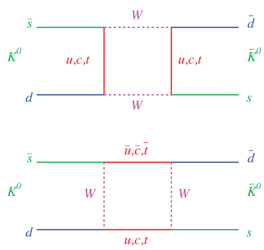
## 3-2. Kaon B parameter $B_K$

$K_0$ - $\bar{K}_0$ mixing parameter (indirect CP violation)




effective 4-fermi operator

$$Q^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d]$$

Weak Matrix Element

$$B_K(\mu) = \frac{\langle \bar{K}^0 | Q_R^{\Delta S=2}(\mu) | K^0 \rangle}{\frac{8}{3} f_K^2 m_K^2}.$$

3-pt correlation function

$$\langle 0|K_0(t_1)Q_R^{\Delta S=2}(t_O)K_0(t_2)|0\rangle = \langle 0|K_0|\bar{K}_0\rangle\langle \bar{K}_0|Q_R^{\Delta S=2}|K_0\rangle\langle K_0|K_0|0\rangle e^{-m_{K_0}(t_2-t_1)} + \cdots$$

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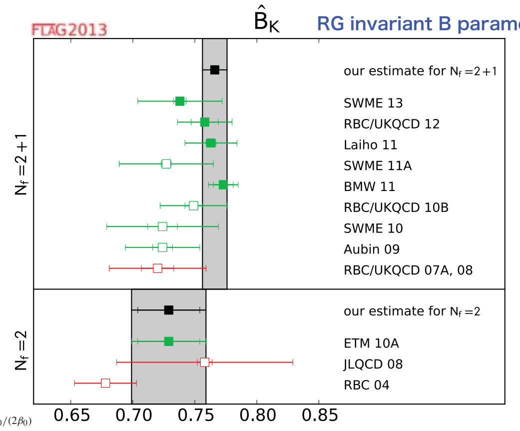
## The latest lattice results

RG invariant B parameter




$$\hat{B}_K = \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} \times \left\{1 + \frac{\bar{g}(\mu)^2}{(4\pi)^2}\left[\frac{\beta_1\gamma_0 - \beta_0\gamma_1}{2\beta_0^2}\right]\right\} B_K(\mu).$$

$N_f = 2+1: \quad \hat{B}_K = 0.7661(99),$

$N_f = 2+1: \quad B_K^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.5596(72).$

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## 3-3. Kaon decays

$K \to \pi\pi$ decays

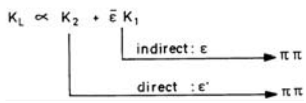
$$A(K^+ \to \pi^+\pi^0) = \sqrt{\frac{3}{2}} A_2 e^{i\delta_2}$$
$$A(K^0 \to \pi^+\pi^-) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} + \sqrt{\frac{1}{3}} A_2 e^{i\delta_2}$$
$$A(K^0 \to \pi^0\pi^0) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} - 2\sqrt{\frac{1}{3}} A_2 e^{i\delta_2}.$$

$\delta_{0,2}$ strong phases

$A_I$: $K \to \pi\pi$ $(I = 0, 2)$ weak decay amplitude

$\Delta I = 1/2$ rule $\quad \frac{\mathrm{Re}\ A_0}{\mathrm{Re}\ A_2} = 22.45(6)$ Experiment

CP violation




direct CP violation $\quad \varepsilon' = \frac{1}{\sqrt{2}}\mathrm{Im}\left(\frac{A_2}{A_0}\right)e^{i\Phi}, \quad \Phi = \pi/2 + \delta_2 - \delta_0,$

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## Some lattice results

$K \to (\pi\pi)_{I=2}$ decay amplitude

T. Blum et al., PRL108(2012)141061
T. Blum et al., PRD86(2012)074513

Lattice

$$\mathrm{Re}A_2 = 1.381(46)_{\mathrm{stat}}(258)_{\mathrm{syst}}10^{-8}\ \mathrm{GeV},$$
$$\mathrm{Im}A_2 = -6.54(46)_{\mathrm{stat}}(120)_{\mathrm{syst}}10^{-13}\ \mathrm{GeV}.$$

Experiment

$\mathrm{Re}A_2 = 1.479(4) \times 10^{-8}\mathrm{GeV}$

$K^+$ decays

$a^{-1} = 1.364$ GeV, $m_\pi = 142$ MeV, $m_K = 506$ MeV

$W_{2\pi} = 486$ MeV

$\Delta I = 1/2$ rule

P. Boyle et al., PRL110(2013)152001

Lattice

$$\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = \begin{cases} 9.1(2.1) & \text{for } m_K = 878\text{ MeV},\ m_\pi = 422\text{ MeV} \\ 12.0(1.7) & \text{for } m_K = 662\text{ MeV},\ m_\pi = 329\text{ MeV}. \end{cases}$$

Experiment

$\frac{\mathrm{Re}\ A_0}{\mathrm{Re}\ A_2} = 22.45(6)$

$a^{-1} = 1.73$ GeV

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## Latest (preliminary) results

Z. Bai et al., arXiv:1505.07863[hep-lat]

### Re $(A_0)$

Lattice

$$\mathrm{Re}(A_0) = 4.66(1.00)(1.21) \times 10^{-7}\ \mathrm{GeV}$$
$$\mathrm{Im}(A_0) = -1.90(1.23)(1.04) \times 10^{-11}\ \mathrm{GeV}$$

Experiment

$\mathrm{Re}\ (A_0) = 3.3201(18) \times 10^{-7}$ GeV

### $\varepsilon'/\varepsilon$

Lattice

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right]\right\}$$
$$= 1.38(5.15)(4.43) \times 10^{-4}.$$

Experiment

$\mathrm{Re}\ (\varepsilon'/\epsilon) = 16.6(2.3) \times 10^{-4}$

2+1 flavor QCD, $a^{-1} = 1.38$ GeV, $m_\pi = 143$ MeV

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# 4. EoS at Finite Temperature QCD

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## Finite temperature QCD

**Phase transition at finite T**

hadrons (quark confinement) → quark-gluon plasma (deconfinement)

$T \to \text{large}$

**Lattice QCD at finite temperature**

$N_s^3 \times N_T, N_T \ll N_s$ → $T = \frac{1}{N_t a}$

**Equation of State (EoS)** $p(T), s(T), \varepsilon(T)$

free energy

$F = -T \log Z$ →

pressure $\frac{p(T)}{T} = \frac{\partial \log Z}{\partial V} \simeq \frac{\ln Z}{V}$

energy density $\varepsilon(T) = -\frac{1}{V}\frac{\partial \log Z}{\partial 1/T}$

entropy density $s(T) = \frac{1}{V}\frac{\partial (T \log Z)}{\partial T} = \frac{p(T)}{T} + \frac{\varepsilon(T)}{T}$

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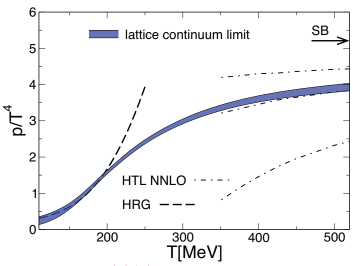
## The latest lattice results

**Equation of states from lattice QCD**

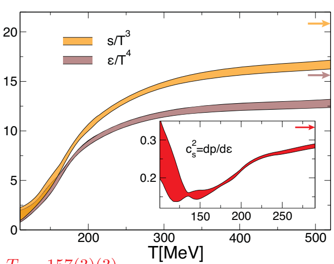
Borsanyi et al.

PoS(LATTICE2013)155. (arXiv:1312.2193[hep-lat], 2+1 flavor QCD)

**Pressure** **Entropy & energy density**




$T_c = 157(3)(3)$




$T_c = 157(3)(3)$

speed of sound

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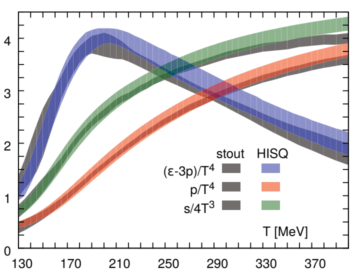
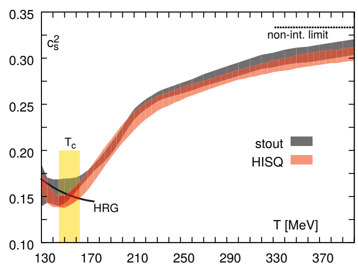
## EoS from other group

Bazavov et al. (HotQCD Collaboration)

PRD90(2014)9,094503. (arXiv:1407.6387[hep-lat], 2+1 flavor QCD)

**Trace anomaly, pressure & entropy** **Speed of sound**

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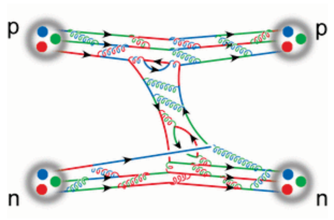
# 5. Hadron interactions




--approaches to nuclear physics from lattice QCD--

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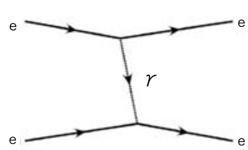
## 5-1. Hadron Interactions

Ex. Nuclear Force

1949 Nobel prize (1st in Japan)

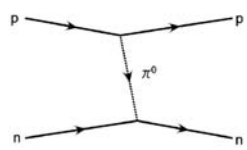
Meson Theory (before quarks) 1935 Hideki Yukawa (1st director of YITP)

- Nucleons interact with each other by exchanging virtual particles.
- the interaction range is proportional to the inverse of the virtual particle's mass
  - -> the virtual particles are heavier than electrons but lighter than nucleons
    - ->($\pi$)"meson"




Coulomb potential

$$V(r) = \frac{e^2}{4\pi}\frac{1}{r}$$




Yukawa potential

$$V(r) = \frac{g^2}{4\pi}\frac{e^{-m_\pi r}}{r}$$

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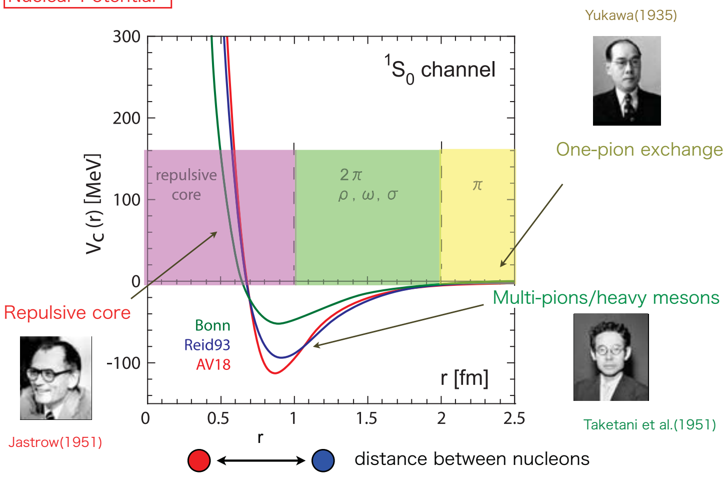
## Modern nuclear forces after Yukawa

Nuclear Potential

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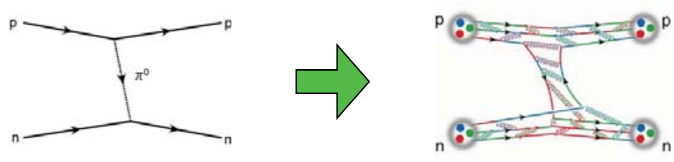
## Nuclear forces in terms of quarks ?




Meson Theory

Quark Theory

Much more difficult than masses.

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## 5-2. Three strategies to nuclear physics

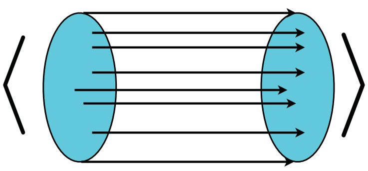
**Extreme** calculate nuclei directly from lattice QCD

Ab-Initio but (almost) impossible.

difficult to extract “physics” from results

difficult to apply results to other systems

nuclei propagator



$$\simeq e^{-m_A t} + \cdots$$

3A quark lines

A: atomic number

large number of contractions/very noisy

some reduction (Doi-Endres, CPC 184(2013)117)

**No reliable results so far in this approach.**

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**Standard** calculate NN phase shift from lattice QCD

Ab-Initio for phase shift. Results can not be directly applied to nuclear physics.

phase shift ⟹ nuclear potential ⟹ nuclear structure

Lüsher’s finite volume method for the phase shift

two particles in the finite box $(V = L^3)$

energy $E = 2\sqrt{\mathbf{k}^2 + m^2}$ ⟹ $\mathbf{k} \neq \frac{2\pi}{L}\mathbf{n}\ (\mathbf{n} \in Z^3)$

due to the interaction between two particles

⟹ phase shift $\delta_l(k_n)$

Formula (Ex.) $k \cot \delta_0(k) = \frac{2}{\sqrt{\pi} L} Z_{00}(1; q^2)$ $\quad k = |\mathbf{k}| \quad q = \frac{kL}{2\pi} \neq \mathbf{Z}$

generalize zeta-function $Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbf{Z}^3} (\mathbf{n}^2 - q^2)^{-s}$

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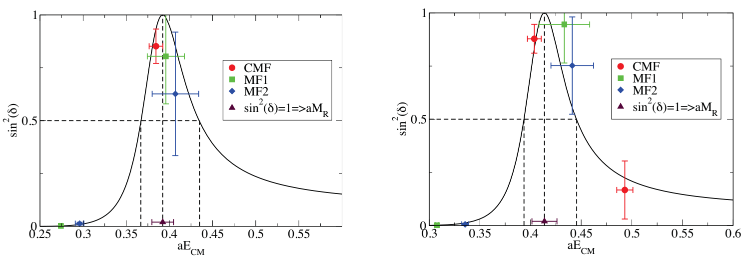
## $\pi^+\pi^-$ scattering ( $\rho$ meson width)

ETMC: Feng-Jansen-Renner, PLB684(2010)




$m_\pi = 290$ MeV $\qquad m_\pi = 330$ MeV

Resonance can be treated in this way.

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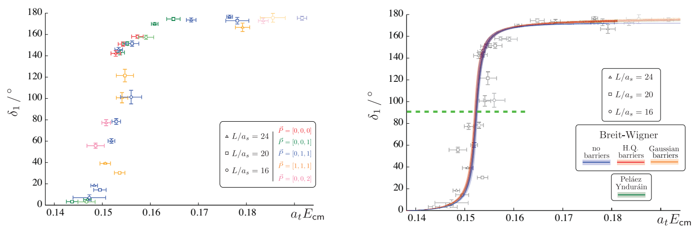
$I = 1\ \pi\pi$ scattering ($\rho$ resonance)

Dudek-Edwards-Thomas, PRD87(2013)034505

$\delta_1(E_{\rm cm})$




2-flavor anisotropic clover fermion

$a_s \sim 0.12$ fm

$m_\pi \sim 400$ MeV

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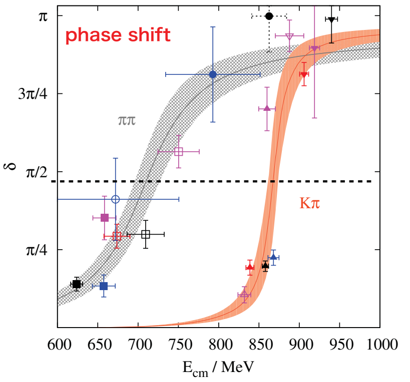
P-wave $\pi\pi$ and $K\pi$ scattering

2-flavor QCD, $a \simeq 0.071\text{fm}$, $m_\pi \simeq 150$ MeV

Bali *et al.*, arXiv:1512.08678[hep-lat]




**Breit-Wigner fit**

$$\tan\delta = \frac{g^2}{6\pi}\frac{p_{cm}^3}{E_{cm}(m_R^2 - E_{cm}^2)}$$

$$\Gamma = \frac{g^2}{6\pi}\frac{p_R^3}{m_R^2} \quad \text{Witdh}$$

$$m_\rho = 715(16)(21)\,\text{MeV}, \quad m_{K^*} = 869(6)(26)\,\text{MeV}$$

$$g_{\rho\pi\pi} = 5.37 \pm 0.79, \quad g_{K^*K\pi} = 4.43 \pm 0.66,$$
$$\Gamma_\rho = 120(32)(4)\,\text{MeV}, \quad \Gamma_{K^*} = 26(8)(1)\,\text{MeV},$$

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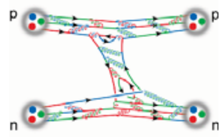
Alternative calculate nuclear potential from lattice QCD — strategy in this talk

Ab-Initio for potential. "Physics" is clear

nuclear potential ⇨ nuclear structure

Difficulties

A. Interactions are much more difficult than masses.




more complicated diagrams,
larger volume,
more Monte-Carlo sampling, etc.

B. Definition of potential in quantum theories ?

classical $V(x)$ ⇨ quantum $V(x)$ — potential is an input

no classical NN potentials — QCD — $V_{NN}(x)$ ? — output from QCD

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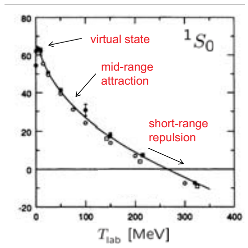
## Potentials in QCD ?

What are "potentials" in quantum field theories such as QCD ?

"Potentials" themselves can NOT be directly measured. cf. running coupling in QCD

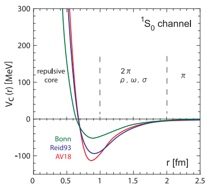
scheme dependent, Unitary transformation

experimental data of scattering phase shifts




inverse scattering ?

potentials, but not unique




useful to "understand" physics

cf. asymptotic freedom

"Potentials" are useful tools to extract observables such as scattering phase shift.

⇨ One may adopt a convenient definition of potentials as long as they reproduce correct physics of QCD.

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## 5-3. Our strategy in lattice QCD

Full details: Aoki, Hatsuda & Ishii, PTP123(2010)89

Consider "elastic scattering"

$NN \to NN$ ~~$NN \to NN + \text{others}$~~ ~~$(NN \to NN + \pi, NN + \bar{N}N, \cdots)$~~

energy $W_k = 2\sqrt{\mathbf{k}^2 + m_N^2} < W_{\text{th}} = 2m_N + m_\pi$ Elastic threshold

**Quantum Field Theoretical consideration**

Unitarity constrains S-matrix below inelastic threshold as

$$S = e^{2i\delta}$$

Ex. Scalar particles

$$\delta(k) = \begin{pmatrix} \delta_0(k) & & & \\ & \delta_1(k) & & \\ & & \delta_2(k) & \\ & & & \cdots \end{pmatrix}$$

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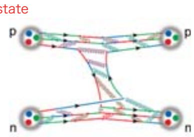
Step 1 define (Equal-time) Nambu-Bethe-Salpeter (NBS) Wave function

Spin model: Balog et al., 1999/2001

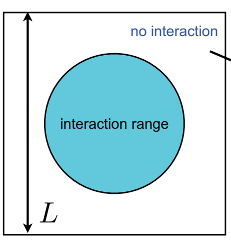
$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0 | N(\mathbf{x}+\mathbf{r}, 0) N(\mathbf{x}, 0) | NN, W_k \rangle$$

QCD eigen-state

$N(x) = \varepsilon_{abc} q^a(x) q^b(x) q^c(x)$: local operator "scheme"




Asymptotic behavior of NBS wave function




$r = |\mathbf{r}| \to \infty$

partial wave

$$\varphi_{\mathbf{k}}^l \to A_l \frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr}$$

NBS wave function ⇔ scattering wave function in quantum mechanics

cf. Luescher's finite volume method — allowed $k$ at $L$ ⇨ $\delta_l(k_n)$

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Step 2 define non-local but energy-independent "potential" as

$$\left[\epsilon_k - H_0\right]\varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, U(\mathbf{x},\mathbf{y})\varphi_{\mathbf{k}}(\mathbf{y})$$

$\mu = m_N/2$ reduced mass

$\epsilon_k = \dfrac{\mathbf{k}^2}{2\mu}$ $\quad H_0 = \dfrac{-\nabla^2}{2\mu}$ $\quad$ non-local potential

(Trivial) proof of "existence"

We can construct a non-local but energy-independent potential easily as

$$U(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'}^{W_k, W_{k'} \le W_{\text{th}}} \left[\epsilon_k - H_0\right]\varphi_{\mathbf{k}}(\mathbf{x})\eta^{-1}_{\mathbf{k},\mathbf{k}'}\varphi^\dagger_{\mathbf{k}'}(\mathbf{y})$$

inner product: $\eta^{-1}_{\mathbf{k},\mathbf{k}'}$: inverse of $\eta_{\mathbf{k},\mathbf{k}'} = (\varphi_{\mathbf{k}}, \varphi_{\mathbf{k}'})$

For $^\forall W_{\mathbf{p}} < W_{\text{th}}$

$$\int d^3y\, U(\mathbf{x},\mathbf{y})\phi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'}\left[\epsilon_k - H_0\right]\varphi_{\mathbf{k}}(x)\eta^{-1}_{\mathbf{k},\mathbf{k}'}\eta_{\mathbf{k}',\mathbf{p}} = \left[\epsilon_p - H_0\right]\varphi_{\mathbf{p}}(x)$$

Remark

Non-relativistic approximation is NOT used. We just take the specific (equal-time) frame.

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Step 3 expand the non-local potential in terms of derivative as

$$U(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\nabla)\delta^3(\mathbf{x}-\mathbf{y})$$

$$V(\mathbf{x},\nabla) = V_0(r) + V_\sigma(r)(\sigma_1\cdot\sigma_2) + V_T(r)S_{12} + V_{LS}(r)\mathbf{L}\cdot\mathbf{S} + O(\nabla^2)$$

LO LO LO NLO NNLO

tensor operator $S_{12} = \frac{3}{r^2}(\sigma_1\cdot\mathbf{x})(\sigma_2\cdot\mathbf{x}) - (\sigma_1\cdot\sigma_2)$ spins

This expansion is a part of our "scheme" for potentials.

Step 4 extract the local potential at LO as

$$V_{\rm LO}(\mathbf{x}) = \frac{[\epsilon_k - H_0]\varphi_{\mathbf{k}}(\mathbf{x})}{\varphi_{\mathbf{k}}(\mathbf{x})}$$

Step 5 solve the Schroedinger Eq. in the infinite volume with this potential.

phase shifts and binding energy below inelastic threshold

We can check a size of errors of the LO in the expansion. (See later).

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## 5.4 Results

Standard method to extract NBS wave function

NBS wave function

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0|N(\mathbf{x}+\mathbf{r},0)N(\mathbf{x},0)|NN,W_k\rangle$$

Potential

$$[\epsilon_k - H_0]\varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, U(\mathbf{x},\mathbf{y})\varphi_{\mathbf{k}}(\mathbf{y})$$

4-pt Correlation function source for NN

$$F(\mathbf{r},t-t_0) = \langle 0|T\{N(\mathbf{x}+\mathbf{r},t)N(\mathbf{x},t)\}\overline{\mathcal{J}}(t_0)|0\rangle$$

complete set for NN

$$F(\mathbf{r},t-t_0) = \langle 0|T\{N(\mathbf{x}+\mathbf{r},t)N(\mathbf{x},t)\}\sum_{n,s_1,s_2}|2N,W_n,s_1,s_2\rangle\langle 2N,W_n,s_1,s_2|\overline{\mathcal{J}}(t_0)|0\rangle + \cdots$$

$$= \sum_{n,s_1,s_2} A_{n,s_1,s_2}\varphi^{W_n}(\mathbf{r})e^{-W_n(t-t_0)}, \quad A_{n,s_1,s_2} = \langle 2N,W_n,s_1,s_2|\overline{\mathcal{J}}(0)|0\rangle.$$

ground state saturation at large t

$$\lim_{(t-t_0)\to\infty} F(\mathbf{r},t-t_0) = A_0\varphi^{W_0}(\mathbf{r})e^{-W_0(t-t_0)} + O(e^{-W_{n\neq 0}(t-t_0)})$$

NBS wave function

This is a standard method in lattice QCD and was employed for our first calculation.

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Improved method Ishii et al. (HALQCD), PLB712(2012) 437

normalized 4-pt function $R(\mathbf{r},t) \equiv F(\mathbf{r},t)/(e^{-m_N t})^2 = \sum_n A_n\varphi^{W_n}(\mathbf{r})e^{-\Delta W_n t}$

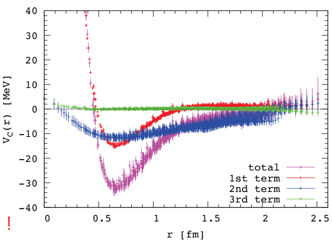
$$\Delta W_n = W_n - 2m_N = \frac{\mathbf{k}_n^2}{m_N} - \frac{(\Delta W_n)^2}{4m_N}$$

$$-\frac{\partial}{\partial t}R(\mathbf{r},t) = \left\{H_0 + U - \frac{1}{4m_N}\frac{\partial^2}{\partial t^2}\right\}R(\mathbf{r},t)$$

potential Leading Order

$$\left\{-H_0 - \frac{\partial}{\partial t} + \frac{1}{4m_N}\frac{\partial^2}{\partial t^2}\right\}R(\mathbf{r},t) = \int d^3r'\, U(\mathbf{r},\mathbf{r}')R(\mathbf{r}',t) = V_C(\mathbf{r})R(\mathbf{r},t) + \cdots$$

1st 2nd 3rd total




3rd term(relativistic correction) is negligible.

Ground state saturation is no more required ! (advantage over finite volume method.)

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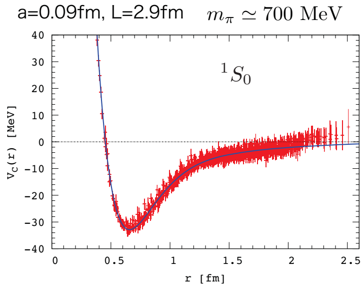
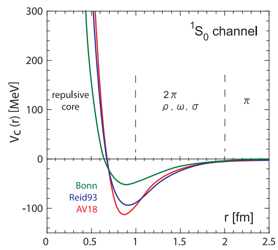
NN potential 2+1 flavor QCD, spin-singlet potential (PLB712(2012)437)

a=0.09fm, L=2.9fm $m_\pi \simeq 700$ MeV




phenomenological potential




Qualitative features of NN potential are reproduced !

(1)attractions at medium and long distances
(2)repulsion at short distance(repulsive core)

1st paper(quenched QCD): Ishii-Aoki-Hatsuda, PRL90(2007)0022001

selected as one of 21 papers in Nature Research Highlights 2007. (One from Physics, Two from Japan, the other is on "iPS" by Sinya Yamanaka et al. )

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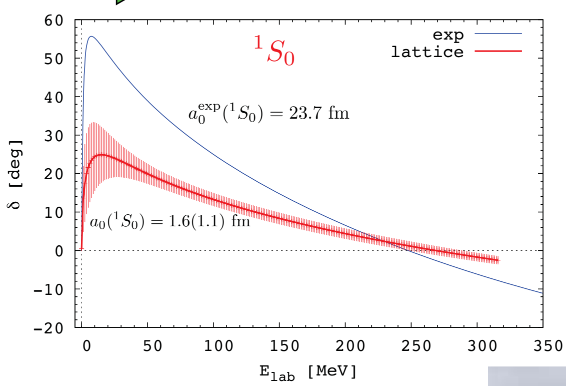
NN potential phase shift




It has a reasonable shape. The strength is weaker due to the heavier quark mass.

Need calculations at physical quark mass on K-computer.

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# 6. Summary

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- Lattice QCD is a very powerful method to investigate dynamics of quarks
- not only hadron masses but also hadron interactions can be investigated from the 1st principle
- the potential (HALQCD) method is new but very useful to investigate not only the nuclear force but also general baryonic interactions in (lattice) QCD.
- the method can be easily applied also to meson-baryon and meson-meson interactions.

## Our strategy

Potentials from lattice QCD → Nuclear Physics with these potentials → Neutron stars Supernova explosion

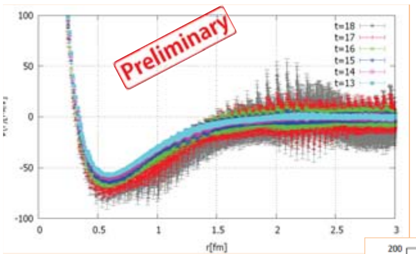

16年1月11日月曜日

# Back-up

16年1月11日月曜日

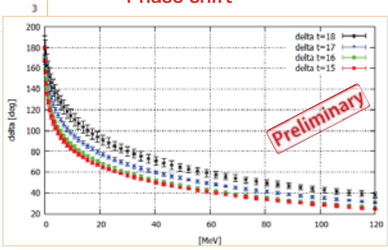
## Preliminary results at almost physical point

2+1 flavor QCD, $a \simeq 0.085$ fm, space-time $\simeq (8\ \text{fm})^4$, $m_\pi \simeq 145$ MeV




$\Omega\Omega$ potential

Phase shift

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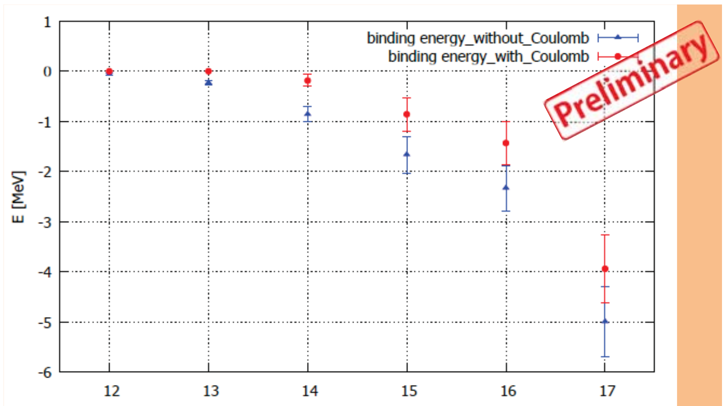
Binding energy

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### Convergence of velocity expansion: estimate 1

If the higher order terms are large, LO potentials determined from NBS wave functions at different energy become different.(cf. LOC of ChPT).

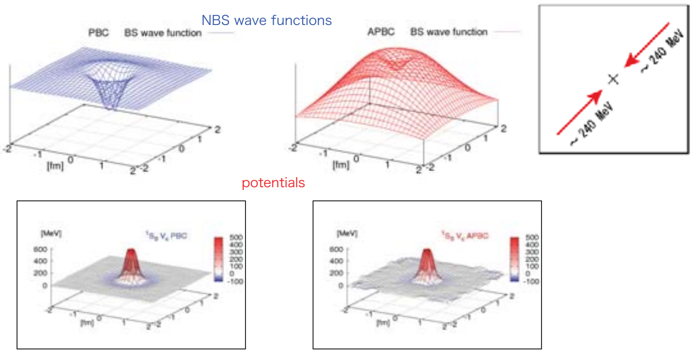
Numerical check in quenched QCD

$m_\pi \simeq 0.53$ GeV

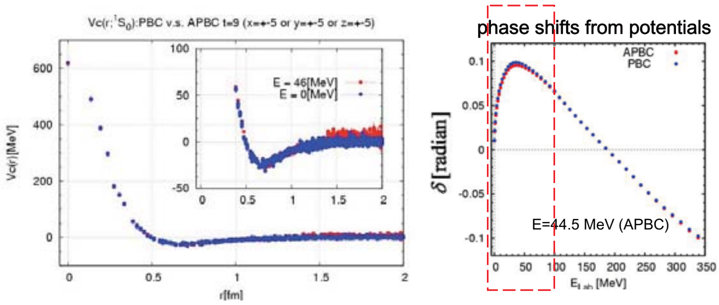
a=0.137fm, L=4.0 fm

K. Murano, N. Ishii, S. Aoki, T. Hatsuda PTP 125 (2011)1225.

• PBC (E∼0 MeV) • APBC (E∼46 MeV)




16年1月11日月曜日




Higher order terms turn out to be very small at low energy in our scheme.

Need to be checked at lighter pion mass in 2+1 flavor QCD.

Note: convergence of the velocity expansion can be checked within this method.

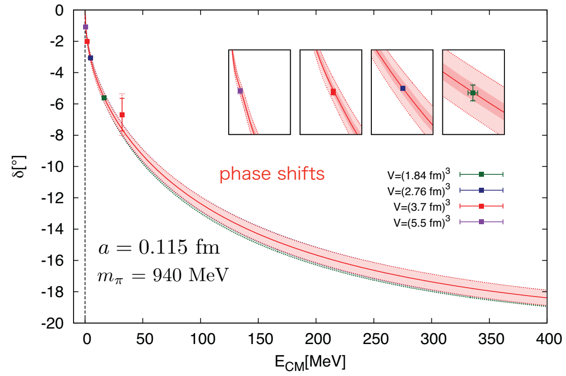
(in contrast to convergence of ChPT, convergence of perturbative QCD)

16年1月11日月曜日

## Convergence of velocity expansion: estimate 2

Kurth, Ishii, Doi, Aoki & Hatsuda, JHEP 1312(2013)015

Potential vs Luescher (I=2 pi-pi scattering. Quenched QCD)




both methods agree very well.

This establishes a validity of the potential method and shows a good convergence of the velocity expansion.

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## More structure at LO — Tensor potential

$$(H_0 + V_C(r) + V_T(r)S_{12})\psi(\mathbf{r};1^+) = E\psi(\mathbf{r};1^+)$$

J=1, S=1

mixing between $^3S_1$ and $^3D_1$ through the tensor force

$$\psi(\mathbf{r};1^+) = \mathcal{P}\psi(\mathbf{r};1^+) + \mathcal{Q}\psi(\mathbf{r};1^+)$$

$^3S_1$: "projection" to L=0 $^3D_1$: "projection" to L=2

$$\begin{aligned} H_0[\mathcal{P}\psi](\mathbf{r}) + V_C(r)[\mathcal{P}\psi](\mathbf{r}) + V_T(r)[\mathcal{P}S_{12}\psi](\mathbf{r}) &= E[\mathcal{P}\psi](\mathbf{r}) \\ H_0[\mathcal{Q}\psi](\mathbf{r}) + V_C(r)[\mathcal{Q}\psi](\mathbf{r}) + V_T(r)[\mathcal{Q}S_{12}\psi](\mathbf{r}) &= E[\mathcal{Q}\psi](\mathbf{r}) \end{aligned}$$

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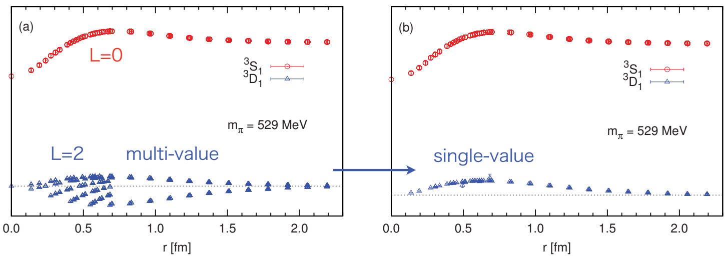
## Wave functions

Quenched

Aoki, Hatsuda, Ishii, PTP 123 (2010)89




divided by $Y_{20}(\theta,\phi)$

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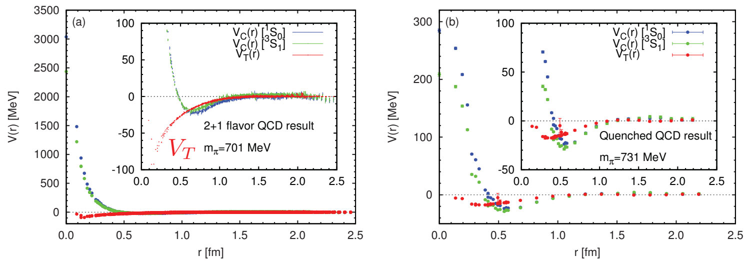
## Potentials




- no repulsive core in the tensor potential.
- the tensor potential is enhanced in full QCD

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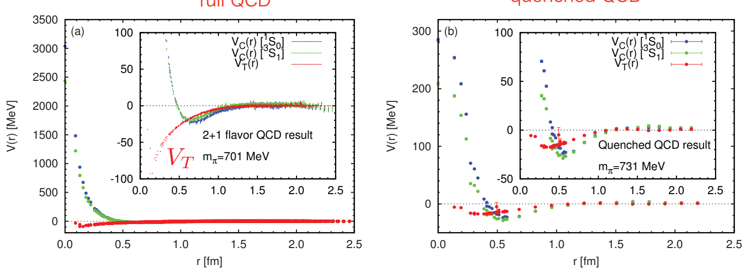
## Potentials




- no repulsive core in the tensor potential.
- the tensor potential is enhanced in full QCD

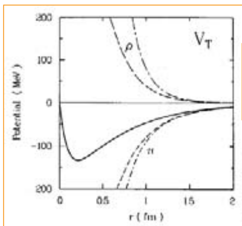



Fig. 3.7. The contributions from π and ρ (dashed) to the T = 0 tensor potential. The solid line is the full potential. The dash-dot lines are obtained when the cutoff is omitted.

from R.Machleidt, Adv.Nucl.Phys.**19**

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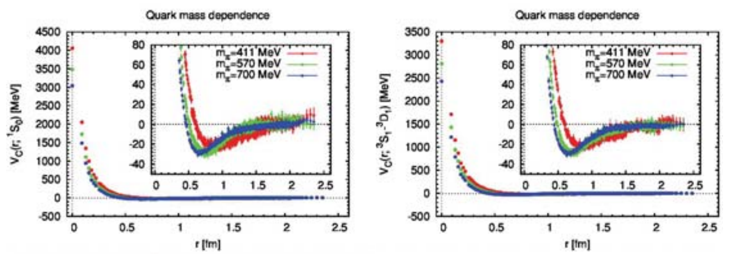
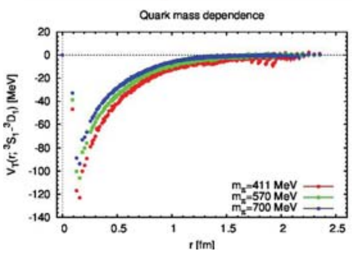
## Quark mass dependence (full QCD)




- the tensor potential increases as the pion mass decreases.
  - manifestation of one-pion-exchange ?
- both repulsive core and attractive pocket are also grow as the pion mass decreases.

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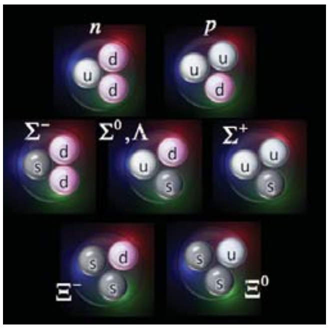
# 2. Hyperon Interactions

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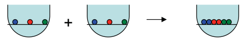
# Origin of the repulsive core ?

quarks are "fermion" → two can not occupy the same position. ("Pauli principle")

they have 3 colors(red,blue,green), 2 spin( ↑ ↓ ), 2 flavors(up,down)

→ 6 quark can occupy the same position



u↑ u↓ d↑ d↓ u↑ u↓

$p\uparrow$ $p\downarrow$

but allowed color combinations are limited + interaction among quarks

? → repulsive core ?

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# What happen if strange quarks are added ?

$\Lambda(uds)$ - $\Lambda(uds)$ interaction

u↑ d↓ s↑ u↓ d↑ s↓

all color combinations are allowed

? → no repulsive core ?

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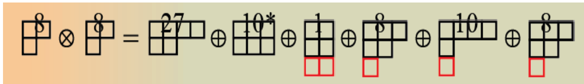
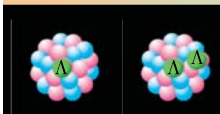
# Octet Baryon interactions



- phase shift available for YN and YY scattering are limited
- plenty of hyper-nucleus data will be soon available at J-PARC


- prediction from lattice QCD
- difference between NN and YN ?

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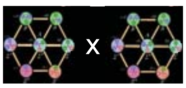
# Baryon Potentials in the flavor SU(3) limit

$m_u = m_d = m_s$

1. First setup to predict YN, YY interactions not accessible in exp.
2. Origin of the repulsive core (universal or not)



$$8 \times 8 = \underline{27 + 8s + 1} + \underline{10^* + 10 + 8a}$$

Symmetric Anti-symmetric

6 independent potentials in flavor-basis

$$V^{(27)}(r),\ V^{(8s)}(r),\ V^{(1)}(r) \leftarrow {}^1S_0$$
$$V^{(10^*)}(r),\ V^{(10)}(r),\ V^{(8a)}(r) \leftarrow {}^3S_1$$

3-flavor QCD a=0.12 fm

Inoue et al. (HAL QCD Coll.), PTP124(2010)591 L=2 fm

Inoue et al. (HAL QCD Coll.), NPA881(2012)28 L=2-4 fm

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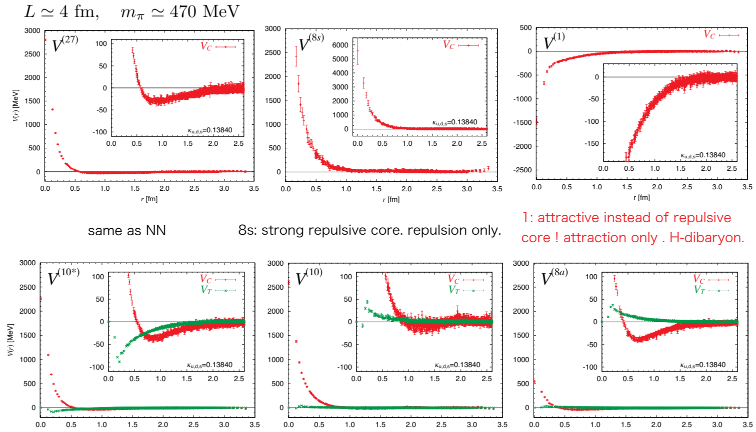
$L \simeq 4$ fm, $m_\pi \simeq 470$ MeV




same as NN

8s: strong repulsive core. repulsion only.

1: attractive instead of repulsive core ! attraction only . H-dibaryon.

same as NN

10: strong repulsive core. weak attraction.

8a: weak repulsive core. strong attraction.

Flavor dependences of BB interactions become manifest in SU(3) limit !

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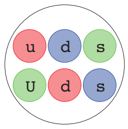
H-dibaryon:
a possible six quark state(uuddss)
predicted by the model but not observed yet.




http://physics.aps.org/synopsis-for/10.1103/PhysRevLett.106.162001

**Binding baryons on the lattice**

April 26, 2011

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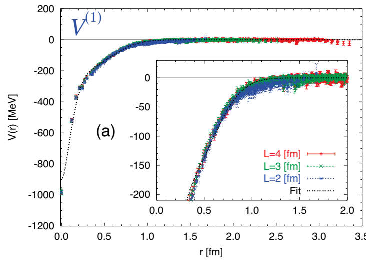
## H-dibaryon in the flavor SU(3) limit

Inoue et al. (HAL QCD Coll.), PRL106(2011)162002

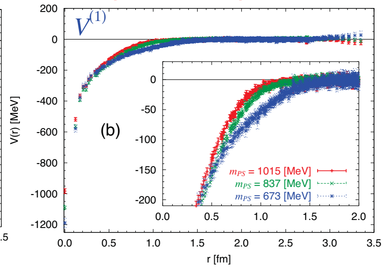
a=0.12 fm

Attractive potential in the flavor singlet channel → possibility of a bound state (H-dibaryon) $\Lambda\Lambda - N\Xi - \Sigma\Sigma$

volume dependence




pion mass dependence




L=3 fm is enough for the potential.

lighter the pion mass, stronger the attraction

fit potentials at L=4 fm by $V(r) = a_1 e^{-a_2 r^2} + a_3 \left(1 - e^{-a_4 r^2}\right)^2 \left(\frac{e^{-a_5 r}}{r}\right)^2$

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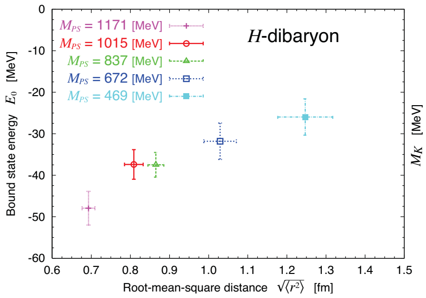
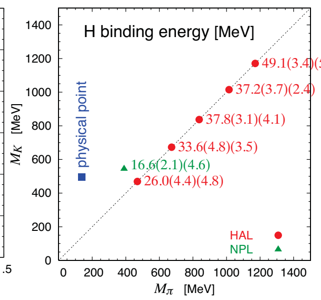
Solve Schroedinger equation in the infinite volume → One bound state (H-dibaryon) exists.




An H-dibaryon exists in the flavor SU(3) limit.
Binding energy = 25-50 MeV at this range of quark mass.
A mild quark mass dependence.

Real world ?

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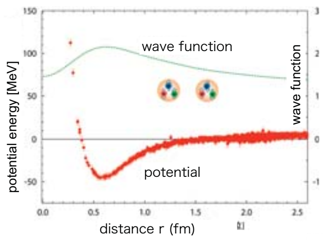
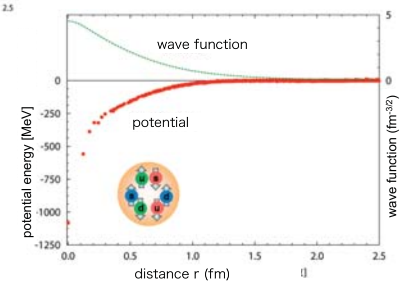
Deuteron




H-dibayon

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# 3. Extensions

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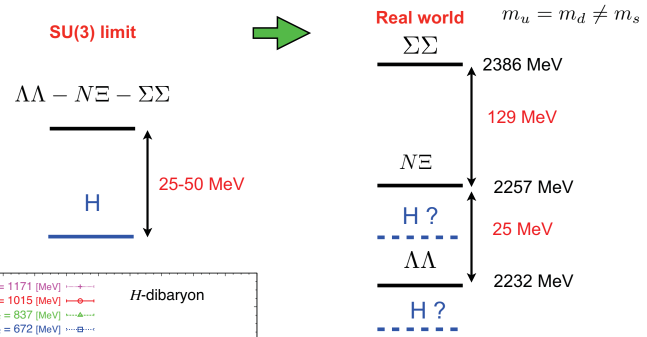
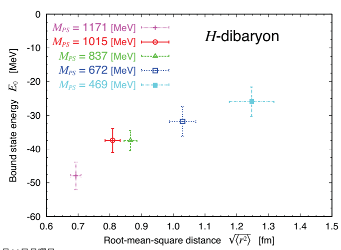
## H-dibaryon with the flavor SU(3) breaking

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## S=-2 "Inelastic" scattering

$m_N = 939$ MeV, $m_\Lambda = 1116$ MeV, $m_\Sigma = 1193$ MeV, $m_\Xi = 1318$ MeV

S=-2 System(I=0)

$M_{\Lambda\Lambda} = 2232$ MeV $< M_{N\Xi} = 2257$ MeV $< M_{\Sigma\Sigma} = 2386$ MeV

The eigen-state of QCD in the finite box is a mixture of them:

$$|S=-2, I=0, E\rangle_L = c_1(L)|\Lambda\Lambda, E\rangle + c_2(L)|\Xi N, E\rangle + c_3(L)|\Sigma\Sigma, E\rangle$$

$$E = 2\sqrt{m_\Lambda^2 + \mathbf{p}_1^2} = \sqrt{m_\Xi^2 + \mathbf{p}_2^2} + \sqrt{m_N^2 + \mathbf{p}_2^2} = 2\sqrt{m_\Sigma^2 + \mathbf{p}_3^2}$$

In this situation, we can not directly extract the scattering phase shift in lattice QCD.

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## Extended method

Let us consider 2-channel problem for simplicity.

NBS wave functions for 2 channels at 2 values of energy:

$$\Psi_\alpha^{\Lambda\Lambda}(\mathbf{x}) = \langle 0|\Lambda(\mathbf{x})\Lambda(\mathbf{0})|E_\alpha\rangle$$
$$\Psi_\alpha^{\Xi N}(\mathbf{x}) = \langle 0|\Xi(\mathbf{x})N(\mathbf{0})|E_\alpha\rangle \qquad \alpha = 1,2$$

They satisfy

$$(\nabla^2 + \mathbf{p}_\alpha^2)\Psi_\alpha^{\Lambda\Lambda}(\mathbf{x}) = 0$$
$$(\nabla^2 + \mathbf{q}_\alpha^2)\Psi_\alpha^{\Xi N}(\mathbf{x}) = 0 \qquad |\mathbf{x}| \to \infty$$

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We define the "potential" from the coupled channel Schroedinger equation:

$$\left(\frac{\nabla^2}{2\mu_{\Lambda\Lambda}} + \frac{\mathbf{p}_\alpha^2}{2\mu_{\Lambda\Lambda}}\right)\Psi_\alpha^{\Lambda\Lambda}(\mathbf{x}) = V^{\Lambda\Lambda\leftarrow\Lambda\Lambda}(\mathbf{x})\Psi_\alpha^{\Lambda\Lambda}(\mathbf{x}) + V^{\Lambda\Lambda\leftarrow\Xi N}(\mathbf{x})\Psi_\alpha^{\Xi N}(\mathbf{x})$$

diagonal off-diagonal

$$\left(\frac{\nabla^2}{2\mu_{\Xi N}} + \frac{\mathbf{q}_\alpha^2}{2\mu_{\Xi N}}\right)\Psi_\alpha^{\Xi N}(\mathbf{x}) = V^{\Xi N\leftarrow\Lambda\Lambda}(\mathbf{x})\Psi_\alpha^{\Lambda\Lambda}(\mathbf{x}) + V^{\Xi N\leftarrow\Xi N}(\mathbf{x})\Psi_\alpha^{\Xi N}(\mathbf{x})$$

off-diagonal diagonal

$\mu$: reduced mass

$$\begin{pmatrix}(E_1 - H_0^X)\Psi_1^X(\mathbf{x})\\(E_2 - H_0^X)\Psi_2^X(\mathbf{x})\end{pmatrix} = \begin{pmatrix}\Psi_1^X(\mathbf{x}) & \Psi_1^Y(\mathbf{x})\\\Psi_2^X(\mathbf{x}) & \Psi_2^Y(\mathbf{x})\end{pmatrix}\begin{pmatrix}V^{X\leftarrow X}(\mathbf{x})\\V^{X\leftarrow Y}(\mathbf{x})\end{pmatrix}$$

$$E_\alpha = \frac{\mathbf{p}_\alpha^2}{2\mu_{\Lambda\Lambda}}, \frac{\mathbf{q}_\alpha^2}{2\mu_{\Xi N}} \qquad X \neq Y \qquad X, Y = \Lambda\Lambda \text{ or } \Xi N$$

$$\begin{pmatrix}V^{X\leftarrow X}(\mathbf{x})\\V^{X\leftarrow Y}(\mathbf{x})\end{pmatrix} = \begin{pmatrix}\Psi_1^X(\mathbf{x}) & \Psi_1^Y(\mathbf{x})\\\Psi_2^X(\mathbf{x}) & \Psi_2^Y(\mathbf{x})\end{pmatrix}^{-1}\begin{pmatrix}(E_1 - H_0^X)\Psi_1^X(\mathbf{x})\\(E_2 - H_0^X)\Psi_2^X(\mathbf{x})\end{pmatrix}$$

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Using the coupled channel potentials:

$$\begin{pmatrix}V^{\Lambda\Lambda\leftarrow\Lambda\Lambda}(\mathbf{x}) & V^{\Xi N\leftarrow\Lambda\Lambda}(\mathbf{x})\\V^{\Lambda\Lambda\leftarrow\Xi N}(\mathbf{x}) & V^{\Xi N\leftarrow\Xi N}(\mathbf{x})\end{pmatrix}$$

we solve the coupled channel Schroedinger equation in the infinite volume with an appropriate boundary condition.

For example, we take the incomming $\Lambda\Lambda$ state by hand.

In this way, we can avoid the mixture of several "in"-states.

$$|S=-2, I=0, E\rangle_L = c_1(L)|\Lambda\Lambda, E\rangle + c_2(L)|\Xi N, E\rangle + c_3(L)|\Sigma\Sigma, E\rangle$$

Lattice is a tool to extract the interaction kernel ("T-matrix" or "potential").

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## Preliminary results from HAL QCD Collaboration

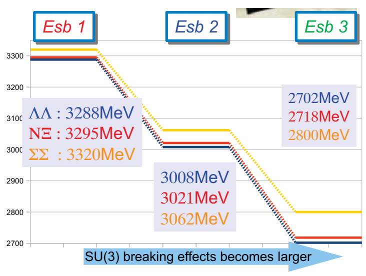
Sasaki for HAL QCD Collaboration

Gauge ensembles

| In unit of MeV | Esb 1 | Esb 2 | Esb 3 |
|---|---|---|---|
| π | 701±1 | 570±2 | 411±2 |
| K | 789±1 | 713±2 | 635±2 |
| $m_\pi/m_K$ | 0.89 | 0.80 | 0.65 |
| N | 1585±5 | 1411±12 | 1215±12 |
| Λ | 1644±5 | 1504±10 | 1351± 8 |
| Σ | 1660±4 | 1531±11 | 1400±10 |
| Ξ | 1710±5 | 1610± 9 | 1503± 7 |

u,d quark masses lighter

thresholds

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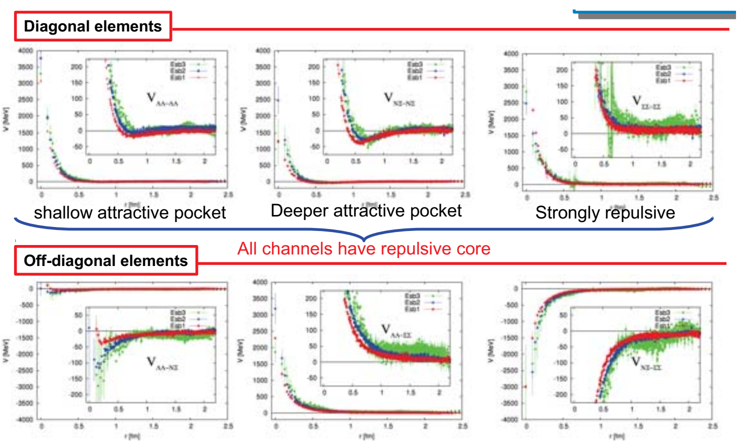
coupled channel 3x3 potentials

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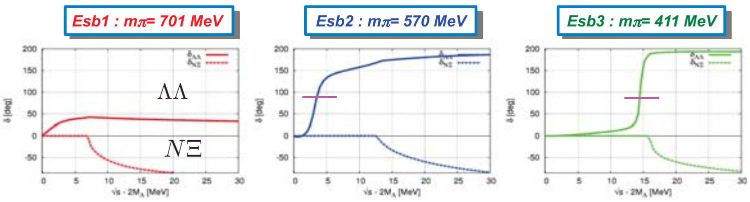
## $\Lambda\Lambda$ and $N\Xi$ phase shift

**Preliminary !**




Bound H-dibaryon coupled to NΞ

H as ΛΛ resonance
H as bound NΞ

H as ΛΛ resonance
H as bound NΞ

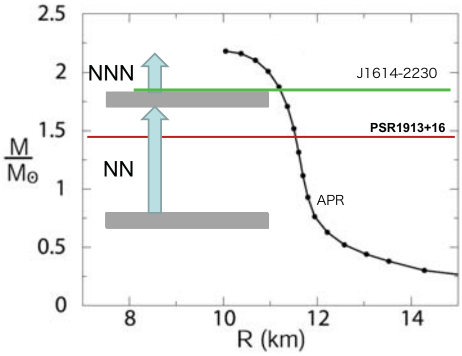
This suggests that H-dibaryon becomes resonance at physical point.
Below or above NΞ ? Need simulation at physical point.

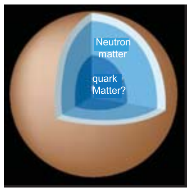
Physically, it is essential that H-dibaryon is a bound state in the flavor SU(3) limit.



## 4. Challenge: Three nucleon force (TNF)

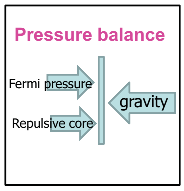
Maximum mass of neutron stars




($\rho_{max}$ ~ $6\rho_0$)

sustains neutron stars against gravitational collapse



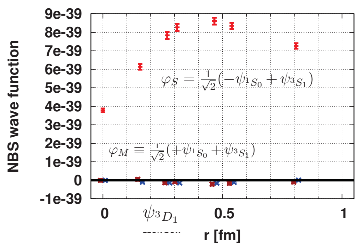



## TNF from lattice QCD

Doi et al. (HAL QCD), PTP 127 (2012) 723

(1,2) pair $\quad {}^1S_0,\ {}^3S_1,\ {}^3D_1 \quad$ S-wave only

$\mathrm{Triton}(I = 1/2,\ J^P = 1/2^+)$

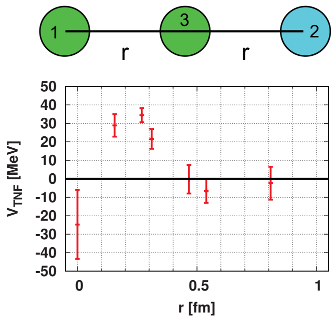



$\varphi_S = \frac{1}{\sqrt{2}}(-\psi_{^1S_0} + \psi_{^3S_1})$

$\varphi_M \equiv \frac{1}{\sqrt{2}}(+\psi_{^1S_0} + \psi_{^3S_1})$

$\psi_{^3D_1}$

Linear setup




scalar/isoscalar TNF is observed at short distance.

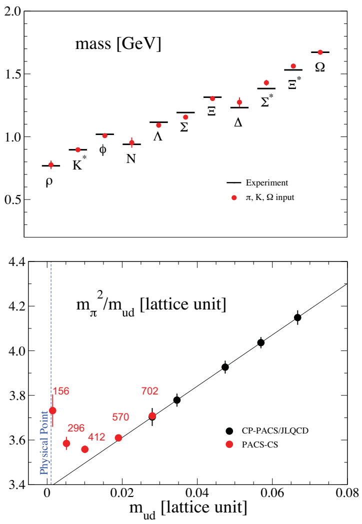
further study is needed to confirm this result.



# More back up



PACS-CS Collaboration




Phys. Rev. D79 (2009) 034503

$a = 0.09$ fm

$L = 2.9$ fm $\quad m_\pi L = 2.3$

$m_\pi^{\rm min.} = 156$ MeV

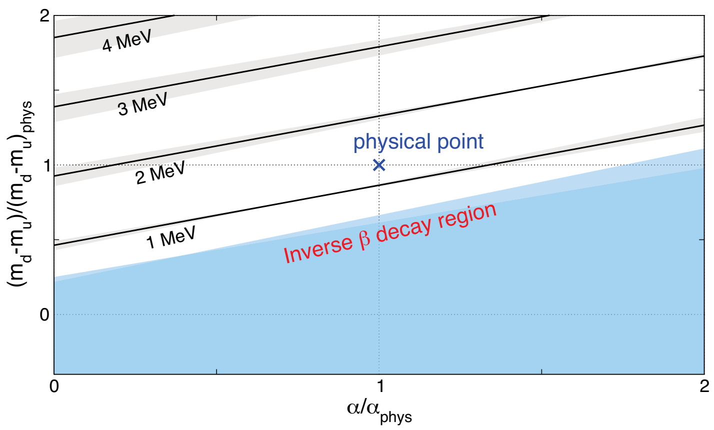
Almost on physical quark mass
(no chiral extrapolation)

chiral extrapolation vs. physical point

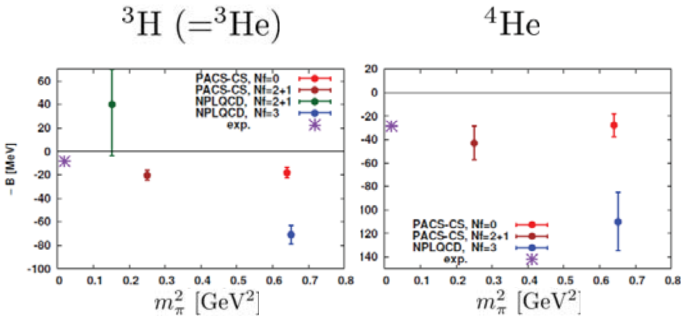
Chiral extrapolation sometimes becomes non-trivial due to the chiral-log, as shown in the figure.



## Fine tuning in Nature ?



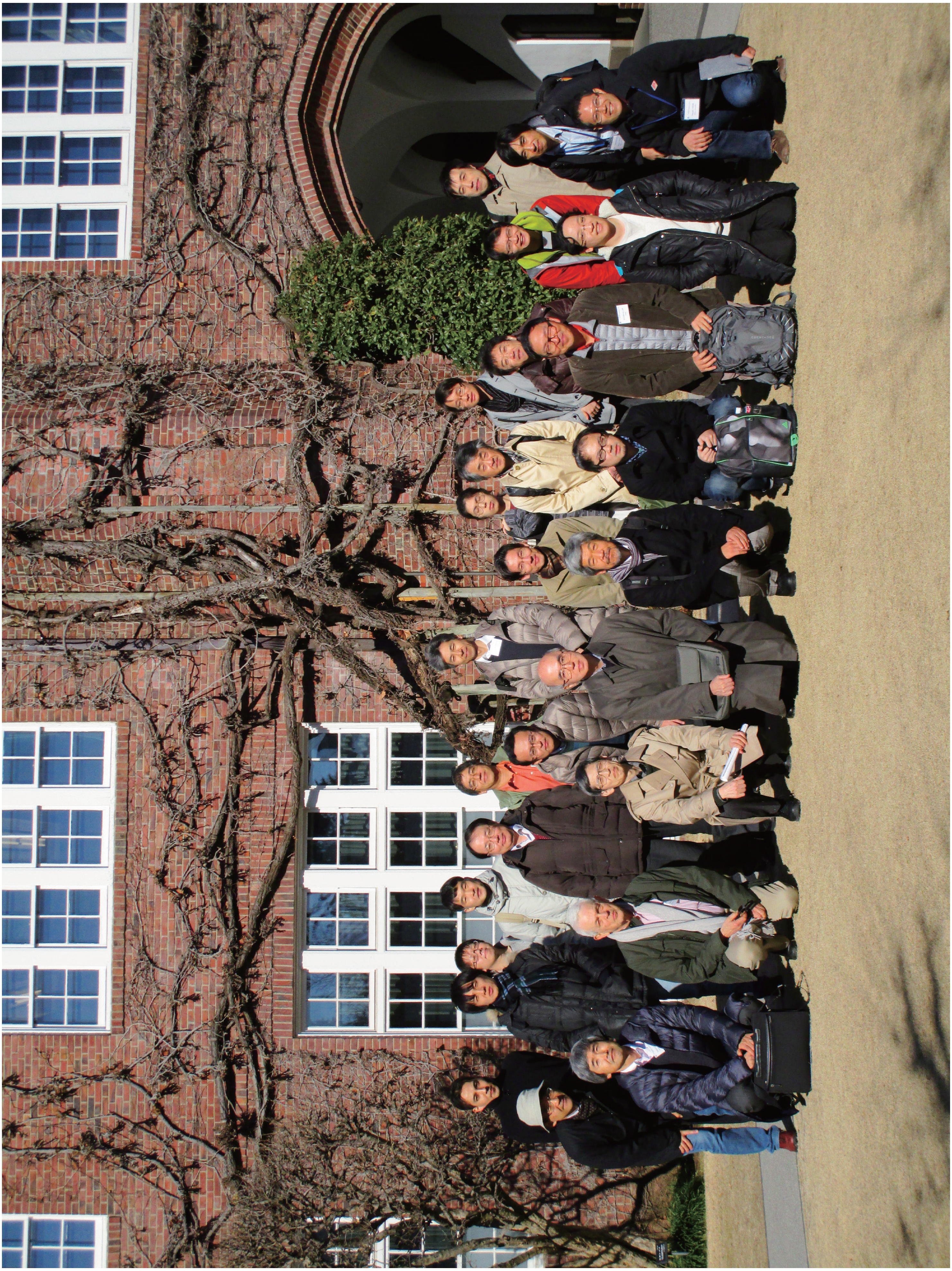
binding energy of A=3,4 nuclei




PACS-CS, PRD81(2010)111504, PRD86 (2012) 074514.
NPLQCD, PPNP66(2011)1, arXiv:1004.2935.

signals can be obtained, though results scatter.

16年1月11日月曜日



| | Given Name | Family Name | Affiliation |
|---|---|---|---|
| 1 | Sinya | Aoki | Yukawa Institute for Theoretical Physics, Kyoto University |
| 2 | Hidetoshi | Awata | Graduate School of Mathematics, Nagoya University |
| 3 | Tetsuo | Deguchi | Ochanomizu University |
| 4 | Richard | Eager | Universität Heidelberg |
| 5 | Tohru | Eguchi | Rikkyo University |
| 6 | Masayuki | Fukuda | The University of Tokyo |
| 7 | Masashi | Hamanaka | Nagoya University |
| 8 | Chika | Hasegawa | Rikkyo University |
| 9 | Yasuyuki | Hatsuda | University of Geneva |
| 10 | Yasuaki | Hikida | Rikkyo University |
| 11 | Kentato | Hori | Kavli IPMU |
| 12 | Takashi | Imamura | Chiba University |
| 13 | Ivan | Ip | Kyoto University |
| 14 | Akihiro | Ishibashi | Kinki University |
| 15 | Katsushi | Ito | Tokyo Institute of Technology |
| 16 | Keiichi R. | Ito | Rikkyo University |
| 17 | Michio | Jimbo | Rikkyo University |
| 18 | Saburo | Kakei | Rikkyo University |
| 19 | Masataka | Kanki | The University of Tokyo |
| 20 | Akishi | Kato | The University of Tokyo |
| 21 | Makoto | Katori | Department of Physics, Chuo University |
| 22 | Yoichi | Kazama | Research Center for Mathematical Physics, Rikkyo University |
| 23 | Yusuke | Kimura | Okayama Institute for Quntum Physics |
| 24 | Kei-Ichi | Kondo | Chiba University |
| 25 | Hitoshi | Konno | Tokyo University of Marine Science and Technology |

| | Given Name | Family Name | Affiliation |
|---|---|---|---|
| 26 | Tomohiko | Konno | Waseda University |
| 27 | Toshiro | Kuwabara | University of Tsukuba |
| 28 | Kimyeong | Lee | Korea Institute for Advanced Study |
| 29 | Makiko | Mase | Tokyo Metropolitan University |
| 30 | Yutaka | Matsuo | The University of Tokyo |
| 31 | Hajime | Moriya | Shibaura Insititute of Technology |
| 32 | Sanefumi | Moriyama | Osaka City University |
| 33 | Hisayoshi | Muraki | University of Tsukuba |
| 34 | Tadahito | Nakajima | Nihon University |
| 35 | Masayuki | Nakamura | Springer Japan KK |
| 36 | Toshio | Nakatsu | Setsunan University |
| 37 | Yu | Nakayama | Kavli IPMU / California Institute of Technology |
| 38 | Hiroshi | Naruse | University of Yamanashi |
| 39 | Tomoki | Nosaka | Yukawa Institute for Theoretical Physics, Kyoto University |
| 40 | Satoru | Odake | Shinshu University |
| 41 | Kazumi | Okuyama | Shinshu University |
| 42 | Kazuhiro | Sakai | Meiji Gakuin University |
| 43 | Noriaki | Sannomiya | The University of Tokyo |
| 44 | Ryu | Sasaki | Shinshu University |
| 45 | Yuji | Satoh | University of Tsukuba |
| 46 | Tokuzo | Shimada | Department of Physics, Meiji University |
| 47 | Hongfei | Shu | Tokyo Institute of Technology |
| 48 | Daniel | Sternheimer | Rikkyo University / University de Bourgogne |
| 49 | Kennichi | Sugiyama | Rikkyo University |
| 50 | Junji | Suzuki | Shizuoka University |

| | Given Name | Family Name | Affiliation |
|---|---|---|---|
| 51 | Norio | Suzuki | Kitami Institute of Technology |
| 52 | Tsukasa | Tada | RIKEN |
| 53 | Fuminobu | Takahashi | Tohoku University |
| 54 | Kanehisa | Takasaki | Kinki University |
| 55 | Takashi | Takebe | National Research University Higher School of Economics |
| 56 | Shintarou | Yanagida | Research Institute for Mathematical Sciences |
| 57 | Shuichi | Yokoyama | Keio University |