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Toda Lattice Hierarchy and Goldstein-Petrich Flows for Plane Curves

by

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Abstract. A relation between the Goldstein-Petrich hierarchy for plane curves and the Toda lattice hierarchy is investigated. A representation formula for plane curves is given in terms of a special class of τ-functions of the Toda lattice hierarchy. A representation formula for discretized plane curves is also discussed.

1. Introduction

Intimate connection between integrable systems and differential geometry of curves and surfaces has been important topic of intense research [1, 23]. Goldstein and Petrich introduced a hierarchy of commuting flows for plane curves that is related to the modified Korteweg-de Vries (mKdV) hierarchy [6]. The second Goldstein-Petrich flow is defined by the modified Korteweg-de Vries equation,

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^3 \kappa}{\partial x^3} + \frac{3}{2}\kappa^2 \frac{\partial \kappa}{\partial x}, \tag{1.1}$$

where $\kappa = \kappa(x,t)$ denotes the curvature and x is the arc-length. This result has been extended and investigated from various viewpoints [3, 4, 5, 9, 10, 11, 16, 21, 22]. In [9, 10], a representation formula for curve motion in terms of the τ function with respect to the second Goldstein-Petrich flow has been presented by means of the Hirota bilinear formulation and determinant expression of solutions. The aim of this article is to generalize the results in [9, 10] to the whole hierarchy. We will show how the Goldstein-Petrich hierarchy is embedded in the Toda lattice hierarchy[24, 28]. We remark that the semi-discrete case, discussed in [10], is not considered in this paper.

An advantage of infinite hierarchical formulation is its relation to integrable discretization. Miwa showed that Hirota's discrete Toda equation [7] can be obtained by applying a change of coordinate to the KP hierarchy [12, 19, 24]. Using a generalization of Miwa's approach, we will show that Matsuura's discretized curve motion [18] can be obtained also from the Toda lattice hierarchy. Another merit of the KP theoretic formulation is Lie algebraic aspect of the hierarchy [12, 20]. We will discuss a relationship between the Goldstein-Petrich hierarchy and a real form of the affine Lie algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$.

2. Goldstein-Petrich flows for Euclidean plane curves

We assume that $\mathbf{r}(x) = {}^t(X(x), Y(x))$ is a curve in Euclidean plane $\mathbb{R}^2$, parameterized by the arc-length x. Define the tangent vector $\hat{\mathbf{t}}$ and the unit normal $\hat{\mathbf{n}}$ by

$$\hat{\mathbf{t}} = \mathbf{r}_x\,, \quad \hat{\mathbf{n}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{\mathbf{t}}\,. \tag{2.1}$$

Here the subscript x indicates differentiation. The Frenet equation for $\mathbf{r}$ is given by

$$\hat{\mathbf{t}}_x = \kappa \hat{\mathbf{n}}\,, \quad \hat{\mathbf{n}}_x = -\kappa \hat{\mathbf{t}}\,, \tag{2.2}$$

where κ is the curvature of the curve $\mathbf{r}$. Goldstein and Petrich [6] considered dynamics of a plane curve described by the equation of the form

$$\frac{\partial \mathbf{r}}{\partial t_n} = f^{(n)} \hat{\mathbf{n}} + g^{(n)} \hat{\mathbf{t}}\,. \tag{2.3}$$

The coefficients $f^{(n)} = f^{(n)}(x,t)$, $g^{(n)} = g^{(n)}(x,t)$ $(t = (t_1, t_2, t_3, \ldots))$ are differential polynomials in κ. We remark that our choice of signature in (2.2) is different from that of [6]. Following the discussion in [6], we choose $f^{(n)}(x,t)$, $g^{(n)}(x,t)$ as

$$\begin{aligned} &f^{(1)} = 0\,, \quad g^{(1)} = 1\,, \quad f^{(2)} = \kappa_x\,, \quad g^{(2)} = \kappa^2/2\,, \\ &g_x^{(n)} = \kappa f^{(n)}\,, \quad f^{(n+1)} = \left(f_x^{(n)} + \kappa g^{(n)} \right)_x\,. \end{aligned} \tag{2.4}$$

We call as Goldstein-Petrich hierarchy the equations defined by (2.1), (2.2), (2.3) and (2.4).

Applying the condition (2.4) to (2.3), we obtain

$$\frac{\partial \hat{\mathbf{t}}}{\partial t_n} = \left(f_x^{(n)} + \kappa g^{(n)} \right) \hat{\mathbf{n}}\,, \quad \frac{\partial \hat{\mathbf{n}}}{\partial t_n} = -\left(f_x^{(n)} + \kappa g^{(n)} \right) \hat{\mathbf{t}}\,. \tag{2.5}$$

The compatibility condition for (2.2) and (2.5) is reduced to

$$\frac{\partial \kappa}{\partial t_n} = \left(f_x^{(n)} + \kappa g^{(n)} \right)_x = f^{(n+1)}\,. \tag{2.6}$$

The case $n = 2$ of (2.6) gives the mKdV equation (1.1). One finds that

$$f^{(n)} = \Omega f^{(n-1)}\,, \quad \Omega = \partial_x^2 + \kappa^2 + \kappa_x \partial_x^{-1} \circ \kappa\,. \tag{2.7}$$

We remark that the operator Ω is the recursion operator for the modified KdV hierarchy [2].

We now introduce complex coordinate via a map $\rho : \mathbb{R}^2 \to \mathbb{C}$ given by

$$\rho(X, Y) = X + \sqrt{-1}\, Y\,. \tag{2.8}$$

and define Z, T, N as

$$Z = \rho(\mathbf{r})\,, \quad T = \rho(\hat{\mathbf{t}})\,, \quad N = \rho(\hat{\mathbf{n}}) = \sqrt{-1}\, T\,. \tag{2.9}$$

Since $|\hat{\mathbf{t}}| = |\hat{\mathbf{n}}| = 1$, the complex variables T and N satisfy $|T| = |N| = 1$. The equations (2.1), (2.2), (2.3) are rewritten as

$$T = Z_x\,, \quad T_x = \sqrt{-1}\kappa T\,, \quad \frac{\partial Z}{\partial t_n} = \left(g^{(n)} + \sqrt{-1}\, f^{(n)} \right) T\,. \tag{2.10}$$

3. Toda lattice hierarchy

In this section, we briefly review the theory of Toda lattice hierarchy using the language of difference operators [24, 28] (See also [13, 25, 26]). We denote as e^{∂_s} the shift operator with respect to s: $e^{\partial_s} f(s) = f(s+1)$. For a difference operator $A(s) = \sum_{-\infty<j<+\infty} a_j(s)e^{j\partial_s}$, we define the non-negative and negative part of $A(s)$ as

$$(A(s))_{\geq 0} = \sum_{0\leq j<+\infty} a_j(s)e^{j\partial_s} , \quad (A(s))_{<0} = \sum_{-\infty<j<0} a_j(s)e^{j\partial_s} . \tag{3.1}$$

Let $L^{(\infty)}(s)$, $L^{(0)}(s)$ be difference operators of the form

$$L^{(\infty)}(s) = e^{\partial_s} + \sum_{-\infty<j\leq 0} b_j(s)e^{j\partial_s} , \quad L^{(0)}(s) = \sum_{-1\leq j<+\infty} c_j(s)e^{j\partial_s} , \tag{3.2}$$

where we assume $c_{-1}(s) \neq 0$ for any s. We introduce two sets of infinitely many variables $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and define the weight of the variables as

$$\text{weight}(x_n) = n , \quad \text{weight}(y_n) = -n \quad (n = 1, 2, \ldots) . \tag{3.3}$$

Each coefficient of $L^{(\infty)}(s)$, $L^{(0)}(s)$ is a function of x, y, i.e. $b_j(s) = b_j(s; x, y)$, $c_j(s) = c_j(s; x, y)$. The Toda lattice hierarchy is defined as the following set of differential equations of Lax-type:

$$\begin{aligned} &\frac{\partial L^{(\infty)}(s)}{\partial x_n} = \left[B_n(s),\ L^{(\infty)}(s)\right] , \quad \frac{\partial L^{(0)}(s)}{\partial x_n} = \left[B_n(s),\ L^{(0)}(s)\right] , \\ &B_n(s) = \left(L^{(\infty)}(s)^n\right)_{\geq 0} \quad (n = 1, 2, 3, \ldots) , \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{\partial L^{(\infty)}(s)}{\partial y_n} = \left[C_n(s),\ L^{(\infty)}(s)\right] , \quad \frac{\partial L^{(0)}(s)}{\partial y_n} = \left[C_n(s),\ L^{(0)}(s)\right] , \\ &C_n(s) = \left(L^{(0)}(s)^n\right)_{<0} \quad (n = 1, 2, 3, \ldots) . \end{aligned} \tag{3.5}$$

PROPOSITION 1 ([28], Proposition 1.4). *Let $L^{(\infty)}$, $L^{(0)}$ be difference operators of the form* (3.2) *and satisfy the differential equations* (3.4), (3.5). *Then there exist difference operators $\hat{W}^{(\infty)}(s)$, $\hat{W}^{(0)}(s)$ of the form,*

$$\begin{aligned} \hat{W}^{(\infty)}(s) &= 1 + \sum_{j=1}^{\infty} \hat{w}_j^{(\infty)}(s)e^{-j\partial_s} , \\ \hat{W}^{(0)}(s) &= \sum_{j=0}^{\infty} \hat{w}_j^{(0)}(s)e^{j\partial_s} \quad (\hat{w}_0^{(0)}(s) \neq 0) , \end{aligned} \tag{3.6}$$

satisfying the following equations:

$$\begin{aligned} L^{(\infty)}(s) &= \hat{W}^{(\infty)}(s)e^{\partial_s}\hat{W}^{(\infty)}(s)^{-1} , \\ L^{(0)}(s) &= \hat{W}^{(0)}(s)e^{-\partial_s}\hat{W}^{(0)}(s)^{-1} , \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
\frac{\partial \hat{W}^{(\infty)}(s)}{\partial x_n} &= B_n(s)\hat{W}^{(\infty)}(s) - \hat{W}^{(\infty)}(s)e^{n\partial_s}\,,\\
\frac{\partial \hat{W}^{(\infty)}(s)}{\partial y_n} &= C_n(s)\hat{W}^{(\infty)}(s)\,,\\
\frac{\partial \hat{W}^{(0)}(s)}{\partial x_n} &= B_n(s)\hat{W}^{(0)}(s)\,,\\
\frac{\partial \hat{W}^{(0)}(s)}{\partial y_n} &= C_n(s)\hat{W}^{(0)}(s) - \hat{W}^{(0)}(s)e^{-n\partial_s}\,.
\end{aligned}
\tag{3.8}
$$

PROPOSITION 2 ([28], (1.2.18)). *The difference operators* $\hat{W}^{(\infty)}(s)$, $\hat{W}^{(0)}(s)$ *in Proposition* 1 *satisfy*

$$
\begin{aligned}
&\hat{W}^{(\infty)}(s; x', y')\exp\left[\sum_{n=1}^{\infty}(x'_n - x_n)e^{n\partial_s}\right]\hat{W}^{(\infty)}(s; x, y)^{-1}\\
&\quad = \hat{W}^{(0)}(s; x', y')\exp\left[\sum_{n=1}^{\infty}(y'_n - y_n)e^{-n\partial_s}\right]\hat{W}^{(0)}(s; x, y)^{-1}
\end{aligned}
\tag{3.9}
$$

for any x, x', y, y' *and any integer* s.

Define $\hat{w}_j^{(\infty)*}(s; x, y)$, $\hat{w}_j^{(0)*}(s; x, y)$ by expanding $\hat{W}^{(\infty)}(s; x, y)^{-1}$, $\hat{W}^{(0)}(s; x, y)^{-1}$ with respect to e^{∂_s}:

$$
\begin{aligned}
\hat{W}_j^{(\infty)}(s; x, y)^{-1} &= \sum_{j=0}^{\infty} e^{-j\partial_s}\hat{w}_j^{(\infty)*}(s+1; x, y)\,,\\
\hat{W}^{(0)}(s; x, y)^{-1} &= \sum_{j=0}^{\infty} e^{j\partial_s}\hat{w}_j^{(0)*}(s+1; x, y)\,.
\end{aligned}
\tag{3.10}
$$

From (3.6), (3.7) and (3.10), we obtain

$$
\begin{aligned}
b_0(s) &= \hat{w}_1^{(\infty)}(s) + \hat{w}_1^{(\infty)*}(s+1) = \hat{w}_1^{(\infty)}(s) - \hat{w}_1^{(\infty)}(s+1)\,,\\
b_{-n}(s) &= \hat{w}_{n+1}^{(\infty)}(s) + \hat{w}_{n+1}^{(\infty)*}(s+1-n) + \sum_{j=1}^{n}\hat{w}_j^{(\infty)}(s)\hat{w}_{n+1-j}^{(\infty)*}(s+1-n) \quad (n \geq 1)\,,\\
c_n(s) &= \sum_{j=0}^{n+1}\hat{w}_j^{(0)}(s)\hat{w}_{n-j+1}^{(0)*}(s+n+1) \quad (n \geq -1)\,.
\end{aligned}
\tag{3.11}
$$

THEOREM 3 ([28], Theorem 1.7). *There exists a function $\tau(s) = \tau(s; x, y)$ satisfying*

$$
\begin{aligned}
\hat{w}_j^{(\infty)}(s; x, y) &= \frac{p_j(-\tilde{\partial}_x)\tau(s; x, y)}{\tau(s; x, y)}, \\
\hat{w}_j^{(0)}(s; x, y) &= \frac{p_j(-\tilde{\partial}_y)\tau(s+1; x, y)}{\tau(s; x, y)}, \\
\hat{w}_j^{(\infty)*}(s; x, y) &= \frac{p_j(\tilde{\partial}_x)\tau(s; x, y)}{\tau(s; x, y)}, \\
\hat{w}_j^{(0)*}(s; x, y) &= \frac{p_j(\tilde{\partial}_y)\tau(s-1; x, y)}{\tau(s; x, y)}
\end{aligned}
\tag{3.12}
$$

where $\tilde{\partial}_x = \left(\partial_{x_1}, \partial_{x_2}/2, \partial_{x_3}/3, \ldots\right)$, $\tilde{\partial}_y = \left(\partial_{y_1}, \partial_{y_2}/2, \partial_{y_3}/3, \ldots\right)$, *and the polynomials* $p_n(t)$ $(n = 0, 1, 2, \ldots)$ *are defined by*

$$
\xi(t, \lambda) = \exp\left[\sum_{j=1}^{\infty} t_n \lambda^j\right] = \sum_{n=0}^{\infty} p_n(t)\lambda^n, \quad t = (t_1, t_2, \ldots). \tag{3.13}
$$

Furthermore, the τ-function $\tau(s; x, y)$ of the Toda lattice hierarchy is determined uniquely by (3.12) *up to a constant multiple factor.*

It follows that

$$
\begin{aligned}
c_{-1}(s) &= \hat{w}_0^{(0)}(s)\hat{w}_0^{(0)*}(s) = \frac{\tau(s+1)\tau(s-1)}{\tau(s)^2}, \\
c_0(s) &= \hat{w}_0^{(0)}(s)\hat{w}_1^{(0)*}(s+1) + \hat{w}_1^{(0)}(s)\hat{w}_0^{(0)*}(s+1) = \frac{\partial}{\partial y_1}\log\frac{\tau(s)}{\tau(s+1)}.
\end{aligned}
\tag{3.14}
$$

THEOREM 4 ([28], Theorem 1.11). *τ-functions of Toda lattice hierarchy satisfy the following equation (bilinear identity):*

$$
\begin{aligned}
&\oint \tau(s'; x' - [\lambda^{-1}], y')\tau(s; x + [\lambda^{-1}], y) e^{\xi(x'-x,\lambda)}\lambda^{s'-s} d\lambda \\
&= \oint \tau(s'+1; x', y' - [\lambda])\tau(s-1; x, y + [\lambda]) e^{\xi(y'-y,\lambda^{-1})}\lambda^{s'-s} d\lambda,
\end{aligned}
\tag{3.15}
$$

where $[\lambda] = \left(\lambda, \lambda^2/2, \lambda^3/3, \ldots\right)$, *and we have used the notation of formal residue,*

$$
\oint \left(\sum_n a_n \lambda^n\right) d\lambda = 2\pi\sqrt{-1}\, a_{-1}. \tag{3.16}
$$

Conversely, if $\tau(s; x, y)$ solves the bilinear identity (3.15), *then* $\hat{W}^{(\infty)}(s; x, y)$ *and* $\hat{W}^{(0)}(s; x, y)$ *defined by* (3.6) *and* (3.12) *satisfy* (3.8).

4. Time-flows with negative weight with 2-reduction condition

4.1. Reduction to Goldstein-Petrich hierarchy

We now impose the 2-reduction condition[28]

$$L^{(\infty)}(s)^2 = e^{2\partial_s}\,, \quad L^{(0)}(s)^2 = e^{-2\partial_s}\,, \tag{4.1}$$

that implies

$$W^{(\infty)}(s+2) = W^{(\infty)}(s)\,, \quad W^{(0)}(s+2) = W^{(0)}(s)\,, \tag{4.2}$$

$$L^{(\infty)}(s+2) = L^{(\infty)}(s)\,, \quad L^{(0)}(s+2) = L^{(0)}(s)\,. \tag{4.3}$$

PROPOSITION 5 ([28], Proposition 1.13). *Let $L^{(\infty)}(s;x,y)$, $L^{(0)}(s;x,y)$ be solutions to the Toda lattice hierarchy* (3.4), (3.5), *which satisfy the 2-reduction conditions* (4.1)*. Then one finds that*

$$\frac{\partial L^{(\infty)}}{\partial x_{2n}} = \frac{\partial L^{(0)}}{\partial x_{2n}} = \frac{\partial L^{(\infty)}}{\partial y_{2n}} = \frac{\partial L^{(0)}}{\partial y_{2n}} = 0 \tag{4.4}$$

for $n = 1, 2, \ldots$.

PROPOSITION 6 ([28], Corollary 1.14). *Suppose $L^{(\infty)}(s;x,y)$, $L^{(0)}(s;x,y)$ be solutions to the Toda lattice hierarchy* (3.4), (3.5), *which satisfy the 2-reduction conditions* (4.1)*. Then there exist suitable difference operators $\hat{W}^{(\infty)}(s;x,y)$, $\hat{W}^{(0)}(s;x,y)$ such that the corresponding τ functions subject to the following conditions*:

$$\begin{aligned}
&\tau(s;x,y) = \tau'(s;x,y)\exp\left(-\sum_{n=1}^{\infty} n x_n y_n\right)\,, \\
&\tau'(s+2;x,y) = \tau'(s;x,y)\,, \\
&\frac{\partial \tau'(s;x,y)}{\partial x_{2n}} = \frac{\partial \tau'(s;x,y)}{\partial y_{2n}} = 0 \quad (n = 1, 2, \ldots)\,.
\end{aligned} \tag{4.5}$$

We consider the time-evolutions with respect to the variables with negative weight $y = (y_1, y_2, \ldots)$ under the 2-reduction condition (4.1). In this case, one can write down the difference operators $C_n(s)$ $(n = 1, 2, \ldots)$ explicitly:

$$C_{2n}(s) = e^{-2n\partial_s}\,, \quad C_{2n-1} = \sum_{j=-1}^{2n-3} c_j(s) e^{(j-2n)\partial_s}\,. \tag{4.6}$$

Applying (4.6) to (3.5) and (3.8), we obtain the following equations $(n = 0, 1, 2, \ldots)$:

$$\frac{\partial w_1(s)}{\partial y_{2n+1}} = c_{2n-1}(s) \tag{4.7}$$

$$\frac{\partial c_{2n}(s)}{\partial y_1} = \frac{\partial c_0(s)}{\partial y_{2n+1}} = c_{-1}(s)c_{2n+1}(s+1) - c_{-1}(s+1)c_{2n+1}(s)\,, \tag{4.8}$$

$$\frac{\partial c_{-1}(s)}{\partial y_{2n+1}} = \frac{\partial c_{2n-1}(s)}{\partial y_1} = c_{-1}(s)\left\{c_{2n}(s+1) - c_{2n}(s)\right\}\,, \tag{4.9}$$

where we have used the property $c_j(s+2) = c_j(s)$.

PROPOSITION 7. *For $n = 0, 1, 2, \dots$, the coefficients $c_n(s; x, y)$ can be represented by $c_{-1}(s; x, y)$. For example, $c_0(s; x, y)$ and $c_1(s; x, y)$ can be written as*

$$\begin{aligned} c_0(s) &= -\frac{1}{2c_{-1}(s)}\frac{\partial c_{-1}(s)}{\partial y_1} = -\frac{1}{2}\frac{\partial}{\partial y_1}\log c_{-1}(s)\,, \\ c_1(s) &= -\frac{c_{-1}(s)}{2}\left\{c_0(s)^2 + \frac{\partial c_0(s)}{\partial y_1}\right\} \\ &= -\frac{c_{-1}(s)}{8}\left[\left\{\frac{\partial}{\partial y_1}\log c_{-1}(s)\right\}^2 - 2\frac{\partial^2}{\partial y_1^2}\log c_{-1}(s)\right]. \end{aligned} \tag{4.10}$$

Proof. From (3.2) and (4.1), we have

$$\begin{aligned} &c_{-1}(s)c_{-1}(s-1) = 1\,, \quad c_0(s) + c_0(s-1) = 0\,, \\ &c_{-1}(s)c_{k+1}(s-1) + c_{-1}(s+k+1)c_{k+1}(s) + \sum_{j=0}^{k} c_j(s)c_{k-j}(s+j) = 0\,. \end{aligned} \tag{4.11}$$

The desired result can be obtained from (4.8), (4.9) and (4.11). □

REMARK. Under the 2-reduction conditions (4.1), the map

$$\sum_{n\in\mathbb{Z}} a_n(s)e^{n\partial_s} \mapsto \sum_{n\in\mathbb{Z}} \begin{bmatrix} a_n(0) & 0 \\ 0 & a_n(1) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \zeta^2 & 0 \end{bmatrix}^n \tag{4.12}$$

gives an algebra isomorphism [28]. For example, the operators $C_1(s)$, $C_3(s)$ are mapped as follows:

$$\begin{aligned} C_1(s) &\mapsto \begin{bmatrix} c_{-1}(0) & 0 \\ 0 & c_{-1}(1) \end{bmatrix}\begin{bmatrix} 0 & \zeta^{-2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_{-1}(0)\zeta^{-2} \\ 1/c_{-1}(0) & 0 \end{bmatrix}, \\ C_3(s) &\mapsto \begin{bmatrix} c_0(0)\zeta^{-2} & c_{-1}(0)\zeta^{-4} + c_1(0)\zeta^{-2} \\ \zeta^{-2}/c_{-1}(0) + c_1(1) & -c_0(0)\zeta^{-2} \end{bmatrix}. \end{aligned} \tag{4.13}$$

Applying this isomorphism to the equations (3.4), (3.5), one obtains the Lax equations of 2×2-matrix form.

For $n = 0, 1, 2, \dots$, define $F^{(n)}(s)$ and $G^{(n)}(s)$ as

$$\begin{aligned} F^{(n)}(s) &= \frac{1}{2}\left\{c_{-1}(s+1)c_{2n-1}(s) - c_{-1}(s)c_{2n-1}(s+1)\right\}\,, \\ G^{(n)}(s) &= \frac{1}{2}\left\{c_{-1}(s+1)c_{2n-1}(s) + c_{-1}(s)c_{2n-1}(s+1)\right\}\,. \end{aligned} \tag{4.14}$$

From (4.7), (4.11) and (4.14), we have

$$\frac{\partial w_1(s)}{\partial y_{2n+1}} = \frac{F^{(n)}(s) + G^{(n)}(s)}{c_{-1}(s+1)} = c_{-1}(s)\left\{F^{(n)}(s) + G^{(n)}(s)\right\}\,. \tag{4.15}$$

It is straightforward to show that

$$\begin{aligned} \frac{\partial F^{(n)}(s)}{\partial y_1} &= 2c_0(s)G^{(n)}(s) + c_{2n}(s+1) - c_{2n}(s)\,, \\ \frac{\partial G^{(n)}(s)}{\partial y_1} &= 2c_0(s)F^{(n)}(s)\,. \end{aligned} \tag{4.16}$$

Next we consider reality condition. Assume $x_j, y_j \in \mathbb{R}$ ($j = 1, 2, \ldots$) and that the τ-function $\tau(s; x, y)$ satisfies

$$\overline{\tau(s; x, y)} = \tau(s+1; x, y)\,, \tag{4.17}$$

where $\overline{\cdot}$ denotes complex conjugation. Under this condition, the following relations hold:

$$\begin{aligned} &\overline{\hat{w}_j^{(\infty)}(s)} = \hat{w}_j^{(\infty)}(s+1)\,, \quad \overline{\hat{w}_j^{(0)}(s)} = \hat{w}_j^{(0)}(s+1)\,, \\ &\overline{b_{-n}(s)} = b_{-n}(s+1)\,, \quad \overline{c_n(s)} = c_n(s+1)\,, \\ &\overline{F^{(n)}(s)} = -F^{(n)}(s)\,, \quad \overline{G^{(n)}(s)} = G^{(n)}(s)\,. \end{aligned} \tag{4.18}$$

Furthermore, it follows from (3.14) that

$$c_{-1}(s)\,\overline{c_{-1}(s)} = 1\,, \quad c_0(s) + \overline{c_0(s)} = 0\,. \tag{4.19}$$

THEOREM 8 (Representation formula in terms of the τ-functions). *If we set*

$$\begin{aligned} &x = 2y_1\,, \quad t_n = 2y_{2n-1} \quad (n = 1, 2, \ldots)\,, \\ &Z = \hat{w}_1^{(\infty)}(s=0; x, y) = -\frac{\partial}{\partial x_1}\log\tau(0; x, y)\,, \\ &T = \frac{1}{2}c_{-1}(s=0; x, y) = \frac{\tau(1; x, y)^2}{2\tau(0; x, y)^2}\,, \\ &\kappa = \sqrt{-1}\,c_0(s=0; x, y) = \sqrt{-1}\,\frac{\partial}{\partial y_1}\log\frac{\tau(0; x, y)}{\tau(1; x, y)}\,, \\ &f^{(n)} = -\sqrt{-1}\,F^{(n-1)}(s=0)\,, \quad g^{(n)} = G^{(n-1)}(s=0)\,, \end{aligned} \tag{4.20}$$

then $Z, T, \kappa, f^{(n)}, g^{(n)}$ *solve the equations* (2.4), (2.10).

Proof. The first equation of (2.10) follows from (4.7). The second and the third are obtained from (4.9), (4.15). The recurrence relations (2.4) follows from (4.8), (4.14) and (4.16). □

4.2. Discrete mKdV flow on discrete curves

We recall a discrete analogue of the mKdV-flow of plane curve introduced by Matsuura [18]. Let $\gamma_n^m : \mathbb{Z}^2 \to \mathbb{C}$ be a map describing the discrete motion of discrete plane curve with segment length a_n:

$$\left|\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}\right| = 1\,,$$
$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = e^{\sqrt{-1}K_n^m}\frac{\gamma_n^m - \gamma_{n-1}^m}{a_{n-1}}\,, \tag{4.21}$$
$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_n} = e^{\sqrt{-1}W_n^m}\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}\,.$$

The compatibility condition for (4.21) implies the existence of the function θ_n^m defined by

$$W_n^m = \frac{\theta_n^{m+1} - \theta_{n+1}^m}{2}\,, \quad K_n^m = \frac{\theta_{n+1}^m - \theta_{n-1}^m}{2}\,. \tag{4.22}$$

Then the isoperimetric condition (the first equation in (4.21)) implies that θ_n^m satisfies the discrete potential mKdV equation [8]

$$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{2}\right) = \frac{b_m + a_n}{b_m - a_n}\tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{2}\right)\,. \tag{4.23}$$

In what follows, we will show that the equations (4.21) can be obtained from the Toda lattice hierarchy. We introduce discrete variables $m, n \in \mathbb{Z}$ and assume $\tilde{y}_k$ depends on m, n as

$$\tilde{y}_k(m,n) = -\sum_{n'}^{n-1}\frac{a_{n'}^k}{k} - \sum_{m'}^{m-1}\frac{b_{m'}^k}{k} \quad (k = 1, 2, 3, \ldots)\,, \tag{4.24}$$

which is a non-autonomous version of Miwa transformation [30]. We remark that if $a_n = a$ and $b_m = b$ for any n, m then (4.24) is reduced to original Miwa transformation [19]:

$$\tilde{y}_k(m,n) = -\frac{na^k}{k} - \frac{mb^k}{k} \quad (k = 1, 2, 3, \ldots)\,. \tag{4.25}$$

To consider the dependence on m, n, we use the following abbreviation:

$$\begin{aligned} \hat{W}^{(\infty)}(s; m, n) &= \hat{W}^{(\infty)}(s; x, y = \tilde{y}(m,n)),\\ \hat{W}^{(0)}(s; m, n) &= \hat{W}^{(0)}(s; x, y = \tilde{y}(m,n))\,. \end{aligned} \tag{4.26}$$

PROPOSITION 9. *$\hat{W}^{(\infty)}(s; m, n)$ and $\hat{W}^{(0)}(s; m, n)$ satisfy*

$$\begin{aligned} &\hat{W}^{(\infty)}(s; m, n+1) = \left\{1 - a_n\tilde{u}(s; m, n)e^{-\partial_s}\right\}\hat{W}^{(\infty)}(s; m, n)\,,\\ &\hat{W}^{(0)}(s; m, n+1)\left(1 - a_n e^{-\partial_s}\right) = \left\{1 - a_n\tilde{u}(s; m, n)e^{-\partial_s}\right\}\hat{W}^{(0)}(s; m, n)\,,\\ &\hat{W}^{(\infty)}(s; m+1, n) = \left\{1 - b_m\tilde{v}(s; m, n)e^{-\partial_s}\right\}\hat{W}^{(\infty)}(s; m, n)\,,\\ &\hat{W}^{(0)}(s; m+1, n)\left(1 - b_m e^{-\partial_s}\right) = \left\{1 - b_m\tilde{v}(s; m, n)e^{-\partial_s}\right\}\hat{W}^{(0)}(s; m, n)\,, \end{aligned} \tag{4.27}$$

where

$$
\begin{aligned}
\tilde{u}(s;m,n) &= \frac{\hat{w}_0^{(0)}(s;m,n+1)}{\hat{w}_0^{(0)}(s-1;m,n)} = \frac{\tau(s-1;m,n)\tau(s+1;m,n+1)}{\tau(s;m,n)\tau(s;m,n+1)}\,,\\
\tilde{v}(s;m,n) &= \frac{\hat{w}_0^{(0)}(s;m+1,n)}{\hat{w}_0^{(0)}(s-1;m,n)} = \frac{\tau(s-1;m,n)\tau(s+1;m+1,n)}{\tau(s;m,n)\tau(s;m+1,n)}\,.
\end{aligned}
\tag{4.28}
$$

Proof. Setting $x'_k = x_k$, $y'_k = \tilde{y}(m,n+1)$, $y_k = \tilde{y}(m,n)$ $(k = 1,2,\ldots)$ in (3.9), we have

$$
\begin{aligned}
&\hat{W}^{(\infty)}(s;m,n+1)\hat{W}^{(\infty)}(s;m,n)^{-1}\\
&\quad = \hat{W}^{(0)}(s;m,n+1)\left(1 - a_n e^{-\partial_s}\right)\hat{W}^{(0)}(s;m,n)^{-1}\,,
\end{aligned}
\tag{4.29}
$$

where we have used the formula $\exp\big(-\sum_{n=0}^{\infty} z^n/n\big) = 1 - z$. Since the left-hand side of (4.29) is of non-positive order with respect to e^{∂_s}, it follows that it is of the form

$$
(4.29) = \tilde{c}_0(s;m,n) + \tilde{c}_{-1}(s;m,n)e^{-\partial_s}\,. \tag{4.30}
$$

Inserting $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ of (3.6) to (4.29) with (4.30), we obtain the first and the second equation of (4.27). The third and the fourth can be obtained in the same fashion. □

REMARK. Tsujimoto [27] proposed and investigated the equations (4.27) as a discrete analogue of (3.8). In our approach, the results in [27] can be obtained directly from (3.9) with the Miwa transformation.

Hereafter in this section, we impose the 2-reduction condition $\tau(s+2;m,n) = \tau(s;m,n)$. From the first and the third equations of (4.27), we obtain

$$
\begin{aligned}
\hat{w}_1^{(\infty)}(s;m,n+1) &= \hat{w}_1^{(\infty)}(s;m,n) - a_n\tilde{u}(s;m,n)\,,\\
\hat{w}_1^{(\infty)}(s;m+1,n) &= \hat{w}_1^{(\infty)}(s;m,n) - b_m\tilde{v}(s;m,n)\,.
\end{aligned}
\tag{4.31}
$$

It follows that

$$
\begin{aligned}
&\frac{\hat{w}_1^{(\infty)}(s;m,n+1) - \hat{w}_1^{(\infty)}(s;m,n)}{a_n}\\
&\quad = \mathcal{K}(s;m,n)\frac{\hat{w}_1^{(\infty)}(s;m,n) - \hat{w}_1^{(\infty)}(s;m,n-1)}{a_{n-1}}\,,\\
&\frac{\hat{w}_1^{(\infty)}(s;m+1,n) - \hat{w}_1^{(\infty)}(s;m,n)}{b_m}\\
&\quad = \mathcal{W}(s;m,n)\frac{\hat{w}_1^{(\infty)}(s;m,n+1) - \hat{w}_1^{(\infty)}(s;m,n)}{a_n}\,,
\end{aligned}
\tag{4.32}
$$

with

$$
\begin{aligned}
\mathcal{K}(s;m,n) &= \frac{\tilde{u}(s;m,n)}{\tilde{u}(s;m,n-1)} = \frac{\tau(s+1;m,n+1)\tau(s;m,n-1)}{\tau(s;m,n+1)\tau(s+1;m,n-1)}\,,\\
\mathcal{W}(s;m,n) &= \frac{\tilde{v}(s;m,n)}{\tilde{u}(s;m,n)} = \frac{\tau(s+1;m+1,n)\tau(s;m,n+1)}{\tau(s;m+1,n)\tau(s+1;m,n+1)}\,.
\end{aligned}
\tag{4.33}
$$

If we introduce $\Theta(s; m, n)$ as

$$\Theta(s; m, n) = \tau(s + 1; m, n)/\tau(s; m, n)\,, \tag{4.34}$$

then $\mathcal{K}(s; m, n)$ and $\mathcal{W}(s; m, n)$ are written as

$$\mathcal{K}(s; m, n) = \frac{\Theta(s; m, n + 1)}{\Theta(s; m, n - 1)}\,, \quad \mathcal{W}(s; m, n) = \frac{\Theta(s; m + 1, n)}{\Theta(s; m, n + 1)}\,. \tag{4.35}$$

We furthermore impose the reality condition (4.17). Under the condition, $\Theta(s; m, n)$ satisfies $|\Theta(s; m, n)| = 1$ and one can set

$$e^{\sqrt{-1}\theta_n^m} = \Theta(s = 0; m, n) = \tau(1; m, n)/\tau(0; m, n)\,. \tag{4.36}$$

THEOREM 10 (Representation formula for discrete curves in terms of the τ-functions). *If we set*

$$\begin{aligned} \gamma_n^m &= \hat{w}_1^{(\infty)}(s = 0; m, n) = -\frac{\partial}{\partial x_1}\log\tau(0; m, n)\,, \\ \theta_n^m &= \frac{1}{\sqrt{-1}}\log\Theta(s = 0; m, n) = \frac{1}{\sqrt{-1}}\log\frac{\tau(1; m, n)}{\tau(0; m, n)}\,, \end{aligned} \tag{4.37}$$

then γ_n^m *and* θ_n^m *solve the equations* (4.21) *and* (4.22).

Proof. From (4.31) and (4.28), it follows that

$$\begin{aligned} &\left|\frac{\hat{w}_1^{(\infty)}(s; m, n + 1) - \hat{w}_1^{(\infty)}(s; m, n)}{a_n}\right| \\ &\quad = \left|\frac{\tau(s - 1; m, n)\tau(s + 1; m, n + 1)}{\tau(s; m, n)\tau(s; m, n + 1)}\right| = 1 \end{aligned} \tag{4.38}$$

under the condition (4.17). This is equivalent to the first equation of (4.21). The remaining equations follow directly from (4.32), (4.35) and (4.36). □

5. Fermionic construction of τ-functions

In [25, 26], Takebe described τ-functions for the Toda hierarchy as expectation values of fermionic operators (See also [24]). We firstly recall the definition of charged free fermions [12, 20].

Let $\mathcal{A}$ be an associative unital $\mathbb{C}$-algebra generated by ψ_i, ψ_i^* $(i \in \mathbb{Z})$ satisfying the relations

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}\,, \quad \psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0\,. \tag{5.1}$$

We consider a class of infinite matrices $A = \left[a_{ij}\right]_{i,j\in\mathbb{Z}}$ that satisfies the following condition:

$$\text{there exists } N > 0 \text{ such that } a_{ij} = 0 \text{ for all } i, j \text{ with } |i - j| > N\,. \tag{5.2}$$

Define the Lie algebra $\mathfrak{gl}(\infty)$ as [12]

$$\mathfrak{gl}(\infty) = \left\{\sum_{i,j\in\mathbb{Z}} a_{ij} :\psi_i\psi_j^*: \;\middle|\; A = \left[a_{ij}\right]_{i,j\in\mathbb{Z}} \text{ satisfies (5.2)}\right\} \oplus \mathbb{C} \tag{5.3}$$

where $:\cdot:$ indicates the normal ordering

$$:\psi_i\psi_j^*: = \begin{cases} \psi_i\psi_j^* & \text{if } i \neq j \text{ or } i = j \geq 0\,, \\ -\psi_j^*\psi_i & \text{if } i = j < 0\,. \end{cases} \tag{5.4}$$

We also define the group $\mathbf{G}$ corresponds to $\mathfrak{gl}(\infty)$ to be

$$\mathbf{G} = \left\{ e^{X_1}e^{X_2}\dots e^{X_k} \;\middle|\; X_i \in \mathfrak{gl}(\infty) \right\}\,. \tag{5.5}$$

Consider a left $\mathcal{A}$-module with a cyclic vector $|\mathrm{vac}\rangle$ satisfying

$$\psi_j|\mathrm{vac}\rangle = 0 \quad (j < 0)\,, \qquad \psi_k^*|\mathrm{vac}\rangle = 0 \quad (k \geq 0)\,. \tag{5.6}$$

The $\mathcal{A}$-module $\mathcal{A}|\mathrm{vac}\rangle$ is called the fermion Fock space $\mathcal{F}$, which we denote $\mathcal{F}$. We also consider a right $\mathcal{A}$-module (the dual Fock space $\mathcal{F}^*$) with a cyclic vector $\langle\mathrm{vac}|$ satisfying

$$\langle\mathrm{vac}|\psi_j = 0 \quad (j \geq 0)\,, \qquad \langle\mathrm{vac}|\psi_k^* = 0 \quad (k < 0)\,. \tag{5.7}$$

We further define the generalized vacuum vectors $|s\rangle$, $\langle s|$ ($s \in \mathbb{Z}$) as

$$\begin{aligned} |s\rangle &= \begin{cases} \psi_s^*\cdots\psi_{-1}^*|\mathrm{vac}\rangle & \text{for } s < 0\,, \\ |\mathrm{vac}\rangle & \text{for } s = 0\,, \\ \psi_{s-1}\cdots\psi_0|\mathrm{vac}\rangle & \text{for } s > 0\,, \end{cases} \\ \langle s| &= \begin{cases} \langle\mathrm{vac}|\psi_{-1}\cdots\psi_s & \text{for } s < 0\,, \\ \langle\mathrm{vac}| & \text{for } s = 0\,, \\ \langle\mathrm{vac}|\psi_0^*\cdots\psi_{s-1}^* & \text{for } s > 0\,. \end{cases} \end{aligned} \tag{5.8}$$

There exists a unique linear map (the vacuum expectation value) $\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{F} \to \mathbb{C}$ such that $\langle\mathrm{vac}| \otimes |\mathrm{vac}\rangle \mapsto 1$. For $a \in \mathcal{A}$ we denote by $\langle\mathrm{vac}|a|\mathrm{vac}\rangle$ the vacuum expectation value of the vector $\langle\mathrm{vac}|a \otimes |\mathrm{vac}\rangle = \langle\mathrm{vac}| \otimes a|\mathrm{vac}\rangle$ in $\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{F}$.

THEOREM 11 ([25] §2, [26] §2). *For $s \in \mathbb{Z}$ and $g \in \mathbf{G}$, define $\tau_g(s; x, y)$ as*

$$\tau_g(s; x, y) = \langle s|e^{H(x)}ge^{-\bar{H}(y)}|s\rangle\,, \tag{5.9}$$

where

$$H(x) = \sum_{n=1}^{\infty} x_n \sum_{j\in\mathbb{Z}} \psi_j\psi_{j+n}^*\,, \quad \bar{H}(y) = \sum_{n=1}^{\infty} y_n \sum_{j\in\mathbb{Z}} \psi_{j+n}\psi_j^*\,. \tag{5.10}$$

Then $\tau_g(s; x, y)$ satisfies the bilinear identity (3.15).

We introduce an automorphism ι_l of $\mathcal{A}$ by

$$\iota_l(\psi_i) = \psi_{i-l}\,, \quad \iota_l(\psi_i^*) = \psi_{i-l}^*\,, \tag{5.11}$$

which satisfies

$$\langle s'|a|s\rangle = \langle s' - l|\iota_l(a)|s - l\rangle \tag{5.12}$$

for any s, s', l and any $a \in \mathcal{A}$.

PROPOSITION 12. *If $g \in \mathbf{G}$ satisfies*

$$\iota_1(g) = \overline{g}\,, \tag{5.13}$$

then the τ-function corresponds to g gives a solution of the Goldstein-Petrich hierarchy.

Proof. From (5.12) and (5.13), it is clear that (4.17) holds. □

To construct soliton-type solutions, we choose g as

$$\begin{aligned} g_N(\{c_j\},\{p_j\},\{q_j\}) &= \prod_{j=1}^{N} e^{c_i\psi(p_i)\psi^*(q_i)}\,, \\ \psi(p) = \sum_{j\in\mathbb{Z}} \psi_j p^j\,, &\quad \psi^*(q) = \sum_{j\in\mathbb{Z}} \psi_j^* q^{-j}\,. \end{aligned} \tag{5.14}$$

We remark that the vacuum expectation value of $e^{c\psi(p)\psi^*(q)}$ makes sense even when $X = c\psi(p)\psi^*(q)$ does not satisfy the condition (5.2):

$$\langle s|e^{c\psi(p)\psi^*(q)}|s\rangle = \langle s|\left\{1 + c\psi(p)\psi^*(q)\right\}|s\rangle = 1 + \left(\frac{p}{q}\right)^s \frac{cq}{p-q}\,. \tag{5.15}$$

We consider the following two types of conditions for the parameters in (5.14):

A. (Soliton solutions)

$$c_j \in \sqrt{-1}\mathbb{R}, \quad p_j \in \mathbb{R}, \quad q_j = -p_j \quad (j = 1, 2, \ldots, N)\,, \tag{5.16}$$

B. (Breather solutions)

$$\begin{aligned} &N = 2M, \quad \overline{c_{2k-1}} = -c_{2k}\,, \quad \overline{p_{2k-1}} = p_{2k} \quad (k = 1, 2, \ldots, M)\,, \\ &q_j = -p_j \quad (j = 1, 2, \ldots, N)\,. \end{aligned} \tag{5.17}$$

A straightforward calculation shows that $g_N\left(\{c_j\},\{p_j\},\{q_j\}\right)$ satisfies (5.13) under each of the conditions (5.16), (5.17). The τ-functions under these conditions provide the solutions given in [9, 10].

We now consider Lie algebraic meaning of the condition (5.13). We recall the facts about a fermionic representation of the affine Lie algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$. The affine Lie algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$ is generated by the Chevalley generators $\{e_0, e_1, f_0, f_1, h_0, h_1\}$ that satisfy

$$\begin{aligned} &\left[h_i, h_j\right] = 0\,, \quad \left[e_i, f_j\right] = \delta_{ij}h_i \ \text{ for all } i, j\,, \\ &\left[h_i, e_j\right] = \begin{cases} 2e_j & \text{if } i = j\,, \\ -2e_j & \text{if } i \neq j\,, \end{cases} \qquad \left[h_i, f_j\right] = \begin{cases} -2e_j & \text{if } i = j\,, \\ 2e_j & \text{if } i \neq j\,, \end{cases} \\ &\left[e_i,\left[e_i,\left[e_i,e_j\right]\right]\right] = \left[f_i,\left[f_i,\left[f_i,f_j\right]\right]\right] = 0 \ \text{ if } i \neq j\,. \end{aligned} \tag{5.18}$$

Define a linear map $\pi : \widehat{\mathfrak{sl}}(2,\mathbb{C}) \to \mathfrak{gl}(\infty)$ as

$$\begin{aligned} \pi\left(e_j\right) &= \sum_{n\equiv j \bmod 2} \psi_{n-1}\psi_n^*\,, \quad \pi\left(f_j\right) = \sum_{n\equiv j \bmod 2} \psi_n\psi_{n-1}^*\,, \\ \pi\left(h_j\right) &= \sum_{n\equiv j \bmod 2} \left(:\psi_{n-1}\psi_{n-1}^*: - :\psi_n\psi_n^*:\right) + \delta_{j0} \quad (j = 0, 1)\,. \end{aligned} \tag{5.19}$$

THEOREM 13 ([12, 20]). *$(\pi, \mathcal{F})$ is a representation of $\widehat{\mathfrak{sl}}(2,\mathbb{C})$.*

Note that ι_1 works as an involutive automorphism:

$$\iota_1(e_0) = e_1\,, \quad \iota_1(f_0) = f_1\,, \quad \iota_1(e_1) = e_0\,, \quad \iota_1(f_1) = f_0\,, \tag{5.20}$$

which defines a real form of $\widehat{\mathfrak{sl}}(2,\mathbb{C})$. Kobayashi [15] classified automorphisms of prime order of the affine Lie algebra $\widehat{\mathfrak{sl}}(n,\mathbb{C})$. The involutive automorphism ι_1 under consideration is labeled as (1a')-type ([15], Theorem 3). We remark that the same real form of $\widehat{\mathfrak{sl}}(2,\mathbb{C})$ appeared also in construction of solutions of a derivative nonlinear Schödinger equation [14].

Appendix: Time-flows with positive weight

So far, we have used the time-evolutions with respect to the variables with negative weight $y = (y_1, y_2, \ldots)$ to derive the Goldstein-Petrich hierarchy. In this appendix, we use $x = (x_1, x_2, \ldots)$ and show that the mKdV hierarchy can be obtained under the 2-reduction condition (4.1). Applying the condition (4.1), one can show that

$$\begin{aligned} B_{2n-1}(s) &= e^{(2n-1)\partial_s} + \sum_{-2(n-1)\le j\le 0} b_j(s) e^{(2n-2+j)\partial_s}\,, \\ B_{2n}(s) &= e^{2n\partial_s} \quad (n = 1, 2, \ldots)\,. \end{aligned} \tag{A.1}$$

From (3.2) and (4.1), we obtain

$$\begin{aligned} &b_0(s+1) + b_0(s) = 0, \\ &b_{-k-1}(s+1) + b_{-k-1}(s) + \sum_{j=0}^{k} b_{-j}(s) b_{j-k}(s-j) = 0 \quad (k = 0, 1, 2, \ldots)\,. \end{aligned} \tag{A.2}$$

Applying (A.1) to (3.4), we obtain

$$\frac{\partial b_0(s)}{\partial x_{2n-1}} = b_{-2n+1}(s+1) - b_{-2n+1}(s)\,. \tag{A.3}$$

Define $L_1(x, y)$, $L_2(x, y)$ by

$$\begin{aligned} L_1(x, y) &= \frac{1}{2}\left\{ L^{(\infty)}(s = 0; x, y) - L^{(\infty)}(s = 1; x, y) \right\}, \\ L_2(x, y) &= \frac{1}{2}\left\{ L^{(\infty)}(s = 0; x, y) + L^{(\infty)}(s = 1; x, y) \right\}, \end{aligned} \tag{A.4}$$

which have the following form:

$$\begin{aligned} &L_1(x, y) = \sum_{n=0}^{\infty} q_n(x, y) e^{-n\partial_s}\,, \quad L_2(x, y) = e^{\partial_s} + \sum_{n=1}^{\infty} r_n(x, y) e^{-n\partial_s}\,, \\ &q_n(x, y) = \frac{b_{-n}(s = 0, x, y) - b_{-n}(s = 1, x, y)}{2} \quad (n = 0, 1, 2, \ldots)\,, \\ &r_n(x, y) = \frac{b_{-n}(s = 0, x, y) + b_{-n}(s = 1, x, y)}{2} \quad (n = 1, 2, 3, \ldots)\,. \end{aligned} \tag{A.5}$$

We remark that q_n and r_n are eigenfunctions of e^{∂_s}:

$$e^{\partial_s} q_n = -q_n\,, \quad e^{\partial_s} r_n = r_n\,. \tag{A.6}$$

Applying the notation (A.5) to (A.3), we have

$$\frac{\partial q_0}{\partial x_{2n-1}} = -2q_{2n-1} \tag{A.7}$$

Since $B_1(0)$, $B_1(1)$ are of the form

$$B_1(0) = e^{\partial_s} + q_0\,, \quad B_1(1) = e^{\partial_s} - q_0\,, \tag{A.8}$$

it follows that

$$\frac{\partial L_1}{\partial x_1} = -2L_1 e^{\partial_s} + \left[q_0,\ L_2\right]\,, \quad \frac{\partial L_2}{\partial x_1} = \left[q_0,\ L_1\right]\,, \tag{A.9}$$

and hence

$$\begin{aligned} \frac{\partial q_{2n-1}}{\partial x_1} &= -2q_{2n} + 2q_0 r_{2n-1}\,, & \frac{\partial q_{2n}}{\partial x_1} &= -2q_{2n+1}\,, \\ \frac{\partial r_{2n-1}}{\partial x_1} &= 2q_0 q_{2n-1}\,, & \frac{\partial r_{2n}}{\partial x_1} &= 0\,. \end{aligned} \tag{A.10}$$

From (A.7) and (A.10), we have

$$\frac{\partial q_0}{\partial x_{2n+1}} = \left(\frac{1}{4}\partial_{x_1}^2 - q_0^2 - \frac{\partial q_0}{\partial x_1}\partial_{x_1}^{-1} \circ q_0\right)\frac{\partial q_0}{\partial x_{2n-1}}\,. \tag{A.11}$$

Especially for the case $n = 1$,

$$\frac{\partial q_0}{\partial x_3} = \frac{1}{4}\frac{\partial^3 q_0}{\partial x_1^3} - \frac{3}{2}q_0^2\frac{\partial q_0}{\partial x_1}\,. \tag{A.12}$$

After suitable scaling, the linear operator appeared in the right-hand side of (A.11) yields the recursion operator Ω in (2.7), and the equation (A.12) yields the mKdV equation (1.1).

We remark that another derivation of the recursion operator Ω in terms of bilinear differential equations of Hirota-type was given in [29]. Here we briefly summarize the approach in [29]. We use the Hirota differential operators D_x, D_y, ..., defined by

$$D_x^m D_y^n f(x, y) \cdot g(x, y) = (\partial_x - \partial_{x'})^m \left(\partial_y - \partial_{y'}\right)^n f(x, y)g(x', y')\big|_{x'=x, y'=y}\,. \tag{A.13}$$

Setting $s' = 0$, $s = 1$ $y'_n = y_n$, $x'_n = x_n + a_n$ $(n = 1, 2, \ldots)$, the bilinear identity (3.15) is reduced to

$$\oint \tau(0; x' - [\lambda^{-1}], y)\tau(1; x + [\lambda^{-1}], y)e^{\xi(x'-x,\lambda)}\lambda^{-1}d\lambda = \tau(1; x', y)\tau(0; x, y)\,, \tag{A.14}$$

or, using the Hirota operators $\tilde{D} = (D_1, D_2/2, D_3/3, \ldots)$, $D_j = D_{x_j}$ $(j = 1, 2, \ldots)$, we can write

$$\sum_{j=0}^{\infty} p_j(-2a)p_j(\tilde{D}) \exp\left(\sum_{k=1}^{\infty} a_k D_k\right)\tau(0) \cdot \tau(1) = \exp\left(\sum_{k=1}^{\infty} a_k D_k\right)\tau(1) \cdot \tau(0) \tag{A.15}$$

for any $a = (a_1, a_2, \ldots)$ (cf. [17]). Expanding (A.15) with respect to the variables $a = (a_1, a_2, \ldots)$, we obtain

$$\left(p_m(\tilde{D}) - D_m\right)\tau(1) \cdot \tau(0) = 0 \tag{A.16}$$

from the coefficient of a_m, and

$$\left(-2p_{m+k}(\tilde{D}) + p_m(\tilde{D})D_k + p_k(\tilde{D})D_m\right)\tau(1) \cdot \tau(0) = 0 \tag{A.17}$$

from the coefficient of $a_m a_k$. Using (A.16) to eliminate the first term in (A.17), we have

$$\left(-2D_{m+k} + p_m(\tilde{D})D_k + p_k(\tilde{D})D_m\right)\tau(1)\cdot\tau(0) = 0\,. \tag{A.18}$$

Hereafter we impose the 2-reduction condition $\partial_{x_{2n}}\tau = 0$ ($n = 1, 2, \ldots$). Setting $k = 2$, the bilinear equations (A.16), (A.18) yield

$$D_1^2\tau(1)\cdot\tau(0) = 0\,, \quad \left(-4D_{m+2} + D_1^2 D_m\right)\tau(1)\cdot\tau(0) = 0\,. \tag{A.19}$$

If we set

$$\psi = \log\left(\tau(1)/\tau(0)\right), \quad \phi = \log\left(\tau(0)\tau(1)\right)\,, \tag{A.20}$$

it follows that

$$(\partial_1\psi)^2 + \partial_1^2\phi = 0\,, \quad -4\partial_{m+2}\psi + \partial_1^2\partial_m\psi + 2(\partial_1\psi)(\partial_1\partial_m\phi) = 0\,, \tag{A.21}$$

from (A.19), where $\partial_n = \partial/\partial x_n$. Setting

$$q_0 = \partial_1\psi = \partial_1\left(\log\frac{\tau(1)}{\tau(0)}\right)\,, \tag{A.22}$$

we have the recursion relation (A.11).

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