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# A Remark on the Wiener–Ikehara Tauberian Theorem

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**Abstract.** In this paper we point out that the proof of Kable's extension of the Wiener–Ikehara Tauberian theorem can be applied to the case where the Dirichlet series has a pole of order "$l/m$" without much modification (Kable proved the case $l = 1$).

## 1. Introduction

We use the notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the sets of positive integers, all integers, real numbers, complex numbers respectively. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. For $x \in \mathbb{R}$ we put $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$.

Let $\{a_n\}$ be a sequence of non-negative real numbers, $d$ a positive real number, and $m, l \in \mathbb{N}$. Suppose that the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > d$. Also suppose that $L^m$ has a meromorphic continuation to an open set containing the closed half-plane $\mathrm{Re}(s) \geq d$ and holomorphic except for a pole of order $l$ at $s = d$. In this paper, if $L^m$ has a pole of order $l$, then we say that $L$ has a pole of order $l/m$. For functions $f$ and $g : \mathbb{R} \to \mathbb{R}$, we denote $f(x) \sim g(x)$ if $\lim_{x\to+\infty} f(x)/g(x) = 1$.

Our purpose is to determine the asymptotic behavior of $\sum_{n \leq X} a_n$ as $X \to \infty$ by properties of $L(s)$.

In the case where $L$ has a simple pole at $s = d$ (i.e., $m = l = 1$), Wiener and Ikehara proved that

$$\sum_{n \leq X} a_n \sim \frac{A}{d} X^d$$

as $X \to \infty$, where $A$ is residue of $L$ at $s = d$. This result was proved in 1932([6]) and is called the Wiener–Ikehara theorem.

In the general case, an extension was given by Delange ([1, p.235, THÉORÈM III]) in 1954. Delange considered the case where the order of the pole is a positive real number

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in some sense. However, to apply Delange's theorem to $L$ satisfying above conditions, an extra condition about zeros of $L^m$ on the line $\mathrm{Re}(s) = d$ is required. Kable has given an extension for the case where the order of the pole is $1/m$ without the condition. In [2], he used the notion of functions of bounded variation.

The result obtained by applying Delange's result to the Dirichlet series $L(s)$ satisfying above conditions is as follows. Let $A^m = \lim_{s\to d} L(s)^m(s-d)^l$, where $A > 0$. If $m \geq 2$, then we assume that the order of the all zeros of $L(s)^m$ on the line $\mathrm{Re}(s) = d$ is divisible by $m$. Then

$$\sum_{n\leq X} a_n \sim \frac{AX^d}{d\Gamma(\alpha)(\log(X))^{1-\frac{l}{m}}} \tag{1.1}$$

as $X \to \infty$. In particular, in the case $m = 1$, we need no assumption about the zeros of $L(s)$.

The result of Kable is having removed this assumption in the case $m \geq 2$ and $l = 1$.

We shall point out in this paper that the proof of Kable's result works for the case $m \geq 2$ and $l \geq 2$ without much modification, and that (1.1) holds without the assumption about the order of the zeros of $L^m$.

The organization of this paper is as follows. In Section 2, we define symbols and functions used in this paper. Then we slightly extend [2, p.140, THEOREM 1]. In Section 3, we apply the result in Section 2 to obtain the same result as (1.1) without the assumption about the order of the zeros of $L^m$. In Section 4, we give two examples of a Dirichlet series which has a pole of rational order and apply the main theorem (Theorem 3.1).

## 2. Preliminaries

Let $\lambda$ be $(2\pi)^{-\frac{1}{2}}$ times the Lebesgue measure on $\mathbb{R}$ and $\mathcal{S}(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$. For $\Phi \in \mathcal{S}(\mathbb{R})$ define the Fourier transform by

$$\mathcal{F}(\Phi)(t) = \int_{\mathbb{R}} \Phi(u)e^{-iut}d\lambda(u)\,.$$

We define the inverse Fourier transform by

$$\mathcal{F}^{-1}(\Phi)(t) = \int_{\mathbb{R}} \Phi(u)e^{iut}d\lambda(u)\,.$$

Then

$$\mathcal{F}^{-1}(\mathcal{F}(\Phi))(t) = \Phi(t) \quad i.e., \quad \Phi(t) = \int_{\mathbb{R}} \mathcal{F}(\Phi)(u)e^{iut}d\lambda(u)\,.$$

For $\Phi, \Psi \in \mathcal{S}(\mathbb{R})$, we define the convolution by

$$(\Phi * \Psi)(t) = \int_{\mathbb{R}} \Phi(t-u)\Psi(u)d\lambda(u)\,.$$

It is well known (see [5, p.183, 7.2 Theorem]) that

$$\mathcal{F}(\Phi * \Psi) = \mathcal{F}(\Phi)\mathcal{F}(\Psi)\,.$$

For $s \in \mathbb{C} - (-\infty, 0]$ we choose the branch of $\log(s)$ so that $-\pi < arg(s) < \pi$. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$. For $s \in \mathbb{C} - \{s \in \mathbb{C} \mid s - x \in (\infty, 0]\}$ we define $(s-x)^\alpha = \exp(\alpha \log(s-x))$. Then $(s-x)^\alpha$ is holomorphic on $\mathbb{C} - \{s \in \mathbb{C} \mid s - x \in (\infty, 0]\}$ and has positive real values for $s \in \{s \in \mathbb{C} \mid s - x \in \mathbb{R}_{>0}\}$.

Let $\mathcal{P}([a,b])$ denote the set of all partitions of $[a,b]$ (i.e., sequences $x_0 = a < x_1 < \cdots < x_n = b$). For a function $f : [a,b] \to \mathbb{R}$ we define $V_a^b f \in \mathbb{R} \cup \{\infty\}$ by

$$V_a^b f = \sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \,\middle|\, P = \{x_i \mid i = 0, 1, \ldots, n\} \in \mathcal{P}([a,b]) \right\} .$$

We say that a function $f : [a,b] \to \mathbb{R}$ is of bounded variation if $V_a^b f < +\infty$. Also we say that a function $f : [a,b] \to \mathbb{C}$ is of bounded variation if the real part and the imaginary part of $f$ are of bounded variation. If a function $f : \mathbb{R} \to \mathbb{C}$ is of bounded variation on any closed-interval $[a,b]$, then we say that $f$ is locally of bounded variation.

The following theorem is an extension of the Wiener–Ikehara theorem and plays a key role to prove the main theorem in Section 3. This theorem slightly extends [2, p.140, THEOREM 1].

THEOREM 2.1. *Let $f : \mathbb{R} \to [0, \infty)$ be a non-decreasing function such that the Laplace transform*

$$F(s) = \int_0^\infty f(u) e^{-su} du$$

*converges absolutely for* $\mathrm{Re}(s) > 1$. *Let $\alpha, \alpha_i \in \mathbb{R}_{>0} (i = 1, 2, \ldots, r)$, $A \in \mathbb{R}_{>0}$, $A_i \in \mathbb{C}$ and $\alpha > \alpha_i$ for all $i$. We define*

$$G(s) = F(s) - \frac{A}{(s-1)^\alpha} - \sum_{i=1}^{r} \frac{A_i}{(s-1)^{\alpha_i}}$$

*for* $\mathrm{Re}(s) > 1$. *Suppose that the function $G$ extends continuously to the set* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\}$. *If $\alpha < 1$ then assume, in addition, that once so extended, the function $t \mapsto G(1+it)$ is locally of bounded variation. Then*

$$\lim_{u \to \infty} u^{1-\alpha} e^{-u} f(u) = \frac{A}{\Gamma(\alpha)} .$$

*Proof.* Define a function $h : \mathbb{R} \to [0, \infty)$ by $h(u) = u^{1-\alpha} e^{-u} f(u)$ and let $C_i = A_i / \Gamma(\alpha_i)$, $C = A/\Gamma(\alpha)$. By [2, p.140, LEMMA 2]

$$\frac{1}{(s-1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(s-1)u} u^{\alpha-1} du$$

for $\mathrm{Re}(s) > 1$. By definition

$$F(s) = \int_0^\infty h(u) e^{-(s-1)u} u^{\alpha-1} du .$$

It follows that

$$G(s) = \int_0^\infty \left( h(u) - C - \sum_{i=1}^{r} \frac{C_i}{u^{\alpha-\alpha_i}} \right) e^{-(s-1)u} u^{\alpha-1} du$$

for $\mathrm{Re}(s) > 1$. Let $\Psi$ be an even Schwartz function with compact support and $\Phi = \mathcal{F}(\Psi)^2$. Since $\mathcal{F}(\Phi) = \Psi * \Psi$, $\mathcal{F}(\Phi)$ is even and has compact support.

Suppose that $\sigma > 1$ and $x > 0$. Since

$$F(\sigma) \geq \int_x^\infty f(u)e^{-\sigma u}du \quad , \quad f(u) \geq 0$$

and $f(u)$ is monotone increasing, we obtain $f(x) \leq \sigma e^{\sigma x} F(\sigma)$. Hence,

$$h(x) \leq \sigma F(\sigma) e^{(\sigma-1)x} x^{1-\alpha} \tag{2.2}$$

for any $x > 0$.

Let $\varepsilon > 0$ be a small number. We choose $\sigma$ so that $0 < \sigma - 1 < \varepsilon$. For any $v \in \mathbb{R}_{>0}$, by using (2.2), we obtain

$$\int_0^\infty h(u)\Phi(v-u)e^{-\varepsilon u}u^{\alpha-1}du \leq \sigma F(\sigma) \int_0^\infty \Phi(v-u)e^{-\left(\varepsilon-(\sigma-1)\right)u}du \,. \tag{2.3}$$

Since $\Phi$ is bounded and non-negative, the integral on the right-hand side of (2.3) converges. Since $h(u)$ is also non-negative, the integral on the left-hand side converges absolutely. Moreover the integral

$$\int_0^\infty \Phi(v-u)e^{-\varepsilon u}u^{\alpha_i-1}du$$

also converges absolutely. Therefore, by Fubini's theorem, the following equation holds.

$$\begin{aligned}
&\int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}}\right)\Phi(v-u)e^{-\varepsilon u}u^{\alpha-1}du \\
&= \int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}}\right)e^{-\varepsilon u}u^{\alpha-1}\int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{i(v-u)t}d\lambda(t)du \\
&= \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}\int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}}\right)e^{-(it+\varepsilon)u}u^{\alpha-1}dud\lambda(t) \\
&= \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+\varepsilon+it)d\lambda(t)\,.
\end{aligned} \tag{2.4}$$

By assumption, $G(s)$ is continuous for $\mathrm{Re}(s) \geq 1$ and the support of $\mathcal{F}(\Phi)$ is compact. Thus, $G(1+\varepsilon+it)$ converges to $G(1+it)$ uniformly on the support of $\mathcal{F}(\Phi)(t)$ as $\varepsilon \to +0$. Therefore,

$$\lim_{\varepsilon\to+0}\int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+\varepsilon+it)d\lambda(t) = \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+it)d\lambda(t)\,. \tag{2.5}$$

As far as the left-hand side of (2.4) is concerned, by using the monotone convergence theorem to each term, we have

$$\begin{aligned}(2.6)\qquad &\lim_{\varepsilon\to+0}\int_0^\infty \left(h(u)-C-\sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}}\right)\Phi(v-u)e^{-\varepsilon u}u^{\alpha-1}du\\ &\qquad = \int_0^\infty \left(h(u)-C-\sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}}\right)\Phi(v-u)u^{\alpha-1}du\,.\end{aligned}$$

We may choose a closed-interval $[a, b]$ containing the support of $\mathcal{F}(\Phi)$. By [2, p.140 LEMMA 3], if $\alpha < 1$

$$\begin{aligned}(2.7)\qquad &\lim_{v\to\infty} v^{1-\alpha}\int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+it)d\lambda(t)\\ &= \lim_{v\to\infty} v^{1-\alpha}\int_a^b \mathcal{F}(\Phi)(t)e^{ivt}G(1+it)d\lambda(t)\\ &= 0\,.\end{aligned}$$

This is the place that the assumption that $G(1+it)$ is locally of bounded variation is used.

If $\alpha \geq 1$, then we have the same equation by [5, p.185, 7.5 Theorem]. Since the right-hand sides of (2.5) and (2.6) are equal, by using (2.7), we obtain

$$(2.8)\qquad \lim_{v\to\infty} v^{1-\alpha}\int_0^\infty \left(h(u)-C-\sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}}\right)\Phi(v-u)u^{\alpha-1}du = 0$$

for any $\alpha > 0$.

By [2, p.139, Lemma 1] we have

$$(2.9)\qquad \lim_{v\to\infty} v^{1-\alpha}\int_0^\infty \Phi(v-u)u^{\alpha-1}du = \int_{\mathbb{R}} \Phi(u)du$$

for any $\alpha > 0$. So,

$$\begin{aligned}(2.10)\qquad &\lim_{v\to\infty} v^{1-\alpha}\int_0^\infty \frac{C_i}{u^{\alpha-\alpha_i}}\Phi(v-u)u^{\alpha-1}du\\ &= \lim_{v\to\infty} \frac{1}{v^{\alpha-\alpha_i}}v^{1-\alpha_i}\int_0^\infty C_i\Phi(v-u)u^{\alpha_i-1}du\\ &= 0\end{aligned}$$

for $i = 1, 2, \ldots, r$. Therefore, by (2.8), (2.9), and (2.10), we have

$$\begin{aligned}\lim_{v\to\infty} v^{1-\alpha}\int_0^\infty h(u)\Phi(v-u)u^{\alpha-1}du &= \lim_{v\to\infty} C\, v^{1-\alpha}\int_0^\infty \Phi(v-u)u^{\alpha-1}du\\ &= C\int_{\mathbb{R}} \Phi(u)du.\end{aligned}$$

The rest of the argument is similar to that in [2, pp.142–143], and we can conclude that

$$\lim_{u\to\infty} h(u) = C.$$

Therefore, Theorem 2.1 is proved. □

## 3. Main Theorem

In this section, we apply Theorem 2.1 to Dirichlet series which are obtained by sequences of non-negative real numbers and satisfy some conditions. The following theorem is the main application of Theorem 2.1.

THEOREM 3.1. *Let $\{a_n\}$ be a sequence of non-negative real numbers, $d \in \mathbb{R}_{>0}$ and $m$ a positive integer. Suppose that the Dirichlet series*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

*converges absolutely for* $\mathrm{Re}(s) > d$. *Also suppose that $L^m$ has a meromorphic continuation to an open set containing the closed half-plane* $\mathrm{Re}(s) \geq d$ *and holomorphic except for a pole of order $l$ at $s = d$ with* $\lim_{s\to d} L(s)^m(s-d)^l = A^m$, *where $A > 0$. Then we have*

$$\sum_{n\leq X} a_n \sim \frac{AX^d}{d\Gamma(l/m)(\log(X))^{1-\frac{l}{m}}}$$

*as $X \to \infty$.*

*Proof.* For a subset $S \subset \mathbb{R}$, let $\phi_S$ be the characteristic function of $S$. Define a function $f : \mathbb{R} \to [0, \infty)$ by

$$f(u) = \sum_{n=1}^{\infty} a_n \phi_{[d\log(n),\infty)}(u).$$

By direct computation, we have

$$F(s) = \int_0^{\infty} f(u)e^{-su}du = \frac{1}{s}L(ds)$$

for $\mathrm{Re}(s) > 1$.

Let $L(s)^m(s-d)^l = Q(s)$. Since $L(s)^m$ has a pole of order $l$ at $s = d$, $Q(s)$ is holomorphic around $s = d$ and $Q(d) = A^m (\neq 0)$. So there exists a holomorphic function $P(s)$ defined on an open disc $D$ with center at $s = d$ such that $P(s)^m = Q(s)$. Since $L(s)(s-d)^{\frac{l}{m}}$ is holomorphic on the set $\mathrm{Re}(s) > d$, there exists an $m$-th root of unity $\zeta$ such that $L(s)(s-d)^{\frac{l}{m}} = \zeta \cdot P(s)$ on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > d\} \cap D$. Therefore, $L(s)(s-d)^{\frac{l}{m}}$ can be extended to a holomorphic function around $s = d$, and $\lim_{s\to d} L(s)(s-d)^{\frac{l}{m}} = A$ ($A \in \mathbb{R}_{>0}$) since $L(s)(s-d)^{\frac{l}{m}}$ has positive real values for $s \in \mathbb{R}_{>d}$. Hence, $\frac{1}{s}L(ds)(ds-d)^{\frac{l}{m}}$ is holomorphic around $s = 1$. Since

$$F(s)(s-1)^{\frac{l}{m}} = \frac{1}{s}L(ds)(s-1)^{\frac{l}{m}} = \frac{1}{sd^{\frac{l}{m}}}L(ds)(ds-d)^{\frac{l}{m}},$$

$F(s)(s-1)^{\frac{l}{m}}$ is holomorphic around $s = 1$.

Therefore, there exists a holomorphic function $B(s)$ defined on an open disc $D'$ with center at $s = 1$ such that

$$F(s)(s-1)^{\frac{l}{m}} = \sum_{i=0}^{r} A_i (s-1)^i + (s-1)^{r+1} B(s)\,,$$

where $r = \lceil l/m \rceil - 1$ and $A_i \in \mathbb{C}$ for $i = 0, 1, \ldots, r$. Then

$$\lim_{s \to 1} F(s)(s-1)^{\frac{l}{m}} = \frac{A}{d^{\frac{l}{m}}} \quad i.e., \quad A_0 = \frac{A}{d^{\frac{l}{m}}}\,.$$

Thus, we have

$$F(s) = \frac{A_0}{(s-1)^{\frac{l}{m}}} + \sum_{i=1}^{r} \frac{A_i}{(s-1)^{\frac{l}{m}-i}} + (s-1)^{r+1-\frac{l}{m}} B(s)$$

for $s \in D' \cap \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

Now, we define

$$\begin{aligned} G(s) &= F(s) - \frac{A_0}{(s-1)^{\frac{l}{m}}} - \sum_{i=1}^{r} \frac{A_i}{(s-1)^{\frac{l}{m}-i}} \\ &= \frac{1}{s} L(ds) - \frac{A_0}{(s-1)^{\frac{l}{m}}} - \sum_{i=1}^{r} \frac{A_i}{(s-1)^{\frac{l}{m}-i}} \end{aligned}$$

for $\mathrm{Re}(s) > 1$.

We shall prove the following (a), (b).

(a) The function $G(s)$ extends continuously to the set $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\}$.

(b) The function $x \mapsto G(1+ix)$ $(x \in \mathbb{R})$ is locally of bounded variation.

Then $f(s)$, $F(s)$, and $G(s)$ satisfy the condition of Theorem 2.1. It is enough to prove (a), (b) locally.

We first consider the neighborhood of $s = 1$. By definition

$$G(s) = (s-1)^{r+1-\frac{l}{m}} B(s)$$

for $s \in D' \cap \{s \in \mathbb{C} \mid \mathrm{Re}(s) > d\}$.

By Theorems 10.2 of [3, p.192], the function $x \mapsto B(1+ix)$ is of bounded variation on any closed-interval $[a, b]$, where $\{1+ix \in \mathbb{C} \mid x \in [a, b]\} \subset D'$. For any $\alpha > 0$ we can obviously extend $(s-1)^{\alpha}$ to a continuous function on $\mathrm{Re}(s) \geq 1$. By a similar argument as in [2, p.144], the function $x \mapsto \big((1+ix)-1\big)^{\alpha}$ $(x \in \mathbb{R})$ is locally of bounded variation.

Therefore, if $r + 1 - \frac{l}{m} > 0$, then $G(s)$ extends continuously to the set $D' \cap \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\}$ and the function $x \mapsto G(1+ix)$ $(x \in \mathbb{R})$ is of bounded variation on any closed-interval $[a, b]$, where $\{1+ix \in \mathbb{C} \mid x \in [a, b]\} \subset D'$. If $r + 1 - \frac{l}{m} = 0$, then $G(s)$ has the same properties since $G(s) = B(s)$ on $D' \cap \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$. Thus, (a), (b) are proved around $s = 1$.

Next we consider the neighborhood of $s = 1 + ix_0$, where $x_0 \in \mathbb{R} - \{0\}$. By the definition of $G(s)$, it is enough to prove (a), (b) for $L(ds)$ instead of $G(s)$.

Since $L^m$ is holomorphic around $s = d + ix_0$, there exists a holomorphic function $R(s)$ defined for $s$ near $d + ix_0$ such that

$$L(s)^m = \bigl(s - (d + ix_0)\bigr)^k R(s), \quad R(d + ix_0) \neq 0, \quad and \quad k \geq 0\,.$$

Since $R(d + ix_0) \neq 0$, there exists a holomorphic function $T(s)$ defined on an open disc $D''$ with center at $s = d + ix_0$ such that $T(s)^m = R(s)$. Then, there exists an $m$-th root of unity $\zeta$ such that

$$L(s) = \zeta \cdot \bigl(s - (d + ix_0)\bigr)^{\frac{k}{m}} T(s)$$

on $D'' \cap \{s \in \mathbb{C} \mid \mathrm{Re}(s) > d\}$.

Now, for any $x_0 \in \mathbb{R} - \{0\}$ we can extend $\bigl(s - (d+ix_0)\bigr)^{\frac{k}{m}}$ to a continuous function on the set $\mathrm{Re}(s) \geq d$ and the function $x \mapsto \bigl(d+ix-(d+ix_0)\bigr)^{\frac{k}{m}}$ is locally of bounded variation also as in [2, p.144]. Thus, $L(s)$ extends continuously to the set $D'' \cap \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq d\}$ and the function $x \mapsto L(d + ix)$ is of bounded variation around $x_0$. It follows that we can extend $L(ds)$ to a continuous function on the set $\mathrm{Re}(s) \geq 1$ except $s = 1$ and the function $x \mapsto L(d+idx)$ is of bounded variation on any closed-interval $[a, b]$ not containing $x = 0$. Thus, we have proved (a), (b).

Therefore, we can apply Theorem 2.1 to $f(s)$, $F(s)$, $G(s)$, and $\alpha = l/m$. So we have

$$\lim_{u\to\infty} u^{1-\frac{l}{m}} e^{-u} f(u) = \frac{A_0}{\Gamma(l/m)} = \frac{A}{d^{\frac{l}{m}}\Gamma(l/m)}\,. \tag{3.11}$$

Since $f(d \log X) = \sum_{n\leq X} a_n$, by substituting $d \log X$ for $u$ in (3.11), we have

$$\lim_{X\to\infty} (d \log X)^{1-\frac{l}{m}} e^{-(d \log X)} f(d \log X) = \frac{A}{d^{\frac{l}{m}}\Gamma(l/m)}\,,$$

$$\lim_{X\to\infty} d^{1-\frac{l}{m}} (\log X)^{1-\frac{l}{m}} X^{-d} \sum_{n\leq X} a_n = \frac{A}{d^{\frac{l}{m}}\Gamma(l/m)}\,.$$

Therefore,

$$\sum_{n\leq X} a_n \sim \frac{AX^d}{d\Gamma(l/m)(\log(X))^{1-\frac{l}{m}}}\,.$$

□

## 4. Examples

We give two examples of a Dirichlet series having a pole of order either "2/3" or "5/3" and apply Theorem 3.1.

If $n = {p_1}^{e_1} {p_2}^{e_2} \cdots {p_r}^{e_r}$ is a prime decomposition of a positive integer $n$ where $p_1, \ldots, p_r$ are distinct primes, then we define $\Omega(n) = \sum_{n=1}^{r} e_i$. We define a Dirichlet series $L$ by

$$L(s) = \prod_{p:prime} \left(1 - \frac{2}{3}p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{\Omega(n)} n^{-s}\,.$$

Then the function $L(s)$ is holomorphic on the set $\mathrm{Re}(s) > 1$ since $(2/3)^{\Omega(n)} \leq 1$, and

$$L'(s) = -\sum_{n=1}^{\infty} \log(n) \left(\frac{2}{3}\right)^{\Omega(n)} n^{-s}$$

for $\mathrm{Re}(s) > 1$. We shall apply Theorem 3.1 to $L(s)$ and $-L'(s)$.

First, we consider $L(s)$. We claim that $L(s)^3$ has a meromorphic continuation to an open set containing the closed half-plane $\mathrm{Re}(s) \geq 1$ and is holomorphic except for a pole of order 2 at $s = 1$.

By computation

$$L(s)^3 = \prod_{p:prime} \left(1 - 2p^{-s} + \frac{4}{3}p^{-2s} - \frac{8}{27}p^{-3s}\right)^{-1},$$

and

$$\begin{aligned}
\frac{L(s)^3}{\zeta(s)^2} &= \prod_{p:prime} \frac{(1-p^{-s})^2}{\left(1 - 2p^{-s} + \frac{4}{3}p^{-2s} - \frac{8}{27}p^{-3s}\right)} \\
&= \prod_{p:prime} \frac{(1-p^{-s})^2(1+p^{-s})^2}{\left(1 - 2p^{-s} + \frac{4}{3}p^{-2s} - \frac{8}{27}p^{-3s}\right)(1+p^{-s})^2} \\
&= \prod_{p:prime} \frac{(1-p^{-2s})^2}{\left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right)} \\
&= \zeta(2s)^{-2} \prod_{p:prime} \left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right)^{-1}.
\end{aligned}$$

Let

$$F(s) = \prod_{p:prime} \left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right).$$

If $\mathrm{Re}(s) \geq 0$, then we have

$$\left|-\frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right| \leq \frac{83}{27}|p^{-2s}|.$$

Thus, if $\mathrm{Re}(s) > 1/2$, then

$$\begin{aligned}
\sum_{p:prime} \left|-\frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right| &\leq \sum_{p:prime} \frac{83}{27}p^{-2\mathrm{Re}(s)} \\
&< +\infty.
\end{aligned}$$

Therefore,

$$\sum_{p:prime} \left|-\frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right|$$

converges absolutely and uniformly on any compact subset of the open half-plane $\mathrm{Re}(s) > 1/2$. It follows that

$$\prod_{p:prime}\left(1-\frac{5}{3}p^{-2s}+\frac{10}{27}p^{-3s}+\frac{20}{27}p^{-4s}-\frac{8}{27}p^{-5s}\right)$$

converges uniformly on any compact subset of the open half-plane $\mathrm{Re}(s) > 1/2$ and is holomorphic on $\mathrm{Re}(s) > 1/2$ (see [4, p.300, 15.6 Theorem ]). Thus, $F(s)$ is holomorphic on the set $\mathrm{Re}(s) > 1/2$.

For any prime number $p$, since $(1-\frac{2}{3}p^{-s})$, $(1+p^{-s}) \neq 0$ for $\mathrm{Re}(s) > 1/2$,

$$\left(1-\frac{5}{3}p^{-2s}+\frac{10}{27}p^{-3s}+\frac{20}{27}p^{-4s}-\frac{8}{27}p^{-5s}\right)=\left(1+p^{-s}\right)^2\left(1-\frac{2}{3}p^{-s}\right)^3 \neq 0$$

for $\mathrm{Re}(s) > 1/2$. Hence $F(s) \neq 0$ on the set $\mathrm{Re}(s) > 1/2$ i.e., $F(s)^{-1}$ is holomorphic on the set $\mathrm{Re}(s) > 1/2$ (see [4, p.300, 15.6 Theorem]). Moreover, since $F(s)$ is holomorphic, $F(s)^{-1}$ has no zeros for $\mathrm{Re}(s) > 1/2$.

It follows that $L(s)^3$ has a meromorphic continuation to $\mathrm{Re}(s) > 1/2$ and is holomorphic except for a pole of order 2 at $s = 1$. Moreover,

$$\begin{aligned}\lim_{s\to 1} L(s)^3(s-1)^2 &= \lim_{s\to 1}\frac{L(s)^3}{\zeta(s)^2}\left(\zeta(s)(s-1)\right)^2\\ &= \prod_{p:prime}\frac{(1-p^{-2})^2}{\left(1-\frac{5}{3}p^{-2}+\frac{10}{27}p^{-3}+\frac{20}{27}p^{-4}-\frac{8}{27}p^{-5}\right)}\\ &= \zeta(2)^{-2}F(1)^{-1}.\end{aligned}$$

By applying Theorem 3.1 to $L(s)$, we have

$$\sum_{n\le X}\left(\frac{2}{3}\right)^{\Omega(n)} \sim \frac{A}{\Gamma(2/3)}\cdot\frac{X}{\log(X)^{1/3}},$$

where

$$A=\left(\zeta(2)^2\prod_{p:prime}\left(1-\frac{5}{3}p^{-2}+\frac{10}{27}p^{-3}+\frac{20}{27}p^{-4}-\frac{8}{27}p^{-5}\right)\right)^{-\frac{1}{3}}.$$

Next, we consider $L'(s)$. By computation, we have

$$L'(s)^3=\frac{((L(s)^3)')^3}{27(L(s)^3)^2}.$$

Since $F(s)^{-1}$ has no zeros for $\mathrm{Re}(s) > 1/2$ and $\zeta(s)$ has no zeros for $\mathrm{Re}(s) \ge 1$, $L(s)^3 = \zeta(s)^2\zeta(2s)^{-1}F(s)^{-1}$ has no zeros for $\mathrm{Re}(s) \ge 1$. Hence $L(s)^{-3}$ is holomorphic on an open set containing the closed half-plane $\mathrm{Re}(s) \ge 1$. Since $L(s)^3$ is holomorphic for $\mathrm{Re}(s) > 1/2$ except for a pole of order 2 at $s = 1$, $(L(s)^3)'$ is holomorphic for $\mathrm{Re}(s) > 1/2$ except for a pole of order 3 at $s = 1$. It follows that $L'(s)^3$ has a meromorphic continuation to an open set containing the closed half-plane $\mathrm{Re}(s) \geq 1$ and is holomorphic except for a pole of order 5 at $s = 1$. Moreover,

$$\lim_{s\to 1}(L(s)^3)'(s-1)^3 = -3A^3$$

where $A = (\zeta(2)^2 F(1)^1)^{-1/3}$, and

$$\begin{aligned}\lim_{s\to 1} L'(s)^3(s-1)^5 &= \lim_{s\to 1} \frac{\left(\left(\left(L(s)^3\right)'\right)(s-1)^3\right)^3}{27\left((L(s)^3)(s-1)^2\right)^2} \\ &= \frac{(-3A^3)^3}{27A^6} \\ &= -A^3 .\end{aligned}$$

By applying Theorem 3.1 to $-L'(s)$, we have

$$\sum_{n\leq X} \log(n)\left(\frac{2}{3}\right)^{\Omega(n)} \sim \frac{A}{\Gamma(5/3)} \cdot X\log(X)^{2/3} .$$

It turns out that we can also apply Delange's result to these two examples. In the first example, $L(s)^3$ has no zeros on the line $\mathrm{Re}(s) = 1$ because $L(s)^3/\zeta(s)^2$ has no zeros for $\mathrm{Re}(s) > 1/2$ and $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$. Therefore, we can apply Delange's result to $L(s)$. In the second example, the order of all zeros of $L'(s)^3 = ((L(s)^3)')^3/27(L(s)^3)^2$ on the line $\mathrm{Re}(s) = 1$ is divisible by 3. Therefore, Delange's result is applicable to this case also.

So these are not exactly examples where Delange's result is not applicable. However, we do not have to worry about the zeros of the Dirichlet series, which is somewhat convenient. Therefore, we included these two examples.

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