

## Capelli Identities of Odd Type and $b$ -Functions

by

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*Dedicated to Professor Fumihiko Sato on the occasion of his 65th birthday*

### 1. Introduction

Let  $T_{ij}$  ( $1 \leq i, j \leq n$ ) be variables, and  $\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) A_{\sigma(1)1} \cdots A_{\sigma(n)n}$  denote the column-determinant. The Capelli identity by Capelli [1, 2] is:

$$\det({}^t T) \det\left(\frac{\partial}{\partial T}\right) = \det\left({}^t T \frac{\partial}{\partial T} + \begin{pmatrix} n-1 & & & 0 \\ & n-2 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}\right), \quad (1)$$

where  $T$  and  $\partial/\partial T$  denote the following  $n \times n$  matrices:

$$T = (T_{ij})_{1 \leq i, j \leq n}, \quad \frac{\partial}{\partial T} = \left(\frac{\partial}{\partial T_{ij}}\right)_{1 \leq i, j \leq n}.$$

Although the Capelli identity (1) has representation-theoretic meaning such as relation to the center of the universal enveloping algebra, the simplest interpretation of the equation (1) is a non-commutative version of the product formula of two determinants.

If there are three determinants on the left-hand side, then we have the following product formula [9]:

$$\det\left(\frac{\partial}{\partial T}\right) \det({}^t T) \det\left(\frac{\partial}{\partial T}\right) = \det\left(\frac{\partial}{\partial T} {}^t T \frac{\partial}{\partial T}\right), \quad (2)$$

which we call *the Capelli identity of odd type*. When  $T$  is symmetric ( $T_{ij} = T_{ji}$ ), the Capelli identity of odd type also holds [10]. The main result of this paper is the Capelli identity of odd type for alternating matrices. Namely, when  $T$  is alternating ( $T_{ij} = -T_{ji}$ ), we have a product formula of pfaffians (see Subsection 2.3 for the definition of differential operators with parameter):

$$\operatorname{pf}\left(\frac{\partial}{\partial T}(u)\right) \operatorname{pf}({}^t T) \operatorname{pf}\left(\frac{\partial}{\partial T}(u+1)\right) = \pm \operatorname{pf}\left(\frac{\partial}{\partial T}(u) {}^t T \frac{\partial}{\partial T}(u+1)\right). \quad (3)$$

In fact, these formulas hold in more generalized forms. See Theorems 1, 2 and 3 for details. It is remarkable that there is no diagonal shift in the Capelli identities of odd type, although the ordinary Capelli identities such as (1) have diagonal shifts (see [3] for example). While

the ordinary Capelli identities for alternating matrices are quite complicated [3, 4, 5], the Capelli identity (3) of odd type is rather simple as seen in above.

As an application we can compute  $b$ -functions of some reducible prehomogeneous vector spaces. Since one of the purposes of the original Capelli identity is to compute the  $b$ -function of the determinant, this is a natural application. Here we list relative invariants of some prehomogeneous vector spaces for which we can compute the  $b$ -functions using the Capelli identities of odd type.

$$\det(X^{(1)}X^{(2)} \dots X^{(l)}) \quad (m_0 = m_l), \quad (4)$$

$$\det(X^{(1)}X^{(2)} \dots X^{(l)} {}^t(X^{(1)}X^{(2)} \dots X^{(l)})), \quad (5)$$

$$\det(X^{(1)}X^{(2)} \dots X^{(l)} S {}^t(X^{(1)}X^{(2)} \dots X^{(l)})) \quad (S \in \text{Sym}(m_l)), \quad (6)$$

$$\text{pf}(X^{(1)}X^{(2)} \dots X^{(l)} A {}^t(X^{(1)}X^{(2)} \dots X^{(l)})) \quad (A \in \text{Alt}(m_l)), \quad (7)$$

where  $X^{(r)} \in \text{Mat}(m_{r-1}, m_r)$ ,  $m_0 \leq m_r$  for any  $r$ , and  $\text{Sym}(m_l)$  and  $\text{Alt}(m_l)$  denote the symmetric matrices and the alternating matrices, respectively. See [9] for the  $b$ -function of (4), and [10] for those of (5) and (6). We compute the  $b$ -function of (7) in Section 3. It should be remarked that the  $b$ -functions of the above relative invariants can be computed by using the result of Sato-Sugiyama [6]. In fact, the  $b$ -functions of (4) and (5) are already computed by Sugiyama [7, 8] and Sato-Sugiyama [6], respectively.

This paper is organized as follows. In Section 2 we give the full formulas of the Capelli identities of odd type, and fix the notation needed for the formulas. The proof is postponed until Section 4. In Section 3 we compute the  $b$ -function of (7) by using the Capelli identity of odd type. In Section 4 we prove the Capelli identity of odd type in the case where  $T_{ij}$  is alternating.

The sets of matrices  $\text{Mat}(m, n)$ ,  $\text{Sym}(n)$ ,  $\text{Alt}(n)$ , the general linear group  $GL(n)$  and its Lie algebra  $\mathfrak{gl}_n$  are defined over the complex number field unless otherwise specified. We denote the  $n \times n$  unit matrix by  $1_n$ , the symmetric group on  $n$  letters by  $S_n$ , and the sign of a permutation  $\sigma$  by  $\text{sgn}(\sigma)$ .

## 2. Capelli identities of odd type

In this section we summarize the Capelli identities of odd type. Theorems 1 and 2 are results of [9] and [10], respectively, and the proofs are omitted. Theorem 3, which is the main result of this paper, is proved in Section 4.

### 2.1. The normal case

This is the case where variables  $T_{ij}$  have no relation, namely, the matrix  $T = (T_{ij})$  is neither symmetric nor alternating. Let  $T_{ij}$  be variables ( $1 \leq i, j \leq n$ ), and set

$$T = (T_{ij})_{1 \leq i, j \leq n}, \quad \frac{\partial}{\partial T} = \left( \frac{\partial}{\partial T_{ij}} \right)_{1 \leq i, j \leq n}, \quad f = \det(T).$$

While the ordinary Capelli identity is an equation in the Weyl algebra  $\mathbb{C}[T_{ij}, \partial/\partial T_{ij}]$ , we work with the localized algebra  $\mathcal{D} = \mathbb{C}[T_{ij}, \partial/\partial T_{ij}, f^{-1}]$  since we need to add parameters

to differential operators. We define an operator in  $\mathcal{D}$  and a matrix in  $\text{Mat}_n(\mathcal{D})$  as

$$\begin{aligned} \frac{\partial}{\partial T_{ij}}(u) &= \frac{\partial}{\partial T_{ij}} + u({}^t T^{-1})_{ij} \in \mathcal{D}, \\ \frac{\partial}{\partial T}(u) &= \frac{\partial}{\partial T} + u{}^t T^{-1} \in \text{Mat}_n(\mathcal{D}). \end{aligned}$$

**THEOREM 1 ([9]).** *With the notation as above, we have*

$$\begin{aligned} &\left| \frac{\partial}{\partial T}(u_1) {}^t T \frac{\partial}{\partial T}(u_2) {}^t T \cdots {}^t T \frac{\partial}{\partial T}(u_l) \right| \\ &= \left| \frac{\partial}{\partial T}(u_1) \right| |{}^t T| \left| \frac{\partial}{\partial T}(u_2) \right| |{}^t T| \cdots |{}^t T| \left| \frac{\partial}{\partial T}(u_l) \right|, \end{aligned}$$

where  $|X|$  denotes  $\det(X)$ , and  $u_1, u_2, \dots, u_l \in \mathbb{C}$ .

*In addition, the entries are commutative in each determinant on both sides, and the matrix on the left-hand side is symmetric with respect to  $u_1, u_2, \dots, u_l$ .*  $\square$

**2.2. The symmetric case**

Let  $T_{ij}$  be variables ( $1 \leq i, j \leq n$ ) satisfying  $T_{ij} = T_{ji}$ , and set

$$T = (T_{ij})_{1 \leq i, j \leq n}, \quad \frac{\bar{\partial}}{\partial T} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial T_{ij}} \right)_{1 \leq i, j \leq n}, \quad \frac{\bar{\partial}}{\partial T}(u) = \frac{\bar{\partial}}{\partial T} + u{}^t T^{-1}.$$

**THEOREM 2 ([10]).** *With the notation as above, for  $u_1, u_2, \dots, u_l \in \mathbb{C}$  we have*

$$\begin{aligned} &\left| \frac{\bar{\partial}}{\partial T}(u_1) {}^t T \frac{\bar{\partial}}{\partial T}(u_2) {}^t T \cdots {}^t T \frac{\bar{\partial}}{\partial T}(u_l) \right| \\ &= \left| \frac{\bar{\partial}}{\partial T}(u_1) \right| |{}^t T| \left| \frac{\bar{\partial}}{\partial T}(u_2) \right| |{}^t T| \cdots |{}^t T| \left| \frac{\bar{\partial}}{\partial T}(u_l) \right|. \end{aligned}$$

*In addition, the entries are commutative in each determinant on both sides, and the matrix on the left-hand side is symmetric with respect to  $u_1, u_2, \dots, u_l$ .*  $\square$

**2.3. The alternating case**

Let  $T_{ij}$  be variables ( $1 \leq i, j \leq n$ ) satisfying  $T_{ij} = -T_{ji}$  and  $n = 2m$ , and set

$$T = (T_{ij})_{1 \leq i, j \leq n}, \quad \frac{\partial}{\partial T} = \left( \frac{\partial}{\partial T_{ij}} \right)_{1 \leq i, j \leq n}, \quad f = \text{pf}(T),$$

where  $\text{pf}(T)$  denotes the pfaffian

$$\text{pf}(T) = \sum_{\substack{\sigma \in S_n, \\ \sigma(1) < \sigma(3) < \dots < \sigma(n-1), \\ \sigma(1) < \sigma(2), \dots, \sigma(n-1) < \sigma(n)}} \text{sgn}(\sigma) T_{\sigma(1), \sigma(2)} T_{\sigma(3), \sigma(4)} \cdots T_{\sigma(n-1), \sigma(n)}$$

of the alternating matrix  $T$ , and  $\partial/\partial T_{ii} = 0$  by definition. We again work with the localized algebra  $\mathcal{D} = \mathbb{C}[T_{ij}, \partial/\partial T_{ij}, f^{-1}]$ . Set

$$\frac{\partial}{\partial T_{ij}}(u) = \frac{\partial}{\partial T_{ij}} + u({}^t T^{-1})_{ij} \in \mathcal{D},$$

$$\frac{\partial}{\partial T}(u) = \frac{\partial}{\partial T} + u^t T^{-1} \in \text{Mat}_n(\mathcal{D}).$$

The next theorem is the main result of this paper. We prove this theorem in Section 4.

**THEOREM 3.** *For any  $u_1, u_2, \dots, u_l \in \mathbb{C}$ , the following is an alternating matrix.*

$$\frac{\partial}{\partial T}(u_1) {}^t T \frac{\partial}{\partial T}(u_2) {}^t T \cdots {}^t T \frac{\partial}{\partial T}(u_l).$$

*This matrix is also symmetric with respect to  $u_1, u_2, \dots, u_l$ , and the entries commute with each other. Moreover, when  $u_r = u_{l-r+1} - 1$  for  $1 \leq r \leq \lfloor l/2 \rfloor$ , we have*

$$\begin{aligned} & \text{pf} \left( \frac{\partial}{\partial T}(u_1) {}^t T \frac{\partial}{\partial T}(u_2) {}^t T \cdots {}^t T \frac{\partial}{\partial T}(u_l) \right) \\ &= (-1)^{m(l-1)} \text{pf} \left( \frac{\partial}{\partial T}(u_1) \right) \text{pf}({}^t T) \text{pf} \left( \frac{\partial}{\partial T}(u_2) \right) \text{pf}({}^t T) \cdots \text{pf}({}^t T) \text{pf} \left( \frac{\partial}{\partial T}(u_l) \right). \end{aligned}$$

□

### 3. *b*-Function

In this section we compute the *b*-function of the polynomial (7). The strategy to compute the *b*-function is as follows. We first change the variables, and then obtain the differential operator appeared in the Capelli identity of odd type. This operator is factored by the Capelli identity of odd type, and the computation is reduced to the *b*-function of the pfaffian.

Let  $l$  be a positive integer. Let  $m_0, m_1, \dots, m_l$  be positive integers such that  $m_0$  is even and  $m_0 \leq m_r$  for any  $r$ . Let  $X^{(r)} \in \text{Mat}(m_{r-1}, m_r)$ ,  $A \in \text{Alt}(m_l)$ , and  $T = X^{(1)} X^{(2)} \cdots X^{(l)} A {}^t (X^{(1)} X^{(2)} \cdots X^{(l)})$ . The goal of this section is to compute the *b*-function of the polynomial

$$f = \text{pf}(T) = \text{pf}(X^{(1)} X^{(2)} \cdots X^{(l)} A {}^t (X^{(1)} X^{(2)} \cdots X^{(l)})).$$

This is a relative invariant of the prehomogeneous vector space  $(G, V)$  defined as

$$\begin{aligned} G &= GL(m_0) \times GL(m_1) \times \cdots \times GL(m_{l-1}) \times GL(m_l), \\ V &= \text{Mat}(m_0, m_1) \oplus \text{Mat}(m_1, m_2) \oplus \cdots \oplus \text{Mat}(m_{l-1}, m_l) \oplus \text{Alt}(m_l), \\ &(g_0, \dots, g_l). (X^{(1)}, \dots, X^{(l)}, A) \\ &= (g_0 X^{(1)} g_1^{-1}, g_1 X^{(2)} g_2^{-1}, \dots, g_{l-1} X^{(l)} g_l^{-1}, g_l A {}^t g_l). \end{aligned}$$

We use the notation of Subsection 2.3. We have defined  $(\partial/\partial T)(u) = \partial/\partial T + u^t T^{-1}$ , and this is rewritten as follows:

**LEMMA 4.** *We define  $g = \log f$ . Then we have*

$$\frac{\partial}{\partial T}(u) = \frac{\partial}{\partial T} + u \frac{\partial g}{\partial T} = f^{-u} \frac{\partial}{\partial T} f^u,$$

where  $\partial g/\partial T$  denotes the  $m_0 \times m_0$  matrix whose  $(i, j)$ -entry is the multiplication operator by the function  $\partial g/\partial T_{ij}$ , and  $f^{-u}(\partial/\partial T)f^u$  denotes the  $m_0 \times m_0$  matrix whose  $(i, j)$ -entry is the differential operator  $f^{-u}(\partial/\partial T_{ij})f^u$ .

In particular, the entries of  $(\partial/\partial T)(u)$  commute with each other.

*Proof.* The second equality easily follows from

$$\left[ \frac{\partial}{\partial T_{ij}}, f^u \right] = \frac{\partial f^u}{\partial T_{ij}} = u f^{u-1} \frac{\partial f}{\partial T_{ij}}.$$

For the first equality, it suffices to show that

$$\frac{\partial g}{\partial T} = {}^t T^{-1}.$$

If there are two rows that coincide with each other, then the pfaffian is equal to zero. Therefore we have  $\sum_i T_{ij}(\partial f/\partial T_{ik}) = 0$  for each  $j \neq k$ . In addition,  $f = \text{pf}(T)$  is of total degree one with respect to each column of  $T$ , and hence we have  $\sum_i T_{ij}(\partial f/\partial T_{ij}) = f$  for each  $j$ . Thus it turns out that  ${}^t T(\partial f/\partial T) = f \cdot 1_{m_0}$ , namely,  ${}^t T(\partial g/\partial T) = 1_{m_0}$ .  $\square$

We define two matrices whose entries are differential operators in order to show Lemma 7 on change of variables.

$$\frac{\partial}{\partial X^{(r)}} = \left( \frac{\partial}{\partial X_{ij}^{(r)}} \right)_{\substack{1 \leq i \leq m_{r-1} \\ 1 \leq j \leq m_r}},$$

$$\frac{\partial}{\partial X^{(r)}}(u) = \frac{\partial}{\partial X^{(r)}} + u \frac{\partial g}{\partial X^{(r)}} = f^{-u} \frac{\partial}{\partial X^{(r)}} f^u,$$

where  $(i, j)$ -entry of  $\partial g/\partial X^{(r)}$  is the multiplication operator by a function  $\partial g/\partial X_{ij}^{(r)}$ . Note that  $X^{(r)}$  is not necessarily a square matrix, and that  $\partial g/\partial X^{(r)}$  is not the inverse of  $X^{(r)}$  in general. Remark that the entries of  $(\partial/\partial X^{(r)})(u)$  commute with each other since it is equal to  $f^{-u}(\partial/\partial X^{(r)})f^u$ .

The next lemma, which is needed in Lemma 6 and Proposition 10, is proved by a straight forward computation, and we omit the proof.

LEMMA 5. (1) Let  $X = (X_{ij})$  be a  $p \times q$  matrix whose entries are (independent) variables. Then we have

$$\frac{\partial}{\partial X} {}^t X - \left( X \frac{\partial}{\partial X} \right) = q 1_p.$$

(2) Let  $T = (T_{ij})$  be an  $n \times n$  matrix whose entries are variables satisfying  $T_{ij} = -T_{ji}$ . Then we have

$$\frac{\partial}{\partial T} {}^t T - \left( T \frac{\partial}{\partial T} \right) = (n-1) 1_n.$$

$\square$

Set  $X^{(a,b)} = X^{(a)} X^{(a+1)} \dots X^{(b)}$ , and we have the following lemma.

LEMMA 6. We have the following equalities of differential operators acting on functions in  $T$ .

- (1)  $\frac{\partial}{\partial X^{(r)}}(u - m_r)^t X^{(1,r)} = {}^t X^{(1,r-1)} \frac{\partial}{\partial T}(u - m_0 + 1)^t T$  for  $1 \leq r \leq l$ .
- (2)  $\frac{\partial}{\partial A}(u - m_l)^t A^t X^{(1,l)} = {}^t X^{(1,l)} \frac{\partial}{\partial T}(u - m_0)^t T$ .
- (3)  $\frac{{}^t \partial}{\partial X^{(r)}}(u - m_{r-1})^t (X^{(1,l)} A^t X^{(r,l)}) = {}^t (X^{(1,l)} A^t X^{(r+1,l)}) \frac{\partial}{\partial T}(u - m_0)^t T$  for  $1 \leq r \leq l$ . Moreover, when  $r = 1$ , this equality implies  $\frac{{}^t \partial}{\partial X^{(1)}}(u - m_0) = {}^t (X^{(1,l)} A^t X^{(2,l)}) \frac{\partial}{\partial T}(u - m_0)$ .

*Proof.* We give the proof of (1) only. The other formulas are proved similarly. Put  $Y = X^{(r+1,l)} A^t X^{(1,l)}$ , and we have  $T = X^{(1,r-1)} X^{(r)} Y$ . For a function  $\phi = \phi(T)$  in  $T$ , it follows from the chain rule that

$$\begin{aligned} \frac{\partial \phi}{\partial X^{(r)}_{ij}} &= \sum_{1 \leq p < q \leq m_0} \frac{\partial \phi}{\partial T_{pq}} \frac{\partial T_{pq}}{\partial X^{(r)}_{ij}} \\ &= \sum_{1 \leq p < q \leq m_0} \frac{\partial \phi}{\partial T_{pq}} \left( X^{(1,r-1)}_{pi} Y_{jq} + {}^t Y_{pj} X^{(1,r-1)}_{iq} \right) \\ &= \sum_{p,q=1}^{m_0} \frac{\partial \phi}{\partial T_{pq}} X^{(1,r-1)}_{pi} Y_{jq} \\ &= \left( {}^t X^{(1,r-1)} \frac{\partial \phi}{\partial T} {}^t Y \right)_{ij}. \end{aligned}$$

Therefore we have the equation of matrices  $\partial \phi / \partial X^{(r)} = {}^t X^{(1,r-1)} (\partial \phi / \partial T) {}^t Y$ , and by multiplying  ${}^t X^{(1,r)}$  from the right we obtain

$$\frac{\partial \phi}{\partial X^{(r)}} {}^t X^{(1,r)} = {}^t X^{(1,r-1)} \frac{\partial \phi}{\partial T} {}^t T.$$

By moving the terms containing  $\phi$  to the right in both sides using transposition, we can drop  $\phi$ . In addition, by the conjugation  $z \mapsto f^{-u} z f^u$ , we can put the parameter to differential operators. We thus have

$${}^t \left( X^{(1,r)} \frac{{}^t \partial}{\partial X^{(r)}}(u) \right) = {}^t X^{(1,r-1)} {}^t \left( T \frac{{}^t \partial}{\partial T}(u) \right).$$

It follows from Lemma 5 that

$${}^t \left( X^{(1,r-1)} {}^t \left( \frac{\partial}{\partial X^{(r)}}(u) {}^t X^{(r)} - m_r 1_{m_{r-1}} \right) \right) = {}^t X^{(1,r-1)} \left( \frac{\partial}{\partial T}(u) {}^t T - (m_0 - 1) 1_{m_0} \right).$$

Since the entries of  ${}^t X^{(1,r-1)}$  and those of  $(\partial / \partial X^{(r)})(u)$  commute with each other, the left-hand side is equal to  ${}^t (\partial / \partial X^{(r)})(u) {}^t X^{(1,r)} - m_r {}^t X^{(1,r-1)}$ . By moving the term  $-m_r {}^t X^{(1,r-1)}$  to the right-hand side we have

$$\frac{{}^t \partial}{\partial X^{(r)}}(u) {}^t X^{(1,r)} = {}^t X^{(1,r-1)} \frac{\partial}{\partial T}(u - m_0 + 1 + m_r) {}^t T.$$

Finally the desired formula is proved by replacing  $u$  with  $u - m_r$ .  $\square$

LEMMA 7. We use the notation as above. Let  $u_r, v, w_r$  be complex numbers. Then we have

$$\begin{aligned} & \frac{\partial}{\partial T}(u_1)^t T \frac{\partial}{\partial T}(u_2) \cdots \frac{\partial}{\partial T}(u_l) \cdot {}^t T \frac{\partial}{\partial T}(v)^t T \cdot \frac{\partial}{\partial T}(w_l)^t T \frac{\partial}{\partial T}(w_{l-1}) \cdots \frac{\partial}{\partial T}(w_1) \\ &= \frac{\partial}{\partial X^{(1)}}(u_1 - m_1 + m_0 - 1) \cdots \frac{\partial}{\partial X^{(l)}}(u_l - m_l + m_0 - 1) \\ & \quad \cdot \frac{\partial}{\partial A}(v - m_l + m_0) \\ & \quad \cdot \frac{\partial}{\partial X^{(l)}}(w_l - m_{l-1} + m_0) \cdots \frac{\partial}{\partial X^{(1)}}(w_1 - m_0 + m_0). \end{aligned}$$

*Proof.* By applying Lemma 6 (1) to the underlined part below, we compute the left-hand side of the lemma as follows.

$$\begin{aligned} & \frac{\partial}{\partial T}(u_1)^t T \frac{\partial}{\partial T}(u_2) \cdots \frac{\partial}{\partial T}(w_1) \\ &= \frac{\partial}{\partial X^{(1)}}(u_1 - m_1 + m_0 - 1) \frac{\partial}{\partial T}(u_2)^t T \cdots \frac{\partial}{\partial T}(w_1) \\ &= \frac{\partial}{\partial X^{(1)}}(u_1 - m_1 + m_0 - 1) \cdot \frac{\partial}{\partial X^{(2)}}(u_2 - m_2 + m_0 - 1) {}^t X^{(1,2)} \cdots \frac{\partial}{\partial T}(w_1) \\ & \quad \vdots \\ &= \frac{\partial}{\partial X^{(1)}}(u_1 - m_1 + m_0 - 1) \cdots \frac{\partial}{\partial X^{(l)}}(u_l - m_l + m_0 - 1) {}^t X^{(1,l)} \cdot \frac{\partial}{\partial T}(v)^t T \\ & \quad \cdot \frac{\partial}{\partial T}(w_l)^t T \cdots \frac{\partial}{\partial T}(w_1). \end{aligned}$$

Next we use Lemma 6 (2), and the above expression is equal to

$$\begin{aligned} & \frac{\partial}{\partial X^{(1)}}(u_1 - m_1 + m_0 - 1) \cdots \frac{\partial}{\partial X^{(l)}}(u_l - m_l + m_0 - 1) \frac{\partial}{\partial A}(v - m_l + m_0)^t A {}^t X^{(1,l)} \\ & \quad \cdot \frac{\partial}{\partial T}(w_l)^t T \cdots \frac{\partial}{\partial T}(w_1). \end{aligned}$$

Next we use Lemma 6 (3) repeatedly, and finally the above expression turns out to be equal to the right-hand side of the desired equation.  $\square$

We can compute the  $b$ -function of  $f = \text{pf}(X^{(1)} X^{(2)} \cdots X^{(l)} A {}^t(X^{(1)} X^{(2)} \cdots X^{(l)}))$  by using Lemma 7. Recall that  $T$  is an alternating  $m_0 \times m_0$  matrix, and  $m_0$  is even.

PROPOSITION 8. The  $b$ -function  $b_f(s)$  of  $f$  is given by

$$b_f(s) = (s + 1)(s + 3) \cdots (s + m_0 - 1) \cdot \prod_{r=1}^l (s + m_r)^{(m_0)},$$

where  $a^{(b)}$  is defined by  $a^{(b)} = a(a - 1)(a - 2) \cdots (a - b + 1)$ .

*Proof.* We first recall the  $b$ -function of the pfaffian of an alternating matrix (see [3] for example):

$$\text{pf}\left(\frac{\partial}{\partial T}\right)(\text{pf}(T)^{s+1}) = b_{\text{alt}}(s) \text{pf}(T)^s,$$

where  $b_{\text{alt}}(s) = (s+1)(s+3)\cdots(s+m_0-1)$ . In view of Lemma 4 we have

$$\text{pf}\left(\frac{\partial}{\partial T}(u)\right)(\text{pf}(T)^{s+1}) = b_{\text{alt}}(s+u) \text{pf}(T)^s.$$

For the  $b$ -function of  $f$ , we have to compute

$$f(\partial)(f^{s+1}) = \text{pf}\left(\frac{\partial}{\partial X^{(1)}} \cdots \frac{\partial}{\partial X^{(l)}} \cdot \frac{\partial}{\partial A} \cdot \left(\frac{\partial}{\partial X^{(1)}} \cdots \frac{\partial}{\partial X^{(l)}}\right)\right)(f^{s+1}). \quad (8)$$

If we change the variables from  $X$  and  $A$  to  $T$  using Lemma 7, then the operator on the right-hand side of (8) comes to be an operator in  $T$  as follows.

RHS of (8)

$$\begin{aligned} &= \text{pf}\left(\frac{\partial}{\partial T}(m_1 - m_0 + 1)^t T \frac{\partial}{\partial T}(m_2 - m_0 + 1) \cdots \frac{\partial}{\partial T}(m_l - m_0 + 1)\right. \\ &\quad \cdot {}^t T \frac{\partial}{\partial T}(m_l - m_0)^t T \\ &\quad \left. \cdot \frac{\partial}{\partial T}(m_{l-1} - m_0)^t T \frac{\partial}{\partial T}(m_{l-2} - m_0) \cdots \frac{\partial}{\partial T}(m_0 - m_0)\right)(f^{s+1}). \quad (9) \end{aligned}$$

The operator on the right-hand side of the above equation factors into  $4l+1$  pfaffians thanks to Theorem 3, where the symmetry in parameters is used. The sign in Theorem 3 is positive in this case. Therefore we first apply  $\text{pf}({}^t T) \text{pf}((\partial/\partial T)(m_0 - m_0))$  to  $f^{s+1}$ , and obtain  $b_{\text{alt}}(s+m_0-m_0)f^{s+1}$ . Second we apply  $\text{pf}({}^t T) \text{pf}((\partial/\partial T)(m_1 - m_0))$  to it, and obtain  $b_{\text{alt}}(s+m_1-m_0)b_{\text{alt}}(s+m_0-m_0)f^{s+1}$ . Similarly we apply the operators repeatedly, and we have

RHS of (9)

$$\begin{aligned} &= b_{\text{alt}}(s+m_1-m_0+1)b_{\text{alt}}(s+m_2-m_0+1)\cdots b_{\text{alt}}(s+m_l-m_0+1) \\ &\quad \cdot b_{\text{alt}}(s+m_l-m_0) \\ &\quad \cdot b_{\text{alt}}(s+m_{l-1}-m_0)b_{\text{alt}}(s+m_{l-2}-m_0)\cdots b_{\text{alt}}(s+m_0-m_0)f^s. \end{aligned}$$

Multiplying  $b_{\text{alt}}(s+m_r-m_0+1)$  and  $b_{\text{alt}}(s+m_r-m_0)$  we obtain  $(s+m_r)^{(m_0)}$  for  $1 \leq r \leq l$ . Together with the remaining factor  $b_{\text{alt}}(s-m_0+m_0)$  the  $b$ -function of  $f$  is

$$\begin{aligned} b_f(s) &= b_{\text{alt}}(s) \cdot \prod_{r=1}^l (s+m_r)^{(m_0)} \\ &= (s+1)(s+3)\cdots(s+m_0-1) \cdot \prod_{r=1}^l (s+m_r)^{(m_0)} \end{aligned}$$

as desired.  $\square$



#### 4. The proof of the main result

##### 4.1. Basic properties of operators in the Capelli identity of odd type

In this subsection we prove that the matrix appeared in the Capelli identity of odd type (Theorem 3) is alternating. The commutativity and the symmetry in Theorem 3 are also proved in Proposition 10.

Let  $T_{ij}$  be variables ( $1 \leq i, j \leq n$ ) satisfying  $T_{ij} = -T_{ji}$ . We use the notation of Subsection 2.3. For any  $u_1, u_2, \dots, u_l$  and  $v$  ( $u_r, v \in \mathbb{C}$ ), set

$$\alpha^l(\underline{u}) = \alpha^l(u_1, u_2, \dots, u_l) = \frac{\partial}{\partial T}(u_1)^t T \frac{\partial}{\partial T}(u_2)^t T \cdots {}^t T \frac{\partial}{\partial T}(u_l),$$

$$\beta(v) = {}^t T \frac{\partial}{\partial T}(v).$$

We define a left action of  $GL(n)$  on  $T_{ij}$  by

$$g \cdot T_{ij} = ({}^t g T g)_{ij} = \sum_{a,b=1}^n g_{ai} T_{ab} g_{bj} \quad (g \in GL(n)),$$

and denote the differentiation of this action by  $\lambda : \mathfrak{gl}_n \rightarrow \mathcal{D}$  (see Subsection 2.3 for the localized Weyl algebra  $\mathcal{D}$ ). Then its explicit form turns out to be

$$\lambda(E_{ij}) = \sum_{a=1}^n T_{ai} \frac{\partial}{\partial T_{aj}} = \left( {}^t T \frac{\partial}{\partial T} \right)_{ij},$$

where  $E_{ij} \in \mathfrak{gl}_n$  denotes the matrix unit. Thanks to this action we have the following lemma.

LEMMA 9. For any  $\underline{u} = (u_1, u_2, \dots, u_l)$  and  $v$ , we have

- (1)  $[\beta(v)_{ij}, \beta(v)_{st}] = \delta_{js} \beta(v)_{it} - \delta_{ti} \beta(v)_{sj}$
- (2)  $[\alpha^l(\underline{u})_{ij}, \beta(v)_{st}] = \delta_{js} \alpha^l(\underline{u})_{it} + \delta_{is} \alpha^l(\underline{u})_{tj}$ ,

where  $\delta_{js}$  denotes the Kronecker delta.

*Proof.* Since  $\beta(v) = \beta(0) + v1_n$ , it suffices to show (1) and (2) when  $v = 0$ . Denote  $\beta(0)$  simply by  $\beta$ .

(1) When  $v = 0$  the equation is nothing but the commutation relation of  $E_{ij}$  and  $E_{st} \in \mathfrak{gl}_n$ , and it holds by  $\beta_{ij} = \lambda(E_{ij})$ .

(2) Note that the left-hand side of the equation corresponds to the action of  $\mathfrak{gl}_n$  on  $\alpha^l(\underline{u})$ .  $GL(n)$  acts on  ${}^t T$  and  $(\partial/\partial T)(u)$  by

$$g \cdot {}^t T = {}^t g^t T g, \quad g \cdot \frac{\partial}{\partial T}(u) = g^{-1} \frac{\partial}{\partial T}(u) {}^t g^{-1} \quad (g \in GL(n)),$$

since  $(\partial/\partial T)(u) = \partial/\partial T + u^t T^{-1}$ . Therefore the action of  $GL(n)$  on  $\alpha^l(\underline{u})$  is

$$g \cdot \alpha^l(\underline{u}) = g^{-1} \alpha^l(\underline{u}) {}^t g^{-1},$$

and this means that the action, which is the transposed inverse of the action of  $GL(n)$  on  ${}^t T$ , is independent of  $l$ . Thus we have only to prove the formula when  $l = 1$ . Moreover we may only consider the action of  $\mathfrak{gl}_n$  on  $\partial/\partial T$ , since  $\partial/\partial T$  and  ${}^t T^{-1}$  subject to the same action. Then the desired formula is just a result of a straight forward computation.  $\square$

PROPOSITION 10. For any  $\underline{u} = (u_1, u_2, \dots, u_l)$  and  $v$ , the following properties hold:

- (1)  $\alpha^l(\underline{u})$  is symmetric with respect to  $u_1, u_2, \dots, u_l$ .
- (2)  $\alpha^l(\underline{u})$  is an alternating matrix.
- (3) The entries of  $\alpha^l(\underline{u})$  commute with each other.

*Proof.* (1) It follows from  $(\partial/\partial T)(u) = \partial/\partial T + u^t T^{-1}$  that

$$\begin{aligned}\alpha^l(\underline{u}) &= \frac{\partial}{\partial T}(u_1)^t T \frac{\partial}{\partial T}(u_2)^t T \cdots {}^t T \frac{\partial}{\partial T}(u_l) \\ &= \alpha^l + e_1 \alpha^{l-1} + \cdots + e_{l-1} \alpha^1 + e_l^t T^{-1},\end{aligned}$$

where  $\alpha^r = \alpha^r(0, 0, \dots, 0)$ , and  $e_r$  is the elementary symmetric polynomial in  $u_1, u_2, \dots, u_l$ . Therefore  $\alpha^l(\underline{u})$  is symmetric with respect to  $u_1, u_2, \dots, u_l$ .

(2) We proceed by induction on  $l$ . When  $l = 1$  (2) is trivial.

Let  $l \geq 1$ , and suppose that (2) holds up to  $l$ . For (2) in  $l + 1$ , by using Lemma 9 (2) we compute as

$$\begin{aligned}\alpha^{l+1}(\underline{u}, v)_{ij} &= \sum_{a=1}^n \alpha^l(\underline{u})_{ia} \beta(v)_{aj} \\ &= \sum_{a=1}^n (\beta(v)_{aj} \alpha^l(\underline{u})_{ia} + \delta_{aa} \alpha^l(\underline{u})_{ij} + \delta_{ia} \alpha^l(\underline{u})_{ja}) \\ &= \sum_{a=1}^n \beta(v)_{aj} \alpha^l(\underline{u})_{ia} + (n-1) \alpha^l(\underline{u})_{ij}.\end{aligned}$$

We use Lemma 5 (2) to the first term, and the above expression equals

$$\sum_{a=1}^n \left( \frac{\partial}{\partial T}(v)^t T \right)_{ja} (-1) \alpha^l(\underline{u})_{ai} = -\alpha^{l+1}(v, \underline{u})_{ji} = -\alpha^{l+1}(\underline{u}, v)_{ji},$$

where we used the induction hypothesis for  $\alpha^l(\underline{u})$  and the symmetry with respect to the parameters in turn.

(3) We proceed by induction on  $l$ . (3) holds by Lemma 4 when  $l = 1$ .

Let  $l \geq 1$ , and suppose that (3) holds up to  $l$ . For (3) in  $l + 1$ , we compute by using the induction hypothesis and Lemma 9 as

$$\begin{aligned}[\alpha^{l+1}(\underline{u}, v)_{ij}, \alpha^{l+1}(\underline{u}, v)_{st}] &= \sum_{a,b} [\alpha^l(\underline{u})_{ia} \beta(v)_{aj}, \alpha^l(\underline{u})_{sb} \beta(v)_{bt}] \\ &= \sum_{a,b} \alpha^l(\underline{u})_{ia} (-\delta_{ba} \alpha^l(\underline{u})_{sj} - \delta_{sa} \alpha^l(\underline{u})_{jb}) \beta(v)_{bt} \\ &\quad + \sum_{a,b} \alpha^l(\underline{u})_{sb} (\delta_{ab} \alpha^l(\underline{u})_{it} + \delta_{ib} \alpha^l(\underline{u})_{ta}) \beta(v)_{aj} \\ &\quad + \sum_{a,b} \alpha^l(\underline{u})_{sb} \alpha^l(\underline{u})_{ia} (\delta_{jb} \beta(v)_{at} - \delta_{ta} \beta(v)_{bj}).\end{aligned}$$

By organizing the expression we have

$$\begin{aligned}
 & -\alpha^l(\underline{u})_{sj}\alpha^{l+1}(\underline{u}, v)_{it} - \alpha^l(\underline{u})_{is}\alpha^{l+1}(\underline{u}, v)_{jt} + \alpha^l(\underline{u})_{it}\alpha^{l+1}(\underline{u}, v)_{sj} \\
 & + \alpha^l(\underline{u})_{si}\alpha^{l+1}(\underline{u}, v)_{tj} + \alpha^l(\underline{u})_{sj}\alpha^{l+1}(\underline{u}, v)_{it} - \alpha^l(\underline{u})_{it}\alpha^{l+1}(\underline{u}, v)_{sj} = 0.
 \end{aligned}$$

□

**4.2. Proof of Theorem 3**

We use the notation of the previous subsection. Let  $n = 2m$ , and  $l$  be a positive integer. Let  $e_i \in \mathbb{C}^n$  be the standard basis, and we work with the exterior algebra  $\bigwedge \mathbb{C}^n$ . For  $\underline{u} = (u_1, u_2, \dots, u_l) \in \mathbb{C}^l$  and  $v \in \mathbb{C}$  we set

$$\begin{aligned}
 \theta^l(\underline{u}) &= \sum_{i,j=1}^n e_i e_j \alpha^l(\underline{u})_{ij} \in \bigwedge \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{D}, \\
 \eta_i(v) &= \sum_{j=1}^n \beta(v)_{ij} e_j \in \bigwedge \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{D}.
 \end{aligned}$$

The following two lemmas can be proved by using Lemma 9.

LEMMA 11. *We have the following commutation relations.*

- (1)  $\eta_s(v)\theta^l(\underline{u}) = \theta^l(\underline{u})\eta_s(v + 2)$ .
- (2)  $\eta_i(v)\eta_s(v + 1) = -\eta_s(v)\eta_i(v + 1)$ . □

LEMMA 12. *For  $l \geq 3$  and  $\underline{u} = (u_1, u'_l, u_l) = (u_1, u_2, \dots, u_{l-1}, u_l)$ , we have*

$$\theta^l(\underline{u}) = \sum_{b,c=1}^n \alpha^{l-2}(\underline{u}')_{bc} \eta_b(u_1) \eta_c(u_l).$$

□

Let us prove Theorem 3 by induction on  $l$ . Recall the condition  $u_r = u_{l-r+1} - 1$  in Theorem 3. When  $l = 1$ , the theorem is clear. Suppose that the theorem is proved up to a positive *odd* integer  $l - 2$ . We prove the theorem first for the *odd* integer  $l$  and then for the *even* integer  $l - 1$ . First we compute  $(\theta^l(\underline{u}))^m$  as

$$(\theta^l(\underline{u}))^m = 2^m m! e_1 e_2 \cdots e_n \text{ pf}(\alpha^l(\underline{u})). \tag{10}$$

Next we compute  $(\theta^l(\underline{u}))^m$  in another way.

$$\begin{aligned}
 (\theta^l(\underline{u}))^m &\stackrel{\text{Lem. 12}}{=} \sum_{b_1, c_1} \alpha^{l-2}(\underline{u}')_{b_1, c_1} \eta_{b_1}(u_1) \eta_{c_1}(u_l) \cdot (\theta^l(\underline{u}))^{m-1} \\
 &\stackrel{\text{Lem. 11 (1)}}{=} \sum_{b_1, c_1} \alpha^{l-2}(\underline{u}')_{b_1, c_1} (\theta^l(\underline{u}))^{m-1} \cdot \eta_{b_1}(u_1 + n - 2) \eta_{c_1}(u_l + n - 2) \\
 &\vdots
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{b_1, \dots, b_m \\ c_1, \dots, c_m}} \alpha^{l-2}(\underline{u})_{b_1, c_1} \cdots \alpha^{l-2}(\underline{u})_{b_m, c_m} \\
 &\quad \cdot \eta_{b_m}(u_1) \eta_{c_m}(u_l) \cdot \eta_{b_{m-1}}(u_1 + 2) \eta_{c_{m-1}}(u_l + 2) \\
 &\quad \cdots \eta_{b_1}(u_1 + n - 2) \eta_{c_1}(u_l + n - 2). \tag{11}
 \end{aligned}$$

By the relation  $u_1 = u_l - 1$  and Lemma 11 (2),  $n$  indices  $b_1, \dots, b_m, c_1, \dots, c_m$  should be distinct. Hence we have

$$\begin{aligned}
 \text{RHS of (11)} &= \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \alpha^{l-2}(\underline{u}')_{\sigma(1), \sigma(2)} \cdots \alpha^{l-2}(\underline{u}')_{\sigma(n-1), \sigma(n)} \\
 &\quad \cdot \eta_1(u_1) \eta_2(u_1 + 1) \cdots \eta_n(u_1 + n - 1). \tag{12}
 \end{aligned}$$

In the above expression, part of  $\alpha$ 's is equal to the pfaffian of the matrix  $\alpha^{l-2}(\underline{u}')$  multiplied by  $2^m m!$ . In addition,  $\eta_1(u_1) \eta_2(u_1 + 1) \cdots \eta_n(u_1 + n - 1)$  is equal to  $e_1 e_2 \cdots e_n$  multiplied by the row-determinant of the matrix  $\beta(u_1) + \text{diag}(0, 1, \dots, n - 1)$ , where  $\text{diag}(0, 1, \dots, n - 1)$  denotes the diagonal matrix with the diagonal entries  $0, 1, \dots, n - 1$ . It is well-known that this row-determinant is equal to the column-determinant of the matrix  $\beta(u_1) + \text{diag}(n - 1, n - 2, \dots, 0)$ , since  $\beta_{ij} = \lambda(E_{ij})$  is the image of a generator of  $\mathfrak{gl}_n$ . Then it follows from (11.3.20) of Howe-Umeda [3] that this column-determinant is equal to  $\det({}^t T) \det((\partial/\partial T)(u_1 + 1)) = \text{pf}({}^t T)^2 \text{pf}((\partial/\partial T)(u_1))^2$ . Note that there is no parameter in [3], but we can add parameters by the conjugation  $z \mapsto f^{-u_1-1} z f^{u_1+1}$ . Thus we have by the induction hypothesis

$$\begin{aligned}
 \text{RHS of (12)} &= 2^m m! e_1 e_1 \cdots e_n \text{pf}(\alpha^{l-2}(\underline{u}')) \\
 &\quad \cdot \text{pf}({}^t T) \text{pf}({}^t T) \text{pf}\left(\frac{\partial}{\partial T}(u_l)\right) \text{pf}\left(\frac{\partial}{\partial T}(u_l)\right) \\
 &= 2^m m! e_1 e_1 \cdots e_n \text{pf}\left(\frac{\partial}{\partial T}(u_2)\right) \text{pf}({}^t T) \cdots \text{pf}({}^t T) \text{pf}\left(\frac{\partial}{\partial T}(u_{l-1})\right) \\
 &\quad \cdot \text{pf}({}^t T) \text{pf}\left(\frac{\partial}{\partial T}(u_1)\right) \text{pf}({}^t T) \text{pf}\left(\frac{\partial}{\partial T}(u_l)\right). \tag{13}
 \end{aligned}$$

Since the parameters  $u_1, \dots, u_l$  have the symmetry in  $\alpha^l(\underline{u})$ , we have proved the theorem for *odd* integer  $l$  by comparing equations (10) and (13).

The remaining part of the proof is to show the theorem for  $l - 1$ . Since  $l$  is odd, the parameter  $u_{(l+1)/2}$  has no relation with the other parameters. We take the coefficient of the top degree of  $u_{(l+1)/2}$ , and obtain

$$\begin{aligned}
 &\text{pf}\left(\frac{\partial}{\partial T}(u_1) \cdots {}^t T \cdot {}^t T^{-1} \cdot {}^t T \cdots \frac{\partial}{\partial T}(u_l)\right) \\
 &= \text{pf}\left(\frac{\partial}{\partial T}(u_1)\right) \cdots \text{pf}({}^t T) \text{pf}({}^t T^{-1}) \text{pf}({}^t T) \cdots \text{pf}\left(\frac{\partial}{\partial T}(u_l)\right).
 \end{aligned}$$

Since  $\text{pf}({}^t T) \text{pf}({}^t T^{-1}) \text{pf}({}^t T) = (-1)^m \text{pf}({}^t T)$ , this proves the theorem for  $l - 1$ . Thus we complete the induction process, and Theorem 3 is proved.

### References

- [ 1 ] Alfredo Capelli. Ueber die Zurückführung der Cayley'schen Operation  $\Omega$  auf gewöhnliche Polar-Operationen. *Math. Ann.*, 29(3): 331–338, 1887.
- [ 2 ] Alfredo Capelli. Sur les Opérations dans la théorie des formes algébriques. *Math. Ann.*, 37 (1): 1–37, 1890.
- [ 3 ] Roger Howe and Tôru Umeda. The Capelli identity, the double commutant theorem, and multiplicity-free actions. *Math. Ann.*, 290 (3): 565–619, 1991.
- [ 4 ] Minoru Itoh. A Cayley-Hamilton theorem for the skew Capelli elements. *J. Algebra*, 242 (2): 740–761, 2001.
- [ 5 ] Kenji Kinoshita and Masato Wakayama. Explicit Capelli identities for skew symmetric matrices. *Proc. Edinb. Math. Soc.* (2), 45 (2): 449–465, 2002.
- [ 6 ] Fumihiro Sato and Kazunari Sugiyama. Multiplicity one property and the decomposition of  $b$ -functions. *Internat. J. Math.*, 17 (2): 195–229, 2006.
- [ 7 ] Kazunari Sugiyama.  $b$ -functions associated with quivers of type A. *Transform. Groups*, 16 (4): 1183–1222, 2011.
- [ 8 ] Kazunari Sugiyama.  $b$ -functions and the representation theory of quivers. In *New developments in group representation theory and non-commutative harmonic analysis*, RIMS Kôkyûroku Bessatsu, B36, pages 1–16. Res. Inst. Math. Sci. (RIMS), Kyoto, 2012.
- [ 9 ] Akihito Wachi. Logarithmic derivative and the Capelli identities. In *New developments in group representation theory and non-commutative harmonic analysis*, RIMS Kôkyûroku Bessatsu, B36, pages 43–52. Res. Inst. Math. Sci. (RIMS), Kyoto, 2012.
- [ 10 ] Akihito Wachi. The  $b$ -function of a reducible prehomogeneous vector space and a scalar generalized Verma module. *Sûrikaiseikikenkyûsho Kôkyûroku*, 1825: 35–54, 2013. Perspectives of Representation Theory and Noncommutative Harmonic Analysis (Japanese) (Kyoto, 2012).

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