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Strong Koszulness of the Toric Ring Associated to a Cut Ideal

by

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Abstract. A cut ideal of a graph was introduced by Sturmfels and Sullivant. In this paper, we give a necessary and sufficient condition for toric rings associated to the cut ideal to be strongly Koszul.

Introduction

Let G be a finite simple graph on the vertex set $V(G) = [n] = \{1, \dots, n\}$ with the edge set $E(G)$. For two subsets A and B of $[n]$ such that $A \cap B = \emptyset$ and $A \cup B = [n]$, the $(0, 1)$-vector $\delta_{A|B}(G) \in \mathbb{Z}^{|E(G)|}$ is defined as

$$\delta_{A|B}(G)_{ij} = \begin{cases} 1 & \text{if } |A \cap \{i, j\}| = 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where ij is an edge of G. Let

$$X_G = \left\{ \begin{pmatrix} \delta_{A_1|B_1}(G) \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \delta_{A_N|B_N}(G) \\ 1 \end{pmatrix} \right\} \subset \mathbb{Z}^{|E(G)|+1} \quad (N = 2^{n-1})\,.$$

As necessary, we consider X_G as the collection of vectors or as the matrix. Let K be a field and

$$K[q] = K[q_{A_1|B_1}, \dots, q_{A_N|B_N}]\,,$$
$$K[s, T] = K[s, t_{ij} \mid ij \in E(G)]$$

be two polynomial rings over K. Then the ring homomorphism is defined as follows:

$$\pi_G \ :\ K[q] \to K[s, T], \quad q_{A_l|B_l} \mapsto s \cdot \prod_{\substack{|A_l \cap \{i,j\}|=1 \\ ij \in E(G)}} t_{ij}$$

for $1 \le l \le N$. The *cut ideal* I_G of G is the kernel of π_G and the *toric ring* R_G of X_G is the image of π_G. We put $u_{A|B} = \pi_G(q_{A|B})$.

In [9], Sturmfels and Sullivant introduced a cut ideal and posed the problem of relating properties of cut ideals to the class of graphs.

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Let R be a semigroup ring and I be the defining ideal of R. We say that R is *compressed* if the initial ideal of I is squarefree with respect to any reverse lexicographic order. For the toric ring R_G and the cut ideal I_G, the following results are known:

THEOREM 0.1 ([9]). *The toric ring R_G is compressed if and only if G has no K_5-minor and every induced cycle in G has length* 3 *or* 4.

THEOREM 0.2 ([1]). *The cut ideal I_G is generated by quadratic binomials if and only if G has no K_4-minor.*

Nagel and Petrović showed that the cut ideal I_G associated with ring graphs has a quadratic Gröbner basis [4]. However we do not know generally when the cut ideal I_G has a quadratic Gröbner basis and when R_G is Koszul except for trivial cases.

On the other hand, the notion of strongly Koszul algebras was introduced by Herzog, Hibi and Restuccia [2]. A strongly Koszul algebra is a stronger notion of Koszulness. In general, it is known that, for a semigroup ring R,

The defining ideal of R has a quadratic Gröbner basis, or R is strongly Koszul

$$\Downarrow$$

R is Koszul

$$\Downarrow$$

The defining ideal of R is generated by quadratic binomials.

We do not know whether the defining ideal of a strongly Koszul semigroup ring has a quadratic Gröbner basis. In [7], Restuccia and Rinaldo gave a sufficient condition for toric rings to be strongly Koszul. In [3], Matsuda and Ohsugi proved that any squarefree strongly Koszul toric ring is compressed.

In this paper, we give a sufficient condition for cut ideals to have a quadratic Gröbner basis and we characterize the class of graphs such that R_G is strongly Koszul.

The outline of this paper is as follows. In Section 1, we show that the set of graphs such that R_G is strongly Koszul is closed under contracting edges, induced subgraphs and 0-sums. In Section 2, we compute a Gröbner basis for the cut ideal without (K_4, C_5)-minor. In Section 3, by using results of Section 1 and Section 2, we prove that the toric ring R_G is strongly Koszul if and only if G has no (K_4, C_5)-minor.

1. Clique sums and strongly Koszul algebras

In this paper, we introduce the equivalent condition as the definition of the strongly Koszul algebra.

Let R be a semigroup ring generated by $u_1, \dots, u_n$. We say that a semigroup ring R is *strongly Koszul* if the ideals $(u_i) \cap (u_j)$ are generated in degree 2 for all $i \neq j$ [2, Proposition 1.4].

PROPOSITION 1.1 ([2, Proposition 2.3]). *Let R and P be semigroup rings over same field, and Q be the tensor product or the Segre product of R and P. Then Q is strongly Koszul if and only if both R and P are strongly Koszul.*

Recall that a graph H is a *minor* of a graph G if H can be obtained by deleting and contracting edges of G. We say that a subgraph H is an *induced subgraph* of a graph G if H contains all the edges $ij \in E(G)$ with $i, j \in V(H)$.

PROPOSITION 1.2. *Let G be a finite simple connected graph. Assume that R_G is strongly Koszul. Then*

(1) *If H_1 is an induced subgraph of G, then R_{H_1} is strongly Koszul.*
(2) *If H_2 is obtained by contracting an edge of G, then R_{H_2} is strongly Koszul.*

Proof. By [5] and [9], R_{H_1} and R_{H_2} are combinatorial pure subrings of R_G. Therefore, by [6, Corollary 1.6], R_{H_1} and R_{H_2} are strongly Koszul. □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs such that $V_1 \cap V_2$ is a clique of both graphs. The new graph $G = G_1 \# G_2$ with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$ is called the *clique sum* of G_1 and G_2 along $V_1 \cap V_2$. If the cardinality of $V_1 \cap V_2$ is $k+1$, then this operation is called a *k-sum* of the graphs. It is clear that if $R_{G_1 \# G_2}$ is strongly Koszul, then both R_{G_1} and R_{G_2} are strongly Koszul because G_1 and G_2 are induced subgraphs of $G_1 \# G_2$.

PROPOSITION 1.3. *The set of graphs G such that R_G is strongly Koszul is closed under the 0-sum.*

Proof. Let G_1 and G_2 be finite simple connected graphs and assume that R_{G_1} and R_{G_2} are strongly Koszul. Then the toric ring $R_{G_1 \# G_2}$, where $G_1 \# G_2$ is the 0-sum of G_1 and G_2, is the usual Segre product of R_{G_1} and R_{G_2}. Thus it follows by Proposition 1.1. □

However the set of graphs G such that R_G is strongly Kosuzl is not always closed under the 1-sum.

Let K_n denote the complete graph on n vertices, C_n denote the cycle of length n and $K_{l_1,\ldots,l_m}$ denote the complete m-partite graph on the vertex set $V_1 \cup \cdots \cup V_m$, where $|V_i| = l_i$ for $1 \le i \le m$ and $V_i \cap V_j = \emptyset$ for $i \neq j$.

EXAMPLE 1.4. Let $G_1 = C_3 \# C_3 (= K_4 \setminus e)$, $G_2 = C_4 \# C_3$ and $G_3 = (K_4 \setminus e) \# C_3$ be graphs shown in Figures 1-3. All of R_{C_3}, R_{C_4} and R_{G_1} are strongly Koszul because R_{C_3} is isomorphic to the polynomial ring and I_{C_4} and I_{G_1} have quadratic Gröbner bases with respect to any reverse lexicographic order, respectively (see [7, 9]). However neither R_{G_2} nor R_{G_3} is strongly Koszul since $(u_{\emptyset|[5]}) \cap (u_{\{1,3,4\}|\{2,5\}})$ is not generated in degree 2.

2. A Gröbner basis for the cut ideal

In this section, we compute a Gröbner basis of I_G such that G has no (K_4, C_5)-minor.

LEMMA 2.1. *Let G be a simple 2-connected graph on the vertex set $V(G)$. Then G has no (K_4, C_5)-minor if and only if G is K_3, $K_{2,n-2}$ or $K_{1,1,n-2}$ for $n \ge 4$.*

Proof. Since G is 2-connected, G contains a cycle. Let C be the longest cycle in G. It follows that $|V(C)| \le 4$ because G has no C_5-minor. If $|V(C)| = 3$, then $G = K_3$ since G is 2-connected. Suppose that $|V(C)| = 4$. If $|V(G)| = |V(C)|$, then G is either $K_{2,2}$

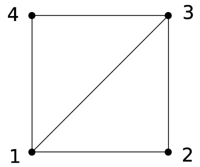

FIGURE 1. $C_3 \# C_3$

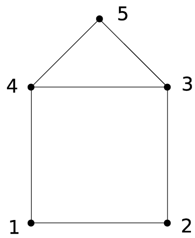

FIGURE 2. $C_4 \# C_3$

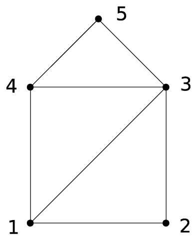

FIGURE 3. $(K_4 \setminus e) \# C_3$

or $K_{1,1,2}$. Next, we assume that $|V(G)| > |V(C)| = 4$. Consider $v \in V(G) \setminus V(C)$. Let P and Q be two paths each with one end in v and another end in $V(C)$, disjoint except for their common end in v and having no internal vertices in C. Such paths exist since G is 2-connected. If $|V(P)| > 2$, or $|V(Q)| > 2$, or the ends of P and Q in C are consecutive in C, then $P \cup Q$ together with a subpath of C form a cycle of length longer than C. Hence every vertex $v \notin V(C)$ has exactly two neighbors in $V(C)$, which are not consecutive. Moreover, if some two vertices $v_1, v_2 \in V(G) \setminus V(C)$ are adjacent to different pairs of vertices in C, then a cycle of length six is induced in G by $\{v_1, v_2\} \cup V(C)$. Therefore there exist $u_1, u_2 \in V(C)$, which are both adjacent to all vertices in $V(G) \setminus \{u_1, u_2\}$. If two vertices in $V(G) \setminus \{u_1, u_2\}$ are adjacent, then together with $\{u_1, u_2\}$ and any other vertex they induce a cycle in G of length five. Therefore G is either $K_{2,n-2}$ or $K_{1,1,n-2}$. It is easy to see that all of K_3, $K_{2,n-2}$ and $K_{1,1,n-2}$ have no (K_4, C_5)-minor. □

It is already known that the cut ideal $I_{K_{1,n-2}}$ for $n \geq 4$ has a quadratic Gröbner basis since $K_{1,n-2}$ is 0-sums of K_2 and $I_{K_2} = \langle 0 \rangle$ [9, Theorem 2.1]. In this paper, to prove Theorem 2.3, we compute the reduced Gröbner basis of $I_{K_{1,n-2}}$. Let $<$ be a reverse lexicographic order on $K[q]$ which satisfies $q_{A|B} < q_{C|D}$ with $\min\{|A|, |B|\} < \min\{|C|, |D|\}$.

LEMMA 2.2. *Let $G = K_{1,n-2}$ be the complete bipartite graph on the vertex set $V_1 \cup V_2$, where $V_1 = \{1\}$ and $V_2 = \{3, \ldots, n\}$ for $n \geq 4$. Then the reduced Gröbner basis of I_G with respect to $<$ consists of*

$$q_{A|B} q_{C|D} - q_{A \cap C | B \cup D} q_{A \cup C | B \cap D} \; (1 \in A \cap C, A \not\subset C, \; C \not\subset A) .$$

The initial monomial of each binomial is the first monomial.

Proof. Let $\mathcal{G}$ be the set of all binomials above. It is easy to see that $\mathcal{G} \subset I_G$. Let $\mathrm{in}(\mathcal{G}) = \langle \mathrm{in}_<(g) \mid g \in \mathcal{G} \rangle$. Let u and v be monomials that do not belong to $\mathrm{in}(\mathcal{G})$:

$$u = \prod_{l=1}^{m} (q_{\{1\} \cup A_l | B_l})^{p_l}, \quad v = \prod_{l=1}^{m'} (q_{\{1\} \cup C_l | D_l})^{p'_l},$$

where $0 < p_l, p'_l \in \mathbb{Z}$ for any l. Since neither u nor v is divided by $q_{A|B} q_{C|D}$, it follows that

$$A_1 \subset A_2 \subset \cdots \subset A_m, \quad C_1 \subset C_2 \subset \cdots \subset C_{m'}.$$

Let

$$A_l = A_{l-1} \cup \{b_1^{l-1}, \ldots, b_{\beta_{l-1}}^{l-1}\},\ B_k = \bigcup_{i=k}^{m} \{b_1^i, \ldots, b_{\beta_i}^i\}$$

$$C_l = C_{l-1} \cup \{d_1^{l-1}, \ldots, d_{\delta_{l-1}}^{l-1}\},\ D_k = \bigcup_{i=k}^{m'} \{d_1^i, \ldots, d_{\delta_i}^i\}$$

for $k \geq 1$ and $l \geq 2$, where $A_1 = V_2 \setminus B_1$, $C_1 = V_2 \setminus D_1$. We suppose that $\pi_G(u) = \pi_G(v)$:

$$\pi_G(u) = s^p \prod_{l=1}^{m} (t_{1b_1^l} \cdots t_{1b_{\beta_l}^l})^{\sum_{k=1}^{l} p_k}, \quad \pi_G(v) = s^{p'} \prod_{l=1}^{m'} (t_{1d_1^l} \cdots t_{1d_{\delta_l}^l})^{\sum_{k=1}^{l} p'_k}.$$

Here we set $p = \sum_{l=1}^{m} p_l$ and $p' = \sum_{l=1}^{m'} p'_l$. Assume that $A_1 \neq C_1$. Then there exists $a \in A_1$ such that $a \notin C_1$. Hence, for some $l_1 \in [m']$, $a \in \{d_1^{l_1}, \ldots, d_{\delta_{l_1}}^{l_1}\}$. However, for any $l \in [m]$, $a \notin \{b_1^l, \ldots, b_{\beta_l}^l\}$. This contradicts that $\pi_G(u) = \pi_G(v)$. Thus $A_1 = C_1$ and $p_1 = p'_1$. By performing this operation repeatedly, it follows that $A_l = C_l$, $B_l = D_l$ and $p_l = p'_l$ for any l. Since $u = v$, $\mathcal{G}$ is a Gröbner basis of I_G. It is trivial that $\mathcal{G}$ is reduced. □

THEOREM 2.3. *Let $G = K_{2,n-2}$ be the complete bipartite graph on the vertex set $V_1 \cup V_2$, where $V_1 = \{1, 2\}$ and $V_2 = \{3, \ldots, n\}$ for $n \geq 4$. Then a Gröbner basis of I_G consists of*

(i) $q_{A|B} q_{E|F} - q_{\emptyset|[n]} q_{\{1,2\}|\{3,\ldots,n\}}$ $(1 \in A, 2 \in B)$,

(ii) $q_{A|B} q_{C|D} - q_{A \cap C|B \cup D} q_{A \cup C|B \cap D}$ $(1 \in A \cap C, 2 \in B \cap D, A \not\subset C, C \not\subset A)$,

(iii) $q_{A|B} q_{C|D} - q_{A \cap C|B \cup D} q_{A \cup C|B \cap D}$ $(1, 2 \in A \cap C, A \not\subset C, C \not\subset A)$,

where $E = (B \cup \{1\}) \setminus \{2\}$ and $F = (A \cup \{2\}) \setminus \{1\}$. The initial monomial of each binomials is the first binomial.

Proof. Let $\mathcal{G}$ be the set of all binomials above. It is easy to see that $\mathcal{G} \subset I_G$. Let u and v be monomials which do not belong to $\mathrm{in}(\mathcal{G})$:

$$u = \prod_{l=1}^{m_1} (q_{\{1\} \cup A_l | \{2\} \cup B_l})^{p_l} \prod_{l=1}^{m_2} (q_{\{1,2\} \cup C_l | D_l})^{r_l},$$

$$v = \prod_{l=1}^{m'_1} (q_{\{1\} \cup A'_l | \{2\} \cup B'_l})^{p'_l} \prod_{l=1}^{m'_2} (q_{\{1,2\} \cup C'_l | D'_l})^{r'_l},$$

where $0 < p_l, r_l, p_l', r_l' \in \mathbb{Z}$ for any l. Since neither u nor v is divided by initial monomials of (ii) and (iii), it follows that

$$A_1 \subset \cdots \subset A_{m_1}, \quad C_1 \subset \cdots \subset C_{m_2},$$
$$A_1' \subset \cdots \subset A'_{m_1'}, \quad C_1' \subset \cdots \subset C'_{m_2'}.$$

Suppose that $\pi_G(u) = \pi_G(v)$:

$$\pi_G(u) = \prod_{l=1}^{m_1} (u_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l} \prod_{l=1}^{m_2} (u_{\{1,2\}\cup C_l|D_l})^{r_l},$$
$$\pi_G(v) = \prod_{l=1}^{m_1'} (u_{\{1\}\cup A_l'|\{2\}\cup B_l'})^{p_l'} \prod_{l=1}^{m_2'} (u_{\{1,2\}\cup C_l'|D_l'})^{r_l'}.$$

Let Y be the matrix consisting of the first $n-2$ rows of $X_{K_{1,n-2}}$. Then X_G is the following matrix:

$$\begin{pmatrix} Y & Y \\ Y & \mathbf{1}_{n-2,2^{n-2}} - Y \\ \mathbf{1} & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1}_{n-2,2^{n-2}}$ is the $(n-2) \times 2^{n-2}$ matrix such that each entry is all ones. Note that

$$\begin{pmatrix} Y \\ Y \end{pmatrix} = \left(\delta_{P_1|Q_1}(K_{2,n-2}) \cdots \delta_{P_{2^{n-2}}|Q_{2^{n-2}}}(K_{2,n-2})\right)$$
$$\begin{pmatrix} Y \\ \mathbf{1}_{n-2,2^{n-2}} - Y \end{pmatrix} = \left(\delta_{R_1|S_1}(K_{2,n-2}) \cdots \delta_{R_{2^{n-2}}|S_{2^{n-2}}}(K_{2,n-2})\right),$$

where $1, 2 \in P_l$, $1 \in R_l$ and $2 \in S_l$ for $1 \le l \le 2^{n-2}$. By elementary row operations of X_G, we have

$$X_G' = \begin{pmatrix} 2Y - \mathbf{1}_{n-2,2^{n-2}} & O \\ O & 2Y - \mathbf{1}_{n-2,2^{n-2}} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

Each column vector of $2Y - \mathbf{1}_{n-2,2^{n-2}}$ is the form ${}^t(\varepsilon_1, \ldots, \varepsilon_{n-2})$, where $\varepsilon_i \in \{1, -1\}$ for $1 \le i \le n-2$. Let $I_{X_G'}$ denote the toric ideal of X_G' (see [8]). Then $u - v \in I_G$ if and only if $u - v \in I_{X_G'}$. Let $\mathbf{a}_{P|Q}$ denote the column vector of $2Y - \mathbf{1}_{n-2,2^{n-2}}$ in X_G' corresponding to the column vector $\delta_{P|Q}(G)$ of X_G. Then

$$\sum_{l=1}^{m_1} p_l \begin{pmatrix} \mathbf{0} \\ \mathbf{a}_{\{1\}\cup A_l|\{2\}\cup B_l} \\ 1 \end{pmatrix} + \sum_{l=1}^{m_2} r_l \begin{pmatrix} \mathbf{a}_{\{1,2\}\cup C_l|D_l} \\ \mathbf{0} \\ 1 \end{pmatrix} = \sum_{l=1}^{m_1'} p_l' \begin{pmatrix} \mathbf{0} \\ \mathbf{a}_{\{1\}\cup A_l'|\{2\}\cup B_l'} \\ 1 \end{pmatrix} + \sum_{l=1}^{m_2'} r_l' \begin{pmatrix} \mathbf{a}_{\{1,2\}\cup C_l'|D_l'} \\ \mathbf{0} \\ 1 \end{pmatrix}.$$

In particular,

$$\sum_{l=1}^{m_1} p_l \mathbf{a}_{\{1\}\cup A_l|\{2\}\cup B_l} = \sum_{l=1}^{m_1'} p_l' \mathbf{a}_{\{1\}\cup A_l'|\{2\}\cup B_l'}, \quad \sum_{l=1}^{m_2} r_l \mathbf{a}_{\{1,2\}\cup C_l|D_l} = \sum_{l=1}^{m_2'} r_l' \mathbf{a}_{\{1,2\}\cup C_l'|D_l'}$$

hold. Let $p = \sum_{l=1}^{m_1} p_l$, $r = \sum_{l=1}^{m_2} r_l$, $p' = \sum_{l=1}^{m_1'} p_l'$ and $r' = \sum_{l=1}^{m_2'} r_l'$. Since neither u nor v is divided by initial monomials of (i), it follows that either $A_1 \neq \emptyset$ or $A_{m_1} \neq [n] \setminus \{1,2\}$ (resp. $A_1' \neq \emptyset$ or $A_{m_2'}' \neq [n] \setminus \{1,2\}$). If $A_1 \neq \emptyset$, then there exists $i \in [n] \setminus \{1,2\}$ such that $i \in A_l$ for any $l \in [m_1]$. If $A_{m_1} \neq [n] \setminus \{1,2\}$, that is, $B_{m_1} \neq \emptyset$, then there exists $i \in [n] \setminus \{1,2\}$ such that $i \in B_{m_1}$, and $i \notin A_l$ for any $l \in [m_1]$. Thus either p or $-p$ appears in the entry of $\sum_{l=1}^{m_1} p_l \mathbf{a}_{\{1\}\cup A_l|\{2\}\cup B_l}$. Similarly, either p' or $-p'$ appears in the entry of $\sum_{l=1}^{m_1'} p_l' \mathbf{a}_{\{1\}\cup A_l'|\{2\}\cup B_l'}$. Therefore $p = p'$. Hence

$$\prod_{l=1}^{m_1} (u_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l} = \prod_{l=1}^{m_1'} (u_{\{1\}\cup A_l'|\{2\}\cup B_l'})^{p_l'}, \quad \prod_{l=1}^{m_2} (u_{\{1,2\}\cup C_l|D_l})^{r_l} = \prod_{l=1}^{m_2'} (u_{\{1,2\}\cup C_l'|D_l'})^{r_l'}$$

hold. Thus

$$\prod_{l=1}^{m_1} (q_{\{1\}\cup A_l|\{2\}\cup B_l})^{p_l} - \prod_{l=1}^{m_1'} (q_{\{1\}\cup A_l'|\{2\}\cup B_l'})^{p_l'} \in I_{Z_1},$$

$$\prod_{l=1}^{m_2} (q_{\{1,2\}\cup C_l|D_l})^{r_l} - \prod_{l=1}^{m_2'} (q_{\{1,2\}\cup C_l'|D_l'})^{r_l'} \in I_{Z_2},$$

where Z_1 (resp. Z_2) is the matrix consisting of the first (resp. last) 2^{n-2} columns of X_G'. Here I_{Z_1} and I_{Z_2} are toric ideals of Z_1 and Z_2. By elementary row operations of Z_1 (resp. Z_2), we have

$$\prod_{l=1}^{m_1} (q_{\{1\}\cup A_l|B_l})^{p_l} - \prod_{l=1}^{m_1'} (q_{\{1\}\cup A_l'|B_l'})^{p_l'}, \quad \prod_{l=1}^{m_2} (q_{\{1\}\cup C_l|D_l})^{r_l} - \prod_{l=1}^{m_2'} (q_{\{1\}\cup C_l'|D_l'})^{r_l'} \in I_{K_{1,n-2}}.$$

By Lemma 2.2, $u = v$ holds. Therefore $\mathcal{G}$ is a Gröbner basis of I_G. □

COROLLARY 2.4. *If G has no (K_4, C_5)-minor, then I_G has a quadratic Gröbner basis.*

Proof. If G is not 2-connected, then there exist 2-connected components $G_1, \ldots, G_s$ of G such that G is 0-sums of $G_1, \ldots, G_s$. By [9] and Lemma 2.1, it is enough to show that, I_{K_2}, I_{K_3}, $I_{K_{2,n-2}}$ and $I_{K_{1,1,n-2}}$ have a quadratic Gröbner basis. It is trivial that I_{K_2} and I_{K_3} have a quadratic Gröbner basis because $I_{K_2} = \langle 0 \rangle$ and $I_{K_3} = \langle 0 \rangle$. Since $K_{1,1,n-2}$ is obtained by 1-sums of C_3, $I_{K_{1,1,n-2}}$ has a quadratic Gröbner basis. Therefore, by Theorem 2.3, I_G has a quadratic Gröbner basis. □

3. Strongly Koszul toric rings of cut ideals

In this section, we characterize the class of graphs whose toric rings associated to cut ideals are strongly Koszul.

PROPOSITION 3.1. *Let $G_1 = K_{1,1,n-2}$ and $G_2 = K_{2,n-2}$ for $n \geq 4$. Then R_{G_1} and R_{G_2} are strongly Koszul.*

Proof. By elementary row operations of X_{G_1}, we have

$$X_{G_1} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ Y & Y \\ Y & \mathbf{1}_{n-2,2^{n-2}} - Y \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ Y & Y \\ Y & -Y \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ Y & Y \\ Y & O \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ O & Y \\ Y & O \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

Hence $R_{G_1} \cong R_{K_{1,n-2}} \otimes_K R_{K_{1,n-2}}$. Since $R_{K_{1,n-2}}$ is Segre products of R_{K_2}, R_{G_1} is strongly Koszul. Next, by the symmetry of $X_{G'}$ in the proof of Theorem 2.3, it is enough to consider the following two cases:

(1) $(u_{\emptyset|[n]}) \cap (u_{\{1\}|\{2,\dots,n\}})$,
(2) $(u_{\emptyset|[n]}) \cap (u_{\{1,2\}\cup A|B})$.

Since $q_{\emptyset|[n]}$ is the smallest variable and $q_{\{1\}|\{2,\dots,n\}}$ is the second smallest variable with respect to the reverse lexicographic order $<$, by [3] and Theorem 2.3, $(u_{\emptyset|[n]}) \cap (u_{\{1\}|\{2,\dots,n\}})$ is generated in degree 2. Assume that $(u_{\emptyset|[n]}) \cap (u_{\{1,2\}\cup A|B})$ is not generated in degree 2. Then there exists a monomial $u_{E_1|F_1} \cdots u_{E_s|F_s}$ belonging to a minimal generating set of $(u_{\emptyset|[n]}) \cap (u_{\{1,2\}\cup A|B})$ such that $s \geq 3$. Since $u_{E_1|F_1} \cdots u_{E_s|F_s}$ is in $(u_{\emptyset|[n]}) \cap (u_{\{1,2\}\cup A|B})$, it follows that

$$q_{\{1,2\}\cup A|B} \prod_{l=1}^{\alpha} q_{\{1,2\}\cup A_l|B_l} \prod_{l=1}^{\beta} q_{\{1\}\cup C_l|\{2\}\cup D_l} - q_{\emptyset|[n]} \prod_{l=1}^{\gamma} q_{\{1,2\}\cup P_l|Q_l} \prod_{l=1}^{\delta} q_{\{1\}\cup R_l|\{2\}\cup S_l} \in I_{G_2}.$$

If one of the monomials appearing in the above binomial is divided by initial monomials of (i) in Theorem 2.3, then $u_{E_1|F_1} \cdots u_{E_s|F_s}$ is divided by $u_{\emptyset|[n]} u_{\{1,2\}|\{3,\dots,n\}}$. This contradicts that $u_{E_1|F_1} \cdots u_{E_s|F_s}$ belongs to a minimal generating set of $(u_{\emptyset|[n]}) \cap (u_{\{1,2\}\cup A|B})$ since, for any $u_{A|B}$ and $u_{C|D}$ with $u_{A|B} \neq u_{C|D}$, $u_{\emptyset|[n]} u_{\{1,2\}|\{3,\dots,n\}}$ belongs to a minimal generating set of $(u_{A|B}) \cap (u_{C|D})$. If one of $\prod_{l=1}^{\beta} q_{\{1\}\cup C_l|\{2\}\cup D_l}$ and $\prod_{l=1}^{\delta} q_{\{1\}\cup R_l|\{2\}\cup S_l}$ is divided by initial monomials of (ii) in Theorem 2.3, the monomial is reduced to the monomial which is not divided by initial monomials of (ii) with respect to $\mathcal{G}$, where $\mathcal{G}$ is a Gröbner basis of I_{G_2}. Thus we may assume that

$$C_1 \subset \cdots \subset C_\beta, \quad R_1 \subset \cdots \subset R_\delta.$$

Similar to what did in the proof of Theorem 2.3, we have

$$u_{\{1,2\}\cup A|B} \prod_{l=1}^{\alpha} u_{\{1,2\}\cup A_l|B_l} = u_{\emptyset|[n]} \prod_{l=1}^{\gamma} u_{\{1,2\}\cup P_l|Q_l},$$

$$\prod_{l=1}^{\beta} u_{\{1\}\cup C_l|\{2\}\cup D_l} = \prod_{l=1}^{\delta} u_{\{1\}\cup R_l|\{2\}\cup S_l}.$$

It follows that $\alpha = \gamma$, $\beta = \delta$, $C_l = R_l$, $D_l = S_l$ for any l, and

$$q_{\{1\}\cup A|B} \prod_{l=1}^{\alpha} q_{\{1\}\cup A_l|B_l} - q_{\emptyset|[n]\setminus\{2\}} \prod_{l=1}^{\alpha} q_{\{1\}\cup P_l|Q_l} \in I_{K_{1,n-2}}.$$

Hence the ideal $(u_{\{1\}\cup A|B}) \cap (u_{\emptyset|[n]\setminus\{2\}})$ of $R_{K_{1,n-2}}$ is not generated in degree 2. However this contradicts that $R_{K_{1,n-2}}$ is strongly Koszul. Therefore R_{G_2} is strongly Koszul. □

LEMMA 3.2. *Let G be a finite simple 2-connected graph with no K_4-minor. If G has C_5-minor, then by only contracting edges of G, we obtain one of C_5, the 1-sum of C_4 and C_3, and the 1-sum of $K_4 \setminus e$ and C_3.*

Proof. Let G be a graph with C_5-minor and C be a longest cycle in G. It follows that $|V(C)| \geq 5$. Then, by contracting edges of G, we obtain a graph $G^{'}$ of five vertices such that C_5 is a subgraph of $G^{'}$. Assume that $G^{'} \neq C_5$. Then there exist $u, v \in V(C_5)$ with $uv \notin E(C_5)$ such that $uv \in E(G^{'})$. Since G has no K_4-minor, there do not exist $\alpha, \beta \in V(C_5) \setminus \{u, v\}$ such that $\alpha\beta \in E(G^{'}) \setminus E(C_5)$. Therefore we obtain one of the 1-sum of C_4 and C_3, and the 1-sum of $K_4 \setminus e$ and C_3. □

THEOREM 3.3. *Let G be a finite simple connected graph. Then R_G is strongly Koszul if and only if G has no (K_4, C_5)-minor.*

Proof. Let G be a graph with no (K_4, C_5)-minor. If G is not 2-connected, then there exist 2-connected components $G_1, \ldots, G_s$ of G such that G is 0-sums of $G_1, \ldots, G_s$. By Lemma 2.1, it is enough to show that R_{K_2}, R_{K_3}, $R_{K_{2,n-2}}$ and $R_{K_{1,1,n-2}}$ are strongly Koszul. It is clear that R_{K_2} and R_{K_3} are strongly Koszul. By Proposition 3.1, $R_{K_{2,n-2}}$ and $R_{K_{1,1,n-2}}$ are strongly Koszul. Next, we suppose that G has K_4-minor. Then the cut ideal I_G is not generated by quadratic binomials [1]. In particular, R_G is not strongly Koszul. Assume that G has no K_4-minor. If G has C_5-minor, then, by Lemma 3.2, we obtain one of C_5, $C_4\#C_3$ and $(K_4 \setminus e)\#C_3$ by contracting edges of G. By Example 1.4, neither $R_{C_4\#C_3}$ nor $R_{(K_4\setminus e)\#C_3}$ is strongly Koszul. By [9, Theorem 1.3], since R_{C_5} is not compressed, R_{C_5} is not strongly Koszul [3, Theorem 2.1]. Therefore, by Proposition 1.2, R_G is not strongly Koszul. □

By using above results, we have

COROLLARY 3.4. *The set of graphs G such that R_G is strongly Koszul is minor closed.*

COROLLARY 3.5. *If R_G is strongly Koszul, then I_G has a quadratic Gröbner basis.*

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