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On a Sakaguchi Type Class of Analytic Functions Associated with Quasi-Subordination

by

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Abstract. In this paper, we introduce a class $\mathcal{G}_{\mathrm{q}}^{\lambda}(\phi, b)$ of analytic functions which is defined in terms of a quasi-subordination. The coefficient estimates including the classical Fekete-Szegö inequality of functions belonging to this class are then derived. We also present certain improved results for the associated classes involving the subordination and majorization. Relevances with known results and extensions of main results involving convolution structures are briefly mentioned.

1. Introduction

Let $\mathcal{A}$ denotes a class of functions analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = 0 = f'(0) - 1$.

For two analytic functions f, g such that $f(0) = g(0)$, we say that f is subordinate to g in $\mathbb{U}$ and write $f(z) \prec g(z),\ z \in \mathbb{U}$, if there exists a Schwarz function $w(z)$ (analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| \leq |z|$, $z \in \mathbb{U}$) such that $f(z) = g(w(z))\ (z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The concept of subordination can be found in [8, p. 226].

Further, f is said to be quasi-subordinate to g in $\mathbb{U}$ and written as $f(z) \prec_q g(z),\ z \in \mathbb{U}$, if there exists an analytic function $\varphi(z)$ with $|\varphi(z)| \leq 1\ \ (z \in \mathbb{U})$ such that $\frac{f(z)}{\varphi(z)}$ is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{\varphi(z)} \prec g(z)\ \ (z \in \mathbb{U}),$$

that is there exists a Schwarz function $w(z)$ such that $f(z) = \varphi(z)g(w(z)), z \in \mathbb{U}$. This definition of quasi-subordination is given by Robertson [20].

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It is observed that if $\varphi(z) \equiv 1 \ (z \in \mathbb{U})$, then the quasi-subordination $\prec_q$ becomes the usual subordination $\prec$, and for the Schwarz function $w(z) = z \ (z \in \mathbb{U})$, the quasi-subordination $\prec_q$ becomes the majorization $' \ll '$. In this case:

$$f(z) \prec_q g(z) \Rightarrow f(z) = \varphi(z) g(z) \Rightarrow f(z) \ll g(z), \quad z \in \mathbb{U}.$$

The concept of majorization is due to MacGregor [6].

Recently, Owa *et al.* [16] introduced and studied a Sakaguchi type class $\mathcal{S}^* (\alpha, b)$ of functions $f \in \mathcal{A}$ which satisfies for $b \neq 1, |b| \leq 1$ and for some $\alpha \ (0 \leq \alpha < 1)$, the condition that

$$\Re\left(\frac{(1-b)\, z f'(z)}{f(z) - f(bz)}\right) > \alpha, \quad z \in \mathbb{U}. \tag{1.1}$$

Obradovic [10] introduced a class of functions $f \in \mathcal{A}$ which for $0 < \lambda < 1$ satisfies the inequality that

$$\Re\left\{ f'(z) \left(\frac{z}{f(z)}\right)^{1+\lambda} \right\} > 0, \quad z \in \mathbb{U}, \tag{1.2}$$

and he calls such functions as functions of non-Bazilevič type.

We also denote by $\mathcal{P}$ the class of functions ϕ analytic in $\mathbb{U}$, such that $\phi(0) = 1$ and $\Re(\phi(z)) > 0, z \in \mathbb{U}$.

Ma and Minda [5] gave a unified presentation of the class of starlike functions by using the method of subordination, and introduced a class $S^* [\phi]$ which is defined by

$$S^* [\phi] = \left\{ h \in \mathcal{A} : \frac{z h'(z)}{h(z)} \prec \phi(z), z \in \mathbb{U} \right\}, \tag{1.3}$$

where $\phi \in \mathcal{P}$ and $\phi(\mathbb{U})$ is symmetrical about the real axis and $\phi'(0) > 0$. A function $f \in S^* [\phi]$ is thus called a Ma and Minda starlike function with respect to ϕ.

Following (1.3) and motivated by the works in [3, 5, 7, 29], we adopt the following definition which defines and introduces a generalization of the above class conditions (1.1) and (1.2) by invoking a quasi-subordination.

DEFINITION 1. Let $\phi \in \mathcal{P}$ be univalent and $\phi(\mathbb{U})$ symmetrical about the real axis and $\phi'(0) > 0$. For $1 \neq b \in \mathbb{C}, |b| \leq 1$ and for $\lambda \geq 0$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}_{\mathrm{q}}^{\lambda} (\phi, b)$ if

$$\left(f'(z) \left(\frac{(1-b)\, z}{f(z) - f(bz)}\right)^{\lambda} - 1 \right) \prec_q (\phi(z) - 1), \quad z \in \mathbb{U}, \tag{1.4}$$

where powers are considered to be having only principal values.

From the above Definition 1, it follows that $f \in \mathcal{G}_{\mathrm{q}}^{\lambda} (\phi, b)$ if and only if there exists an analytic function $\varphi(z)$ with $|\varphi(z)| \leq 1 \ (z \in \mathbb{U})$ such that

$$\frac{f'(z) \left(\frac{(1-b)z}{f(z)-f(bz)}\right)^{\lambda} - 1}{\varphi(z)} \prec (\phi(z) - 1) \ (z \in \mathbb{U}). \tag{1.5}$$

If in the subordination condition (1.5), $\varphi(z) \equiv 1 \ (z \in \mathbb{U})$, then the class $\mathcal{G}_{\mathrm{q}}^{\lambda}(\phi, b)$ is denoted by $\mathcal{G}^{\lambda}(\phi, b)$ and the functions therein satisfy the condition that

$$f'(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} \prec \phi(z), \quad z \in \mathbb{U}.$$

Here, we note that the functions in the class $\mathcal{G}^{2}(\phi, 0)$ if $\phi(z) = \frac{1+z}{1-z}$, $z \in \mathbb{U}$ are univalent [9]. However, $\mathcal{G}^{\lambda}(\phi, b)$ is a Ma-Minda type class of close-to-convex functions if $g(z) := z\left(\frac{f(z)-f(bz)}{(1-b)z}\right)^{\lambda}$ $(1 \neq b \in \mathbb{C}, |b| \leq 1,\ \lambda \geq 0)$ is starlike in $\mathbb{U}$.

It may also be noted that the classes $\mathcal{G}_{\mathrm{q}}^{1}(\phi, 0) = \mathcal{S}_{\mathrm{q}}^{*}(\phi)$ and $\mathcal{G}_{\mathrm{q}}^{0}(\phi, 0) = \mathcal{R}_{\mathrm{q}}(\phi)$ were earlier studied by Mohd. and Darus in [7]. Also, we note for real b and for $\lambda = 1$, the class $\mathcal{G}^{1}(\phi, b)$ was earlier studied by Goyal and Goswami in [3].

It is worth mentioning here that for $\phi(z) = \frac{1+z}{1-z}$, the class $\mathcal{G}^{1}(\phi, -1) = S_{s}^{*}$ is a class of functions starlike with respect to symmetric points which was introduced by Sakaguchi [21]. On the other hand, for $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$, $0 < \lambda < 1$, the class $\mathcal{G}^{1+\lambda}(\phi, 0)$ was studied by Tuneski and Darus [29].

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Initially, a sharp bound of the functional $\left|a_3 - c a_2^2\right|$ for univalent functions $f \in \mathcal{A}$ of the form (2.1) with real c $(0 \leq c \leq 1)$ was obtained by Fekete and Szegö [2] in 1933. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \mathcal{A}$ with any complex c is generally known as the classical Fekete-Szegö problem or inequality. Fekete-Szegö problem for several subclasses of $\mathcal{A}$ have been studied by many authors (see [5], [17]–[19], [22]–[25] and [27]–[29]).

In this paper, we mainly concentrate ourselves in determining the coefficient estimates including a Fekete-Szegö inequality of functions belonging to the classes $\mathcal{G}_{\mathrm{q}}^{\lambda}(\phi, b)$, $\mathcal{G}^{\lambda}(\phi, b)$ and the class involving majorization. Some consequences of the main results involving real parameters are also given. It is also mentioned how a convolution structure can be used to extend the main results.

2. Main Results

Let $f \in \mathcal{A}$ be of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.1}$$

then (for $1 \neq b \in \mathbb{C}, |b| \leq 1$)

$$\frac{f(z)-f(bz)}{1-b} = z + \sum_{n=2}^{\infty} \mu_n a_n z^n,$$

where

$$\mu_n = \frac{1-b^n}{1-b} = 1 + b + b^2 + \cdots + b^{n-1}, \quad n \in \mathbb{N}. \tag{2.2}$$

Hence, for $\lambda \geq 0$, we get

$$\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} = 1-\lambda\mu_2 a_2\,z+\lambda\left\{\frac{(1+\lambda)}{2}\mu_2^2 a_2^2-\mu_3 a_3\right\}z^2+\cdots \quad . \tag{2.3}$$

Throughout the paper, we assume the values of λ and b are such that $\lambda\mu_n \neq n$, and for real b, $\lambda\mu_n < n$, $n = 2, 3, \ldots$.

Also, let the function $\phi \in \mathcal{P}$ be of the form:

$$\phi\,(z) = 1 + B_1 z + B_2 z^2 + \cdots \quad (B_1 > 0)\,, \tag{2.4}$$

and $\varphi(z)$ analytic in $\mathbb{U}$ be of the form:

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + \cdots \quad . \tag{2.5}$$

In proving our results, we use an inequality of Keogh and Merkes [4, p. 10] which is given in the following lemma.

LEMMA 1. *Let the Schwarz function $w(z)$ be given by*

$$w\,(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathbb{U})\,, \tag{2.6}$$

then

$$|w_1| \leq 1,\ \left|w_2 - t w_1^2\right| \leq 1 + (|t|-1)\,|w_1|^2 \leq \max\{1, |t|\}\ ,$$

where $t \in \mathbb{C}$. The result is sharp for the function $w\,(z) = z$ or $w\,(z) = z^2$.

Our first main result is contained in the following:

THEOREM 1. *Let $f \in \mathcal{A}$ of the form* (2.1) *belong to the class $\mathcal{G}_{\mathrm{q}}^{\lambda}\,(\phi, b)$, then*

$$|a_2| \leq \frac{B_1}{|2-\lambda\mu_2|}\,, \tag{2.7}$$

and for some $c \in \mathbb{C}$:

$$\left|a_3 - c a_2^2\right| \leq \frac{B_1}{|3-\lambda\mu_3|}\max\left\{1, \left|\frac{B_2}{B_1} - B_1 K\right|\right\}\,, \tag{2.8}$$

where

$$K = \frac{c\,(3-\lambda\mu_3)}{(2-\lambda\mu_2)^2} - \frac{\lambda\left(1+\frac{2-\mu_2}{2-\lambda\mu_2}\right)\mu_2}{2\,(2-\lambda\mu_2)} \tag{2.9}$$

and μ_n $(n \in \mathbb{N})$ is given by (2.2). *The result is sharp.*

Proof. Let $f \in \mathcal{G}_{\mathrm{q}}^{\lambda}\,(\phi, b)$, then for a Schwarz function $w\,(z)$ given by (2.6) and for an analytic function $\varphi(z)$ given by (2.5), we have

$$f'(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} - 1 = \varphi(z)\,(\phi\,(w\,(z)) - 1)\,, \quad z \in \mathbb{U}\,. \tag{2.10}$$

In view of (2.4), we obtain

$$\begin{aligned}\varphi(z)\,(\phi\,(w\,(z))-1) &= \left(d_0 + d_1 z + d_2 z^2 + \cdots\right)\left(B_1 w_1 z + \left(B_1 w_2 + B_2 w_1^2\right)z^2 + \cdots\right)\\ &= d_0 B_1 w_1 z + \left\{d_0\left(B_1 w_2 + B_2 w_1^2\right) + d_1 B_1 w_1\right\}z^2 + \cdots \quad . \end{aligned} \tag{2.11}$$

Using the series expansion of $f^{'}(z)$ from (2.1), and the expansion given by (2.3), we get

$$f^{'}(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda}-1$$
$$=(2-\lambda\mu_2)\,a_2 z+\left[(3-\lambda\mu_3)\,a_3-\lambda\left\{2-\frac{(1+\lambda)}{2}\mu_2\right\}\mu_2 a_2^2\right]z^2+\cdots\ . \tag{2.12}$$

From the expansions (2.11) and (2.12), on equating the coefficients of z and z^2 in (2.10), we find that

$$(2-\lambda\mu_2)\,a_2=d_0 B_1 w_1\,, \tag{2.13}$$
$$(3-\lambda\mu_3)\,a_3-\lambda\left\{2-\frac{(1+\lambda)}{2}\mu_2\right\}\mu_2 a_2^2$$
$$=d_0\left(B_1 w_2+B_2 w_1^2\right)+d_1 B_1 w_1\,. \tag{2.14}$$

Now (2.13) gives

$$a_2=\frac{d_0 B_1 w_1}{2-\lambda\mu_2}, \tag{2.15}$$

which in view of (2.14) yields that

$$(3-\lambda\mu_3)\,a_3=\frac{\lambda\left\{4-(1+\lambda)\,\mu_2\right\}\mu_2}{2\,(2-\lambda\mu_2)^2}d_0^2 B_1^2 w_1^2+d_0\left(B_1 w_2+B_2 w_1^2\right)+d_1 B_1 w_1\,,$$

and therefore,

$$a_3=\frac{B_1}{3-\lambda\mu_3}\left[d_1 w_1+d_0\left\{w_2+\left(\frac{d_0\lambda\left(1+\frac{2-\mu_2}{2-\lambda\mu_2}\right)\mu_2 B_1}{2\,(2-\lambda\mu_2)}+\frac{B_2}{B_1}\right)w_1^2\right\}\right]. \tag{2.16}$$

For some $c\in\mathbb{C}$, we obtain from (2.15) and (2.16):

$$a_3-ca_2^2=\frac{B_1}{3-\lambda\mu_3}\left[d_1 w_1+\left(w_2+\frac{B_2}{B_1}w_1^2\right)d_0-B_1 K w_1^2 d_0^2\right], \tag{2.17}$$

where K is given by (2.9). Since, $\varphi(z)$ given by (2.5) is analytic and bounded in $\mathbb{U}$, therefore, on using [8, p. 172], we have for some y $(|y|\le 1)$:

$$|d_0|\le 1 \text{ and } d_1=\left(1-d_0^2\right)y\,. \tag{2.18}$$

On putting the value of d_1 from (2.18) into (2.17), we get

$$a_3-ca_2^2=\frac{B_1}{3-\lambda\mu_3}\left[yw_1+\left(w_2+\frac{B_2}{B_1}w_1^2\right)d_0-\left(B_1 K w_1^2+yw_1\right)d_0^2\right]. \tag{2.19}$$

If $d_0=0$ in (2.19), we at once get

$$\left|a_3-ca_2^2\right|\le\frac{B_1}{|3-\lambda\mu_3|}. \tag{2.20}$$

But if $d_0\neq 0$, let us then suppose that

$$F(d_0):=yw_1+\left(w_2+\frac{B_2}{B_1}w_1^2\right)d_0-\left(B_1 K w_1^2+yw_1\right)d_0^2\,,$$

which is a polynomial in d_0 and hence analytic in $|d_0| \le 1$, and maximum of $|F(d_0)|$ is attained at $d_0 = e^{i\theta}$ $(0 \le \theta < 2\pi)$. We find that $\max_{0\le\theta<2\pi} \left|F(e^{i\theta})\right| = |F(1)|$ and

$$\left|a_3 - ca_2^2\right| \le \frac{B_1}{|3-\lambda\mu_3|}\left|w_2 - \left(B_1 K - \frac{B_2}{B_1}\right) w_1^2\right| , \tag{2.21}$$

which on using Lemma 1 shows that

$$\left|a_3 - ca_2^2\right| \le \frac{B_1}{|3-\lambda\mu_3|} \max\left\{1, \left|B_1 K - \frac{B_2}{B_1}\right|\right\} ,$$

and this last above inequality together with (2.20) thus establishes the result (2.8). Sharpness of this result can be verified for the functions $f(z)$ given by (for $1 \ne b \in \mathbb{C}$, $|b| \le 1$, $\lambda \ge 0$ and $z \in \mathbb{U}$)

$$f^{'}(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} = \phi(z) \quad \text{or} \quad f^{'}(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} = \phi\left(z^2\right) , \tag{2.22}$$

or

$$f^{'}(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} - 1 = z\,(\phi(z)-1) \; .$$

This completes the proof of Theorem 1. □

For the case when $b = 0$, we set, respectively, $\lambda = 1$ and $\lambda = 0$ in Theorem 1 to obtain the following sharp results for the known subclasses $\mathcal{S}_{\mathrm{q}}^*(\phi)$ and $\mathcal{R}_{\mathrm{q}}(\phi)$.

COROLLARY 1. *Let* $f \in \mathcal{A}$ *of the form* (2.1) *belong to the class* $\mathcal{S}_{\mathrm{q}}^*(\phi)$, *then*

$$|a_2| \le B_1 ,$$

and for some $c \in \mathbb{C}$:

$$\left|a_3 - ca_2^2\right| \le \frac{B_1}{2} \max\left\{1, \left|\frac{B_2}{B_1} + (1-2c)\,B_1\right|\right\} .$$

The result is sharp.

COROLLARY 2. *Let* $f \in \mathcal{A}$ *of the form* (2.1) *belong to the class* $\mathcal{R}_{\mathrm{q}}(\phi)$, *then*

$$|a_2| \le \frac{B_1}{2},$$

and for $c \in \mathbb{C}$:

$$\left|a_3 - ca_2^2\right| \le \frac{B_1}{3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3c}{4} B_1\right|\right\} ,$$

The result is sharp.

REMARK 1. We note that the Fekete-Szegö inequality obtained above for the classes $\mathcal{S}_{\mathrm{q}}^*(\phi)$ and $\mathcal{R}_{\mathrm{q}}(\phi)$ improve the results obtained earlier by Mohd. and Darus [7, Theorems 2.1 and 2.6].

We next mention the following result for the class $\mathcal{G}^{\lambda}(\phi, b)$.

THEOREM 2. *Let $f \in \mathcal{A}$ of the form* (2.1) *belong to the class $\mathcal{G}^{\lambda}(\phi, b)$, then*

$$|a_2| \leq \frac{B_1}{|2 - \lambda\mu_2|},$$

and for some $c \in \mathbb{C}$:

$$\left|a_3 - ca_2^2\right| \leq \frac{B_1}{|3 - \lambda\mu_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 K\right|\right\},$$

where K is given by (2.9) *and μ_n $(n \in \mathbb{N})$ is given by* (2.2). *The result is sharp.*

Proof. Let $f \in \mathcal{G}^{\lambda}(\phi, b)$. Similar to the proof of Theorem 1, if $\varphi(z) \equiv 1$, then (2.5) evidently implies that $d_0 = 1$ and $d_n = 0, n \in \mathbb{N}$, hence, in view of (2.15) and (2.17) and Lemma 1, we obtain the desired result of Theorem 2. Sharpness can be verified for the functions $f(z)$ given by

$$f^{'}(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\lambda} = \phi(z) \quad \text{or} \quad f^{'}(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\lambda} = \phi\left(z^2\right).$$

□

Our next result is devoted to the majorization and the result pertaining to it is contained in the following.

THEOREM 3. *Let $1 \neq b \in \mathbb{C}$, $|b| \leq 1$. If a function $f \in \mathcal{A}$ of the form* (2.1) *satisfies*

$$\left(f^{'}(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\lambda} - 1\right) \ll (\phi(z) - 1), \quad z \in \mathbb{U}, \tag{2.23}$$

then

$$|a_2| \leq \frac{B_1}{|2 - \lambda\mu_2|},$$

and for some $c \in \mathbb{C}$:

$$\left|a_3 - ca_2^2\right| \leq \frac{B_1}{|3 - \lambda\mu_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 K\right|\right\},$$

where μ_n $(n \in \mathbb{N})$ is given by (2.2) *and K is given by* (2.9). *The result is sharp.*

Proof. Following the proof of Theorem 1, if $w(z) \equiv z$ in (2.6), so that $w_1 = 1$ and $w_n = 0, n = 2, 3, ..,$ then in view of (2.15) and (2.17), we get

$$|a_2| \leq \frac{B_1}{|2 - \lambda\mu_2|}$$

and

$$a_3 - ca_2^2 = \frac{B_1}{3 - \lambda\mu_3}\left[d_1 + \frac{B_2}{B_1} d_0 - B_1 K d_0^2\right]. \tag{2.24}$$

On putting the value of d_1 from (2.18) in (2.24), we get

$$a_3 - ca_2^2 = \frac{B_1}{3 - \lambda\mu_3}\left[y + \frac{B_2}{B_1} d_0 - (B_1 K + y) d_0^2\right]. \tag{2.25}$$

If $d_0 = 0$ in (2.25), we get

$$\left|a_3 - ca_2^2\right| \leq \frac{B_1}{|3-\lambda\mu_3|}\,, \tag{2.26}$$

and if $d_0 \neq 0$, let

$$G(d_0) := y + \frac{B_2}{B_1}d_0 - (B_1K + y)\, d_0^2\,,$$

which being a polynomial in d_0 is analytic in $|d_0| \leq 1$, and maximum of $|G(d_0)|$ is attained at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. We thus find that $\max\limits_{0\leq\theta<2\pi} \left|G(e^{i\theta})\right| = |G(1)|$ and consequently

$$\left|a_3 - ca_2^2\right| \leq \frac{B_1}{|3-\lambda\mu_3|}\left|B_1K - \frac{B_2}{B_1}\right|$$

which together with (2.26) establishes the desired result of Theorem 3. Sharpness can be verified for the function given by

$$f^{'}(z)\left(\frac{(1-b)\,z}{f(z)-f(bz)}\right)^{\lambda} = \phi\,(z)\ .$$

□

We turn our attention to obtain the bounds of the functional $\left|a_3 - ca_2^2\right|$ for real values of c and b. We obtain the following result for the class $\mathcal{G}_q^{\lambda}\,(\phi, b)$ which involves a quasi-subordination (by applying Theorem 1). Similar forms of results would arise from Theorems 2 and 3. But we omit these considerations.

COROLLARY 3. *Let $f \in \mathcal{A}$ of the form* (2.1) *belong to the class* $\mathcal{G}_q^{\lambda}\,(\phi, b)$, *then for real values of c and b* :

$$\left|a_3 - ca_2^2\right| \leq \begin{cases} \frac{B_1}{3-\lambda\mu_3}\left[B_1\left(\frac{\lambda\{4-(1+\lambda)\mu_2\}\mu_2 - 2c(3-\lambda\mu_3)}{2(2-\lambda\mu_2)^2}\right) + \frac{B_2}{B_1}\right] & \text{if} \quad c \leq \rho, \\ \frac{B_1}{3-\lambda\mu_3} & \text{if} \quad \rho \leq c \leq \rho + 2\sigma, \\ \frac{B_1}{3-\lambda\mu_3}\left[B_1\left(\frac{2c(3-\lambda\mu_3) - \lambda\{4-(1+\lambda)\mu_2\}\mu_2}{2(2-\lambda\mu_2)^2}\right) - \frac{B_2}{B_1}\right] & \text{if} \quad c \geq \rho + 2\sigma, \end{cases} \tag{2.27}$$

where

$$\rho = \frac{\lambda\,\{4-(1+\lambda)\,\mu_2\}\,\mu_2}{2\,(3-\lambda\mu_3)} - \frac{(2-\lambda\mu_2)^2}{3-\lambda\mu_3}\left(\frac{1}{B_1} - \frac{B_2}{B_1^2}\right), \tag{2.28}$$

$$\sigma = \frac{(2-\lambda\mu_2)^2}{(3-\lambda\mu_3)\,B_1}\,, \tag{2.29}$$

and μ_n $(n \in \mathbb{N})$ *is given by* (2.2). *The result is sharp.*

Proof. For real values of c and b, we get from (2.8) the above bounds, respectively, under the following cases:

$$B_1K - \frac{B_2}{B_1} \leq -1\,, \quad -1 \leq B_1K - \frac{B_2}{B_1} \leq 1 \quad \text{and} \quad B_1K - \frac{B_2}{B_1} \geq 1\,,$$

where K is given by (2.9). This establishes the inequality (2.27).

(1) For the extreme range of c, i.e. when $c < \rho$ or $c > \rho + 2\sigma$, the equality holds if and only if $w(z) = z$, or one of its rotations.
(2) For the middle range of c, i.e. when $\rho < c < \rho + 2\sigma$, the equality holds if and only if $w(z) = z^2$, or one of its rotations.
(3) Equality holds for $c = \rho$ if and only if $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations, while for $c = \rho + 2\sigma$, the equality holds if and only if $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations.

□

The bounds of the functional $\left|a_3 - ca_2^2\right|$ for real values of c and b for the middle range of the parameter c can be improved further. This new form of the result is contained in the following theorem.

THEOREM 4. *Let $f \in \mathcal{A}$ of the form* (2.1) *belong to the class $\mathcal{G}_q^\lambda(\phi, b)$, then for real values of c and b* (*when $\rho < c < \rho + 2\sigma$*) :

$$\left|a_3 - ca_2^2\right| + (c - \rho)\left|a_2\right|^2 \le \frac{B_1}{3 - \lambda\mu_3} \quad (\rho < c \le \rho + \sigma) \tag{2.30}$$

and

$$\left|a_3 - ca_2^2\right| + (\rho + 2\sigma - c)\left|a_2\right|^2 \le \frac{B_1}{3 - \lambda\mu_3} \quad (\rho + \sigma < c < \rho + 2\sigma), \tag{2.31}$$

where ρ and σ are given, respectively, by (2.28) *and* (2.29) *and μ_3 is given by* (2.2) (*when $n = 3$ therein*).

Proof. For the real values of c and b, let $f \in \mathcal{G}_q^\lambda(\phi, b)$, then from (2.15) and (2.21) (when $\rho < c < \rho + 2\sigma$), we get if $\rho < c \le \rho + \sigma$:

$$\begin{aligned} &\left|a_3 - ca_2^2\right| + (c - \rho)\left|a_2\right|^2 \\ &\le \frac{B_1}{3 - \lambda\mu_3}\left[\left|w_2\right| - \frac{B_1(3 - \lambda\mu_3)}{(2 - \lambda\mu_2)^2}(c - \rho - \sigma)\left|w_1\right|^2 + \right. \\ &\left. \frac{B_1(3 - \lambda\mu_3)}{(2 - \lambda\mu_2)^2}(c - \rho)\left|w_1\right|^2\right]. \end{aligned}$$

Hence, by applying Lemma 1, we get

$$\left|a_3 - ca_2^2\right| + (c - \rho)\left|a_2\right|^2 \le \frac{B_1}{3 - \lambda\mu_3}\left[1 - \left|w_1\right|^2 + \left|w_1\right|^2\right],$$

which yields the estimate (2.30). If $\rho + \sigma < c < \rho + 2\sigma$, then again from (2.15) and (2.21) by Lemma 1, we get

$$\left|a_3 - ca_2^2\right| + (\rho + 2\sigma - c)\left|a_2\right|^2$$

$$\leq \frac{B_1}{3-\lambda\mu_3}\left[|w_2| + \frac{B_1(3-\lambda\mu_3)}{(2-\lambda\mu_2)^2}(c-\rho-\sigma)|w_1|^2 + \right.$$
$$\left. \frac{B_1(3-\lambda\mu_3)}{(2-\lambda\mu_2)^2}(\rho+2\sigma-c)|w_1|^2\right]$$
$$\leq \frac{B_1}{3-\lambda\mu_3}\left[1-|w_1|^2+|w_1|^2\right],$$

which gives the other estimate (2.31). □

REMARK 2. We observe that on choosing $\lambda = 1$, the inequality (2.27) and its subsequent improved result given by Theorem 4 coincide with the results of Goyal and Goswami [3, Theorems 2.1, 2.2].

3. Concluding Remarks

Lastly, we make concluding remarks by observing that several differential as well as integral linear operators for the class $\mathcal{A}$ defined in literature can be expressed by means of a convolution structure, see (amongst others), the works in the papers of [11]–[15] and [26]; see also [1].

Since, the convolution of $f \in \mathcal{A}$ of the form (2.1) and $h \in \mathcal{A}$ of the form:

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \quad h_n \neq 0, \quad z \in \mathbb{U} \tag{3.1}$$

is defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n h_n z^n = (h * f)(z),$$

therefore, we can apply Theorems 1, 2 and 3 for the functions involved in the convolution $(f * h)(z)$, and obtain the associated results for such a convolution structure. These considerations being quite straightforward, we omit mentioning of such results here.

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