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Continued Fraction Expansions with Even Period and Primary Symmetric Parts with Extremely Large End

by

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Abstract. For a non-square positive integer d with $4 \nmid d$, put $\omega(d) := (1+\sqrt{d})/2$ if d is congruent to 1 modulo 4 and $\omega(d) := \sqrt{d}$ otherwise. Let $a_1, a_2, \ldots, a_{\ell-1}$ be the symmetric part of the simple continued fraction expansion of $\omega(d)$. We say that the sequence $a_1, a_2, \ldots, a_{[\ell/2]}$ is the primary symmetric part of the simple continued fraction expansion of $\omega(d)$. The main purposes of this article are to introduce a notion of "extremely large end (ELE)" for a finite sequence, and to study properties for a non-square positive integer d such that the primary symmetric part of the simple continued fraction expansion of $\sqrt{d}$ with even period is of ELE type.

Introduction

Let d be a non-square positive integer and put $\alpha = \sqrt{d}$ or $\alpha = (1+\sqrt{d})/2$. Then it is known that the simple continued fraction expansion is of the form

$$\alpha = [a_0, \overline{a_1, a_2, \ldots, a_\ell}] \text{ (the periodic part begins with } a_1) ,$$
$$a_n = a_{\ell-n} \ (1 \le n \le \ell-1) \text{ (the symmetric property holds)} .$$

Here, ℓ is the minimal period. Then we say that the sequence $a_1, a_2, \ldots, a_{\ell-1}$ is the *symmetric part* of the simple continued fraction expansion of α. Moreover, putting $L := [\ell/2]$, we say that the sequence $a_1, a_2, \ldots, a_L$ is the *primary symmetric part* of the simple continued fraction expansion of α, where $[x]$ denotes the largest integer $\le x$ for a real number x. For a non-square positive integer d with $4 \nmid d$, put $\omega(d) := (1+\sqrt{d})/2$ if d is congruent to 1 modulo 4 and $\omega(d) := \sqrt{d}$ otherwise. Then the canonical integral basis of a real quadratic field $\mathbb{Q}(\sqrt{d})$ is given by $\{1, \omega(d)\}$ when d is square-free. In this paper, we examine primary symmetric parts of the simple continued fraction expansions of $\omega(d)$.

The class number one problem for real quadratic fields is a mysterious classical problem. The class number is closely related to the fundamental unit. For instance, by Siegel's

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Theorem, the fundamental units of real quadratic fields with class number 1 are relatively large. It is known that there exist only finitely many real quadratic fields of extended Richaud–Degert type (call simply ERD type; see Mollin [18, Definition 3.2.2] for the definition) with class number 1 and they are determined (see also [18, Theorem 5.4.3]) with one more possible exception. We easily see that the fundamental units of real quadratic fields $\mathbb{Q}(\sqrt{d})$ of ERD type are $< d$, namely, they are small, by using their explicit form. Moreover the minimal periods of $\omega(d)$ are ≤ 12 (cf. [18, Section 3.2]). According to results of Sasaki [21] and Lachaud [15], for any positive integers ℓ and h, there exist at most finitely many real quadratic fields with period ℓ of class number h. Yamamoto [23], Halter-Koch [5, 6], Williams [22] and others examined a construction of infinite families of real quadratic fields with large fundamental units (see [22] for the history). We can observe that these infinite families consist of real quadratic fields with various periods. Mollin [19], McLaughlin [17], and [12] examined a construction of infinite families of real quadratic fields with a given even period. However the fundamental unit of them is relatively small.

In [11], on the other hand, it was proved that there exist exactly 51 real quadratic fields of class number 1 that are not of minimal type (we give the definition later), with one more possible exception. This was shown by using the fact that if a real quadratic field $\mathbb{Q}(\sqrt{d})$ is not of minimal type then the Yokoi invariant m_d of d (see Remark 1.4 (2) for the definition) is ≤ 3 (see [11, Proposition 4.2] and [13, Proposition 4.2]). Hence a real quadratic field with large fundamental unit is of minimal type. Thus we have to examine a construction of real quadratic fields with non-fixed period ℓ of minimal type in order to find many real quadratic fields of class number 1.

Here, let d_ℓ be the smallest integer d such that the minimal periods of the simple continued fraction expansions of $\omega(d)$ are equal to a fixed positive integer ℓ where d runs through square-free positive integers with $d \equiv 2, 3 \pmod 4$. Then the following hold for each even positive integer ℓ with $8 \leq \ell \leq 73478$; i) the class number of $\mathbb{Q}(\sqrt{d_\ell})$ is equal to 1, ii) $\mathbb{Q}(\sqrt{d_\ell})$ is of minimal type, iii) the primary symmetric part of the simple continued fraction expansion of $\omega(d_\ell)$ is of ELE type (see Section 6 for more detail). In the next section, we introduce a notion of "extremely large end (ELE)" for a finite sequence of positive integers.

From now on, we shall state the definition of "minimal type". For a symmetric sequence of $\ell - 1$ positive integers $a_1, a_2, \ldots, a_{\ell-1}$, we define nonnegative integers q_n, r_n by using a_n $(1 \leq n \leq \ell - 1)$:

$$\begin{cases} q_0 = 0, \quad q_1 = 1, \quad q_n = a_{n-1}q_{n-1} + q_{n-2}\ (2 \leq n \leq \ell)\,, \\ r_0 = 1, \quad r_1 = 0, \quad r_n = a_{n-1}r_{n-1} + r_{n-2}\ (2 \leq n \leq \ell)\,. \end{cases} \tag{0.1}$$

For brevity, we put

$$A := q_\ell,\ B := q_{\ell-1},\ C := r_{\ell-1}\,,$$

and define linear polynomials $g(x), h(x)$ and a quadratic polynomial $f(x)$ by

$$g(x) = Ax - (-1)^\ell BC,\ h(x) = Bx - (-1)^\ell C^2,\ f(x) = g(x)^2 + 4h(x)\,.$$

Furthermore, let s_0 be the least integer x for which $g(x) > 0$.

We consider three cases separately:

$$\text{(I) } A \equiv 1 \pmod 2, \quad \text{(II) } (A, C) \equiv (0, 0) \pmod 2, \quad \text{(III) } (A, C) \equiv (0, 1) \pmod 2\,.$$

The following theorem was shown in [11, Theorem 3.1] which is an improvement of results of Friesen [1, Theorem] and of Halter-Koch [7, Theorem 1A, Corollary 1A].

THEOREM 0.1. *Let* $\ell \geq 2$ *be a fixed positive integer and* $a_1, \dots, a_{\ell-1}$ *any symmetric sequence of* $\ell - 1$ *positive integers.*

When Case (I) *or Case* (II) *occurs, we let* s *be any integer with* $s \geq s_0$, *and put* $d := f(s)/4$ *and* $a_0 := g(s)/2$. *Here, we choose an even integer* s *in Case* (I), *and assume that*

$$g(s) > a_1, \dots, a_{\ell-1} . \tag{0.2}$$

Then, d *and* a_0 *are positive integers,* d *is non-square,* $a_0 = [\sqrt{d}]$ *and the simple continued fraction expansion of* $\sqrt{d}$ *is*

$$\sqrt{d} = [a_0, \overline{a_1, \dots, a_{\ell-1}, 2a_0}] \tag{0.3}$$

with minimal period ℓ. *Also, in Case* (III), *there is no positive integer* d *such that* (0.3) *is the simple continued fraction expansion of* $\sqrt{d}$.

When Case (I) *or Case* (III) *occurs, we let* s *be any integer with* $s \geq s_0$, *and put* $d := f(s)$ *and* $a_0 := (g(s) + 1)/2$. *Here, we choose an odd integer* s *in Case* (I), *and assume that* (0.2) *holds. Then,* d *and* a_0 *are positive integers,* d *is non-square,* $d \equiv 1 \pmod 4$, $a_0 = [(1 + \sqrt{d})/2]$ *and the simple continued fraction expansion of* $(1 + \sqrt{d})/2$ *is*

$$\frac{1 + \sqrt{d}}{2} = [a_0, \overline{a_1, \dots, a_{\ell-1}, 2a_0 - 1}] \tag{0.4}$$

with minimal period ℓ. *Also, in Case* (II), *there is no positive integer* d *such that* $d \equiv 1 \pmod 4$ *and* (0.4) *is the simple continued fraction expansion of* $(1 + \sqrt{d})/2$.

Conversely, we let d *be any non-square positive integer. By using a quadratic polynomial* $f(x)$ *and an integer* s_0 *obtained as above from the symmetric part of the simple continued fraction expansion of* $\sqrt{d}$, d *can be written uniquely as* $d = f(s)/4$ *with some integer* $s \geq s_0$, *and* (0.2) *holds. If* $d \equiv 1 \pmod 4$ *in addition then the same thing is true for* $(1 + \sqrt{d})/2$.

DEFINITION 0.1 ([11, Definition 3.1]). Let d be a non-square positive integer. By Theorem 0.1, d can be written uniquely as $d = f(s)/4$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \dots, a_{\ell-1}$ of the simple continued fraction expansion of $\sqrt{d}$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)/4$ holds, then we say that d is a *positive integer with period* ℓ *of minimal type for* (*the simple continued fraction expansion of*) $\sqrt{d}$. When $d \equiv 1 \pmod 4$ in addition, d can be written uniquely as $d = f(s)$ with some integer $s \geq s_0$, where $f(x)$ and s_0 are obtained as above from the symmetric part $a_1, a_2, \dots, a_{\ell-1}$ of the simple continued fraction expansion of $(1 + \sqrt{d})/2$ and ℓ is the minimal period. If $s = s_0$, that is, $d = f(s_0)$ holds, then we say that d is a *positive integer with period* ℓ *of minimal type for* (*the simple continued fraction expansion of*) $(1 + \sqrt{d})/2$.

Furthermore, for a square-free positive integer $d > 1$, we say that $\mathbb{Q}(\sqrt{d})$ is a *real quadratic field with period* ℓ *of minimal type*, if d is a positive integer with period ℓ of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \pmod 4$, and if d is a positive integer with period ℓ of minimal type for $(1 + \sqrt{d})/2$ when $d \equiv 1 \pmod 4$.

In [10], following [11], [12] and [14], we calculated s_0, $g(s_0)$, $h(s_0)$. By using this result, we construct a real quadratic field $\mathbb{Q}(\sqrt{d})$ of minimal type such that the primary symmetric part of the simple continued fraction expansion of $\omega(d)$ is of ELE type.

1. Introduction to sequences of ELE type and main results

In this section, we introduce a notion of "extremely large end" for a finite sequence of positive integers and describe our main theorems (Theorems 1, 2). Theorem 1 contains great pioneering works of Golubeva [3, 4] (see Remark 1.2). We let d be a non-square positive integer and assume that the simple continued fraction expansion of $\sqrt{d}$ is

$$\sqrt{d} = [a_0, \overline{a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1, 2a_0}]$$

with minimal even period $2L$ (≥ 4). Then it is known by a classical result (see Perron [20, Satz 3.14]) that both

$$a_n < \frac{2a_0}{3} \ (1 \leq {}^{\forall}n \leq L-1)\,,$$

and

$$a_L = a_0,\ a_L = a_0 - 1 \text{ or } a_L \leq \frac{2a_0}{3}$$

hold. When the condition

$$a_L = a_0 \text{ or } a_L = a_0 - 1 \tag{1.1}$$

holds, we see that the value of a_L is relatively larger than that of the former partial quotients a_n $(1 \leq n \leq L-1)$. We will give new conditions which are equivalent to the condition (1.1). For this, we consider the conditions

$$\text{“}a_L \geq 2 \text{ and } \mu = a_L\text{” or “}a_L \geq 4 \text{ and } \mu = a_L + 2\text{”}\,. \tag{1.2}$$

Here we define an integer $\mu \geq 0$ as follows by using the results of [10]. From the primary symmetric part $a_1, \dots, a_L$, we calculate nonnegative integers q_n, r_n $(1 \leq n \leq L+1)$ by using (0.1), and define integers u_1, u_2, w, v_1, v_2, z, δ by

$$(r_L^2 - (-1)^L)(r_{L+1} + r_{L-1}) = q_L v_1 + u_1 \ (0 \leq u_1 < q_L), \tag{1.3}$$

$$(-1)^L (r_L - q_{L-1}) r_L = q_L z + w \ (0 \leq w < q_L), \tag{1.4}$$

$$(-1)^L (q_L - r_{L+1}) + z = q_L v_2 + u_2 \ (0 \leq u_2 < q_L)\,, \tag{1.5}$$

$$\delta = \begin{cases} 0 & \text{if } u_1 \leq u_2\,, \\ 1 & \text{if } u_1 > u_2\,. \end{cases}$$

We put

$$\gamma := q_L(\delta q_L + u_2 - u_1) + w, \tag{1.6}$$

$$\mu := \frac{1}{q_L}\{\gamma(q_{L+1} + q_{L-1}) + 2(q_{L-1} - r_L)\} \tag{1.7}$$

which is the first term of the right hand-side of (2.16) in Section 2. We determine quadratic irrationals ω_n $(0 \leq n \leq 2L)$ such that

$$\omega_0 := \sqrt{d}\,, \quad \omega_n = a_n + \frac{1}{\omega_{n+1}}, \quad a_n = [\omega_n]\,,$$

where $a_n = a_{n-L}$ $(L+1 \leq n \leq 2L-1)$ and $a_{2L} = 2a_0$. Then we can write uniquely $\omega_n = (P_n + \sqrt{d})/Q_n$ with some positive integers P_n, Q_n for each $n \geq 1$ (cf. [11, Section 2]).

THEOREM 1. *Under the above setting, assume that* $L \geq 3$ *and* $d \neq 19$. *Then the following four conditions are equivalent*:

(i) *d is of minimal type for $\sqrt{d}$ and the condition* (1.2) *holds*;

(ii) *d is of minimal type for $\sqrt{d}$, and either*

$$r_L = 2q_{L-1},\ a_L \equiv (-1)^{L-1} q_{L-1} r_{L-1} \pmod{q_L} \text{ and } a_L \geq 2$$

or

$$r_L = 2q_{L-1} - q_L,\ a_L \equiv (-1)^{L-1} q_{L-1}(q_{L-1} + r_{L-1}) \pmod{q_L} \text{ and } a_L \geq 4$$

holds;

(iii) $Q_L = 2$;

(iv) $a_L = a_0$ *or* $a_L = a_0 - 1$.

In particular, Theorem 1 leads to the following corollary which gives a family of real quadratic fields of minimal type.

COROLLARY 1. *Let p be a prime number with* $p \equiv 3 \pmod 4$. *Then if the minimal period of the simple continued fraction expansion of $\sqrt{p}$ is less than or equal to* 4, *then $\mathbb{Q}(\sqrt{p})$ is not of minimal type. On the other hand, if it is greater than or equal to* 6 *then $\mathbb{Q}(\sqrt{p})$ is of minimal type.*

REMARK 1.1. Let $d = 19$. Then, $\sqrt{d} = [4, \overline{2, 1, 3, 1, 2, 8}]$, $L = 3$, $a_L = a_0 - 1 = 3$, $Q_L = 2$, and we have the following table:

n	0	1	2	3	4
q_n	0	1	2	3	11
r_n	1	0	1	1	4

We easily see that $u_1 = 1$, $v_1 = 3$; $w = 1$, $z = 0$; $u_2 = 1$, $v_2 = 0$; $\delta = 0$, $\gamma = 1$, $\mu = a_L + 2 = 5$; $r_L = 2q_{L-1} - q_L = 1$, and $a_L \equiv (-1)^{L-1} q_{L-1}(q_{L-1} + r_{L-1}) \pmod{q_L}$. Moreover $d = 19$ is of minimal type for $\sqrt{d}$ because of $s = s_0 = 2$. Thus all conditions of Theorem 1 hold with one exception "$a_L \geq 4$".

REMARK 1.2. Golubeva proved that (iii) yields the equation and the congruence in (ii) when d is a prime number congruent to 3 modulo 4 ([4, Theorem 1]). However her ingenious proof also works for any non-square positive integer d as in Theorem 1 (cf. Section 4.4). The implication (iii) $\Rightarrow$ (iv) is shown in the proof of [20, Satz 3.14] or [4, p.1279].

Now we see by Theorem 1 that the condition (1.2) is a necessary condition for the condition (1.1) under some conditions. So we define the following notion.

DEFINITION 1.1. Let $L \geq 2$ and let $a_1, a_2, \dots, a_L$ be a sequence of positive integers. If the above condition (1.2) holds, we say that $a_1, a_2, \dots, a_L$ is a *sequence with extremely large end* (we also write that $a_1, a_2, \dots, a_L$ is of *ELE type*). Specially $a_1, a_2, \dots, a_L$ is said to be of *ELE_1 type* (resp. *ELE_2 type*) if $a_L \geq 2$ and $\mu = a_L$ (resp. $a_L \geq 4$ and $\mu = a_L + 2$) hold.

REMARK 1.3. We consider a sequence a_1, a_2. Using the calculation results in [10, Example 1], we have

$$\mu = \begin{cases} 0 & \text{if } a_1 \mid a_2, \\ (a_1 - r)(a_1 a_2 + 2) & \text{if } a_1 \nmid a_2, \end{cases}$$

where r is the remainder of the division of a_2 by a_1. We see that if $a_1 \mid a_2$,

$$\mu = 0 < a_2 < a_2 + 2$$

and if $a_1 \nmid a_2$,

$$\mu = (a_1 - r)(a_1 a_2 + 2) \geq a_1 a_2 + 2 > a_2 + 2 > a_2$$

because of $a_1 > 1$. Hence we obtain $\mu \neq a_2$ and $\mu \neq a_2 + 2$. Therefore, there is no sequence of ELE type with length 2.

Theorem 2 (2) stated below gives a way of constructing every positive integer d satisfying the condition (i) of Theorem 1, namely, a positive integer d of minimal type such that the primary symmetric part of the simple continued fraction expansion of $\sqrt{d}$ with even period is of ELE type (see the proof of the implication (i) ⇒ (iv) in Section 4.2).

THEOREM 2. *Assume that a sequence $a_1, a_2, \dots, a_L$ $(L \geq 3)$ is of ELE type. In addition, we assume*

$$2a_L > a_1, a_2, \dots, a_{L-1} \tag{1.8}$$

$$(\mathit{resp.}\ 2a_L + 2 > a_1, a_2, \dots, a_{L-1}), \tag{1.9}$$

and put $\varepsilon := 0$ (resp. $\varepsilon := 1$) if $a_1, a_2, \dots, a_L$ is of ELE_1 type (resp. ELE_2 type).

(1) *There does not exist a positive integer d, $d \equiv 1 \pmod 4$, with period $2L$ of minimal type for $(1 + \sqrt{d})/2$ whose simple continued fraction expansion has the symmetric part $a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1$.*

(2) *Put $a_0 := g(s_0)/2$, $d := f(s_0)/4$. Then a_0 and d are positive integers with*

$$a_0 = a_L + \varepsilon \ \text{ and } \ d = (a_L + \varepsilon)^2 + \frac{2r_{L+1} + \varepsilon r_L}{q_L} \equiv \begin{cases} 2 \pmod 4 & \text{if } a_L \text{ is even}, \\ 3 \pmod 4 & \text{if } a_L \text{ is odd}. \end{cases}$$

Furthermore, the simple continued fraction expansion of $\sqrt{d}$ is

$$\sqrt{d} = [a_L + \varepsilon, \overline{a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1, 2a_L + 2\varepsilon}]$$

and d is a positive integer with period $2L$ of minimal type for $\sqrt{d}$.

(3) *Let d be as in (2). Then we have*

$$(-1)^n Q_n = -\frac{2r_{L+1} + \varepsilon r_L}{q_L} q_n^2 + 2(a_L + \varepsilon) q_n r_n + r_n^2 \ (1 \leq n \leq 2L - 1). \tag{1.10}$$

In particular, we have

$$\begin{aligned} Q_L &= 2\,, \\ Q_{L-1} &= \frac{1}{2}\left(\frac{2r_{L+1}+\varepsilon r_L}{q_L} + \varepsilon(2a_L+1)\right)\,, \\ Q_1 &= \frac{2r_{L+1}+\varepsilon r_L}{q_L}\,. \end{aligned}$$

Moreover, let m_d *be the Yokoi invariant of* d *defined below. Then we have* $m_d = 2q_L^2$ *if* L *is even, and* $m_d = 2q_L^2 - 1$ *if* L *is odd.*

REMARK 1.4. (1) The values of Q_n are related to the class number one problem (cf. Louboutin [16]). They will be studied on another occasion.

(2) Let d be a non-square positive integer with $d \equiv 2,\ 3 \pmod 4$. We let $d = d_1 d_2^2$ be a factorization of d into positive integers with d_1 square-free, and consider a real quadratic field $K = \mathbb{Q}(\sqrt{d_1})$. Let $\mathcal{O}_{d_2}$ be the order of conductor d_2 in K, that is, the subring of the ring $\mathcal{O}_K$ of integers in K, containing 1, with finite index $(\mathcal{O}_K : \mathcal{O}_{d_2}) = d_2$. By [13, Lemma 2.3], the discriminant of $\mathcal{O}_{d_2}$ is $4d$. Thus we consider the real quadratic order of discriminant $4d$ (cf. [13, Remark 2.4]). We denote by $E_d > 1$ the fundamental unit of $\mathcal{O}_{d_2}$. Then we can write uniquely $E_d = (T + U\sqrt{d})/2$ with positive integers T, U. We define an integer $m_d\ (\geq 0)$ by $m_d = [U^2/T]$ and call it the *Yokoi invariant of* d ([13, Definition 2.1]). By a theorem of Yokoi ([13, Theorem 2.1 [B]]) for a non-square positive integer, it holds that $m_d d < E_d < (m_d+1)d$ if $d > 13$. Thus the quantity m_d gives a size of the fundamental unit E_d for d. The value of m_d gives a rough size of E_d instead of the regulator $\log E_d$.

This paper is organized as follows. After preparations in Section 2, we prove Theorem 2 in Section 3. By using Theorem 2, we prove Theorem 1 in Section 4. In Section 5, we prove Corollary 1. In Section 6, we state motives which came to consider the notion of "ELE", and then give numerical examples.

In [9], we will examine a construction of sequences of ELE type.

2. Preparations

We let d be a non-square positive integer and assume that the simple continued fraction expansion of $\sqrt{d}$ is $\sqrt{d} = [a_0, \overline{a_1, \ldots, a_{\ell-1}, 2a_0}]$ with minimal period $\ell\ (\geq 2)$. In order to prove our theorems, we collect the facts on the simple continued fraction expansions with even period. For basic properties of continued fractions, we refer the reader to an excellent book of Halter-Koch [8]. From the symmetric part $a_1, \ldots, a_{\ell-1}$, we define nonnegative integers q_n, r_n by (0.1) and define positive integers p_n by a recurrence equation:

$$p_0 = 1, \quad p_1 = a_0, \quad p_n = a_{n-1}p_{n-1} + p_{n-2}\ (2 \leq n \leq \ell)\,. \tag{2.1}$$

Then the following hold (not necessary the condition "ℓ even").

LEMMA 2.1. *Let the notation be as above. For* $0 \leq n \leq \ell - 1$, *the following hold*:

$$q_{n+1}r_n - q_n r_{n+1} = (-1)^n, \tag{2.2}$$

$$p_n = a_0 q_n + r_n, \tag{2.3}$$

$$P_{n+1} = P_{\ell-n}, \quad Q_n = Q_{\ell-n}, \tag{2.4}$$

$$P_{n+1} + P_n = a_n Q_n, \tag{2.5}$$

$$d = P_{n+1}^2 + Q_n Q_{n+1}, \tag{2.6}$$

$$0 < P_{n+1} \le a_0 < \sqrt{d}, \quad 0 < Q_{n+1} < 2\sqrt{d}, \tag{2.7}$$

$$Q_n > 1 \ (n \neq 0), \tag{2.8}$$

$$p_n^2 - dq_n^2 = (-1)^n Q_n. \tag{2.9}$$

Proof. For (2.2), see for example [12, (2.3)]; For (2.3), see [12, (2.4)]; For (2.4), see [12, (3.7)]; For (2.5), see [12, (2.16)]; For (2.6), see [12, (2.18)]; For (2.7), see [11, p.871]; For (2.8), see [11, Lemma 2.2]; For (2.9), see [12, Lemma 2.7]. ☐

From now on, we suppose that ℓ is even. We write $\ell = 2L$ with some integer L and define Q and R by

$$Q := q_{L+1} + q_{L-1} (= a_L q_L + 2q_{L-1}),$$
$$R := r_{L+1} + r_{L-1} (= a_L r_L + 2r_{L-1}),$$

respectively, for convenience.

LEMMA 2.2. *Let the notation be as above. Then we have*

$$A = q_\ell = Q q_L, \tag{2.10}$$

$$B = q_{\ell-1} = Q r_L - (-1)^L, \tag{2.11}$$

$$C = r_{\ell-1} = R r_L, \tag{2.12}$$

$$p_\ell = p_L q_{L+1} + p_{L-1} q_L, \tag{2.13}$$

$$p_L = \frac{Q_L Q}{2}, \tag{2.14}$$

$$Q r_L - q_L R = (-1)^L 2, \tag{2.15}$$

$$g(s_0) = \frac{1}{q_L}\{\gamma Q + 2(q_{L-1} - r_L)\} + a_L, \tag{2.16}$$

$$q_L s_0 = r_L C - (-1)^L r_{L-1} + (\delta q_L + u_2 - u_1 - z). \tag{2.17}$$

Proof. For (2.10), (2.11) and (2.12), see [12, Lemma 2.2 (i)]; For (2.13), see [12, (2.12)]; For (2.14), see [12, (3.5)]; For (2.15), see [10, (2.14)]; For (2.16), see [10, (2.6)]; For (2.17), see [10, (2.19)]. ☐

3. Proof of Theorem 2

In this section, we will prove Theorem 2 which gives positive integers d of minimal type for $\sqrt{d}$ such that the primary symmetric parts of the simple continued fraction expansions of $\sqrt{d}$ are of ELE type. For this, we first analyze the value of μ defined by (1.7):

$$\mu := \frac{1}{q_L}\{\gamma(q_{L+1}+q_{L-1})+2(q_{L-1}-r_L)\}\,,$$

where γ is as in (1.6).

Proposition 3.1. *Let $L \geq 2$. For a sequence $a_1, a_2, \dots, a_L$, the following hold.*

(1) *Assume $u_1 = u_2$ and $w = 1$. Then we have*

$$\mu = a_L \iff r_L = 2q_{L-1}\,,$$
$$\mu = a_L + 2 \iff r_L = 2q_{L-1} - q_L\,.$$

(2) *If $q_L > 1$, $a_L \geq 2$ and $\mu = a_L$, then $r_L = 2q_{L-1}$, $u_1 = u_2$, $w = 1$, $2 \nmid q_L$, $q_L \mid r_{L+1}$ and $z = (-1)^L r_{L-1}$.*

(3) *If $q_L > 1$, $a_L \geq 4$ and $\mu = a_L + 2$, then $r_L = 2q_{L-1} - q_L$, $u_1 = u_2$, $w = 1$, $2 \nmid q_L$, $q_L \mid (2r_{L+1} + r_L)$ and $z = (-1)^L(r_{L-1} - r_L)$.*

Before proving this, we will show the following lemma.

Lemma 3.1. (1) *If $r_L \equiv 2q_{L-1} \pmod{q_L}$, then $u_1 \equiv (-1)^L(r_{L+1}+r_{L-1}) \pmod{q_L}$.*

(2) *If $r_L = 2q_{L-1}$, then $w = 1$ and $z = (-1)^L r_{L-1}$. If $r_L = 2q_{L-1} - q_L$, then $w = 1$ and $z = (-1)^L(r_{L-1} - r_L)$.*

(3) *We have $u_2 \equiv (-1)^{L-1} r_{L+1} + z \pmod{q_L}$.*

Proof. First we remark that the relation

$$q_L r_{L-1} - q_{L-1} r_L = (-1)^{L-1} \tag{3.1}$$

holds by (2.2), which yields the congruence

$$q_{L-1} r_L \equiv (-1)^L \pmod{q_L}\,. \tag{3.2}$$

(1) We assume $r_L \equiv 2q_{L-1} \pmod{q_L}$. Then by (3.2), we have

$$q_{L-1} r_L^2 \equiv (-1)^L 2q_{L-1} \pmod{q_L}\,.$$

Since $\gcd(q_L, q_{L-1}) = 1$, we get $r_L^2 \equiv (-1)^L 2 \pmod{q_L}$. From this together with (1.3), we have

$$u_1 \equiv (r_L^2 - (-1)^L)(r_{L+1} + r_{L-1}) \equiv (-1)^L(r_{L+1} + r_{L-1}) \pmod{q_L}\,.$$

(2) If $r_L = 2q_{L-1}$, then by (3.1) we have

$$(-1)^L(r_L - q_{L-1})r_L = (-1)^L q_{L-1} r_L = (-1)^L(q_L r_{L-1} - (-1)^{L-1}) = (-1)^L r_{L-1} \cdot q_L + 1\,.$$

Hence we get $w = 1$ and $z = (-1)^L r_{L-1}$ by (1.4).

If $r_L = 2q_{L-1} - q_L$, then by (3.1) we have

$$(-1)^L(r_L - q_{L-1})r_L = (-1)^L(q_{L-1} - q_L)r_L = (-1)^L(r_{L-1} - r_L) \cdot q_L + 1\,.$$

Hence we get $w = 1$ and $z = (-1)^L(r_{L-1} - r_L)$ by (1.4).

(3) This congruence is given by (1.5) immediately. □

Proof of Proposition 3.1. Since $a_L q_L = q_{L+1} - q_{L-1}$, it follows from (1.7) that we have

$$\mu - a_L = \frac{1}{q_L}\{(q_{L+1} + q_{L-1})(\gamma - 1) + 2(2q_{L-1} - r_L)\} \tag{3.3}$$

and

$$\mu - a_L - 2 = \frac{1}{q_L}\{(q_{L+1} + q_{L-1})(\gamma - 1) + 2(2q_{L-1} - q_L - r_L)\}\,. \tag{3.4}$$

Here we recall (1.6):

$$\gamma = \begin{cases} q_L(u_2 - u_1) + w & \text{if } u_1 \le u_2\,, \\ q_L(q_L + u_2 - u_1) + w & \text{if } u_1 > u_2\,. \end{cases}$$

In the case $u_1 = u_2$, we easily see $\gamma = w$. In the case $u_1 \neq u_2$ and $q_L > 1$, we have

$$\gamma \ge q_L + w > 1,$$

because of $-q_L < u_2 - u_1$. Thus we have

$$\gamma = 1 \iff u_1 = u_2,\ w = 1$$

under the condition $q_L > 1$.

(1) Assume $u_1 = u_2$ and $w = 1$. Then we have $\gamma = 1$. Hence by (3.3) and (3.4), we have

$$\mu - a_L = \frac{2}{q_L}(2q_{L-1} - r_L)$$

and

$$\mu - a_L - 2 = \frac{2}{q_L}(2q_{L-1} - q_L - r_L)\,,$$

respectively. Thus we obtain

$$\mu = a_L \iff r_L = 2q_{L-1}\,,$$
$$\mu = a_L + 2 \iff r_L = 2q_{L-1} - q_L\,.$$

(2) Assume $q_L > 1$, $a_L \ge 2$ and $\mu = a_L$. Since $a_L \ge 2$ and $L \ge 2$, we have

$$q_{L+1} + q_{L-1} - 2(2q_{L-1} - r_L) = a_L q_L - 2q_{L-1} + 2r_L \ge 2(q_L - q_{L-1} + r_L) > 0\,, \tag{3.5}$$

$$q_{L+1} + q_{L-1} + 2(2q_{L-1} - r_L) = a_L q_L - 2r_L + 6q_{L-1} \ge 2(q_L - r_L + 3q_{L-1}) > 0\,. \tag{3.6}$$

Suppose that $u_1 \neq u_2$. Since $q_L > 1$, we have $\gamma > 1$. Then by (3.3) and (3.6) we get $\mu > a_L$, which contradicts the assumption $\mu = a_L$. Hence we have $u_1 = u_2$. Then we have $\gamma = w$. If $w \ge 2$, then we also have $\gamma > 1$ and hence $\mu > a_L$. If $w = 0$, then by (3.3) and (3.5) we have $\mu < a_L$. Therefore, it must hold that $w = 1$. Then by (1) of this proposition, we have $r_L = 2q_{L-1}$. Hence by Lemma 3.1 (2), we have $z = (-1)^L r_{L-1}$. From this together with Lemma 3.1 (3), we have

$$u_2 \equiv (-1)^{L-1} r_{L+1} + z = (-1)^L(-r_{L+1} + r_{L-1}) \pmod{q_L}\,.$$

On the other hand, by Lemma 3.1 (1), we have

$$u_1 \equiv (-1)^L (r_{L+1} + r_{L-1}) \pmod{q_L}\,.$$

Then by $u_1 = u_2$, we obtain $2r_{L+1} \equiv 0 \pmod{q_L}$. Since r_L is even and $q_L r_{L-1} - q_{L-1} r_L = (-1)^{L-1}$, q_L is odd. This implies to $q_L \mid r_{L+1}$.

(3) Assume $q_L > 1$, $a_L \geq 4$ and $\mu = a_L + 2$. Since $a_L \geq 4$ and $L \geq 2$, we have

$$q_{L+1} + q_{L-1} - 2(2q_{L-1} - q_L - r_L) = (a_L + 2)q_L - 2q_{L-1} + 2r_L \geq 2(q_L - q_{L-1} + r_L) > 0\,, \tag{3.7}$$

$$q_{L+1} + q_{L-1} + 2(2q_{L-1} - q_L - r_L) = (a_L - 2)q_L - 2r_L + 6q_{L-1} \geq 2(q_L - r_L + 3q_{L-1}) > 0\,. \tag{3.8}$$

Suppose that $u_1 \neq u_2$. Since $q_L > 1$, we have $\gamma > 1$. Then by (3.4) and (3.8) we get $\mu > a_L + 2$, which contradicts the assumption $\mu = a_L + 2$. Hence we have $u_1 = u_2$. Then we have $\gamma = w$. If $w \geq 2$, then we also have $\gamma > 1$ and hence $\mu > a_L + 2$. If $w = 0$, then by (3.4) and (3.7) we have $\mu < a_L + 2$. Therefore, it must hold that $w = 1$. Then by (1) of this proposition, we have $r_L = 2q_{L-1} - q_L$. Hence by Lemma 3.1 (2), we have $z = (-1)^L (r_{L-1} - r_L)$. From this together with Lemma 3.1 (3), we have

$$u_2 \equiv (-1)^{L-1} r_{L+1} + z = (-1)^L (-r_{L+1} + r_{L-1} - r_L) \pmod{q_L}\,.$$

On the other hand, by Lemma 3.1 (1), we have

$$u_1 \equiv (-1)^L (r_{L+1} + r_{L-1}) \pmod{q_L}\,.$$

Then by $u_1 = u_2$, we obtain $2r_{L+1} + r_L \equiv 0 \pmod{q_L}$. Finally, since

$$(-1)^{L-1} = q_L r_{L-1} - q_{L-1} r_L = q_L r_{L-1} - q_{L-1}(2q_{L-1} - q_L) = q_L(r_{L-1} + q_{L-1}) - 2q_{L-1}^2,$$

we see that q_L is odd. The proof is completed. □

From now on, we assume $L \geq 3$, because there is no sequence of ELE type with length 2, as we have seen in Remark 1.3.

PROPOSITION 3.2. *Under the above setting, we assume that* $u_1 = u_2$. *Then the following hold*:

$$s_0 = \frac{1}{q_L^2}\{q_L r_L^2 (r_{L+1} + r_{L-1}) - (-1)^L r_L^2 + w + 1\}\,, \tag{3.9}$$

$$f(s_0) = \frac{w+1}{q_L^2}\{(w+1)(q_{L+1} + q_{L-1})^2 - (-1)^L 4\}\,. \tag{3.10}$$

Proof. We recall $Q = q_{L+1} + q_{L-1}$, $R = r_{L+1} + r_{L-1}$. By the assumption $u_1 = u_2$, we have $\delta = 0$. Then by (2.17), (2.12), (1.4) and (3.1), we have

$$\begin{aligned}
q_L^2 s_0 &= q_L(r_L C - (-1)^L r_{L-1} - z) \\
&= q_L r_L^2 R - (-1)^L q_L r_{L-1} - \{(-1)^L (r_L - q_{L-1}) r_L - w\} \\
&= q_L r_L^2 R - (-1)^L (q_L r_{L-1} + r_L^2 - q_{L-1} r_L) + w \\
&= q_L r_L^2 R - (-1)^L (r_L^2 + (-1)^{L-1}) + w \\
&= q_L r_L^2 R - (-1)^L r_L^2 + 1 + w\,.
\end{aligned}$$

This gives (3.9).

By [10, Proposition], we have $f(x) = f_1(x)f_2(x)$, where

$$f_1(x) := q_L^2 x - r_L^2(q_L R - (-1)^L)\,,$$
$$f_2(x) := Q^2 x - R^2(Q r_L + (-1)^L)\,.$$

It follows from (3.9) and (2.15) that

$$\begin{aligned}
f_1(s_0) &= q_L r_L^2 R - (-1)^L r_L^2 + w + 1 - r_L^2(q_L R - (-1)^L)\\
&= w + 1\,,\\
f_2(s_0) &= Q^2 \cdot \frac{1}{q_L^2}(q_L r_L^2 R - (-1)^L r_L^2 + w + 1) - R^2(Q r_L + (-1)^L)\\
&= \frac{1}{q_L^2}\{q_L Q^2 r_L^2 R - (-1)^L Q^2 r_L^2 + (w+1)Q^2 - q_L^2 Q r_L R^2 - (-1)^L q_L^2 R^2\}\\
&= \frac{1}{q_L^2}\{(w+1)Q^2 + q_L Q r_L R(Q r_L - q_L R) - (-1)^L(Q^2 r_L^2 + q_L^2 R^2)\}\\
&= \frac{1}{q_L^2}\{(w+1)Q^2 + q_L Q r_L R \cdot 2(-1)^L - (-1)^L(Q^2 r_L^2 + q_L^2 R^2)\}\\
&= \frac{1}{q_L^2}\{(w+1)Q^2 - (-1)^L(Q r_L - q_L R)^2\}\\
&= \frac{1}{q_L^2}\{(w+1)Q^2 - (-1)^L 4\}\,.
\end{aligned}$$

Therefore we obtain (3.10). □

Proof of Theorem 2. It follows from $L \geq 3$ that $q_L > 1$. Moreover we have $a_L \geq 2$ (resp. $a_L \geq 4$) by the definition of ELE type if $a_1, a_2, \ldots, a_L$ is of ELE_1 type (resp. ELE_2 type). Then by Proposition 3.1 (2), (3), we have

$$r_L = 2q_{L-1} - \varepsilon q_L,\ \ u_1 = u_2,\ \ w = 1\ \ 2 \nmid q_L,\ \ q_L \mid (2r_{L+1} + \varepsilon r_L)\,.$$

(1) When a_L is even, we see from [12, Lemma 2.2 (ii)] that Case (II) occurs for $a_1, \ldots, a_L$. When a_L is odd, since q_L is also odd, we see from [12, Lemma 2.2 (iii)] that Case (I) occurs for $a_1, \ldots, a_L$. Furthermore, since $u_1 = u_2$, $w = 1$, it follows from (3.9) that

$$s_0 = \frac{1}{q_L^2}\{q_L r_L^2(r_{L+1} + r_{L-1}) - (-1)^L r_L^2 + 2\}\,.$$

Since q_L is odd, we have

$$s_0 \equiv r_L(r_{L+1} + r_{L-1}) + r_L = r_L(a_L r_L + 2r_{L-1}) + r_L \equiv a_L r_L + r_L \pmod 2\,.$$

Thus s_0 is even if a_L is odd. Therefore only "Case (I) and s_0 even" or Case (II) occurs for $a_1, \ldots, a_L$ with our assumptions. By Theorem 0.1, therefore, there is no positive integer d, $d \equiv 1 \pmod 4$, with period $2L$ of minimal type for $(1+\sqrt{d})/2$ so that the primary symmetric part of the simple continued fraction expansion of $(1+\sqrt{d})/2$ is such $a_1, a_2, \ldots, a_L$.

(2) We recall

$$\mu = a_L + 2\varepsilon$$

by the definition of ELE type. By (2.16), it holds that $g(s_0) = \mu + a_L$. Then we get

$$g(s_0) = 2a_L + 2\varepsilon\,, \tag{3.11}$$

and hence, we have

$$a_0 = \frac{g(s_0)}{2} = a_L + \varepsilon \in \mathbb{Z}\,. \tag{3.12}$$

It follows from $w = 1, 2 \nmid q_L$, $f(s_0) \in \mathbb{Z}$ and (3.10) that $f(s_0)$ is divisible by 4, that is,

$$d = \frac{f(s_0)}{4} \in \mathbb{Z}\,.$$

(This also follows from the assertion (1) and Theorem 0.1.) By $w = 1$ and (3.10), we have

$$d = \frac{f(s_0)}{4} = \frac{1}{q_L^2}\{(q_{L+1} + q_{L-1})^2 - (-1)^L 2\}\,. \tag{3.13}$$

Since q_L is odd, we have

$$\begin{aligned} d &\equiv (q_{L+1} + q_{L-1})^2 - (-1)^L 2 = (a_L q_L + 2q_{L-1})^2 - (-1)^L 2 \\ &\equiv a_L^2 + 2 \equiv \begin{cases} 2 \pmod 4 & \text{if } a_L \text{ is even,} \\ 3 \pmod 4 & \text{if } a_L \text{ is odd.} \end{cases} \end{aligned}$$

Now by recalling $Q = q_{L+1} + q_{L-1}$, we have

$$a_L q_L + \varepsilon q_L = (q_{L+1} - q_{L-1}) + (2q_{L-1} - r_L) = Q - r_L\,, \tag{3.14}$$

and hence

$$\begin{aligned} q_L^2(a_L + \varepsilon)^2 + q_L(2r_{L+1} + \varepsilon r_L) &= (a_L q_L + \varepsilon q_L)^2 + 2q_L r_{L+1} + \varepsilon q_L r_L \\ &= (Q - r_L)^2 + 2q_L r_{L+1} + (2q_{L-1} - r_L) r_L \\ &= Q^2 - 2Q r_L + 2q_L r_{L+1} + 2q_{L-1} r_L \\ &= Q^2 - 2q_{L+1} r_L + 2q_L r_{L+1} \\ &= Q^2 - 2(q_{L+1} r_L - q_L r_{L+1}) \\ &= Q^2 - (-1)^L 2\,. \end{aligned}$$

From this together with (3.13), we have

$$d = (a_L + \varepsilon)^2 + \frac{2r_{L+1} + \varepsilon r_L}{q_L}\,.$$

Now we see from (3.11) that the assumption (0.2) of Theorem 0.1 holds:

$$g(s_0) = 2a_L + 2\varepsilon > a_1, \dots, a_{L-1}, a_L\,.$$

By Theorem 0.1, therefore, we get the desired simple continued fraction expansion of $\sqrt{d}$.

(3) For brevity, we put $\ell := 2L$. From the above integer $a_0 = a_L + \varepsilon$ and the symmetric sequence of $\ell - 1$ positive integers $a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1$, we define nonnegative integers q_n, r_n, p_n $(0 \le n \le \ell)$ by using the recurrence equations (0.1) and (2.1).

Let $1 \le n \le 2L - 1$. By (2.9) and (2.3), we have

$$(-1)^n Q_n = p_n^2 - dq_n^2 = (a_0^2 - d)q_n^2 + 2a_0 q_n r_n + r_n^2$$

and hence by (2) of this theorem, we obtain (1.10).

Substituting $n = L$ into (1.10) and using $\varepsilon q_L = 2q_{L-1} - r_L$ and (3.1), we have

$$\begin{aligned}(-1)^L Q_L &= -2q_L r_{L+1} - \varepsilon q_L r_L + 2(a_L r_L)q_L + 2\varepsilon q_L r_L + r_L^2\\ &= -2q_L r_{L+1} + \varepsilon q_L r_L + 2(r_{L+1} - r_{L-1})q_L + r_L^2\\ &= (\varepsilon q_L)r_L - 2q_L r_{L-1} + r_L^2\\ &= (2q_{L-1} - r_L)r_L - 2q_L r_{L-1} + r_L^2\\ &= 2(q_{L-1}r_L - q_L r_{L-1})\\ &= -(-1)^{L-1}2\,.\end{aligned}$$

Therefore, we get $Q_L = 2$. Similarly, Q_{L-1} and Q_1 can be calculated.

Next we consider the Yokoi invariant. Since $d \equiv 2,\ 3 \pmod 4$, it follows from [13, Proposition 3.3] that the Yokoi invariant m_d of d is

$$m_d = \left[\frac{2{q_\ell}^2}{p_\ell}\right].$$

Now by using (2.3), (3.12) and $r_L = 2q_{L-1} - \varepsilon q_L$, we have

$$p_L = a_0 q_L + r_L = (a_L + \varepsilon)q_L + 2q_{L-1} - \varepsilon q_L = a_L q_L + 2q_{L-1} = q_{L+1} + q_{L-1} = Q\,,$$

$$p_{L-1} = a_0 q_{L-1} + r_{L-1} = (a_L + \varepsilon)q_{L-1} + r_{L-1}\,.$$

By substituting them into (2.13) and by using (3.14) and (3.1), we have

$$\begin{aligned}p_\ell &= Qq_{L+1} + \{(a_L + \varepsilon)q_{L-1} + r_{L-1}\}q_L\\ &= Qq_{L+1} + (Q - r_L)q_{L-1} + q_L r_{L-1}\\ &= Q(q_{L+1} + q_{L-1}) + q_L r_{L-1} - q_{L-1}r_L\\ &= Q^2 - (-1)^L\,.\end{aligned}$$

From this together with (2.10), we have

$$\frac{2{q_\ell}^2}{p_\ell} = \frac{2Q^2q_L^2}{p_\ell} = \frac{2(p_\ell + (-1)^L)q_L^2}{p_\ell} = 2q_L^2 + \frac{(-1)^L 2q_L^2}{p_\ell}\,.$$

Here, we note that $a_L \geq 2$. Then we have $q_{L+1} = a_L q_L + q_{L-1} > 2q_L$, and hence $p_\ell > 2q_L^2$. Then we get inequalities

$$0 < \frac{2q_L^2}{p_\ell} < 1 \ \text{ and } \ 0 < 1 - \frac{2q_L^2}{p_\ell} < 1\,.$$

Therefore, we obtain

$$m_d = \left[\frac{2{q_\ell}^2}{p_\ell}\right] = \begin{cases}\left[2q_L^2 + \dfrac{2q_L^2}{p_\ell}\right] = 2q_L^2 & \text{if } L \text{ is even,}\\[2ex] \left[2q_L^2 - 1 + \left(1 - \dfrac{2q_L^2}{p_\ell}\right)\right] = 2q_L^2 - 1 & \text{if } L \text{ is odd.}\end{cases}$$

Theorem 2 is now proved. □

4. Proof of Theorem 1

As we have stated in Remark 1.2, the implication (iii) $\Rightarrow$ (iv) was shown. In this section, we will prove (iv) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv), (i) $\Leftrightarrow$ (ii), and (iii) $\Rightarrow$ (ii).

4.1. Proof of the implication (iv) ⇒ (iii)

Proof of the implication (iv) $\Rightarrow$ (iii). We easily see that if the simple continued fraction expansions of $\sqrt{d}$ with even period $2L$ satisfies $a_0 \leq 3$, that is, $d \leq 15$, then $L \leq 2$. Hence, when $L \geq 3$, we have $a_0 \geq 4$. Assume that $a_L = a_0$, or $a_L = a_0 - 1$. Then, $a_L \geq a_0 - 1$. Now it follows from (2.4) that $P_{L+1} = P_L$. Then by (2.5) we have

$$2P_L = a_L Q_L . \tag{4.1}$$

From this together with (2.7) and $a_0 \geq 4$, we have

$$Q_L = \frac{2P_L}{a_L} \leq \frac{2a_0}{a_L} \leq \frac{2a_0}{a_0 - 1} = 2 + \frac{2}{a_0 - 1} < 3 .$$

On the other hand, we have $Q_L > 1$ from (2.8). Therefore, $Q_L = 2$. The proof is completed. ☐

4.2. Proof of the implication (i) ⇒ (iv)

Proof of the implication (i) $\Rightarrow$ (iv). Let d be a non-square positive integer such that the simple continued fraction expansion of $\sqrt{d}$ is $\sqrt{d} = [a_0, \overline{a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1, 2a_0}]$ with even period $2L$ (≥ 6). Assume that d is of minimal type for $\sqrt{d}$ and the primary symmetric part $a_1, \dots, a_L$ is of ELE type. Then, since the inequality (0.2) holds by Theorem 0.1, the inequality (1.8) or (1.9) of Theorem 2 holds, because these conditions are equivalent to each other as we have seen in the proof of Theorem 2. Therefore we see that d is obtained as in Theorem 2 (2). Hence the assertion (iv) follows. (When (i) holds, the assertion (iii) also follows from Theorem 2 (3).) ☐

4.3. Proof of the equivalence (i) ⇔ (ii)

The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 4.1.

PROPOSITION 4.1. *Let $L \geq 3$. Then it is a sufficient and necessary condition for a sequence $a_1, a_2, \dots, a_L$ to be of ELE$_1$ type (resp. ELE$_2$ type) that three conditions*

$$r_L = 2q_{L-1},\ a_L \equiv (-1)^{L-1} q_{L-1} r_{L-1} \pmod{q_L} \text{ and } a_L \geq 2 \tag{4.2}$$

(4.3)

$$(\textit{resp.}\ r_L = 2q_{L-1} - q_L,\ a_L \equiv (-1)^{L-1} q_{L-1}(q_{L-1} + r_{L-1}) \pmod{q_L} \text{ and } a_L \geq 4)$$

hold.

Proof. It follows from $L \geq 3$ that $q_L > 1$. Suppose that (4.2) (resp. (4.3)) holds. Then by Lemma 3.1 (2), we have $w = 1$ and $z = (-1)^L r_{L-1}$ (resp. $z = (-1)^L (r_{L-1} - r_L) \equiv (-1)^L (r_{L-1} - 2q_{L-1}) \pmod{q_L}$), and by (3.2), we have

$$a_L r_L \equiv (-1)^{L-1} q_{L-1} r_L r_{L-1} \equiv -r_{L-1} \pmod{q_L}$$

$$(\text{resp. } a_L r_L \equiv (-1)^{L-1} q_{L-1} r_L (q_{L-1} + r_{L-1}) \equiv -q_{L-1} - r_{L-1} \pmod{q_L})\,.$$

Then by $r_{L+1} = a_L r_L + r_{L-1}$ and Lemma 3.1 (1), (3), we have

$$u_1 \equiv (-1)^L (a_L r_L + 2r_{L-1}) \equiv (-1)^L r_{L-1} \pmod{q_L},$$

$$u_2 \equiv (-1)^{L-1} (a_L r_L + r_{L-1}) + z \equiv z = (-1)^L r_{L-1} \pmod{q_L}$$

$$(\text{resp. } u_1 \equiv (-1)^L (a_L r_L + 2r_{L-1}) \equiv (-1)^L (-q_{L-1} + r_{L-1}) \pmod{q_L},$$

$$u_2 \equiv (-1)^{L-1} (a_L r_L + r_{L-1}) + z \equiv (-1)^L q_{L-1} + z \equiv (-1)^L (-q_{L-1} + r_{L-1}) \pmod{q_L}).$$

Then we have $u_1 \equiv u_2 \pmod{q_L}$ and so $u_1 = u_2$. Hence by $u_1 = u_2$, $w = 1$, $r_L = 2q_{L-1}$ (resp. $r_L = 2q_{L-1} - q_L$) and Proposition 3.1 (1), we have $\mu = a_L$ (resp. $\mu = a_L + 2$), that is, $a_1, a_2, \ldots, a_L$ is of ELE$_1$ type (resp. ELE$_2$ type).

Conversely, we assume that $a_1, a_2, \ldots, a_L$ is of ELE$_1$ type (resp. ELE$_2$ type) and put $\varepsilon := 0$ (resp. $\varepsilon := 1$). Then by Proposition 3.1 (2), (3) , we have $r_L = 2q_{L-1} - \varepsilon q_L$. Moreover, $u_1 = u_2$, $2 \nmid q_L$ and $z = (-1)^L (r_{L-1} - \varepsilon r_L)$ hold. It follows from $z = (-1)^L (r_{L-1} - \varepsilon r_L)$ and Lemma 3.1 (3) that

$$u_2 \equiv (-1)^{L-1} r_{L+1} + (-1)^L (r_{L-1} - \varepsilon r_L) = (-1)^L (-r_{L+1} + r_{L-1} - \varepsilon r_L) \pmod{q_L}.$$

Then by Lemma 3.1 (1) and $u_1 = u_2$, we have $2r_{L+1} \equiv -\varepsilon r_L \pmod{q_L}$. Since $r_L = 2q_{L-1} - \varepsilon q_L$ and $2 \nmid q_L$, we have $r_{L+1} \equiv -\varepsilon q_{L-1} \pmod{q_L}$. Then by $r_{L+1} = a_L r_L + r_{L-1}$, we have $a_L r_L \equiv -\varepsilon q_{L-1} - r_{L-1} \pmod{q_L}$. By (3.2), therefore, we obtain

$$a_L \equiv (-1)^L a_L q_{L-1} r_L \equiv (-1)^{L-1} q_{L-1} (\varepsilon q_{L-1} + r_{L-1}) \pmod{q_L}$$

as desired. The inequality $a_L \geq 2$ (resp. $a_L \geq 4$) follows from the definition of ELE type. □

4.4. Proof of the implication (iii) ⇒ (ii)

The argument of this subsection depends heavily on that of the proof of Golubeva [4, Theorem 1], in which Golubeva [3, Theorem 1] is utilized. Since we can prove that Theorem 0.1 leads to [3, Theorem 1], we use Theorem 0.1 in behalf of [3, Theorem 1].

Let d be a non-square positive integer such that the simple continued fraction expansion of $\sqrt{d}$ is $\sqrt{d} = [a_0, \overline{a_1, \ldots, a_{L-1}, a_L, a_{L-1}, \ldots, a_1, 2a_0}]$ with even period $2L$ (≥ 4). Then it follows from Theorem 0.1 that Case (I) or Case (II) occurs for this symmetric part and d can be written uniquely as $d = f(s)/4$ and $a_0 = g(s)/2$ with some integer $s \geq s_0$. Furthermore, when Case (I) occurs, s must be even.

LEMMA 4.1 (cf. [3, Theorem 1, Lemma 4]). *Under the above setting, we have the following relations*:

(1) $Q_L = q_L^2 s - q_L R r_L^2 + (-1)^L r_L^2$.

(2) $Q_L q_{L+1} q_{L-1} - Q_{L-1} q_L^2 = (-1)^L$.

Proof. By (2.9), we have

$$p_L^2 - d q_L^2 = (-1)^L Q_L. \tag{4.4}$$

(1) It follows from (2.3) that

$$p_L = a_0 q_L + r_L = \frac{g(s)}{2} \cdot q_L + r_L\,. \tag{4.5}$$

Moreover we see from the definition of $f(x)$ that

$$d = \frac{f(s)}{4} = \frac{g(s)^2}{4} + h(s)\,. \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we get

$$(g(s)r_L - h(s)q_L)q_L + r_L^2 = (-1)^L Q_L\,.$$

By using (2.10), (2.11), (2.12) and (2.15), it follows from the definition of $g(x)$, $h(x)$ that

$$\begin{aligned} g(s)r_L - h(s)q_L &= (As - BC)r_L - (Bs - C^2)q_L \\ &= (Ar_L - Bq_L)s - (Br_L - Cq_L)C \\ &= (Qq_Lr_L - Qq_Lr_L + (-1)^L q_L)s - (Qr_L^2 - (-1)^L r_L - Rq_Lr_L)Rr_L \\ &= (-1)^L q_L s - ((Qr_L - q_LR)r_L - (-1)^L r_L)Rr_L \\ &= (-1)^L q_L s - ((-1)^L \cdot 2r_L - (-1)^L r_L)Rr_L \\ &= (-1)^L q_L s - (-1)^L Rr_L^2\,. \end{aligned}$$

Hence we obtain

$$(-1)^L q_L^2 s - (-1)^L q_L R r_L^2 + r_L^2 = (-1)^L Q_L\,,$$

which gives the desired equation.

(2) By (2.6) and (4.1), we have

$$d = P_L^2 + Q_{L-1}Q_L = \left(\frac{a_LQ_L}{2}\right)^2 + Q_{L-1}Q_L\,. \tag{4.7}$$

Substituting (2.14) and (4.7) into (4.4), we get

$$Q_L(Q^2 - a_L^2 q_L^2) - 4Q_{L-1}q_L^2 = (-1)^L 4\,.$$

Since

$$Q^2 - a_L^2q_L^2 = (a_Lq_L + 2q_{L-1})^2 - a_L^2q_L^2 = 4(a_Lq_L + q_{L-1})q_{L-1} = 4q_{L+1}q_{L-1},$$

we obtain $Q_Lq_{L+1}q_{L-1} - Q_{L-1}q_L^2 = (-1)^L$. The lemma is proved. □

PROPOSITION 4.2 ([4, pp.1279–1280]). *Under the above setting, assume that* $Q_L = 2$. *Then* $2 \nmid q_L$ *and either* $r_L = 2q_{L-1}$ *or* $r_L = 2q_{L-1} - q_L$ *hold. Furthermore we have the following.*

(1) *If* $r_L = 2q_{L-1}$, *then* $a_L \equiv (-1)^{L-1}q_{L-1}r_{L-1} \pmod{q_L}$.

(2) *If* $r_L = 2q_{L-1} - q_L$, *then* $a_L \equiv (-1)^{L-1}q_{L-1}(q_{L-1} + r_{L-1}) \pmod{q_L}$.

Proof. Put $\alpha := (-1)^{L-1}(q_Ls - Rr_L^2)$. Then by the assumption and Lemma 4.1 (1), we have

$$2 = Q_L = q_L^2 s - q_LRr_L^2 + (-1)^L r_L^2 = (-1)^{L-1}(q_L\alpha - r_L^2)\,,$$

and hence

$$q_L\alpha - r_L^2 = (-1)^{L-1}2\,. \tag{4.8}$$

Here we assume $2 \mid q_L$. Then by (4.8), r_L is even. This is a contradiction to $\gcd(q_L, r_L) = 1$. Hence we have $2 \nmid q_L$.

Now it follows from (3.1) that

$$q_L \cdot 2r_{L-1} - r_L \cdot 2q_{L-1} = (-1)^{L-1}2\,. \tag{4.9}$$

Two equations (4.8) and (4.9) yield that

$$q_L(\alpha - 2r_{L-1}) = r_L(r_L - 2q_{L-1})\,. \tag{4.10}$$

Since $\gcd(q_L, r_L) = 1$, there is some integer ε such that

$$r_L - 2q_{L-1} = -\varepsilon q_L\,. \tag{4.11}$$

First we assume $\varepsilon \le -1$. Then (4.11) implies that

$$r_L = 2q_{L-1} - \varepsilon q_L \ge 2q_{L-1} + q_L > q_L$$

because of $q_{L-1} > 0$ when $L \ge 2$. This contradicts that $r_L \le q_L$. Next we assume $\varepsilon \ge 2$. Then (4.11) implies that

$$r_L = 2q_{L-1} - \varepsilon q_L \le 2q_{L-1} - 2q_L\,.$$

Since it follows from $L \ge 2$ that $0 < r_L$, we have $q_L < q_{L-1}$ and this is a contradiction. Thus we have $\varepsilon = 0$ or 1, so, $r_L = 2q_{L-1}$ or $r_L = 2q_{L-1} - q_L$ holds.

(1) Assume that $r_L = 2q_{L-1}$. Then by the assumption and (3.1), we have

$$\begin{aligned} Q_L q_{L+1} q_{L-1} &= 2q_{L+1}q_{L-1} = q_{L+1}r_L \\ &= a_L q_L r_L + q_{L-1}r_L = a_L q_L r_L + q_L r_{L-1} + (-1)^L\,. \end{aligned}$$

Substituting this into the equation in Lemma 4.1 (2), we get

$$a_L r_L - Q_{L-1}q_L = -r_{L-1}$$

and hence

$$r_L((-1)^{L-1}a_L) - q_L((-1)^{L-1}Q_{L-1}) = (-1)^L r_{L-1}\,.$$

On the other hand, by (3.1), we have

$$r_L(q_{L-1}r_{L-1}) - q_L(r_{L-1}^2) = (-1)^L r_{L-1}\,.$$

These two equations yield that

$$r_L((-1)^{L-1}a_L - q_{L-1}r_{L-1}) = q_L((-1)^{L-1}Q_{L-1} - r_{L-1}^2)\,.$$

Since $\gcd(q_L, r_L) = 1$, we obtain $a_L \equiv (-1)^{L-1}q_{L-1}r_{L-1} \pmod{q_L}$.

(2) Assume that $r_L = 2q_{L-1} - q_L$. Then by the assumption and (3.1), we have

$$\begin{aligned} Q_L q_{L+1} q_{L-1} &= 2q_{L+1}q_{L-1} = q_{L+1}(q_L + r_L) \\ &= q_{L+1}q_L + q_{L+1}r_L = q_{L+1}q_L + q_L r_{L+1} + (-1)^L\,. \end{aligned}$$

Substituting this into the equation in Lemma 4.1 (2), we get

$$q_{L+1} + r_{L+1} - Q_{L-1}q_L = 0\,.$$

Note that $q_{L+1} = a_L q_L + q_{L-1}$ and $r_{L+1} = a_L r_L + r_{L-1} = 2a_L q_{L-1} - a_L q_L + r_{L-1}$. Then we have

$$(2a_L + 1)q_{L-1} - Q_{L-1}q_L = -r_{L-1}$$

and hence

$$q_{L-1}\{(-1)^{L-1}(2a_L + 1)\} - q_L((-1)^{L-1}Q_{L-1}) = (-1)^L r_{L-1}\,.$$

On the other hand, by (3.1), we have

$$q_{L-1}(r_L r_{L-1}) - q_L(r_{L-1}^2) = (-1)^L r_{L-1}\,.$$

These two equations yield that

$$q_{L-1}\{(-1)^{L-1}(2a_L+1) - r_L r_{L-1}\} = q_L((-1)^{L-1}Q_{L-1} - r_{L-1}^2)\,.$$

Since $\gcd(q_{L-1}, q_L) = 1$, we obtain

$$2a_L + 1 \equiv (-1)^{L-1} r_L r_{L-1} \pmod{q_L}\,.$$

Now, $r_L \equiv 2q_{L-1} \pmod{q_L}$ and $q_{L-1}r_L \equiv (-1)^L \pmod{q_L}$ hold by (3.1). Therefore we have

$$\begin{aligned}(-1)^{L-1}2q_{L-1}(q_{L-1}+r_{L-1}) &\equiv (-1)^{L-1}r_L(q_{L-1}+r_{L-1})\\ &= -1 + (-1)^{L-1}r_L r_{L-1}\\ &\equiv 2a_L \pmod{q_L}\,.\end{aligned}$$

As $2 \nmid q_L$, we obtain $a_L \equiv (-1)^{L-1}q_{L-1}(q_{L-1}+r_{L-1}) \pmod{q_L}$. This completes the proof. □

Proof of the implication (iii) ⇒ (ii). Assume that $L \geq 3$, $Q_L = 2$ and $d \neq 19$.

First we consider the case $d < 25$. In this case, we easily see that $L \geq 3$ and $Q_L = 2$ hold only for $d = 22$. Then, $\sqrt{22} = [4, \overline{1, 2, 4, 2, 1, 8}]$, $L = 3$, $a_L = 4 \geq 2$, $Q_L = 2$ and we have the following table:

n	0	1	2	3	4
q_n	0	1	1	3	13
r_n	1	0	1	2	9

Therefore, $r_L = 2q_{L-1} = 2$ and $a_L \equiv (-1)^{L-1}q_{L-1}r_{L-1} \pmod{q_L}$ hold. Moreover $d = 22$ is of minimal type for $\sqrt{d}$ because of $s = s_0 = 14$. Thus the assertion (ii) holds for $d = 22$.

Next we assume $d \geq 25$. Then we see from the implication (iii) ⇒ (iv) of Theorem 1 that

$$a_L = a_0 \text{ or } a_L = a_0 - 1\,,$$

and hence

$$a_L \geq a_0 - 1 = [\sqrt{d}] - 1 \geq 5 - 1 = 4\,.$$

It follows from the assumption $Q_L = 2$ that either

$$r_L = 2q_{L-1},\ a_L \equiv (-1)^{L-1}q_{L-1}r_{L-1} \pmod{q_L}$$

or

$$r_L = 2q_{L-1} - q_L,\ a_L \equiv (-1)^{L-1}q_{L-1}(q_{L-1}+r_{L-1}) \pmod{q_L}$$

holds by Proposition 4.2. Then by Lemma 3.1 (2), we have $w = 1$. By (3.9) in Proposition 3.2, therefore, we obtain

$$q_L^2 s_0 = q_L R r_L^2 - (-1)^L r_L^2 + 2\,. \tag{4.12}$$

Now we see from (4.10) and (4.11) that

$$\alpha = 2r_{L-1} - \varepsilon r_L\,.$$

Then by the definition of α, we have

$$q_L s = Rr_L^2 + (-1)^{L-1}\alpha = Rr_L^2 + (-1)^{L-1}(2r_{L-1} - \varepsilon r_L)\,. \tag{4.13}$$

By (4.13), (3.1) and (4.12), we have

$$\begin{aligned} q_L^2 s &= q_L Rr_L^2 + (-1)^{L-1}2q_L r_{L-1} - (-1)^{L-1}\varepsilon q_L r_L \\ &= q_L Rr_L^2 + (-1)^{L-1}2(q_{L-1}r_L + (-1)^{L-1}) - (-1)^{L-1}\varepsilon q_L r_L \\ &= q_L Rr_L^2 + (-1)^{L-1}2q_{L-1}r_L + 2 - (-1)^{L-1}\varepsilon q_L r_L \\ &= q_L^2 s_0 + (-1)^L r_L^2 + (-1)^{L-1}2q_{L-1}r_L - (-1)^{L-1}\varepsilon q_L r_L \\ &= q_L^2 s_0 + (-1)^L r_L(r_L - 2q_{L-1} + \varepsilon q_L) \\ &= q_L^2 s_0 \end{aligned}$$

and hence we obtain $s = s_0$. Therefore d is of minimal type for $\sqrt{d}$. Thus the proof is completely proved. □

REMARK 4.1. Let d be a non-square positive integer such that the simple continued fraction expansion of $\sqrt{d}$ is $\sqrt{d} = [a_0, \overline{a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1, 2a_0}]$ with even period $2L$ (≥ 4). If we assume $Q_L = 2$ then, by Proposition 4.2, we have $r_L = 2q_{L-1}$ (resp. $r_L = 2q_{L-1} - q_L$). Hence by putting $\varepsilon := 0$ (resp. $\varepsilon := 1$), $r_L = 2q_{L-1} - \varepsilon q_L$ holds. Then we claim that $a_L = a_0 - \varepsilon$ holds. As $Q_L = 2$, we see by Proposition 4.2 that $2 \nmid q_L$. Since

$$p_L^2 - dq_L^2 = (-1)^L Q_L \equiv 0 \pmod 2$$

from (2.9), we have $p_L \equiv d \pmod 2$. Since $p_L = a_0 q_L + r_L$ from (2.3), we obtain

$$d \equiv a_0 + r_L \pmod 2\,. \tag{4.14}$$

Moreover, since $d = a_L^2 + 2Q_{L-1}$ from (4.7), we have

$$d \equiv a_L \pmod 2\,. \tag{4.15}$$

By (4.11) and $2 \nmid q_L$, we have

$$r_L \equiv -\varepsilon \pmod 2\,.$$

From this, together with (4.14) and (4.15), we obtain $a_L \equiv a_0 - \varepsilon \pmod 2$. On the other hand, it follows from the implication (iii) ⇒ (iv) of Theorem 1 that

$$a_L = a_0 \text{ or } a_L = a_0 - 1\,.$$

Since $\varepsilon = 0$ or 1, the equality $a_L = a_0 - \varepsilon$ must hold. This proves our claim.

5. Proof of Corollary 1

Let p be a prime number with $p \equiv 3 \pmod 4$ and ℓ the minimal period of simple continued fraction expansion of $\sqrt{p}$. Then it is known that ℓ is even, which is shown by using $(2.9)_{n=\ell}$. We write $\ell = 2L$. First, we claim that $Q_L = 2$. Since this is true for $p = 3$, we may assume $p \geq 4$. By (4.7), we have

$$4p = Q_L(a_L^2 Q_L + 4Q_{L-1})\,, \tag{5.1}$$

and hence $Q_L \in \{1, 2, 4, p, 2p, 4p\}$. Since $1 < Q_L < 2\sqrt{p}$ by (2.7), (2.8) and $p \geq 4$, it must hold that $Q_L = 2$ or 4. If $Q_L = 4$, then $p = 4a_L^2 + 4Q_{L-1}$ so that $4 \mid p$. This contradicts that p is a prime number. Hence we obtain $Q_L = 2$ (cf. [3, p.2071]). Thus our claim is true.

Assume that $\ell \geq 6$. Since $Q_L = 2$, the implication (iii) ⇒ (i) of Theorem 1 and Remark 1.1 yield that p is of minimal type for $\sqrt{p}$. Hence, $\mathbb{Q}(\sqrt{p})$ is of minimal type.

Assume that $\ell \leq 4$. In the case $\ell = 2$, $\mathbb{Q}(\sqrt{p})$ is not of minimal type by [11, Example 3.5]. So we consider the case $\ell = 4$ and write the simple continued fraction expansion of $\sqrt{p}$ by

$$\sqrt{p} = [a_0, \overline{a_1, a_2, a_1, 2a_0}]\,.$$

From the symmetric part a_1, a_2, a_1, we calculate linear polynomials $g(x), h(x)$, the quadratic polynomial $f(x)$ and the integer s_0 by using the following table:

n	0	1	2	3	4
q_n	0	1	a_1	$a_1a_2 + 1$	$a_1^2a_2 + 2a_1$
r_n	1	0	1	a_2	$a_1a_2 + 1$

By Theorem 0.1, there exists uniquely an integer s with $s \geq s_0$ such that $p = f(s)/4$ and $a_0 = g(s)/2$. Since $Q_2 = 2$, we see by Proposition 4.2 that either $r_2 = 2q_1$ or $r_2 = 2q_1 - q_2$ holds. Since $r_2 = 1$, the latter equation must hold, and hence $a_1 = 1$. For brevity, we put $t := a_2$. Then we obtain

$$\begin{aligned} g(x) &= (t+2)x - (-1)^4 t(t+1) = (t+2)x - t(t+1)\,, \\ h(x) &= (t+1)x - (-1)^4 t^2 = (t+1)x - t^2 \end{aligned}$$

by the above table. Therefore, on the one hand, we have

$$\begin{aligned} g(t-1) &= (t+2)(t-1) - t(t+1) = -2 < 0\,, \\ g(t) &= (t+2)t - t(t+1) = t > 0\,, \end{aligned}$$

and hence $s_0 = t$. On the other hand, by Lemma 4.1 (1), we have

$$2 = Q_2 = q_2^2 s - q_2(r_3 + r_1)r_2^2 + (-1)^2 r_2^2 = s - t + 1\,,$$

and hence $s = t + 1$. Thus we obtain $s > s_0$, which gives that $\mathbb{Q}(\sqrt{p})$ is not of minimal type. Corollary 1 is now proved.

REMARK 5.1. We give several remarks on interesting properties of a prime number p with $p \equiv 3 \pmod 4$.

(1) Let $\sqrt{p} = [a_0, \overline{a_1, \ldots, a_{L-1}, a_L, a_{L-1}, \ldots, a_1, 2a_0}]$ be the simple continued fraction expansion of $\sqrt{p}$. From the symmetric part, we calculate linear polynomials $g(x)$, $h(x)$, the quadratic polynomial $f(x)$ and the integer s_0 by using (0.1). Then by Theorem 0.1, there exists uniquely an integer s with $s \geq s_0$ such that $p = f(s)/4$ and $a_0 = g(s)/2$. Under the situation of Corollary 1, since $p = a_L^2 + 2Q_{L-1}$ from (5.1), a_L is odd. Therefore it follows from [12, Lemma 2.2] and Theorem 0.1 that Case (I) occurs for this symmetric part and s must be even.

We see by $Q_L = 2$ and Remark 4.1 that $a_L = a_0 - \varepsilon$ holds. Since a_L is odd, hence, according to whether a_0 is even or odd, we have $r_L = 2q_{L-1} - q_L$ or $r_L = 2q_{L-1}$.

(2) In the case $\ell = 4$, as we have seen in the above proof, $s = t + 1$ holds, where $t = a_2$ is an odd integer. Then we have

$$\begin{aligned} g(s) &= (t+2)(t+1) - t(t+1) = 2(t+1)\,, \\ h(s) &= (t+1)^2 - t^2 = 2t + 1\,. \end{aligned}$$

Hence the prime number p such that the minimal period of the simple continued fraction expansion of $\sqrt{p}$ is 4, is of the form

$$p = f(s)/4 = (g(s)/2)^2 + h(s) = (t+1)^2 + 2t + 1 = t^2 + 4t + 2\,,$$

and then

$$\sqrt{p} = [t+1, \overline{1, t, 1, 2t+2}]\,.$$

The form of p is already found in Golubeva [2, Theorem]. (See the set P_4 in that theorem.)

6. Numerical examples

In this section, we explain the source of the notion of "ELE" by using some graphs. Let d be a non-square positive integer with $4 \nmid d$ and put

$$\omega(d) := \begin{cases} (1+\sqrt{d})/2 & \text{if } d \equiv 1 \pmod 4\,, \\ \sqrt{d} & \text{if } d \equiv 2, 3 \pmod 4\,. \end{cases}$$

For a positive integer ℓ, we define

$$\mathrm{CF}_\ell := \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} d \text{ is not a square, } 4 \nmid d\text{, the minimal period of the} \\ \text{simple continued fraction expansion of } \omega(d) \text{ is } \ell \end{array} \right\}.$$

We denote the smallest element of CF_ℓ by d_ℓ and arrange all elements of CF_ℓ in order of size:

$$d_\ell = d_\ell^{(0)} < d_\ell^{(1)} < \cdots < d_\ell^{(i)} < \cdots\,.$$

Moreover, we denote the simple continued fraction expansion of $\omega(d_\ell^{(i)})$ by

$$\omega(d_\ell^{(i)}) = [a_0^{(i)}, \overline{a_1^{(i)}, \ldots, a_\ell^{(i)}}]\,.$$

Here we plot $(x, y, z) = (i, j, a_j^{(i)})$ for $0 \le i \le n-1$, $1 \le j \le [\ell/2]$ in three dimensional space and connect them for each i. The figures (a)-(d) are the cases when $\ell = 100, 101, 102, 103$ and $n = 100$.

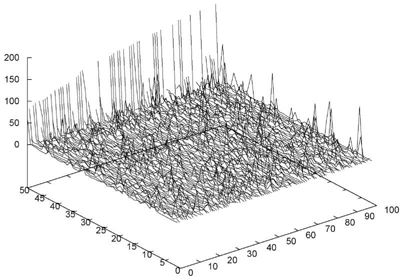
We can observe that the graphs of even cases are characteristic. Our motivation is to investigate why the ends of graphs are extremely large. Dividing the graph in (c) into the case of ELE type and the case of not ELE type (see Figs. (e) and (f)), we expect that "ELE type" has caught the graphs whose ends are extremely large.

Secondly, we have the following numerical results. For $\delta \in \{1, 2, 3\}$, we define

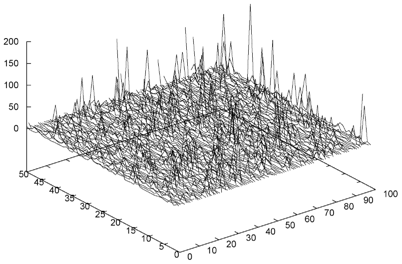
$$\mathrm{CF}_{\ell,\delta} := \{d \in \mathrm{CF}_\ell \mid d \equiv \delta \pmod 4\}\,.$$

Then we have

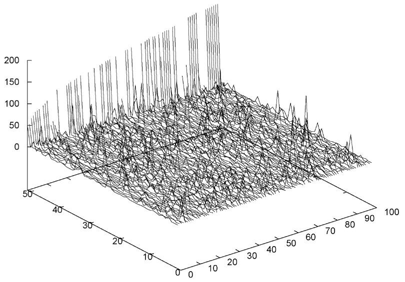
$$\mathrm{CF}_\ell = \mathrm{CF}_{\ell,1} \cup \mathrm{CF}_{\ell,2} \cup \mathrm{CF}_{\ell,3}\,.$$



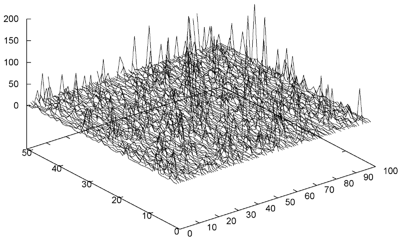
(a) $\ell = 100$, $n = 100$



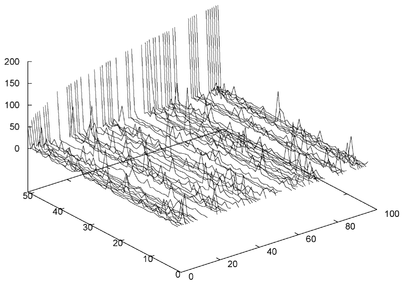
(b) $\ell = 101$, $n = 100$



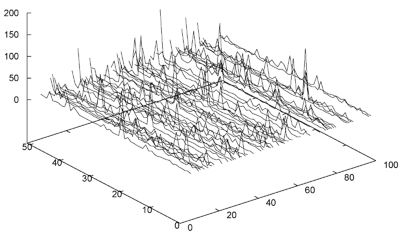
(c) $\ell = 102$, $n = 100$



(d) $\ell = 103$, $n = 100$



(e) $\ell = 102$, $n = 100$, ELE type



(f) $\ell = 102$, $n = 100$, not ELE type

By Theorem 0.1, we can prove $\mathrm{CF}_{\ell,\delta} \neq \emptyset$ for each δ and ℓ (cf. [13, Proposition 4.3]). Here we assume that ℓ is even if $\delta = 3$. Now we consider the smallest element d_ℓ of CF_ℓ for each positive integer ℓ with $1 \leq \ell \leq 69868$. Then the following hold:

(A) d_ℓ is square-free except for $\ell = 1032$. (We have $d_{1032} = 366961 = 7489 \cdot 7^2$.)

(B) The class number of (the maximal order in) $\mathbb{Q}(\sqrt{d_\ell})$ is equal to 1 except for $\ell = 7, 11, 49, 225, 299, 1032$. (For $\ell = 7, 11, 49, 225, 299$, the class number of $\mathbb{Q}(\sqrt{d_\ell})$ is equal to 2. The class number of the order of conductor 7 in $\mathbb{Q}(\sqrt{d_{1032}}) = \mathbb{Q}(\sqrt{7489})$ is equal to 1.)

(C) $\mathbb{Q}(\sqrt{d_\ell})$ is a real quadratic field with period ℓ of minimal type except for $\ell = 1, 2, 3, 4, 7, 1032$.

Thus, as the first step of getting real quadratic fields of class number 1, we will have to know how to get the smallest element d_ℓ, and so we study a real quadratic field of minimal type. Furthermore, we consider the smallest element d'_ℓ of $\mathrm{CF}_{\ell,2} \cup \mathrm{CF}_{\ell,3}$ for each even positive integer ℓ with $8 \leq \ell \leq 73478$, because of Theorem 2 (1), (2). Then the following hold without exception:

(D) d'_ℓ is square-free.

(E) The class number of $\mathbb{Q}(\sqrt{d'_\ell})$ is equal to 1.

(F) $\mathbb{Q}(\sqrt{d'_\ell})$ is a real quadratic field with period ℓ of minimal type.

(G) The primary symmetric part of the simple continued fraction expansion of $\mathbb{Q}(\sqrt{d'_\ell})$ is of ELE type.

(As we have seen in Remark 1.1, the property (G) does not hold for $d_6^{(0)} = 19$, but it holds for $d_6^{(1)} = 22$.) From these, primary symmetric parts of ELE type should be investigated in order to find many real quadratic fields of class number 1.

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