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The MWL-Algorithms for Constructing Cubic Surfaces with Preassigned 27 Lines

by

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In memory of late Professor Jun-ichi Igusa

Abstract. We formulate the algorithms, based on the Mordell-Weil lattices, which can be applied to the construction of a cubic surface together with 27 lines. This construction contains six free parameters, and in particular, it allows to construct 6-parameter families of cubic surfaces and 27 lines on them, all defined over a given field, e.g. over the rational number field **Q**.

1. Introduction

Fix any field K of characteristic $\neq 2, 3$; for example, you can take $K = \mathbf{Q}$ (the field of rational numbers).

We formulate two algorithms (M) and (A). For both algorithms, the input is a six or seven-tuple of elements of K, called the *data* below, while the output is a six-tuple of elements of K, called the *parameter*: $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$.

$Algorithm$	(M)
$Input$	$\xi = (s_1, \ldots, s_6; r)$ s.t. $s_1 \cdots s_6 = r^3$
$Output$	$\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$ and $\{P_1, \ldots, P_{27}\}$

For the algorithm (A), the input data is: $u = (u_1, \ldots, u_6)$, and the output is of the same form as above.

The parameter λ is attached to the following *cubic* Weierstrass equations

$$(M)\ y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2)x + q_0 + q_1 t + q_2 t^2 + t^3 \tag{1}$$

$$(A)\ y^2 - 2t^2 y = x^3 + (p_0 + p_1 t + p_2 t^2)x + q_0 + q_1 t + q_2 t^2 \tag{2}$$

(of total degree at most 3), as the case may be. They have a singular fibre at $t = \infty$ as an elliptic surface over $\mathbf{P}^1$, which is of Kodaira type I_3 for (M) and type IV for (A).

A solution $P = (x, y)$ of the Weierstrass equation will be called a *linear solution* if it is of the form

$$P:\ x = at + b, \quad y = dt + e \quad (a, b, d, e \in K). \tag{3}$$

Our algorithms will give a down-to-earth approach to write down the Weierstrass equations such that the 27 linear solutions $\{P_n\}$ are effectively described in terms of the given data. For the algorithm (M), for example, the coefficients a of the first six linear solutions P_i have the preassigned value:

$$a = -\frac{1}{s_i} \tag{4}$$

determined by the given data, which partly explains the title of the paper. The precise statement of algorithm (M) will be given in §3.

Geometrically, each of the Weierstrass equation will define the following closely related objects:

- E: an elliptic curve over the rational function field $K(t)$,
- S: a smooth projective rational elliptic surface $f : S \to \mathbb{P}^1$ over the projective t-line with zero section,
- X: an affine surface in the affine 3-space $\mathbf{A}^3$ with coordinates (x, y, t), and
- V: a cubic surface in the projective 3-space $\mathbb{P}^3$ with homogeneous coordinates $(X : Y : T : Z) = (x : y : t : 1)$. Explicitly, for (M):

$$V_\lambda : Y^2Z + TXY = X^3 + (p_0Z^2 + p_1TZ + p_2T^2)X + q_0Z^3 + q_1TZ^2 + q_2T^2Z + T^3\,. \tag{5}$$

They will be denoted by E_λ, S_λ, X_λ, V_λ, in case the parameter λ should be indicated. For generic parameter λ, the Mordell-Weil lattice (MWL) of S_λ is known to be isomorphic to E_6^*, the dual lattice of the root lattice E_6. For any parameter λ, S_λ (or λ) will be said to be *non-degenerate* if the MWL of S_λ stays isomorphic to E_6^*. We shall prove a necessary and sufficient condition for non-degeneracy in terms of the input data (§8).

The linear solutions of the Weierstrass equation will then give, in the non-degenerate case,

- the 27 $K(t)$-rational points of the Mordell-Weil group $E_\lambda(K(t))$ of the elliptic curve $E_\lambda/K(t)$ such that suitable six among them, e.g. $\{P_i | 1 \le i \le 5\}$ and P_7, form a set of free generators of the Mordell-Weil group;
- the 27 sections $\{P_n | 1 \le n \le 27\}$ of the Mordell-Weil lattice (MWL) of the elliptic surface S_λ, corresponding to the half of the 54 minimal vectors of height 4/3, and
- the 27 lines on the cubic surface V_λ:

$$l_n : X = aT + bZ, Y = dT + eZ \tag{6}$$

  Moreover our construction automatically incorporates the *double six structure* since the first six lines l_i and the second six lines l_{6+j} form two sets of 6 lines satisfying the "double six" condition in the sense of Schläfli.

Furthermore the map $(t, x, y) \to (t, x)$ defines a double cover of the plane $\mathbb{P}^2$ (with homogeneous coordinates $(T : X : W) = (t : x : 1)$) whose branch locus is a plane quartic curve, say Γ_λ: for(M), it is given by

$$\begin{aligned} X^3W + \frac{1}{4}T^2X^2 + XW(p_0W^2 + p_1TW + p_2T^2) \\ + (q_0W^3 + q_1TW^2 + q_2T^2W + T^3)W = 0 \end{aligned} \tag{7}$$

For generic λ, it is a smooth genus 3 curve, and we get as a by-product:

- the 28 bitangents to Γ_λ are given by the 27 lines
$$X = aT + bW \tag{8}$$
and one more: the line at infinity $W = 0$.

The algorithms (M), (A) are based on the results on Mordell-Weil lattices of the elliptic surface S (or of the elliptic curve $E/k(t)$, $k = \bar{K}$); see §5 below, and cf. [9], [16] for algorithm (M), and [10, 11, 13] for algorithm (A). The chosen letter (M) or (A) refers to the *multiplicative* or *additive* nature of the algorithm, reflecting the structure of the singular fibre at $t = \infty$ of the elliptic surface S_λ, which is of Kodaira type I_3 for (M) and type IV for (A). The smooth part of the singular fibre is an algebraic group whose identity component is the multplicative group $\mathbf{G}_m$ for (M) and the additive group $\mathbf{G}_a$ for (A).

Both algorithms can be formulated in elementary terms only, without use of algebraic geometric terms. Thus the algorithm could be used by any interested reader. For example, it will allow you to write down equations of a cubic surface with all 27 lines having the rational (hence real) coefficients at your will, and to draw the picture of such if you like. See the appendix for some illustration.

Now the subject "cubic surfaces and 27 lines" is one of the most classical, famous and well-studied topics in algebraic geometry since its discovery in the middle 19th century, and there are excellent books on the subject (e.g. [6], [3]). Hopefully the present article will make some useful contribution to the subject, by affording a systematic method to give the defining equation of a cubic surface, together with explicit equations of 27 lines on it. Compared with our old work [11, 12, 13] (based on the idea of algorithm (A)), the present article gets much improved, as it is based on the more recent progress on what we call *mutplicative excellent families* studied in [16] and [5]. The latter paper treats not only MWL of type E_6 but also of types E_7 and E_8, and reveals a close connection of MWL-theory to the representation theory of exceptional Lie algebras (or groups) of type E_6, E_7, E_8. Thus similar application can be expected to the study of del Pezzo surfaces of degree 1 or 2 (cf. [6]) and the exceptional curves on them.

CONTENTS

2. Preliminary remarks

Before formulating the algorithms, we write down the obvious condition for a linear solution (3) of the Weierstrass equation (1) or (2).

First, substituting (3) into (1), we have formally a polynomial of t of degree 3, whose coefficients should be identically zero for (3) to be a solution of (1). Thus in the case (M), the following relations should hold among a, b, d, e:

$$\begin{cases} ad = a^3 + ap_2 + 1 \\ ae = (3a^2 - d + p_2)b + (ap_1 + q_2 - d^2) \\ 0 = 3ab^2 - be - 2de + ap_0 + bp_1 + q_1 \\ 0 = b^3 - e^2 + bp_0 + q_0 \end{cases} \tag{9}$$

Similarly, substituting (3) into (2), we see that, in the case (A), the following relations should hold among a, b, d, e:

$$\begin{cases} -2d = a^3 + ap_2 \\ -2e = 3a^2b - d^2 + ap_1 + bp_2 + q_2 \\ 2de = 3ab^2 + ap_0 + bp_1 + q_1 \\ e^2 = b^3 + bp_0 + q_0 \end{cases} \tag{10}$$

In both cases, one can uniquely determine d, e in terms of a, b from the first two relations of (9) or (10). Then the last two relations of (9) or (10) become two polynomial relations of b of degree 2 and 3, say

$$\varphi_2(b) = 0, \varphi_3(b) = 0 \tag{11}$$

with coefficients in $K[\lambda, a, a^{-1}]$ or in $K[\lambda, a]$.

By eliminating b, i.e. by taking the resultant of $\varphi_2(b), \varphi_3(b)$ with respect to b, we obtain a monic polynomial

$$\Phi(a) = a^{27} + \cdots \tag{12}$$

of degree 27 in a with coefficients in $\mathbf{Q}[\lambda]$, cf. [16] and [10]. For later use, we note an obvious remark about the resultant that there exists at least one common solution b of $\varphi_1(b) = \varphi_2(b) = 0$ if and only if a is a root of the resultant $\Phi(a) = 0$:

$$\exists b \,|\varphi_1(b) = 0\,, \quad \varphi_2(b) = 0 \Leftrightarrow \Phi(a) = 0\,. \tag{13}$$

3. Algorithm (M)

3.1. Input

Fix any field K of characteristic $\neq 2, 3$. The input data of the algorithm(M) is a 7-tuple of non-zero elements in K

$$\xi = (s_1, s_2, \ldots, s_6; r) \quad s_i \neq 0 \tag{14}$$

such that the product of s_i's equals the third power of r:

$$s_1 s_2 \cdots s_6 = r^3 \tag{15}$$

As ξ is uniquely determined by the 6-tuple $\xi' = (s_1, s_2, \ldots, s_5, r)$, to give a data is equivalent to giving a 6-tuple of non-zero elements.

3.2. Notation

To state the algorithm, we fix some notation. Define

$$s'_i := \frac{s_i}{r} \ (1 \leq i \leq 6)\,, \quad s''_{ij} := \frac{r}{s_i s_j} \quad (i \neq j) \tag{16}$$

and consider the finite ordered set with 27 elements (a 27-set in short):

$$\begin{aligned} \Omega : &= \{s_1, s_2, \ldots, s_6, s'_1, s'_2, \ldots, s'_6, s''_{12}, s''_{13}, \ldots, s''_{56}\} \\ &= \{s_1, \ldots, s_{27}\} \end{aligned} \tag{17}$$

with fixed ordering. We write $\Omega = \Omega_\xi$ if necessary. We also need the following set of 36 elements (a 36-set):

$$\Pi = \Pi_\xi = \left\{ \frac{1}{r},\ \frac{s_i}{s_j}\ (i < j \leq 6),\ \frac{r}{s_i s_j s_k}\ (i < j < k \leq 6) \right\} \tag{18}$$

Let

$$\epsilon_n \quad (\text{or} \quad \epsilon_{-n}) \tag{19}$$

denote the n-th elementary symmetric polynomial of $\{s_1, \ldots, s_{27}\}$ (or $\{s_1^{-1}, \ldots, s_{27}^{-1}\}$). Note that $\epsilon_{-n} = \epsilon_{27-n}$ since $\prod_{i=1}^{27} s_i = 1$ as is easily verified. For the algorithm, it is enough to prepare $\epsilon_n, \epsilon_{-n} (n = 1, 2, 3)$. Further we set

$$\delta_1 = r + \frac{1}{r} + \sum_{i \neq j} \frac{s_i}{s_j} + \sum_{i<j<k} \left(\frac{r}{s_i s_j s_k} + \frac{s_i s_j s_k}{r} \right) \tag{20}$$

which is the sum of 36 elements of the set Π above and their inverses.

3.3. Output

Fix an input data (14)

$$\xi = (s_1, s_2, \ldots, s_6; r)\,.$$

The first output is the parameter (Weierstrass coefficients):

$$\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$$

and the second output is the ordered set of 27 linear solutions

$$\{P_n : x = at + b,\ y = dt + e\ (1 \le n \le 27)\}$$

of the form (3), in which a has the value:

$$a = -\frac{1}{s_n}\ (1 \le n \le 27)\,. \tag{21}$$

First, using the six quantities $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}, \delta_1$ given above, we define the output parameter $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$ by the following *key formula*:

$$\begin{cases} p_2 = \frac{1}{6}\,\epsilon_1 \\ p_1 = -\frac{1}{2}\,\delta_1 \\ p_0 = \epsilon_2 - \frac{1}{3}\,\epsilon_1^2 - \epsilon_{-1} \\ q_2 = -\epsilon_{-1} + \frac{1}{36}\,\epsilon_1^2 \\ q_1 = -\epsilon_1 + \epsilon_{-2} - \frac{1}{6}\delta_1\epsilon_1 \\ q_0 = 9 + 3\delta_1 + \frac{1}{4}\delta_1^2 - \frac{1}{3}\epsilon_{-1}\epsilon_1 - \frac{2}{27}\,\epsilon_1^3 + \frac{1}{3}\epsilon_1\epsilon_2 - \epsilon_3 \end{cases} \tag{22}$$

(cf. [16, (25)]). Look at the resulting Weierstrass equation (1):

$$(M)\ y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2)x + q_0 + q_1 t + q_2 t^2 + t^3$$

Note that, if a and b are given, d and e are uniquely determined by the first two relations of (9).

$$\begin{cases} d = a^{-1} + a^2 + p_2 \\ e = ((3a^2 - d + p_2)b + (ap_1 + q_2 - d^2))/a\,. \end{cases} \tag{23}$$

Thus P_n will be uniquely defined by giving $a = -1/s_n$ and finding a suitable b.

We distinguish three cases: (i) $n \le 6$, (ii) $7 \le n \le 12$ and (iii) $n > 12$. In the following, the indices i, j, k are in the set $I := \{1, 2, 3, 4, 5, 6\}$.

3.3.1. Case (i) $n = i$

For $n = i \le 6$, $P_n = P_i$ has

$$a = -\frac{1}{s_i} \tag{24}$$

$$b = s_i + \frac{1}{s_i^2} + \sum_{j \ne i}(s'_j + s''_{i,j}) - \frac{1}{3}\epsilon_1 \tag{25}$$

3.3.2. Case (ii) $n = 6 + i$

We have $s_n = s'_i$ and $P_n = P'_i$ has

$$a = -\frac{1}{s'_i} = -\frac{r}{s_i} \tag{26}$$

$$b = s'_i + \frac{1}{s'^2_i} + \sum_{j \neq i}(s_j + s''_{i,j}) - \frac{1}{3}\epsilon_1. \tag{27}$$

3.3.3. Case (iii) $n > 12$

In this case, we have $s_n = s''_{i,j}$ for some $i \neq j$, and $P_n = P''_{ij}$ has

$$a = -\frac{1}{s''_{ij}} = -\frac{s_i s_j}{r} \tag{28}$$

$$b = s''_{ij} + \frac{1}{s''^2_{ij}} + (s_i + s'_i + s_j + s'_j) + \sum_{k<l} \sum_{\{kl\}\cap\{ij\}=\emptyset} s''_{k,l} - \frac{1}{3}\epsilon_1 . \tag{29}$$

4. Proof

Given the key formula (22), it is straightforward to verify (with the help of computer) that the above formulas for a and b are correct, by plugging in $x = at + b$, $y = dt + e$ into the Weierstrass equation (1). *q.e.d.*

Instead, let us indicate below the idea of proof leading to the above formula for the coefficient b of P_n.

4.1. Idea

Fix n, and take $a = -1/s_n$ and d, e by (23). Then, by (11), b satisfies two polynomial relations of degree 2 and 3 in b,

$$\varphi_2(b) = 0\,, \quad \varphi_3(b) = 0 \tag{30}$$

which have at least one common solution b. In fact, the key formula in the algorithm (M) is so arranged that the resultant $\Phi(a)$, (12) is a degree 27 polynomial in a such that $\Phi(X)$ is equal to the following polynomial:

$$\prod_{n=1}^{27}\left(X + \frac{1}{s_n}\right); \tag{31}$$

cf. [16, (34)], and we have only to apply the remark (13).

In this way, we obtain 27 distinct linear solutions provided that 27 s_n are all distinct. But this condition is too restrictive. To be instructive, let us insert a numerical example here, before continuing the proof.

4.2. A numerical example

Let $K = \mathbf{Q}$ and take the data:

$$\xi = \left(1, 2, -\frac{1}{2}, 3, -\frac{1}{3}, 8; 2\right). \tag{32}$$

By following the above indication (with the help of computer), we obtain 27 distinct linear solutions. In this example, however, the 27 s_n are not necessarily distinct. For example, we have $s_n = 1$ for $n = 1, 8$ or 13, i.e. 1 appears three times in the 27-set Ω. Looking at the list of P_n given in §9.1, we find that the three linear sections P_1, P_8 and P_{13} are indeed distinct and have the same coefficient $a = -1$. There arises a natural question: how to decide the order among them.

There are several methods for this type of question, but the most useful way is the following: first work out the generic case and then consider specializing the results in the case under consideration.

4.3. Proof (continued)

Suppose that the data ξ is generic. By this we mean that the field $K_0(\xi) := K_0(s_1, \ldots, s_6, r)$ has transcendence degree 6 over the prime field K_0 contained in K, and we may replace K by $K_0(\xi)$.

In this case, the polynomials φ_2, φ_3 in (30) have coefficients in the ring of Laurent polynomials in $K_0(\xi)$:

$$K_0[\xi, \xi^{-1}] := K_0[s_1, s_1^{-1}, \ldots, s_6, s_6^{-1}, r, r^{-1}].$$

We claim that the common solution $b = b_n$ of (30) is unique and it is contained in this ring. Indeed, as the degrees of φ_2, φ_3 in b are 2 and 3, b_n is a rational function of $K_0(\xi)$. Furthermore it is integral over $K_0[\xi, \xi^{-1}]$ since $\varphi_3 = 0$ is a monic equation over this ring. This implies our claim, because $K_0[\xi, \xi^{-1}]$ is clearly normal (it is equal to the polynomial ring $K_0[s_1, \ldots, s_5, r]$ in 6 variables, localized by $(s_1 \cdots s_5 r)$.)

Therefore the common solution $b = b_n$ of (30) for $a = -1/s_n$ is an element of the ring $K_0[\xi, \xi^{-1}]$. A direct computation gives this element, which is then identified with the formulas stated in (25), (27) and (29).

Once the generic case is established, we may specialize the generic data, say $\tilde{\xi}$ to any data ξ. Then the generic linear solutions $\tilde{P}_n$'s naturally specialize to P_n's, since all the coefficients involved are contained in the ring of Laurent polynomials. *q.e.d.*

REMARK. There are 27 distinct linear solutions precisely under the *non-degeneracy condition* that

$$r \neq 1, \quad s_i \neq s_j (i < j), \quad s_i s_j s_k \neq r (i < j < k) \tag{33}$$

which is equivalent to saying that the 36-set Π_ξ of (18) does not contain 1:

$$1 \notin \Pi_\xi \tag{34}$$

(see §8). The data (32) satisfies this condition, as shown by (84).

5. Algorithm (A)

For the sake of the comparison, we briefly include a set-up for the algorithm (A). This is based on our old work [10, 12, 13], improved by the simplified construction of the lattices E_6 and E_6^* in [14].

5.1. Input

Assume that K is any field of characteristic $\neq 2, 3, 5, 7$. The input data of the algorithm (A) is simply any 6-tuple of elements in K

$$u = (u_1, u_2, \ldots, u_6)\,. \tag{35}$$

Letting

$$v_0 = \frac{1}{3}(u_1 + u_2 + \cdots + u_6)\,, \tag{36}$$

the 7-tuple $(u_1, u_2, \ldots, u_6; v_0)$ may look more parallel to the data ξ of the algorithm (M).

5.2. Notation

In this case, define

$$u_i' := u_i - v_0 \; (1 \le i \le 6)\,, \quad u_{ij}'' := v_0 - u_i - u_j \;\; (i < j) \tag{37}$$

and consider the finite ordered set with 27 elements (a 27-set in short):

$$\begin{aligned} \Omega := \Omega_u &= \{u_1, u_2, \ldots, u_6, u_1', u_2', \ldots, u_6', u_{12}'', u_{13}'', \ldots, u_{56}''\} \\ &= \{u_1, \ldots, u_{27}\} \end{aligned} \tag{38}$$

as before. We also consider the following set of 36 elements (a 36-set):

$$\Pi = \Pi_u = \{-v_0,\; u_i - u_j \; (i < j \le 6),\; v_0 - u_i - u_j - u_k \; (i < j < k \le 6)\} \tag{39}$$

Next let

$$\epsilon_n \tag{40}$$

denote the n-th elementary symmetric polynomial of $\{u_1, \ldots, u_{27}\}$.

5.3. Output

Then we define the output parameter $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$ by the following formula:

$$\begin{cases} p_2 = \frac{1}{12}\epsilon_2 \\ p_1 = \frac{1}{48}\epsilon_5 \\ q_2 = \frac{1}{96}(-168p_2^3 + \epsilon_6) \\ p_0 = \frac{1}{480}(-294p_2^4 - 528p_2q_2 + \epsilon_8) \\ q_1 = \frac{1}{1344}(-1008p_1p_2^2 + \epsilon_9) \\ q_0 = \frac{1}{17280}(-608p_1^2p_2 - 4768p_0p_2^2 - 252p_2^6 - 1200p_2^3q_2 + 1248q_2^2 + \epsilon_{12}) \end{cases} \tag{41}$$

(cf. [10, (2.15)]). For generic data u, these six polynomials form a system of fundamental invariants of the Weyl group $W(E_6)$, generating the graded ring of the invariants in the classical sense.

Now look at the resulting Weierstrass equation (2):

$$(A)\ y^2 - 2t^2 y = x^3 + (p_0 + p_1 t + p_2 t^2)x + q_0 + q_1 t + q_2 t^2$$

Then there are 27 linear solutions of the form (3) and one can determine the coefficients a, b, d, e as follows.

Take

$$a = u_n \quad (1 \le n \le 27)\,. \tag{42}$$

Then determine d and e from the first two relations of (10): i.e.

$$\begin{cases} d = -\frac{1}{2}(a^3 + ap_2) \\ e = -\frac{1}{2}((3a^2 + p_2)b + (ap_1 + q_2 - d^2))\,. \end{cases} \tag{43}$$

Then the remaining two relations of (10) have at least one (and at most three) common solution(s) b. This follows from (13) and the fact that the resultant $\Phi(x)$, (12) in the case (A), is equal to the following polynomial:

$$\Phi(x) = \prod_{n=1}^{27} (x - u_n)\,; \tag{44}$$

cf. [10, §10].

In this way, we obtain 27 linear solutions of the Weierstrass equation (2), if all u_n $(1 \le n \le 27)$ are distinct, and more generally, under the non-degeneracy condition in the case (A) that no element of the 36-set Π, (39), vanish (cf. §8). We omit the closed formula for the linear solutions P_n here (cf. [13]).

A numerical example is given in §9, (87).

In the rest of the paper, we mostly focus on the multiplicative case (M) only, but there are parallel facts for the additive case (A).

6. Background from the MWL-theory

In general, consider a rational elliptic surface with a zero-section (defined over an algebraically closed field k)

$$f : S \to \mathbf{P}^1 \tag{45}$$

and let M denote the Mordell-Weil lattice ([9]). It is known that (i) if f has no reducible fibres, then M is isomorphic to the root lattice E_8. (ii) If f has a single reducible fibre with two irreducible components (Kodaira type I_2 or III), then M is isomorphic to E_7^*, the dual lattice of the root lattice E_7. (iii) If f has a single reducible fibre with three irreducible components (Kodaira type I_3 or IV), then M is isomorphic to E_6^*, the dual lattice of the root lattice E_6.

As an abstract lattice, the three lattices E_8, E_7^* and E_6^* can be uniformly described as follows (cf. [14]). Let L_r be a free $\mathbf{Z}$-module of rank $r = 6, 7$ or 8, with free generators $u_1, \ldots, u_r$, equipped with bilinear pairing

$$\langle u_i, u_j \rangle = \delta_{i,j} + \frac{1}{9-r}\,, \tag{46}$$

where $\delta_{i,j}$ is the Kronecker's delta, and let $\tilde{L_r}$ be the lattice spanned by $u_1, \ldots, u_r$ and v_0, where

$$v_0 = \frac{1}{3}(u_1 + \cdots + u_r) .$$

Then, for $r = 8, 7$ or 6, the lattice $\tilde{L_r}$ is isomorphic to E_8, E_7^* or E_6^* respectively. The minimal vectors of these lattices and the roots in the root lattices E_r can be described in a relatively simple form in terms of $u_1, \ldots, u_r$ and v_0 (cf. [14]).

In this paper, we are dealing with the case (iii) above. Thus, for any rational elliptic surface having MWL $M \cong E_6^*$, there exist six sections $P_i \in M$ and one $R \in M$, corresponding to u_i and v_0, such that the height pairing satisfies

$$\langle P_i, P_j \rangle = \delta_{i,j} + \frac{1}{3} , \tag{47}$$

and

$$\sum_{i=1}^{6} P_i = 3R . \tag{48}$$

Now consider the elliptic surface defined by the Weierstrass equation (1) with generic parameter λ. It has a fibre of type I_3 at $t = \infty$. Consider the specialization homomorphism

$$sp_\infty : M \to k^* . \tag{49}$$

It is the unique homomorphism from M to k^* such that $sp_\infty(P) = -1/a$ for any linear section $P : x = at + b$, $y = dt + e$ (cf. [7]). If we set

$$s_i = sp_\infty(P_i) , \quad r = sp_\infty(R) , \tag{50}$$

then the relation (48) implies

$$s_1 s_2 \cdots s_6 = r^3 . \tag{51}$$

The algorithm (M) is based on the idea to reverse the above process. Namely, starting from the data $\xi = (s_1, \ldots, s_6; r)$, we recover first the parameter λ, by using the key formula (22), which involves the multiplicative invariants of the Weyl group $W(E_6)$, and then we determine the 27 linear sections $\{P_n\}$ on the elliptic surface S_λ, as indicated in §4.

The MWL M_λ is generated by $P_1, \ldots, P_6$ and R. Furthermore, corresponding to (16), we have

$$P_i' = P_i - R, \quad P_{ij}'' = R - P_i - P_j \ (i < j) \tag{52}$$

where $P_i' = P_{i+6}$ and $\{P_{ij}''\} = \{P_{13}, \ldots, P_{27}\}$. These relations hold true from the start of our construction (47, 48), but of course they can be confirmed, for instance, by the addition formula on an elliptic curve.

Next we look at the sections of height 2 in M^0 (the narrow MWL) corresponding to the 72 roots of E_6. The set of 36 positive roots in $M^0 \simeq E_6$ (w.r.t. a suitable basis) is given by

$$\{-R, A_{ij} := P_i - P_j \ (i < j), B_{ijk} := R - P_i - P_j - P_k \ (i < j < k)\} . \tag{53}$$

More precisely, letting $A_i = A_{i,i+1}$, the six roots

$$\{A_1, \ldots, A_5, B_{123}\} \tag{54}$$

form a Dynkin basis of $M^0 \cong E_6$, and the above 36 elements of (53) are positive roots with respect to this basis.

In general, a root Q of M^0 is of the form:

$$Q = (x, y) : \ x = gt^2 + at + b \,, \quad y = ht^3 + ct^2 + dt + e \,. \tag{55}$$

By a direct computation (the addition theorem, in principle), we have for

$$Q = R : \qquad g = \frac{r}{(r-1)^2} \,, \qquad h = \frac{r}{(r-1)^3} \tag{56}$$

$$Q = A_{ij} : \qquad g = \frac{s_i s_j}{(s_i - s_j)^2} \,, \qquad h = \frac{s_i s_j^2}{(s_i - s_j)^3} \tag{57}$$

$$Q = B_{ijk} : \qquad g = \frac{r s_i s_j s_k}{(r - s_i s_j s_k)^2} \,, \qquad h = \frac{r s_i^2 s_j^2 s_k^2}{(r - s_i s_j s_k)^3} \tag{58}$$

In terms of the specialization homomorphism at infinity, restricted to M^0,

$$sp_\infty : M^0 \to \Theta_0^\# = \mathbf{G}_m \,, \quad sp_\infty(Q) = \frac{g+h}{h} \,, \tag{59}$$

we can write

$$sp_\infty(R) = r, \quad sp_\infty(A_{ij}) = \frac{s_i}{s_j} \,, \quad sp_\infty(B_{ijk}) = \frac{r}{s_i s_j s_k} \tag{60}$$

Hence the above formulas for (g, h) of a root Q can be summarized as follows:

PROPOSITION 6.1. *For any root $Q \in M^0$, let $\rho = sp_\infty(Q)$. Then we have*

$$g = \frac{\rho}{(\rho-1)^2} \,, \quad h = \frac{\rho}{(\rho-1)^3} \,. \tag{61}$$

Moreover we have for $Q = (x, y)$

$$(\rho-1)^2 x \,, \quad (\rho-1)^3 y \in K_0[\xi, \xi^{-1}][t] \,. \tag{62}$$

The 36-set Π (18) is equal to the set of $sp_\infty(Q)$ for the 36 roots $Q \in M^0$ in (53).

The root R can be computed as the difference $P_i - P_i'$ for any $i \leq 6$. For example, the x-coordinate of the root R is given by the following:

$$x = \frac{r}{(r-1)^2}\, t^2 - \frac{r(r+1)\,\gamma_{-1}}{(r-1)^2}\, t + \frac{r^2\,\gamma_{-1}^2}{(r-1)^2} + \ r\,\gamma_{-2} - \frac{1}{3}\epsilon_1 \tag{63}$$

where γ_i (resp. γ_{-i}) denotes i-th elemenratry symmetric polynomial of $\{s_1, \ldots, s_6\}$ (resp. $\{s_1^{-1}, \ldots, s_6^{-1}\}$).

7. Digression: a refined algorithm (M′)

In this section, we consider the following Weierstrass equation

$$(M')\ y^2 + txy = x^3 + m_0 x^2 + (p_0 + p_1 t)x + q_0 + q_1 t + q_2 t^2 + t^3 \tag{64}$$

with a new parameter:

$$\lambda' = (m_0, p_0, p_1, q_0, q_1, q_2) \,. \tag{65}$$

The base field K can be any field of charcteristic $\neq 2$.

Consider the relations in a, b, d, e for a linear solution (3) to satisfy the new Weierstrass equation (64), similar to (9). By eliminating d, e and b successively, we obtain the algebraic equation $\Phi(a) = 0$ of degree 27 with coefficients in $\mathbf{Z}[m_0, p_0, p_1, q_0, q_1, q_2]$.

$$\begin{aligned}\Phi(X) := X^{27} - q_2 X^{26} + (q_1 - m_0)X^{25} + (9 - 6p_1 + p_1^2 - q_0)X^{24} + \cdots \\ + (9 - 6p_1 + p_1^2 - q_0)X^3 + (p_0 - q_2)X^2 - m_0 X + 1 \end{aligned} \tag{66}$$

Then comparing the coefficients of X in (66) and (31), we find the key formula for the algorithm (M′):

$$\begin{cases} m_0 = -\epsilon_1 \\ q_2 = -\epsilon_{-1} \\ p_0 = \epsilon_2 - \epsilon_{-1} \\ q_1 = \epsilon_{-2} - \epsilon_1 \\ q_0 = -\epsilon_3 + \frac{1}{4}(\delta_1 + 6)^2 \\ p_1 = -\frac{1}{2}\,\delta_1 \end{cases} \tag{67}$$

where $\epsilon_n, \epsilon_{-n}$ or δ_1 are the same as before; cf. (19) and (20). It is slightly simpler than the previous one (22).

Accordingly, the coefficients a, b, d, e of linear solutions P_n are simplified.

(i) For $n = i \leq 6$, $P_n = P_i$ has:

$$a = -\frac{1}{s_i}, \quad b = s_i + \frac{1}{s_i^2} + \sum_{j \neq i}(s'_j + s''_{i,,j}) \tag{68}$$

(ii) For $n = 6 + i$, we have $s_n = s'_i$ and $P_n = P'_i$ has:

$$a = -\frac{1}{s'_i}, \quad b = s'_i + \frac{1}{s'^2_i} + \sum_{j \neq i}(s_j + s''_{i,j}) \tag{69}$$

(iii) For $n > 12$, we have $s_n = s''_{i,j}$ for some $i \neq j$, and $P_n = P''_{ij}$ has

$$a = -\frac{1}{s''_{ij}}, \quad b = s''_{ij} + \frac{1}{s''^2_{ij}} + (s_i + s'_i + s_j + s'_j) + \sum_{k<l}\ \sum_{\{kl\}\cap\{ij\}=\emptyset} s''_{k,l}\,. \tag{70}$$

The algorithm (M′) is formulated, using the same input as (M). The output is the new parameter λ', defined by (67), and 27 linear solutions P_n determined by the above (i)–(iii).

The two Weierstrass equations, (64) and (1), are actually related by the coordinate transformation

$$x \to x + \frac{m_0}{3}, \quad y \to y - \frac{m_0}{6}t\,. \tag{71}$$

Thus there is no essential improvement, *except* that the new formula (67) works in every characteristic p different from 2, especially it works even in case $p = 3$.

Personally the idea behind the above refinement is inspired by the work of late Professor Jun-ichi Igusa (especially his paper "Arithmetic variety of moduli for genus two"), in whose memory this paper is dedicated.

8. Non-degeneracy condition and vanishing roots

Some of the following results (Theorem 8.1, Proposition 8.5) have been stated without proof before ([16]).

THEOREM 8.1. *Let $\xi = (s_1, s_2, \ldots, s_6; r)$ be any data and let λ be the corresponding Weierstrass parameter. Then the elliptic surface S_λ has Mordell-Weil lattice E_6^* if and only if the 36-set Π_ξ defined by (18) is free from 1.*

Proof. First let us prove the only-if part. Suppose that $S = S_\lambda$ has Mordell-Weil lattice E_6^*. Then there are 72 sections of height 2, say $Q_m (m \leq 72)$, coresponding to the 72 roots of the root lattice E_6. By the height formula for any section Q of height 2, (Q) is disjoint from (O) on S and it intersects the identity component Θ_0 at $t = \infty$. Thus the values of the specialization homomorphism $sp_v(Q)$ and $sp_v(O)$ differ at every point v of the base curve, and especially at $t = \infty$. Hence

$$sp_\infty(Q) \neq sp_\infty(O) = 1$$

for each $Q = Q_m$. Hence, by Proposition 6.1, the set Π_ξ is free from 1. This proves the only-if part.

Conversely, assume that $1 \notin \Pi_\xi$ and that $S = S_\lambda$ has a degenerate MWL of rank less than 6. This is the case if and only if there is a new reducible fibre at some $v \neq \infty$ (cf. [8, 9]). Let us derive a contradiction.

Take a non-identity component Θ' of this fibre. Then we have $\Theta'^2 = -2$ and the class of Θ' in the Néron-Severi lattice $NS(S)$ belongs to the E_6-frame V, i.e. V is the orthogonal complement of $U \oplus T_\infty$ in $NS(S) \cong U \oplus E_8^-$, which is isomorphic to the negative root lattice E_6^-. [Here U denotes the unimodular rank 2 sublattice spanned by (O), F.]

At this point, we recall (cf. [15]) that the set of negative roots in the Néron-Severi lattice $NS(S)$ (S: a rational elliptic surface)

$$\mathcal{D} = \mathcal{D}_S = \{cl(D) | D^2 = -2, D \perp (O), F\} \subset NS(S) \tag{72}$$

is a finite set of 240 elements. The intersection of $\mathcal{D}$ with the E_6-frame in the present situation consists of the 72 negative roots in $V \cong E_6^-$.

Now take a generic data $\tilde{\xi} = (\tilde{s_1}, \ldots, \tilde{s_6}; \tilde{r})$ and consider the specialization $\sigma : \tilde{\xi} \to \xi$ of $\tilde{\xi}$ to a given data $\xi = (s_1, \ldots, s_6; r)$; practically, this means just replacing $\tilde{s_i}, \tilde{r}$ by s_i, r. Thus if $\tilde{\lambda}$ (resp. λ) is the parameter corresponding to the data $\tilde{\xi}$ (resp. ξ), then the specialization of $\tilde{\lambda}$ under $\sigma : \tilde{\xi} \to \xi$ is equal to λ, i.e. $\sigma(\tilde{\lambda}) = \lambda$.

Applying the results discussed above to the generic $(\tilde{\xi}, \tilde{\lambda})$, we consider the specialization of the elliptic surface $\tilde{S} = S_{\tilde{\lambda}}$ and the 72 roots $\tilde{Q}_m$ on $\tilde{S}$ under $\sigma : \tilde{\xi} \to \xi$. Then the set $\mathcal{D}_{\tilde{S}}$ consists of the classes of 72 elements

$$D_m = (\tilde{Q}_m) - (O) - F .$$

When $\tilde{S}$ is specialized to S via σ, the set of negative roots $\mathcal{D}_{\tilde{S}}$ is specialized bijectively to the set $\mathcal{D}_S$. But $\mathcal{D}_S$ contains the class of Θ', and this shows that some $\tilde{Q}_m$ specializes to O. Hence we have $sp_\infty(Q_m) = 1$, which implies that Π_ξ contains 1 (cf. Lemma below): a contradiction. Thus Theorem 8.1 is proved. *q.e.d.*

DEFINITION. We say that a root $\tilde{Q}$ becomes a *vanishing root* on S_λ if the specialization $Q = \sigma(\tilde{Q})$ is equal to the zero-section O. [An obvious example is this: the root $A_{ij} = P_i - P_j (i < j)$ becomes a vanishing root if $s_i = s_j$ holds in the data ξ.]

LEMMA 8.2. *With the above notation, we have: (i) the specialization $P_n = \sigma(\tilde{P}_n)$ is a linear section on S_λ for any λ and any $n \le 27$. (ii) The specialization $Q = \sigma(\tilde{Q})$ of a root $\tilde{Q}$ on $\tilde{S}$ is the zero-section $Q = O$ on S_λ, if and only if $sp_\infty(Q)$ is equal to 1.*

Proof. (i) The coefficients a, b, d, e of any linear section $\tilde{P}_n$ are contained in the ring $K_0[\tilde{s_1}^{\pm1}, \dots, \tilde{s_6}^{\pm1}, \tilde{r}^{\pm1}]$. Hence they have well-defined specialization under $\sigma : \tilde{\xi} \to \xi$, which gives the specialized linear section P_n.

(ii) Any root $\tilde{Q}$ on $\tilde{S}$ is of the form (55) where the coefficients $g, h, a, b, \dots$ are contained in the ring $K_0[\tilde{s_1}^{\pm1}, \dots, \tilde{s_6}^{\pm1}, \tilde{r}^{\pm1}][1/(\tilde{\rho} - 1)]$, where $\tilde{\rho} = sp_\infty(\tilde{Q})$; see Prop. 6.1. Note that the specialization $\sigma(\tilde{\rho})$ is equal to $\rho = sp_\infty(Q)$. Then, by the formula (61), it is clear that we have $Q = O$ iff $\rho = 1$. [Recall that the zero O of a Weierstrass cubic is the point at infinity : $(x, y) = (\infty, \infty)$.] *q.e.d.*

COROLLARY 8.3. *With the above notation, there are no vanishing roots on S_λ iff the 36-set Π_ξ is free from 1.*

DEFINITION. We call a data ξ *non-degenerate* if the 36-set Π_ξ is free from 1.

Now let us consider, with the notation in §1, the elliptic surface S_λ, the affine surface X_λ or the cubic surface V_λ, defined by the Weierstrass equation (1) with the parameter λ which is determined by a given data ξ by the algorithm (M). Then we have:

THEOREM 8.4. *The following conditions are equivalent to each other*:

(1) *ξ is non-degenerate, i.e. $1 \notin \Pi_\xi$.*

(2) *the elliptic surface S_λ has Mordell-Weil lattice isomorphic to E_6^*.*

(2′) *S_λ has 27 distinct linear sections.*

(3) *S_λ has no reducible fibres other than the I_3-fibre at $t = \infty$.*

(4) *the affine surface X_λ has no singular points.*

(5) *the cubic surface V_λ is smooth.*

(5′) *V_λ contains 27 distinct lines.*

(6) *the plane quartic Γ_λ is smooth.*

(6′) *Γ_λ has 28 distinct bitangents.*

NOTATION. Let $\Theta_i (i = 0, 1, 2)$ be the three irreducible components of the I_3-fibre at $t = \infty$ where Θ_0 intersects the zero-section (O); $\{\Theta_1, \Theta_2\}$ span the (negative) root lattice $T_\infty \simeq A_2$. The linear sections intersect one and the same component which is labeled as Θ_1.

The cubic surface V_λ is defined by the equation:

$$V_\lambda : Y^2Z + TXY = X^3 + (p_0Z^2 + p_1TZ + p_2T^2)X + q_0Z^3 + q_1TZ^2 + q_2T^2Z + T^3 . \quad (73)$$

The plane quartic Γ_λ is defined by the equation:

$$\Gamma_\lambda : x^3 + \frac{1}{4}t^2x^2 + (p_0 + p_1t + p_2t^2)x + q_0 + q_1t + q_2t^2 + t^3 = 0 \tag{74}$$

with $(x : t : 1)$ replaced by $(X : T : W)$.

Proof. (Outline) We assume the basic facts on elliptic surfaces ([4], [17]) and Mordell-Weil lattices (cf. [7], [9], [8]).

The equivalence of (1) through (3) is already covered in Theorem 8.1 and the discussion leading to it.

Let S'_λ denote the open subset of S_λ which is the complement of the union of the zero-section and the fibre at $t = \infty$. By construction of Kodaira-Néron model (or Tate algorithm), S'_λ can be identified with the minimal resolution of the affine surface X_λ. This shows (3) $\Leftrightarrow$ (4).

Further, X_λ is isomorphic to the open set $\{Z \neq 0\}$ of V_λ by definition. Then we see that the birational morphism $S'_\lambda \to X_\lambda$ extends to a morphism $S_\lambda \to V_\lambda$ which blows down three curves (O), Θ_0 and Θ_2 to a point of V, and maps Θ_1 to the nodal rational curve which is cut out on the cubic surface V by the plane at infinity $\{Z = 0\}$. It is easy now to check (4) $\Leftrightarrow$ (5). The equivalence (5) $\Leftrightarrow$ (5′), (6) $\Leftrightarrow$ (6′) are classically well-known, and the equivalence of (2′), (5′) and (6′) can be seen from the explicit construction of the linear sections in our algorithms (M). *q.e.d.*

More generally, we denote by $\nu = \nu(\xi)$ (resp. $\nu(u)$) the number of times 1 (resp. 0) occurs in Π_ξ (resp. Π_u), and we call it *the number of vanishing roots* (logically, 2ν should be called by that name (cf. the proof of the if-part of Theorem 8.1), but we prefer the present one for simplicity. It is obvious that a data ξ (or u) is non-degenerate iff $\nu = 0$. Including the degenerate case $\nu > 0$, we have:

PROPOSITION 8.5. *Suppose that S_λ has new reducible fibres at $t \neq \infty$ and let $T_{new} := \oplus_{v \neq \infty} T_v$ be the new part of the trivial lattice*; *it is a direct sum of root lattices of ADE-type. Then*

(i) 2ν *is equal to the number of roots in T_{new}.*

(ii) *the affine surface X_λ has precisely the ADE-singularities indicated by T_{new}.*

(iii) *the cubic surface V_λ has (only) the same ADE-singularities as above.*

Proof. (Outline) (i) 2ν is the number of $D \in \mathcal{D}_S$ which is mapped to O in the MW M_S. Since the latter is isomorphic to the quotient $NS(S)/Triv(S)$ and $Triv(S) = T_{new} \oplus T_\infty$, (i) follows. (ii) is a wellknown fact on the Kodaira-Néron model, and (iii) then follows from the same argument as the equivalence of (4) and (5) of Theorem 8.4. *q.e.d.*

Applying the algorithm (M) to the degenerate data (i.e. ξ with $\nu(\xi) > 0$), one gets all the 21 possible OS-types for which the trivial lattice contains A_2 as a direct factor. Moreover one can show the existence of **Q**-split examples in all except one inevitable type; for this, cf. [16, Table 1].

9. Examples

Let the base field $K = \mathbf{Q}$. The singular fibres are determined by using [4, 17]

9.1. A non-degenerate example of algorithm (M)

Take the input data

$$\xi = \left(1, 2, -\frac{1}{2}, 3, -\frac{1}{3}, 8; 2\right). \tag{75}$$

The 27-set Ω is:

$$\Omega = \Big\{1, 2, -\frac{1}{2}, 3, -\frac{1}{3}, 8, \frac{1}{2}, 1, -\frac{1}{4}, \frac{3}{2}, -\frac{1}{6}, 4, 1, -4, \frac{2}{3}, -6, \frac{1}{4}, -2, \frac{1}{3}, -3, \frac{1}{8}, -\frac{4}{3}, 12, -\frac{1}{2}, -2, \frac{1}{12}, -\frac{3}{4}\Big\} \tag{76}$$

and the 36-set Π:

$$\Pi = \Big\{\frac{1}{2}, \frac{1}{2}, -2, \frac{1}{3}, -3, \frac{1}{8}, -4, \frac{2}{3}, -6, \frac{1}{4}, -\frac{1}{6}, \frac{3}{2}, -\frac{1}{16}, -9, \frac{3}{8}, -\frac{1}{24}, -2, \frac{1}{3}, -3, \frac{1}{8}, -\frac{4}{3}, 12, -\frac{1}{2}, -2, \frac{1}{12}, -\frac{3}{4}, -\frac{2}{3}, 6, -\frac{1}{4}, -1, \frac{1}{24}, -\frac{3}{8}, 4, -\frac{1}{6}, \frac{3}{2}, -\frac{1}{4}\Big\} \tag{77}$$

Observe that the above Π is free from 1. Hence the chosen data ξ is non-degenerate and we should have MWL $\simeq E_6^*$ and 27 distinct linear sections. Let us check this more directly.

By the algorithm (M), the output parameter is given by

$$\lambda = \left(-\frac{26221}{192}, \frac{1177}{288}, \frac{39}{16}, \frac{46567453}{82944}, -\frac{11609}{256}, -\frac{2223}{256}\right) \tag{78}$$

so that the Weierstrass equation is:

$$y^2 + txy = x^3 + \left(\frac{39t^2}{16} + \frac{1177t}{288} - \frac{26221}{192}\right)x + t^3 - \frac{2223t^2}{256} - \frac{11609t}{256} + \frac{46567453}{82944} \tag{79}$$

The j-invariant is given by

$$\begin{aligned} j &= -27648\big(12t^4 - 1404t^2 - 2354t + 78663\big)^3/D, \\ D &= 47775744t^9 - 698720256t^8 - 11503005696t^7 + 181503783936t^6 \\ &\quad + 889688786688t^5 - 16425389923488t^4 - 14515817491600t^3 \\ &\quad + 501036184456704t^2 - 351749903932440t - 1282516588562325 \end{aligned} \tag{80}$$

which has 9 simple poles at $t \neq \infty$ and a pole of order 3 at $t = \infty$. This shows that there is a reducible fibre at $t = \infty$, which is of Kodaira type I_3, and no others.

The 27 linear sections $P_n = (at + b, dt + e)(1 \le n \le 27)$ are as follows:

$$\left(-t - \frac{39}{8}, \frac{39t}{16} + \frac{9601}{288}\right), \left(-\frac{t}{2} - \frac{7}{12}, \frac{11t}{16} + \frac{7291}{288}\right), \left(2t + \frac{77}{8}, \frac{111t}{16} + \frac{3391}{288}\right),$$
$$\left(\frac{77}{72} - \frac{t}{3}, \frac{17635}{864} - \frac{65t}{144}\right), \left(3t + \frac{259}{24}, \frac{565t}{48} + \frac{5345}{288}\right), \left(\frac{947}{192} - \frac{t}{8}, \frac{12895}{4608} - \frac{355t}{64}\right),$$
$$\left(\frac{89}{24} - 2t, \frac{95t}{16} + \frac{2965}{288}\right), \left(\frac{19}{4} - t, \frac{39t}{16} + \frac{1285}{288}\right), \left(4t + \frac{689}{24}, \frac{299t}{16} + \frac{41035}{288}\right),$$
$$\left(\frac{359}{72} - \frac{2t}{3}, \frac{199t}{144} + \frac{1823}{864}\right), \left(6t + \frac{1073}{24}, \frac{1853t}{48} + \frac{83381}{288}\right), \left(\frac{57}{16} - \frac{t}{4}, -\frac{3t}{2} - \frac{6313}{576}\right),$$
$$\left(\frac{73}{8} - t, \frac{39t}{16} - \frac{2495}{288}\right), \left(\frac{t}{4} - \frac{637}{48}, \frac{13t}{2} + \frac{3485}{576}\right), \left(\frac{119}{12} - \frac{3t}{2}, \frac{193t}{48} - \frac{3889}{288}\right),$$
$$\left(\frac{t}{6} - \frac{473}{36}, \frac{1219t}{144} - \frac{8087}{864}\right), \left(\frac{231}{8} - 4t, \frac{291t}{16} - \frac{41429}{288}\right), \left(\frac{t}{2} - \frac{97}{8}, \frac{75t}{16} - \frac{6005}{288}\right),$$
$$\left(\frac{311}{24} - 3t, \frac{533t}{48} - \frac{8959}{288}\right), \left(\frac{t}{3} - \frac{727}{72}, \frac{799t}{144} - \frac{26077}{864}\right),$$
$$\left(\frac{883}{12} - 8t, \frac{1061t}{16} - \frac{179609}{288}\right), \left(\frac{3t}{4} - \frac{493}{48}, \frac{13t}{3} - \frac{17093}{576}\right),$$
$$\left(\frac{1201}{144} - \frac{t}{12}, \frac{2771}{1728} - \frac{86t}{9}\right), \left(2t + \frac{7}{8}, \frac{111t}{16} - \frac{6059}{288}\right), \left(\frac{t}{2} - \frac{31}{4}, \frac{75t}{16} - \frac{9785}{288}\right),$$
$$\left(\frac{3689}{24} - 12t, \frac{7025t}{48} - \frac{547285}{288}\right), \left(\frac{4t}{3} + \frac{167}{72}, \frac{715t}{144} - \frac{13855}{864}\right) \tag{81}$$

It is immediate to translate the above results into the classical setting of the 27 lines on the smooth cubic surface V_λ, or of the 28 bitangents of the plane quartic Γ_λ, with everything defined over rational numbers $\mathbf{Q}$.

See the appendix for the illustration of the drawings for them.

9.2. A non-degenerate example of algorithm (M′)

Take the input data

$$\xi = \left(1, 2, 3, 4, -1, -\frac{1}{3}; 2\right). \tag{82}$$

The 27-set Ω is:

$$\Omega = \Big\{1, 2, 3, 4, -1, -\frac{1}{3}, \frac{1}{2}, 1, \frac{3}{2}, 2, -\frac{1}{2}, -\frac{1}{6}, 1, \frac{2}{3}, \frac{1}{2}, -2, -6, \frac{1}{3}, \frac{1}{4}, -1, -3, \frac{1}{6}, -\frac{2}{3}, -2, -\frac{1}{2}, -\frac{3}{2}, 6\Big\} \tag{83}$$

and the 36-set Π:

$$\Pi = \Big\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, -1, -3, \frac{2}{3}, \frac{1}{2}, -2, -6, \frac{3}{4}, -3, -9, -4, -12, 3, \frac{1}{3}, \frac{1}{4}, -1, -3,$$

$$\frac{1}{6}, -\frac{2}{3}, -2, -\frac{1}{2}, -\frac{3}{2}, 6, \frac{1}{12}, -\frac{1}{3}, -1, -\frac{1}{4}, -\frac{3}{4}, 3, -\frac{1}{6}, -\frac{1}{2}, 2, \frac{3}{2}\Big\} \quad (84)$$

Observe that the above Π is free from 1.

By the algorithm (M$'$), the output parameter is given by

$$\lambda' = \left(-\frac{21}{4}, -\frac{175}{3}, \frac{68}{9}, \frac{105625}{324}, -\frac{175}{3}, -\frac{21}{4}\right) \quad (85)$$

so that the Weierstrass equation is:

$$y^2 + txy = x^3 - \frac{21x^2}{4} + \left(\frac{68t}{9} - \frac{175}{3}\right)x + \frac{105625}{324} - \frac{175t}{3} - \frac{21t^2}{4} + t^3 \quad (86)$$

Computing the j-invariant as before, we can check that it has 9 simple poles at $t \neq \infty$ and a pole of order 3 at $t = \infty$. Hence there is a I_3-fibre at $t = \infty$, and no other reducible fibres.

The 27 linear sections $P_n = (at + b, dt + e)$ are given as follows. The first six:

$$P_1 = \left(-t, \frac{325}{18}\right), \quad P_2 = \left(\frac{19}{6} - \frac{t}{2}, \frac{395}{36} - \frac{7t}{4}\right), \quad P_3 = \left(\frac{40}{9} - \frac{t}{3}, \frac{385}{54} - \frac{26t}{9}\right),$$

$$P_4 = \left(\frac{85}{16} - \frac{t}{4}, \frac{2435}{576} - \frac{63t}{16}\right), \quad P_5 = \left(t + \frac{20}{3}, 2t - \frac{5}{18}\right),$$

$$P_6 = \left(3t + \frac{20}{3}, \frac{28t}{3} + \frac{5}{18}\right),$$

The second six:

$$P_7 = \left(\frac{19}{3} - 2t, \frac{7t}{2} - \frac{1}{9}\right), \quad P_8 = \left(\frac{25}{4} - t, -\frac{25}{36}\right), \quad P_9 = \left(\frac{55}{9} - \frac{2t}{3}, -\frac{19t}{18} - \frac{35}{27}\right),$$

$$P_{10} = \left(\frac{35}{6} - \frac{t}{2}, -\frac{7t}{4} - \frac{85}{36}\right), \quad P_{11} = \left(2t + 15, \frac{9t}{2} + \frac{365}{9}\right),$$

$$P_{12} = \left(6t + \frac{115}{3}, \frac{217t}{6} + \frac{1945}{9}\right)$$

The remaining fifteen:

$$P_{13} = \left(8 - t, -\frac{107}{18}\right), \quad P_{14} = \left(\frac{55}{6} - \frac{3t}{2}, \frac{19t}{12} - \frac{395}{36}\right), \quad P_{15} = \left(\frac{35}{3} - 2t, \frac{7t}{2} - \frac{205}{9}\right),$$

$$P_{16} = \left(\frac{t}{2} - \frac{15}{2}, \frac{9t}{4} + \frac{245}{36}\right), \quad P_{17} = \left(\frac{t}{6} - \frac{115}{18}, \frac{217t}{36} - \frac{1615}{108}\right),$$

$$P_{18} = \left(\frac{40}{3} - 3t, \frac{26t}{3} - \frac{565}{18}\right), \quad P_{19} = \left(\frac{85}{4} - 4t, \frac{63t}{4} - \frac{715}{9}\right),$$

$$P_{20} = \left(t - \frac{20}{3}, 2t - \frac{245}{18}\right), \quad P_{21} = \left(\frac{t}{3} - \frac{20}{9}, \frac{28t}{9} - \frac{1105}{54}\right),$$

$$P_{22} = \left(\frac{125}{3} - 6t, \frac{215t}{6} - \frac{2225}{9}\right), \quad P_{23} = \left(\frac{3t}{2} - \frac{25}{6}, \frac{35t}{12} - \frac{725}{36}\right),$$

$$P_{24} = \left(\frac{t}{2} + \frac{1}{2}, \frac{9t}{4} - \frac{619}{36}\right), \quad P_{25} = \left(2t - 1, \frac{9t}{2} - \frac{175}{9}\right),$$
$$P_{26} = \left(\frac{2t}{3} + \frac{25}{9}, \frac{35t}{18} - \frac{325}{27}\right), \quad P_{27} = \left(\frac{125}{18} - \frac{t}{6}, \frac{175}{108} - \frac{215t}{36}\right)$$

9.3. A non-degenerate example of algorithm (A)

$$u = (1, 2, 3, 4, 5, -12), \quad v_0 = 1. \tag{87}$$

The 27-set Ω and the 36-set Π:

$$\Omega = \{1, 2, 3, 4, 5, -12, 0, 1, 2, 3, 4, -13, -2, -3, -4, -5, 12, -4, -5, -6, 11, -6, -7, 10, -8, 9, 8\} \tag{88}$$

$$\Pi_u = \{-1, -1, -2, -3, -4, 13, -1, -2, -3, 14, -1, -2, 15, -1, 16, 17, -5, -6, -7, 10, -7, -8, 9, -9, 8, 7, -8, -9, 8, -10, 7, 6, -11, 6, 5, 4\} \tag{89}$$

Observe that the above Π_u does not contain 0. Hence the chosen data u is non-degenerate and we should have MWL $\simeq E_6^*$ and 27 distinct linear sections. Let us check this more directly.

By the algorithm (A), the output parameter is given by

$$\lambda = \left(-\frac{444675}{256}, -360, -\frac{99}{2}, \frac{1013154175}{2048}, \frac{178695}{2}, \frac{142373}{32}\right) \tag{90}$$

so that the Weierstrass equation is:

$$y^2 - 2t^2 y = x^3 + \left(-\frac{99t^2}{2} - 360t - \frac{444675}{256}\right)x + \frac{142373t^2}{32} + \frac{178695t}{2} + \frac{1013154175}{2048} \tag{91}$$

The j-invariant is given by

$$j = -\left(27\left(4224t^2 + 30720t + 148225\right)^3\right)/16D,$$
$$D = 4096t^8 - 37151488t^6 - 873861120t^5 + 65706368816t^4 + 3115481950080t^3 + 50047294335775t^2 + 360113812047000t + 999242951788125$$

This shows that there is a unique reducible fibre at $t = \infty$, which is of type IV.

The 27 linear sections $P_n = (at + b, dt + e)(1 \leq n \leq 27)$ are as follows:

$$
\begin{aligned}
&\left(t+\frac{4975}{16}, \frac{97t}{4}+\frac{21915}{4}\right), \left(2t+\frac{3595}{16}, \frac{91t}{2}+\frac{54135}{16}\right),\\
&\left(3t+\frac{2255}{16}, \frac{243t}{4}+\frac{6985}{4}\right), \left(4t+\frac{979}{16}, 67t+\frac{12573}{16}\right),\\
&\left(5t-\frac{209}{16}, \frac{245t}{4}+\frac{2871}{4}\right), \left(\frac{15635}{16}-12t, 567t-\frac{488435}{16}\right),\\
&\left(\frac{3971}{16}, \frac{62689}{16}\right), \left(t+\frac{2959}{16}, \frac{97t}{4}+\frac{10197}{4}\right),\\
&\left(2t+\frac{1915}{16}, \frac{91t}{2}+\frac{22635}{16}\right), \left(3t+\frac{815}{16}, \frac{243t}{4}+\frac{2935}{4}\right),\\
&\left(4t-\frac{365}{16}, 67t+\frac{11565}{16}\right), \left(\frac{25135}{16}-13t, \frac{3107t}{4}-\frac{248985}{4}\right),\\
&\left(\frac{715}{16}-2t, -\frac{91t}{2}-\frac{11385}{16}\right), \left(\frac{335}{16}-3t, -\frac{243t}{4}-\frac{2735}{4}\right),\\
&\left(-4t-\frac{461}{16}, -67t-\frac{11547}{16}\right), \left(-5t-\frac{1361}{16}, -\frac{245t}{4}-\frac{657}{4}\right),\\
&\left(12t+\frac{16115}{16}, -567t-\frac{511115}{16}\right), \left(\frac{115}{16}-4t, -67t-\frac{11115}{16}\right),\\
&\left(-5t-\frac{689}{16}, -\frac{245t}{4}-\frac{2799}{4}\right), \left(-6t-\frac{1381}{16}, \frac{643}{16}-\frac{81t}{2}\right),\\
&\left(11t+\frac{9295}{16}, -\frac{1573t}{4}-\frac{55935}{4}\right), \left(-6t-\frac{1045}{16}, -\frac{81t}{2}-\frac{9185}{16}\right),\\
&\left(-7t-\frac{1265}{16}, \frac{1485}{4}-\frac{7t}{4}\right), \left(10t+\frac{4651}{16}, -\frac{505t}{2}-\frac{79281}{16}\right),\\
&\left(-8t-\frac{605}{16}, 58t+\frac{11385}{16}\right), \left(9t+\frac{1775}{16}, -\frac{567t}{4}-\frac{5165}{4}\right),\\
&\left(8t+\frac{355}{16}, -58t-\frac{10935}{16}\right)
\end{aligned}
$$

9.4. A cubic surface with four A_1-singular points

Take the input data

$$\xi = \left(-1, -1, 2, 2, \frac{1}{2}, \frac{1}{2}; 1\right). \tag{92}$$

The 27-set Ω and the 36-set Π:

$$\Omega = \Bigg\{ -1, -1, 2, 2, \frac{1}{2}, \frac{1}{2}, -1, -1, 2, 2, \overset{1}{-2}, \frac{1}{2},$$

$$1, -\frac{1}{2}, -\frac{1}{2}, -2, -2, -\frac{}{1}2, -\frac{1}{2}, -2, -2, \frac{1}{4}, 1, 1, 1, 1, 4\Big\} \quad (93)$$

$$\Pi = \Big\{1, 1, -\frac{1}{2}, -\frac{1}{2}, -2, -2, -\frac{1}{2}, -\frac{1}{2}, -2, -2, 1, 4, 4, 4, 4, 1, \frac{1}{2}, \frac{1}{2},$$
$$2,\ 2, -\frac{1}{4}, -1, -1, -1, -1, -4, -\frac{1}{4}, -1, -1, -1,\ -1, -4, \frac{1}{2}, \frac{1}{2}, 2, 2\Big\} \quad (94)$$

The number of vanishing roots $\nu = 4$. This holds only if either $T_{new} = 4A_1$ or $= A_1 \oplus A_2$.

Let us check that we have $T_{new} = 4A_1$; OS41 (i.e. No. 41 in [8]).

By the algorithm (M), the output parameter is given by

$$\lambda = \left(-\frac{483}{16}, 4, \frac{7}{8}, \frac{2009}{32}, -14, -\frac{287}{64}\right) \quad (95)$$

and the Weierstrass equation is:

$$y^2 + txy = x^3 + \left(\frac{7t^2}{8} + 4t - \frac{483}{16}\right)x + t^3 - \frac{287t^2}{64} - 14t + \frac{2009}{32} \quad (96)$$

The j-invariant is computed as

$$j = -\frac{4\left(t^4 - 42t^2 - 192t + 1449\right)^3}{(t-5)^2(t-3)^2(t+1)^2(t+7)^2(4t-21)}. \quad (97)$$

It shows that there are four I_2-fibres at $t = 5, 3, -1$ and -7, in addition to the I_3- fibre at $t = \infty$. Further, the four singular points of V_λ and X_λ are as follows;

$$(t, x, y) = \left(5, -\frac{11}{4}, \frac{55}{8}\right), \left(3, \frac{5}{4}, -\frac{15}{8}\right), \left(-1, \frac{13}{4}, \frac{13}{8}\right), \left(-7, -\frac{35}{4}, -\frac{245}{8}\right) \quad (98)$$

The 27 lines $l_n (1 \le n \le 27)$ in the generic case reduce to the 6 lines with multiplicity 4:

$$\left(t - \frac{7}{4}, \frac{23t}{8} - \frac{21}{2}\right), \quad \left(\frac{11}{4} - \frac{t}{2}, \frac{3}{4} - \frac{7t}{8}\right), \quad \left(\frac{29}{4} - 2t, \frac{35t}{8} - 15\right),$$
$$\left(2t + \frac{21}{4}, \frac{43t}{8} + 7\right), \quad \left(\frac{t}{2} - \frac{21}{4}, \frac{25t}{8} - \frac{35}{4}\right), \quad \left(\frac{9}{4} - t, \frac{7t}{8} + \frac{5}{2}\right), \quad (99)$$

and the 3 lines with multiplicity 1:

$$\left(\frac{9}{2} - t, \frac{7t}{8} - \frac{17}{4}\right), \quad \left(\frac{39}{2} - 4t, \frac{133t}{8} - 83\right), \quad \left(\frac{57}{16} - \frac{t}{4}, \frac{43}{64} - \frac{49t}{16}\right). \quad (100)$$

9.5. Totally degenerare case with E_6-singularity

Finally take the input data

$$\xi = (1^6; 1)\,. \tag{101}$$

We write 1^n for n-times 1. The 27-set Ω and the 36-set Π are simply

$$\Omega = \{1^{27}\}\,, \quad \Pi = \{1^{36}\}\,. \tag{102}$$

The number of vanishing roots is $\nu = 36$, which holds only if $T_{new} = E_6$, i.e. we should have No. 69 in [8]. The MW-group will be $\mathbf{Z}/3\mathbf{Z}$, there is only one line on the cubic surface, which (as well as the affine surface X) will have a unique E_6-singularity.

Let us check these facts directly, verifying that we are in the case No. 69.

By the algorithm (M), the output parameter is given by

$$\lambda = \left(81, -36, \frac{9}{2}, 54, 0, -\frac{27}{4}\right) \tag{103}$$

and the Weierstrass equation is:

$$y^2 + txy = x^3 + t^3 + \left(\frac{9t^2}{2} - 36t + 81\right)x - \frac{27t^2}{4} + 54 \tag{104}$$

The discriminant and j-invariant is given by

$$\Delta = -(t-6)^8(t+21)\,, \quad j = -\frac{(t-6)(t+18)^3}{t+21} \tag{105}$$

which shows that there is a singular fibre of type IV^* at $t = 6$, in addition to the I_3- fibre at $t = \infty$. Further, the unique singular point of V_λ and X_λ below

$$(t, x, y) = (6, -3, 9)$$

is an E_6-singularity, which can be resolved using only curves defined over rational numbers (i.e. without using any irrational numbers in the resolution process).

The MW group is generated by an order 3 point:

$$P_1 : (x, y) = \left(-t + 3, \frac{9}{2}t - 18\right)$$

which gives the unique line on V_λ (multiplicity 27) passing through the singular point.

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Appendix

Picture of a Cubic Surface with 27 Lines and a Plane Quartic with 28 Bitangents, All Over Q

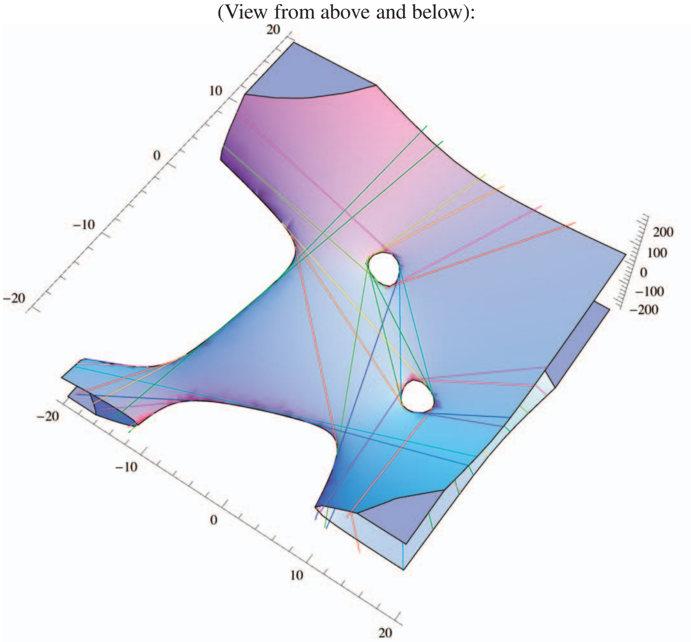
by

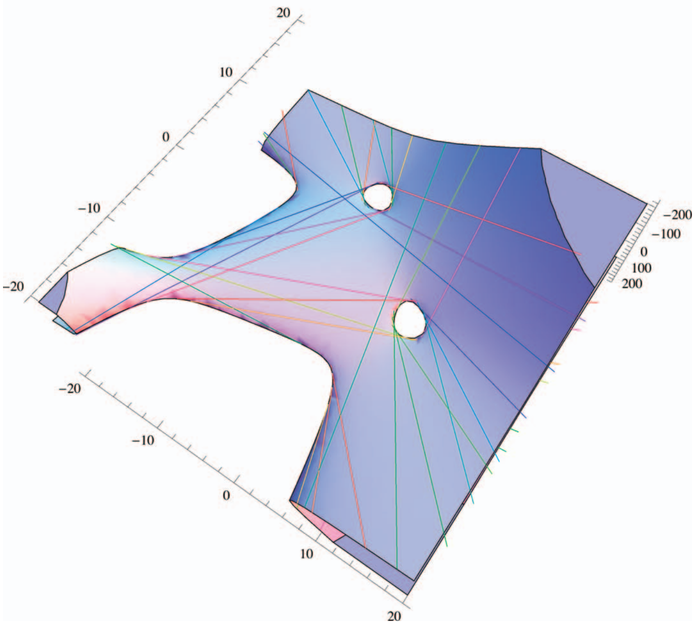
Tetsuji SHIODA

Graphics are drawn with Mathematica, using the examples of outcome of the algorithms in Section 9. The 3D-pictures are in the (t, x, y)-space and the others are in the (t, x)-plane.

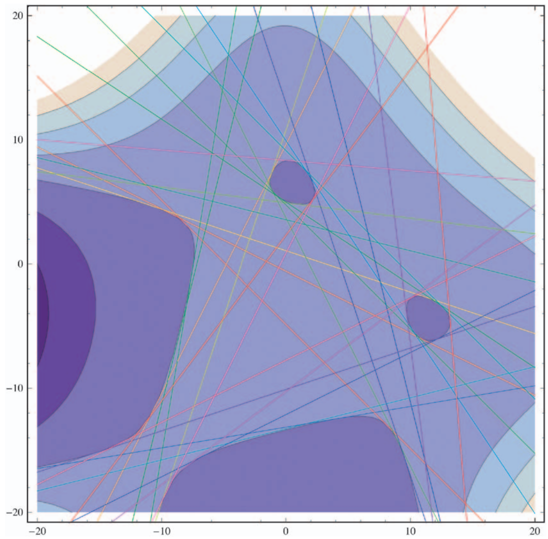
EXAMPLE 9.1 a cubic surface with 27 lines

(View from above and below):

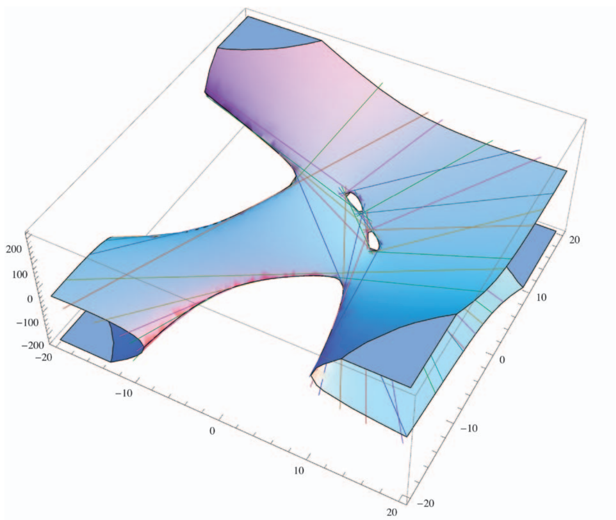




contour map of a plane quartic with 28 bitangents:



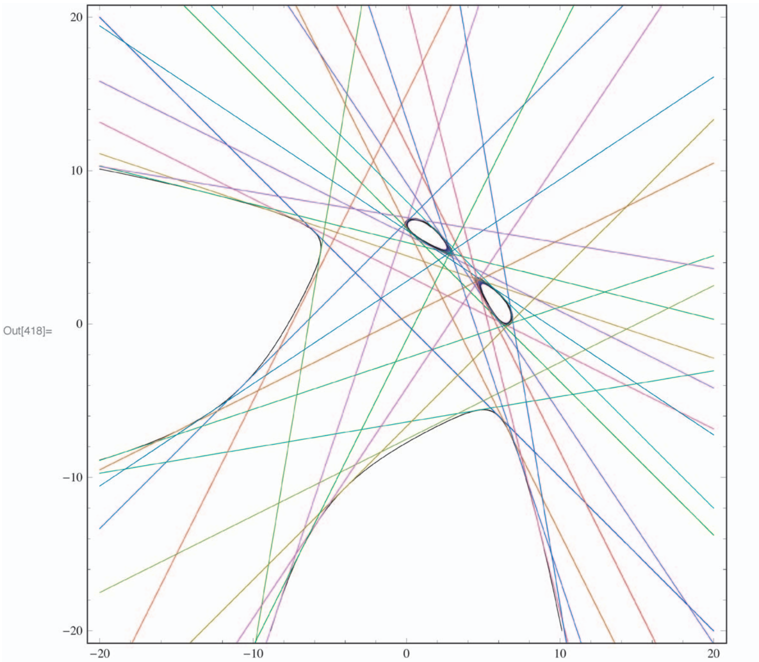
EXAMPLE 9.2 a cubic surface with 27 lines:



Weierstrass equation (M') :

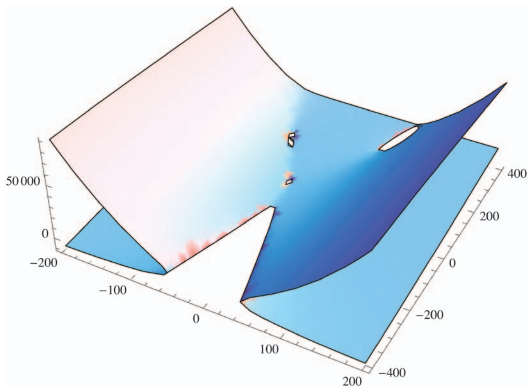
$$\frac{105625}{324} - \frac{175\,t}{3} - \frac{21\,t^2}{4} + t^3 + \left(-\frac{175}{3} + \frac{68\,t}{9}\right) x - \frac{21\,x^2}{4} + x^3 - t\,x\,y - y^2 = 0$$

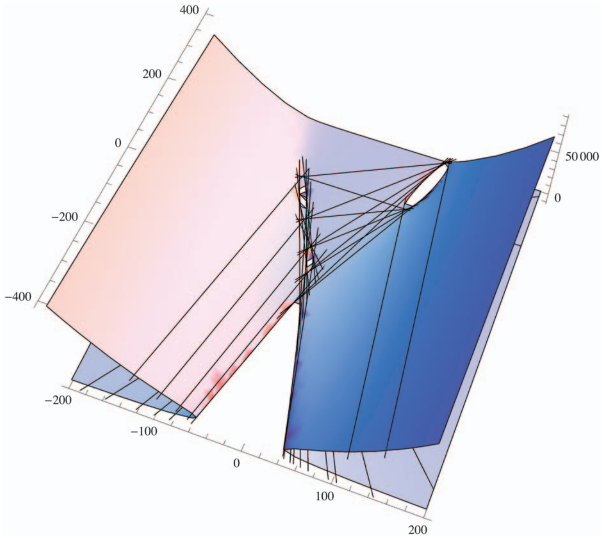
a plane quartic with 28 bitangents:



$$\frac{105625}{324} - \frac{175\,t}{3} - \frac{21\,t^2}{4} + t^3 + \left(-\frac{175}{3} + \frac{68\,t}{9}\right) x + \frac{1}{4}\left(-21 + t^2\right) x^2 + x^3 = 0$$

EXAMPLE 9.3

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200
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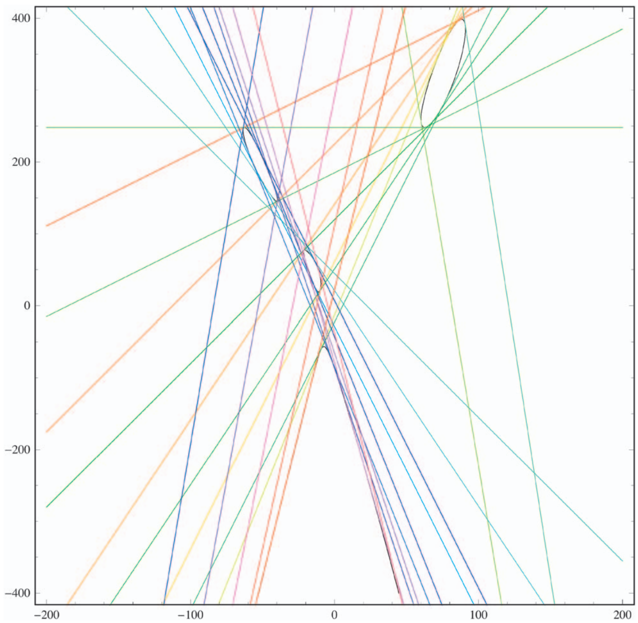
0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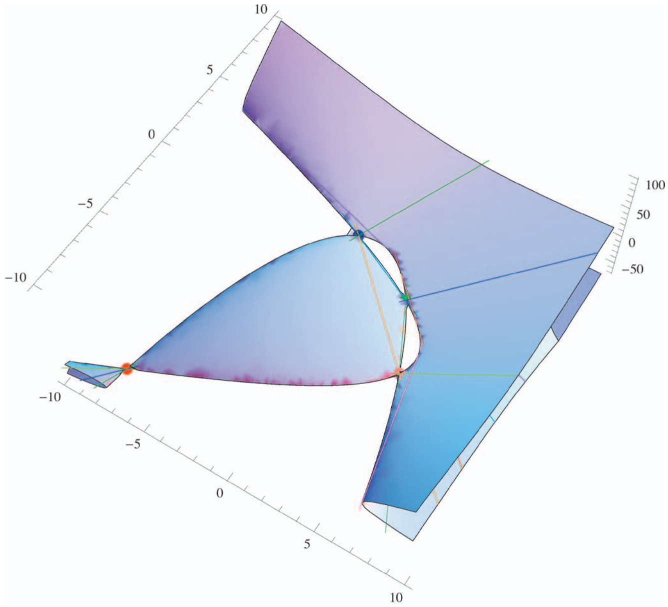
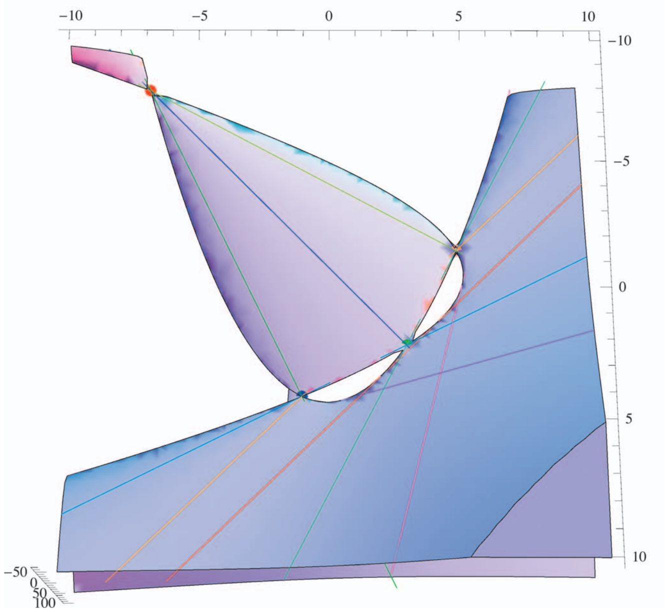
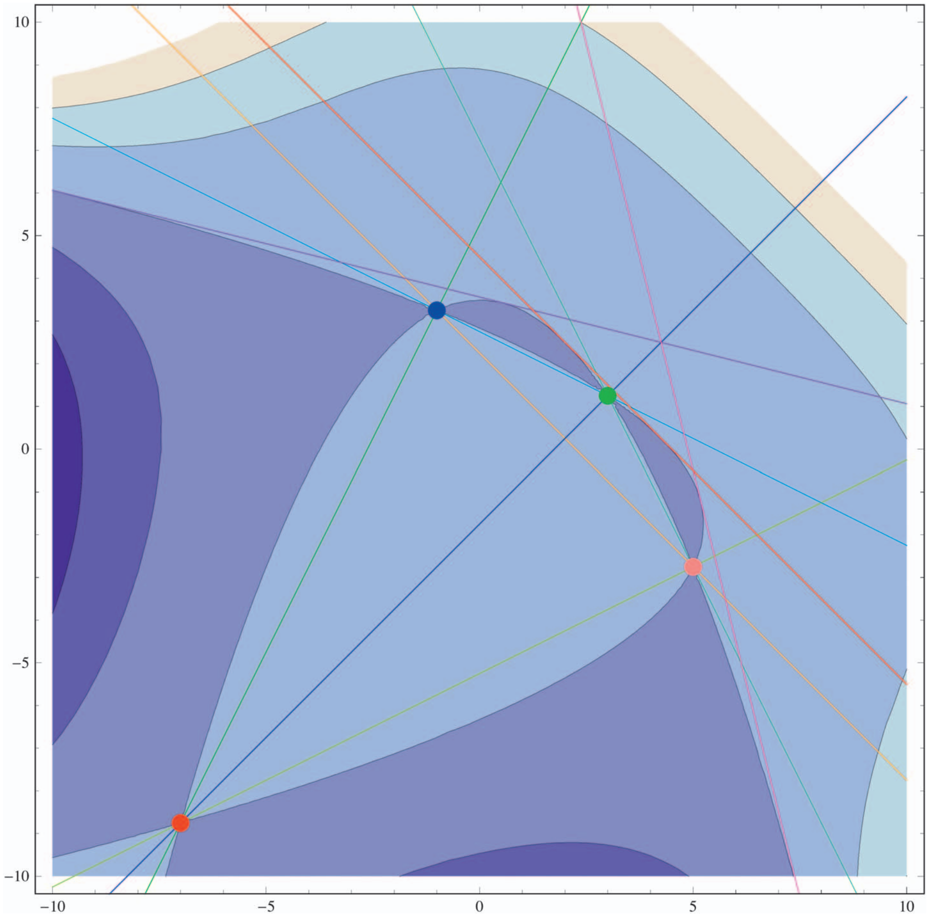
−100
0
100
200

EXAMPLE 9.4 a cubic surface with four A1-nodes:



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