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Discontinuous Stationary Solution to Generalized Eikonal-Curvature Equation and Its Stability

by

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Dedicated to Professor Fumihiro Sato on the occasion of his 65th birthday

1. Introduction

In this paper we consider stability of stationary solutions to an eikonal-curvature flow equation

$$V = C + \mathcal{K} \quad \text{on } \Gamma_t \tag{1.1}$$

for an evolving interface Γ_t in a domain $\Omega \subset \mathbb{R}^N$ with a level set method, where C is a nonnegative constant, and V, $\mathcal{K} = \sum_{j=1}^N \kappa_j$ and κ_j, respectively, is the normal velocity, mean and principal curvature of Γ_t defined with the outer continuous unit normal vector field $\mathbf{n} \in S^{N-1}$ of Γ_t; we call $\mathbf{n}$ the orientation of the evolution. Note that "interface" means the boundary of an open set called "interior" so that $\mathcal{K}$ is not positive for smooth boundary of a convex open set. When Ω has smooth boundary $\partial\Omega$, we impose the right angle condition

$$\Gamma_t \perp \partial\Omega \tag{1.2}$$

between Γ_t and $\partial\Omega$ to (1.1).

A level set method, which is introduced by Osher and Sethian [30] in the numerical analysis on evolving interfaces, describes the evolving interface Γ_t by

$$\Gamma_t = \{x \in \overline{\Omega};\ u(t,x) = c\} \tag{1.3}$$

with an auxiliary function $u\colon [0,\infty) \times \overline{\Omega} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$. The orientation of the evolution is given as

$$\mathbf{n} = -\frac{\nabla u}{|\nabla u|}$$

by setting an interior of Γ_t as the superlevel set $\{x \in \Omega;\ u(t,x) > c\}$, where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_N)$. Then we obtain the level set equation of (1.1) as

$$u_t - |\nabla u| \left\{ \operatorname{div} \frac{\nabla u}{|\nabla u|} + C \right\} = 0 \quad \text{in } (0,T) \times \Omega\,, \tag{1.4}$$

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where

$$\operatorname{div} P(x) = \sum_{j=1}^{N} \frac{\partial p_j}{\partial x_j}(x)$$

for $P(x) = (p_1(x), \dots, p_N(x)) \in \mathbb{R}^N$. The right angle condition (1.2) is represented as the Neumann boundary condition of u;

$$\langle \vec{\nu}, \nabla u \rangle = 0 \quad \text{in } (0, T) \times \partial\Omega\,, \tag{1.5}$$

where $\vec{\nu} \in S^{N-1}$ is the outer unit normal vector field of $\partial\Omega$. The system of equations (1.4)–(1.5) in Ω is formally stronger than (1.1)–(1.2) since (1.4)–(1.5) imposes that all level sets of u evolve with (1.1)–(1.2) although (1.1)–(1.2) holds only on Γ_t. See [16] for details.

A level set method is powerful to study the evolution of Γ_t including singularities, i.e., vanishing of Γ_t in finite time, or collision with each other. It is well-known that the evolving planer closed simple curve moved by (1.1) with $C = 0$ becomes convex in finite time and converges to a single point; see [13, 20]. Although the first behavior does not hold for closed compact hypersurface in $\mathbb{R}^3$ (see [21]), the vanishing property is easily extended to the evolving closed compact hypersurface in $\mathbb{R}^N$ by a level set method. Y.-H. R. Tsai, Y. Giga and the author [29] extend the idea of the level set method to spiral curves on the plane based on an adjusted level set method for spiral [28]. One can find in [29], moreover, their formulation works well even if the topological change of the curves occurs during the evolution of spirals.

However, this method also has difficulties caused by the implicit representation of interfaces, which is the main topic of this paper. In this paper we consider the existence and stability of stationary solutions to (1.1) with the level set formulation. One can easily find some stationary interfaces of (1.1) as the hypersurface with the constant mean curvature. We first prove that there exist discontinuous (so that weak) stationary solutions to (1.4) describing an interface with a constant mean curvature which is a boundary of N-dimensional submanifold. We next prove that there are no continuous stationary solutions describing a sphere in $\mathbb{R}^N$ provided $C \neq 0$, or a hyperplane contacting to $\partial\Omega$ of a sandglass-type domain $\Omega \subset \mathbb{R}^N$ at its neck with the right angle provided that $C = 0$. From the proof of these facts we also deduce that the first stationary solution is unstable which was proved in [19], and the second one is asymptotically stable with set-theoretic approach.

The stationary solution to (1.1) with $C \neq 0$ means an interface with a constant mean curvature. In this paper we treat only a sphere or a cylindrical surface; however there exist various hypersurfaces with constant mean curvature; see [33, 25, 26]. A compact hypersurface with a constant mean curvature is characterized as a stable solution to the variational problem minimizing surface area provided that the measure of the domain enclosed by the surface is a constant. The result of this paper means that a boundary of a domain with a constant mean curvature is unstable from a view point of eikonal-curvature flow (1.1) even if the domain and boundary are compact.

From a view point of stability of stationary solution to an evolution equation, Ei and Yanagida [10, 11] or Ei, Sato and Yanagida [9] investigate the stability or instability of stationary solutions to a generalized mean curvature flow including (1.1) with contact angle condition. Their method is extended to a surface diffusion flow by [14]. As in these papers we often consider a linearization of problems around a stationary solution: see [2] or [27] for details of linearization of nonlinear ordinary or partial differential equations, respectively. In particular, the existence of solutions to (1.1) in [9] is guaranteed with aid of level set method. Giga and Yama-Uchi [19] proved Lyapunov instability of stationary interfaces evolving by an evolution equation depending on the second fundamental form of the interface, which includes (1.1). In this paper we concentrate our attention to (1.1) with $C \neq 0$ and an unstable stationary sphere, or (1.1)–(1.2) with $C = 0$ and stable stationary hyperplane at the neck of a sandglass-type domain. Our method is close to [19] and completely different from [10, 11, 9]; we construct a supersolution with a quadratic function and appeal to the comparison principle. For the results on the unstable stationary sphere, the difference of our results from [19] is to prove that there are no continuous stationary solutions but there exists a discontinuous stationary solution to the level set equation describing the stationary sphere. This fact is not mentioned in [19]. On the other hand, a nonstationary solution u keeps the stationary sphere as a level set $\{x;\ u(t,x) = c\}$ with $c \in \mathbb{R}$ if the sphere is given as $\{x;\ u(0,x) = c\}$ by the existence result and the comparison principle as in [7]. By combining these facts our result means that the stationary solution given as in [9] which is a center of linearization is given by nonstationary continuous or discontinuous stationary viscosity solution. Moreover, our results point out that there may be no suitable center of linearization for (1.4) around a stationary solution if we consider the stability or instability of a stationary interface by a level set method. We also refer [17] for the result of convergence to stationary solution in cylindrical domain by the strong maximum principle of a stationary problem to (1.4)–(1.5) with $C = 0$.

We also mention on a discontinuous solution from another view point. The discontinuous stationary solutions we find are based on characteristic function of an open set, and then they are examples of set-theoretic solution which is introduced in [16]. There is a pioneering work on set-theoretic approach to (1.1) including anisotropic evolution by Soner [32], which is based on the signed distance function. The definition by [16] corresponds to some approximation algorithms to (1.1): see [5, 1, 4, 6, 12]. One can find a lot of characterization of set-theoretic solution in [16]. We calculate weak derivatives of characteristic function in viscosity solution sense in this paper. Moreover, we observe that discontinuous solutions to (1.4) play important role to describe stationary interfaces to (1.1).

This paper is organized as follows. We first summarize the definition and some properties of viscosity solutions in §2. We also give some properties of the characteristic functions for an open set from the view point of viscosity solutions. We next consider the stationary ball solution to (1.1) with $C \neq 0$ in $\Omega = \mathbb{R}^N$ in §3. We finally consider the stationary hyperplane to (1.1) with $C = 0$ in the axis-symmetric domain in §4.

2. Viscosity solutions

In this section we summarize the theory of viscosity solutions to the degenerate parabolic equation, and show some fundamental properties which are used in the following sections. See [16] for details.

2.1. Definitions.

Let $\Omega \subset \mathbb{R}^N$ be a domain and $T > 0$. We consider a geometric degenerate parabolic equation of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0, T) \times \Omega \tag{2.1}$$

including (1.1). We also consider the Neumann boundary condition

$$\langle \vec{\nu}, \nabla u \rangle = 0 \quad \text{in } (0, T) \times \partial\Omega \tag{2.2}$$

if $\partial\Omega \neq \emptyset$. The equation (1.4) is represented with

$$F(p, X) = F_0(p, X) - C|p|, \tag{2.3}$$

$$F_0(p, X) = -\text{trace}\left\{\left(I_N - \frac{p \otimes p}{|p|^2}\right) X\right\} \tag{2.4}$$

for $p \in \mathbb{R}^N \setminus \{0\}$ and $X \in \mathbb{S}^N$, where $\mathbb{S}^N$ is the space of real $N \times N$ symmetric matrices, and $p \otimes q = (p_i q_j)_{1 \le i,j \le N}$ for $p = (p_1, \ldots, p_N)$, $q = (q_1, \ldots, q_N) \in \mathbb{R}^N$.

Note that the equation (2.1) for (1.1) is not defined where $\nabla u = 0$ though we have to handle this situation. In the theory of viscosity solutions we often extend equations or solutions with upper or lower semicontinuous envelope to overcome such a difficulty. For $f : \mathbb{R}^d \supset D \to \mathbb{R}$ define $f^* : \overline{D} \to \mathbb{R} \cup \{\infty\}$ or $f_* : \overline{D} \to \mathbb{R} \cup \{-\infty\}$ as

$$f^*(x) = \lim_{r \to 0} \sup\{f(y);\ |y - x| < r\}, \qquad f_*(x) = \lim_{r \to 0} \inf\{f(y);\ |y - x| < r\},$$

respectively. We call f^* or f_* upper or lower semicontinuous envelope, respectively. Note that $f_* \le f \le f^*$ in D, and f is upper (resp. lower) semicontinuous if and only if $f^* = f$ (resp. $f_* = f$).

We now list the assumptions for F.

(F1) $F : \mathcal{J} := (\mathbb{R}^N \setminus \{0\}) \times \mathbb{S}^N \to \mathbb{R}$ is continuous.

(F2) $-\infty < F_*(0, O) = F^*(0, O) < +\infty$, where O is the zero matrix.

(F3) F is degenerate elliptic, i.e.,

$$F(p, X + Y) \le F(p, X)$$
$$\text{for } (p, X) \in \mathcal{J},\ Y \in \mathbb{S}^N \text{ provided that } Y \ge O,$$

(F4) F is geometric, i.e.,

$$F(\lambda p, \lambda X + \mu p \otimes p) = \lambda F(p, X) \quad \text{for } (p, X) \in \mathcal{J},\ \lambda > 0,\ \mu \in \mathbb{R},$$

(F5) There exist positive constants K_1, K_2, K_3, K_4 such that, if $X, Y \in \mathbb{S}^N$ and non-negative constants $\gamma_1, \gamma_2, \gamma_3$ satisfying

$$\langle X\xi, \xi \rangle + \langle Y\eta, \eta \rangle$$
$$\le \gamma_1 |\xi - \eta|^2 + \gamma_2(|\xi|^2 + |\eta|^2) + \gamma_3 |\xi - \eta|(|\xi| + |\eta|)$$

for $\xi, \eta \in \mathbb{R}^N$, then

$$F(p, X) - F(q, -Y)$$
$$\ge -K_1\gamma_1|\bar{p} - \bar{q}|^2 - K_2\gamma_2 - K_3\gamma_3|\bar{p} - \bar{q}| - K_4|p - q|$$

for $p, q \in \mathbb{R}^N \setminus \{0\}$, where $\bar{p} = p/|p|$, Note that (2.4) and then (2.3) satisfy (F1)–(F5). Moreover, note that F_*, F^* also satisfy (F3) or (F4) if F is so, respectively.

We now recall the definition of viscosity solution, which is a weak solution to a degenerate parabolic or elliptic equation based on the maximum principle of C^2 function.

DEFINITION 2.1. We say $u\colon (0,T)\times\Omega\to\mathbb{R}$ is a viscosity sub- (resp. super-) solution to (2.1) if the followings hold;

(S1) $u^* < \infty$ (resp. $u_* > -\infty$) in $[0,T]\times\overline{\Omega}$.

(S2) for each $(\hat{t},\hat{x})\in(0,T)\times\Omega$ and $\varphi\in C^2((0,T)\times\Omega)$ satisfying

$$u^*(t,x)-\varphi(t,x)\le u^*(\hat{t},\hat{x})-\varphi(\hat{t},\hat{x})$$
$$(\text{resp. } u_*(t,x)-\varphi(t,x)\ge u_*(\hat{t},\hat{x})-\varphi(\hat{t},\hat{x}))$$

for $(t,x)\in(0,T)\times\Omega$,

$$\varphi_t(\hat{t},\hat{x})+F_*(\nabla\varphi(\hat{t},\hat{x}),\nabla^2\varphi(\hat{t},\hat{x}))\le 0 \tag{2.5}$$

$$(\text{resp. } \varphi_t(\hat{t},\hat{x})+F^*(\nabla\varphi(\hat{t},\hat{x}),\nabla^2\varphi(\hat{t},\hat{x}))\ge 0) \tag{2.6}$$

holds.

We say u is a viscosity sub- (resp. super-) solution to (2.1)–(2.2) if u satisfies (S1) and the following (S2)$'$ instead of (S2) hold;

(S2)$'$ for each $(\hat{t},\hat{x})\in(0,T)\times\overline{\Omega}$ and $\varphi\in C^2((0,T)\times\overline{\Omega})$ satisfying

$$u^*(t,x)-\varphi(t,x)\le u^*(\hat{t},\hat{x})-\varphi(\hat{t},\hat{x})$$
$$(\text{resp. } u_*(t,x)-\varphi(t,x)\ge u_*(\hat{t},\hat{x})-\varphi(\hat{t},\hat{x}))$$

for $(t,x)\in(0,T)\times\overline{\Omega}$,

(a) (2.5) (resp. (2.6)) holds if $\hat{x}\in\Omega$,

(b) either (2.5) (resp. (2.6)) or

$$\langle\vec{\nu}(\hat{x}),\nabla\varphi(\hat{t},\hat{x})\rangle\le 0 \quad (\text{resp. } \langle\vec{\nu}(\hat{x}),\nabla\varphi(\hat{t},\hat{x})\rangle\ge 0)$$

holds if $\hat{x}\in\partial\Omega$.

We say u is a viscosity solution if u is viscosity sub- and supersolution.

See [8] for details of viscosity solutions, or [7] for theory of viscosity solutions to degenerate parabolic equations. Note that C^2 classical sub- or supersolution to (2.1) is viscosity sub- or supersolution if (F3) holds. These theory is extended to the Neumann boundary value problem in a bounded domain of geometric and degenerate parabolic equation; see [18] or [31]. In this paper we consider all solutions in viscosity solution sense so that we omit the word of "viscosity" here and hereafter.

It is convenient to introduce an equivalent definition to the conditions of (S2) by [24].

LEMMA 2.2 ([3], [16]). *Assume that* (F3) *holds. Then*, *the condition* (S2) *is equivalent to the following conditions.*

(i) *If* $|\nabla\varphi(\hat{t},\hat{x})| \neq 0$, *then*

$$\varphi_t + F(\nabla\varphi, \nabla^2\varphi) \leq 0 \ (\textit{resp. } \varphi_t + F(\nabla\varphi, \nabla^2\varphi) \geq 0) \quad \textit{at } (\hat{t},\hat{x}).$$

(ii) *If* $|\nabla\varphi(\hat{t},\hat{x})| = 0$, *then*

$$\varphi_t + F_*(0, O) \leq 0 \ (\textit{resp. } \varphi_t + F^*(0, O) \geq 0) \quad \textit{at } (\hat{t},\hat{x})$$

provided that $\nabla\varphi(\hat{t},\hat{x}) = 0$ *and* $\nabla^2\varphi(\hat{t},\hat{x}) = O$.

Note that the paper [3] derives the above for a geometric evolution equation and then the condition (ii) is

$$\varphi_t(\hat{t},\hat{x}) \leq 0 \quad (\text{resp. } \varphi_t(\hat{t},\hat{x}) \geq 0)$$

provided that $\nabla\varphi(\hat{t},\hat{x}) = 0$ and $\nabla^2\varphi(\hat{t},\hat{x}) = O$; in fact $F^*(0, O) = F_*(0, O) = 0$. However, we can prove this lemma without the assumption (F4) by revising the conclusion of the case (ii) as the above; see [16, Proposition 2.2.8].

2.2. Remarks on the existence and uniqueness.

The existence of solution to (2.1) is established by Perron's method due to H. Ishii [22]. This method is based on the following two propositions.

PROPOSITION 2.3. *Let* $\mathcal{S}$ *be a non-empty set of subsolutions* (*resp. supersolutions*) *to* (2.1) *in* $(0, T) \times \Omega$. *Assume that functions in* $\mathcal{S}$ *are locally uniformly bounded in* $(0, T) \times \Omega$. *Then*,

$$u(t, x) = \sup\{v(t, x);\ v \in \mathcal{S}\} \quad (\textit{resp. } u(t, x) = \inf\{v(t, x);\ v \in \mathcal{S}\})$$

is still a subsolution (*resp. supersolution*) *to* (2.1) *in* $(0, T) \times \Omega$.

PROPOSITION 2.4 (Perron's method). *Assume that* (F3) *holds.*
Let $f, g\colon (0, T) \times \Omega \to \mathbb{R}$ *be a locally bounded sub- and supersolution to* (2.1) *in* $(0, T) \times \Omega$ *satisfying* $f \leq g$ *in* $(0, T) \times \Omega$. *Then*,

$$u(t, x) = \sup\left\{ v(t, x);\ \begin{array}{l} v \textit{ is a subsolution to } (2.1) \textit{ in } (0, T) \times \Omega\,, \\ f \leq v \leq g \quad \textit{in } [0, T] \times \overline{\Omega} \end{array} \right\}$$

is a solution to (2.1) *in* $(0, T) \times \Omega$.

This method is established by [22] for Hamilton-Jacobi equations, and is extended to second order degenerate elliptic equation in [23]. This method is also extended to the degenerate parabolic equation in [7] with an interpretation of (2.1) to a second order degenerate elliptic equation

$$E(Du, D^2u) = u_t + F(\nabla u, \nabla^2 u)$$

with a differential operator

$$D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right).$$

It is also extended to the Neumann boundary value problem in [31]. One also can find in [7] or [31] the detailed construction of a solution with initial data $u_0 \in BUC(\overline{\Omega})$ to (2.1), where $BUC(\overline{\Omega})$ denotes the space of bounded and uniformly continuous functions on $\overline{\Omega}$. By combining the comparison principle, which is explained later, one observes that the solution is continuous.

The uniqueness of solutions to (2.1) with respect to the initial data $u|_{t=0} = u_0$ is derived from the following comparison principle.

Comparison principle(CP) : Let u and v be a sub- and supersolution to (2.1) in $(0, T) \times \Omega$, respectively. Under suitable assumptions from boundary condition, if

$$u^*(0, x) \leq v_*(0, x) \quad \text{for} \quad x \in \overline{\Omega}\,, \tag{2.7}$$

then

$$u^*(t, x) \leq v_*(t, x) \quad \text{for} \quad (t, x) \in [0, T) \times \overline{\Omega}.$$

We now summarize the known results on the comparison principle. In the following sections we consider two types of domains;

- $\Omega = \mathbb{R}^N$ (in §3),
- axis-symmetric nonconvex domain (in §4):

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \ ; \ |x'| < r(x_N)\} \tag{2.8}$$

with a smooth function $r : \mathbb{R} \to (0, \infty)$ satisfying

(Ω1) $r' < 0$ in $(-\infty, 0), r' > 0$ in $(0, \infty)$,

(Ω2) there exists $\delta_0 > 0$ such that

$$\overline{B_{\delta_0}(x + \delta_0 \vec{\nu}(x))} \cap \overline{\Omega} = \overline{B_{\delta_0}(x - \delta_0 \vec{\nu}(x))} \cap \Omega^c = \{x\}$$

for $x \in \partial\Omega$, where $B_\delta(x_0) = \{x \in \mathbb{R}^N;\ |x - x_0| < \delta\}$.

If Ω is bounded, (CP) is established by [7] at least (2.1) with (F1)–(F3) and additional assumption from Dirichlet boundary condition as

$$u^*(t, x) \leq v_*(t, x) \quad \text{in } (0, T) \times \partial\Omega\,.$$

Their proof is easily extend to the case $\Omega = \mathbb{R}^N$ with (F1)–(F3) and

$$\begin{cases} \text{there exist } \alpha\,, \beta \in \mathbb{R} \text{ and } R > 0 \text{ such that} \\ u(t, x) = \alpha\,, v(t, x) = \beta \quad \text{if } |x| > R \text{ and } t \in [0, T)\,. \end{cases} \tag{2.9}$$

The above condition is essentially same as the Dirichlet boundary condition. Note that $\alpha \leq \beta$ by (2.7). In general (CP) for unbounded domains is established by [15] with additional assumptions of asymptotic behavior for sub- and supersolutions as $|x| \to \infty$ and boundedness assumptions for F; see [15] and [16] for details. For the Neumann boundary value problem on a bounded nonconvex domain (CP) is established by [18] for (2.1) satisfying (F1)–(F3) and (F5). When we consider the axis-symmetric domain, we additionally assume that

$$(2.10)\qquad \begin{cases} \text{there exist } \alpha_j\,, \quad \beta_j \in \mathbb{R} \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad R > 0 \quad \text{such that} \\ u(t, x) = \alpha_1\,, \quad v(t, x) = \alpha_2 \quad \text{if } x_N < -R\,, \\ u(t, x) = \beta_1\,, \quad v(t, x) = \beta_2 \quad \text{if} \quad x_N > R \\ \text{for} \quad t \in [0, T) \quad \text{and } x = (x', x_N) \in \overline{\Omega}\,. \end{cases}$$

Then (CP) is established by applying the proof in [18]. By (CP) we obtain the solution constructed from $u_0 \in BUC(\overline{\Omega})$ which satisfies (2.9) or (2.10) is not only unique but also continuous in $[0, \infty) \times \overline{\Omega}$.

The level set method establishes the evolution of interfaces by extracting implicitly described Γ_t as in (1.3) with a solution to the level set equation of the evolution equation. However, we remark that the initial data u_0 is not unique for given Γ_0 and implicit description (1.3); for example u_0^3 still describes Γ_0 if u_0 describes it with $c = 0$ level set. Thus, even if solution to the level set equation is unique with respect to initial data u_0, one can obtain several level sets started from Γ_0. It is very important property if Γ_t is determined uniquely with respect to Γ_0.

We shall conclude this section with mentioning the uniqueness of level sets. It is obtained from the comparison of interior or exterior sets. In the level set method we regard a set $\{x \in \overline{\Omega}; u(t, x) > c\}$ or $\{x \in \overline{\Omega}; u(t, x) < c\}$ is interior or exterior of Γ_t given by (1.3), respectively. We now deduce a comparison principle of interior or exterior set from (CP). We first recall the stability result of solutions.

PROPOSITION 2.5. *Let $u_k \colon (0, T) \times \Omega \to \mathbb{R}$ be a sub- (resp. super-) solution to* (2.1) *in $(0, T) \times \Omega$ for $k \in \mathbb{N}$. Assume that u_k converges to $u \colon (0, T) \times \Omega \to \mathbb{R}$ locally uniformly in $(0, T) \times \Omega$ as $k \to \infty$. If $u^* < \infty$ (resp. $u_* > -\infty$) in $(0, T) \times \Omega$, then u is a sub- (resp. super-) solution to* (2.1) *in $(0, T) \times \Omega$.*

Proposition 2.5 is established for more general equations in [7] and extended to the Neumann boundary problem in [31]. We now show the following rescaling invariance of dependent variable for geometric evolution equations.

LEMMA 2.6. *Assume that $(F1)$–$(F4)$ hold. Let u be a sub- (resp. super-) solution to* (2.1) *in $(0, T) \times \Omega$ and $G \colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous and nondecreasing function. Then, $G(u^*(t, x))$ (resp. $G(u_*(t, x))$) is still a sub- (resp. super-) solution to* (2.1) *in $(0, T) \times \Omega$.*

Proof. In this proof we demonstrate only that $w(t, x) = G(u^*(t, x))$ is a subsolution if u is so. Note that w is upper semicontinuous and thus $w^* = w$ in this case.

We first demonstrate that w is a subsolution to (2.1) provided that $G \in C^2(\mathbb{R})$ and $G' > 0$ in $\mathbb{R}$. Let $\varphi \in C^2((0, \infty) \times \Omega)$ and assume that

$$w(t, x) - \varphi(t, x) \leq w(\hat{t}, \hat{x}) - \varphi(\hat{t}, \hat{x}) \quad \text{for} \quad (t, x) \in (0, \infty) \times \Omega$$

We may assume that $w(\hat{t},\hat{x}) - \varphi(\hat{t},\hat{x}) = 0$ by considering $\varphi(t,x) - (w(\hat{t},\hat{x}) - \varphi(\hat{t},\hat{x}))$ instead of φ.

Since $G' > 0$ there exists $H = G^{-1} \in C^2(\mathbb{R})$ and $H' > 0$ in $\mathbb{R}$. We now define $\psi(t,x) = H(\varphi(t,x))$. Then,

$$u^*(t,x) - \psi(t,x) \le u^*(\hat{t},\hat{x}) - \psi(\hat{t},\hat{x}) = 0 \quad \text{for} \quad (t,x) \in (0,\infty) \times \Omega\,.$$

In fact, since $H' > 0$ and $w(t,x) = G(u^*(t,x)) \le \varphi(t,x)$ we obtain

$$u^*(t,x) \le H(\varphi(t,x)) = \psi(t,x)\,.$$

Moreover, we have

$$\psi(\hat{t},\hat{x}) = H(\varphi(\hat{t},\hat{x})) = H(G(u^*(\hat{t},\hat{x}))) = u^*(\hat{t},\hat{x})\,,$$

which implies

$$u^*(t,x) - \psi(t,x) \le 0 = u^*(\hat{t},\hat{x}) - \psi(\hat{t},\hat{x})$$

for $(t,x) \in (0,\infty) \times \Omega$.

By straightforward calculation we obtain

$$\psi_t = H'(\varphi)\varphi_t\,, \quad \nabla\psi = H'(\varphi)\nabla\varphi\,,$$

$$\nabla^2\psi = H'(\varphi)\nabla^2\varphi + H''(\varphi)\nabla\varphi \otimes \nabla\varphi\,.$$

Since u^* is a subsolution to (2.1) then we have

$$\psi_t + F_*(\nabla\psi, \nabla^2\psi) \le 0 \quad \text{at} \quad (\hat{t},\hat{x})\,,$$

which implies

$$H'(\varphi)(\varphi_t + F_*(\nabla\varphi, \nabla^2\varphi)) \le 0 \quad \text{at } (\hat{t},\hat{x})$$

by (F4). Since $H' > 0$ we obtain

$$\varphi_t + F_*(\nabla\varphi, \nabla^2\varphi)) \le 0 \quad \text{at } (\hat{t},\hat{x})\,.$$

Let G is uniformly continuous and nondecreasing. We now approximate G with smooth and strictly increasing G_ε. Let $G_\varepsilon = (G * \rho_\varepsilon)(s) + \varepsilon \tanh s$, where $G * \rho_\varepsilon$ is a convolution between G and ρ_ε, and $\rho_\varepsilon \in C^\infty(\mathbb{R})$ is a mollifier, i.e., $\rho_\varepsilon \ge 0$, $\int_{\mathbb{R}} \rho_\varepsilon = 1$ and $\text{supp}\rho_\varepsilon \subset [-\varepsilon, \varepsilon]$. Then, $G_\varepsilon \in C^\infty(\mathbb{R})$, $G' > 0$ and $\lim_{\varepsilon\to 0} G_\varepsilon = G$ uniformly in $\mathbb{R}$.

We now define $w_\varepsilon(t,x) = G_\varepsilon(u^*(t,x))$. Then, $\lim_{\varepsilon\to 0} w_\varepsilon(t,x) = w(t,x)$ uniformly on $[0,\infty) \times \Omega$ and thus w is a subsolution to (2.1) by Proposition 2.5. □

We are now in the position to state the comparison principle of interior and exterior sets.

THEOREM 2.7. *Either following (I) or (II) holds.*

(I) *For the case $\Omega = \mathbb{R}^N$, assume $(F1)$–$(F4)$ hold. Let u and v be a sub- and supersolution to* (2.1) *in* $(0,T) \times \Omega$ *satisfying* (2.9), *respectively.*

(II) *For the case Ω given by* (2.8) *satisfying* $(\Omega 1)$–$(\Omega 2)$, *assume that* $(F1)$–$(F5)$ *hold. Let u and v be a sub- and supersolution to* (2.1)–(2.2) *in* $(0,T) \times \overline{\Omega}$ *satisfying* (2.10), *respectively.*

If

(2.11) $\{x \in \overline{\Omega};\ u^*(0,x) > c_1\} \subset \{x \in \overline{\Omega};\ v_*(0,x) > c_2\}$

(2.12) (*resp.* $\{x \in \overline{\Omega};\ u^*(0,x) < c_1\} \supset \{x \in \overline{\Omega};\ v_*(0,x) < c_2\}$)

for some $c_1, c_2 \in \mathbb{R}$, *then*

(2.13) $\{x \in \overline{\Omega};\ u^*(t,x) > c_1\} \subset \{x \in \overline{\Omega};\ v_*(t,x) > c_2\}$

(2.14) (*resp.* $\{x \in \overline{\Omega};\ u^*(t,x) < c_1\} \supset \{x \in \overline{\Omega};\ v_*(t,x) < c_2\}$)

for $t \in [0, T]$.

Note that (CP) is available for the situations Theorem 2.7 considered. However, Theorem 2.7 requires no relation between u^* and v_* except (2.11) or (2.12) though (CP) is crucial to prove that. This comparison principle is the generalized result of [7, Theorem 7.1]. Moreover no regularity assumptions for initial data $u(0,\cdot)$ and $v(0,\cdot)$ are required in Theorem 2.7, which is the advantage to the result in [7, §7]. The proof is similar to that in [7], but we verify it here because we relax the assumptions; see also [16, Chapter 4].

Proof. We may assume that $c_1 = c_2 = 0$ without loss of generality by considering $u - c_1$ and $v - c_2$ instead of u and v, respectively. We here mention only the case (I) since the argument for the case (II) is parallel.

We divide the proof into three steps.

Step 1. We construct a rescaling function to apply the comparison principle.

We now define

$$G(s) = \sup\{(u^*(0,y))_+;\ v_*(0,y) \le s\},$$

where $(a)_+ = \max\{0, a\}$. Then, the followings hold.

(i) G is monotone nondecreasing, $G \ge 0$,
(ii) $G(s) = 0$ if $s \le 0$,
(iii) $u^*(0,x) \le G(v_*(0,x))$ for $x \in \overline{\Omega}$,
(iv) G is upper semicontinuous in $\mathbb{R}$, and continuous on $(-\infty, 0]$.

We now demonstrate these properties. The property (i) is derived directly from the definition of G. The property (ii) is derived from (2.11). In fact, from (2.11) we have $u^*(0,x) \le 0$ if $v_*(0,x) \le 0$, which implies

$$(u^*(0,x))_+ = 0 \quad \text{for } x \in \{y \in \overline{\Omega};\ v_*(0,y) \le 0\}$$

and thus $G(s) = 0$ for $s \le 0$. The property (iii) follows from definition of G. In fact, for fixed $x \in \overline{\Omega}$ we have $x \in \{y \in \overline{\Omega};\ v_*(0,y) \le v_*(0,x)\}$, which implies

$$u^*(0,x) \le (u^*(0,x))_+ \le G(v_*(0,x)).$$

Finally, we demonstrate (iv). Let

$$s^* := \sup\{\bar{s};\ G(s) = 0 \text{ for } s \in (-\infty, \bar{s})\},$$

then it is clear that G is continuous in $(-\infty, s^*)$ since $G = 0$ in $(-\infty, s^*)$. Let $\hat{s} \geq s^*$. Then, for each $k \in \mathbb{N}$ there exists $y_k \in \overline{\Omega}$ such that

$$u^*(0, y_k) > G(\hat{s} + k^{-1}) - k^{-1}, \quad v_*(0, y_k) \leq \hat{s} + k^{-1} \tag{2.15}$$

by the definition of G and s^* since $G(\hat{s} + k^{-1}) > 0$. We now divide the case into two cases.

Case 1. Assume that there exists k_0 such that $|y_k| > R$ for $k > k_0$, where $R > 0$ is such that $u^*(0, y) = \alpha$ and $v_*(0, y) = \beta$ if $|y| > R$. Then, $v_*(0, y_k) = \beta \leq \hat{s} + k^{-1}$ provided $k > k_0$, which implies $v_*(0, y_k) \leq \hat{s}$ for $k > k_0$ so that we have

$$u^*(0, y_k) \leq G(\hat{s})$$

provided that $k > k_0$. Then, for $\varepsilon > 0$ we choose $k \in (k_0, \infty)$ satisfying $k^{-1} < \varepsilon$. If $r \in (0, k^{-1})$ then

$$G(\hat{s} + r) \leq G(\hat{s} + k^{-1}) < u^*(0, y_k) + k^{-1} < G(\hat{s}) + \varepsilon\,. \tag{2.16}$$

Case 2. Assume that there exists a subsequence of y_k, which we denoted also by y_k, satisfying $|y_k| \leq R$. Then, we may assume that $\lim_{k\to\infty} y_k = y_0 \in \mathbb{R}^N$ by taking a subsequence of y_k if necessary. Then the second inequality of (2.15) and lower semicontinuity of v_* imply that

$$v_*(0, y_0) \leq \hat{s}$$

and thus $u^*(0, y_0) \leq G(\hat{s})$. For $\varepsilon > 0$ we choose $k \in (0, \infty)$ such that $k^{-1} < \varepsilon/2$ and $u^*(0, y_k) < u^*(0, y_0) + \varepsilon/2$ by the upper semicontinuity of u^*. If $r \in (0, k^{-1})$ then

$$\begin{aligned} G(\hat{s} + r) &\leq G(\hat{s} + k^{-1}) < u^*(0, y_k) + k^{-1} < u^*(0, y_0) + \varepsilon \\ &\leq G(\hat{s}) + \varepsilon\,. \end{aligned} \tag{2.17}$$

The inequalities (2.16) and (2.17) imply $\overline{\lim}_{s\to\hat{s}+0} G(s) = G(\hat{s})$: here we have used $G(\hat{s} + r) \geq G(\hat{s})$. On the other hand $\overline{\lim}_{s\to\hat{s}-0} G(s) \leq G(\hat{s})$ by (i). Hence, we obtain the upper semicontinuity of G.

It remains the continuity of G at $s = 0$. Note that $s^* \geq 0$ by (ii), thus it suffices to prove the continuity of G at $s = 0$ provided that $s^* = 0$. In this case there exists sequence $y_k \in \mathbb{R}^N$ satisfying (2.15) with $\hat{s} = 0$, since $G(k^{-1}) > 0$. Then, the parallel argument of the above Case 1 or Case 2 yields that either the following Case 3 or 4 holds;

Case 3. There exists $k_0 > 0$ such that $v_*(0, y_k) \leq 0$ provided that $k > k_0$,

Case 4. There exists $y_0 = \lim_{k\to\infty} y_k$ satisfying $v_*(0, y_0) \leq 0$ by taking subsequence of y_k.

If Case 3 holds then $u^*(0, y_k) \leq 0$ provided that $k > k_0$, and if Case 4 holds then $u^*(0, y_0) \leq 0$ by (2.11). This implies that $\overline{\lim}_{s\to+0} G(s) \leq 0$ by the parallel argument of the above Case 1 and 2. Since $G \geq 0$ in $\mathbb{R}$ and $G = 0$ on $(-\infty, 0]$ we obtain $\lim_{s\to 0} G(s) = 0 = G(0)$.

Step 2. We now mollify G to uniformly continuous, monotone nondecreasing function.

By (2.9) and upper semicontinuity of u^* there exists $K > 0$ such that $u^*(0,x) \le K$ for $x \in \mathbb{R}^N$. We first define

$$\bar{G}(2^k) := G(2^{k+1}) \quad \text{if} \quad k \in \mathbb{Z} \cap (-\infty, 0]\,,$$
$$\bar{G}(2) := K \ .$$

We next define $\bar{G}$ in $[2^k, 2^{k+1}]$ for $k \in \mathbb{Z} \cap (-\infty, 0]$ with linear interpolation, i.e.,

$$\bar{G}(s) := \bar{G}(2^k)\frac{2^{k+1}-s}{2^{k+1}-2^k} + \bar{G}(2^{k+1})\frac{s-2^k}{2^{k+1}-2^k} \quad \text{for} \quad s \in [2^k, 2^{k+1}]$$

for $k \in \mathbb{Z} \cap (-\infty, 0]$. Finally, we define

$$\bar{G}(s) := \begin{cases} \bar{G}(2) & \text{if } s \in [2, \infty)\,, \\ 0 & \text{if } s \in (-\infty, 0]\,. \end{cases}$$

Then, $\bar{G}$ satisfies

(v) $\bar{G}$ is monotone nondecreasing, $\bar{G} \ge 0$ in $\mathbb{R}$,
(vi) $\bar{G}(s) = 0$ if $s \le 0$,
(vii) $u^*(0,x) \le \bar{G}(v_*(0,x))$ for $x \in \mathbb{R}^N$,
(viii) $\bar{G}$ is uniform continuous on $\mathbb{R}$.

We now demonstrate the above properties. The properties (v), (vi) and (viii) follows directly from the definition of $\bar{G}$.

It remains to prove (vii). Note that $G \le \bar{G}$ on $\mathbb{R}$ so that (vii) follows from (iii). In fact, if $s \in (0,1]$ then there exists $k \in \mathbb{Z} \cap (-\infty, 0]$ such that $s \in [s_{k-1}, s_k]$, which implies

$$\bar{G}(s) \ge \bar{G}(s_{k-1}) = G(s_k) \ge G(s)\,.$$

If $s \le 0$, then $\bar{G}(s) = 0 = G(s)$. If $s \ge 2$, then $\bar{G}(s) = K \ge G(s)$ by definition of G . If $s \in [1,2]$, then

$$\bar{G}(s) = (K - G(2))(s-1) + G(2) \ge G(2) \ge G(s)\,.$$

Step 3. We are in the position to prove (2.13).

Let $w(t,x) = \bar{G}(v_*(t,x))$. Then, w is a supersolution to (2.1) by Lemma 2.6, and lower semicontinuous i.e., $w_* = w$. Moreover, we observe that

$$u^*(0,x) \le w(0,x) \quad \text{for} \quad x \in \mathbb{R}^N$$

by (vii). Then, we obtain

$$u^*(t,x) \le w(t,x) \quad \text{for} \quad (t,x) \in (0.T) \times \mathbb{R}^N$$

by (CP). The above implies

$$\{x \in \mathbb{R}^N;\ u^*(t,x) > 0\} \subset \{x \in \mathbb{R}^N;\ w(t,x) > 0\}$$

for $t \in [0,T)$. By (vi) we have

$$\{x \in \mathbb{R}^N;\ w(t,x) > 0\} \subset \{x \in \mathbb{R}^N;\ v_*(t,x) > 0\}$$

for $t \in [0,T)$, so that we obtain (2.13).

To derive (2.14) from (2.12) we construct a rescaled subsolution $\bar{H}(u^*(t,x))$ with

$$H(s) = \inf\{(v_*(0,y))_-;\ u^*(0,y) \geq s\}\,,$$

where $(a)_- = \min\{0,a\}$, and its relaxation $\bar{H}$ by the parallel argument of the above Step 1, 2, and 3. Then, $\bar{H}$ satisfies

- $\bar{H}$ is monotone nondecreasing, $\bar{H} \leq 0$ in $\mathbb{R}$,
- $\bar{H}(s) = 0$ if $s \geq 0$,
- $\bar{H}(u^*(0,x)) \leq v_*(0,x)$ for $x \in \mathbb{R}^N$,
- $\bar{H}$ is uniform continuous on $\mathbb{R}$.

Hence, we obtain (2.14). □

2.3. A set theoretic solution.

A set-theoretic solution for a geometric evolution equation is introduced in [16]; we say that D_t is a set-theoretic sub- (resp. super-) solution to (1.1) if

$$w(t,x) := \chi_{D_t}(x) = \begin{cases} 1 & \text{if } x \in D_t \\ 0 & \text{otherwise} \end{cases}$$

is a sub- (resp. super-) solution to the level set equation of (1.1), i.e., (2.1) with (2.3).

The characterizations of the set-theoretic solution are also introduced in [16]; see for details. In this section we derive a direct calculation of derivatives of a characteristic function in viscosity solution sense.

THEOREM 2.8. *Let $D \subset \mathbb{R}^N$ be a N-dimensional submanifold with smooth boundary $\Gamma = \partial D$. If $\varphi(x) \in C^2(\mathbb{R}^N)$ and $\hat{x} \in \Gamma$ satisfies*

$$(\chi_D)^*(x) - \varphi(x) \leq (\chi_D)^*(\hat{x}) - \varphi(\hat{x})$$
$$(\textit{resp. } (\chi_D)_*(x) - \varphi(x) \geq (\chi_D)_*(\hat{x}) - \varphi(\hat{x})) \quad \textit{for} \quad x \in \mathbb{R}^N$$

and $|\nabla\varphi(\hat{x})| \neq 0$, *then*

$$-\mathrm{div}\frac{\nabla\varphi}{|\nabla\varphi|} + \mathcal{K}_\Gamma \leq 0 \ \left(\textit{resp.}\ -\mathrm{div}\frac{\nabla\varphi}{|\nabla\varphi|} + \mathcal{K}_\Gamma \geq 0\right) \quad \textit{at } \hat{x},$$

where $\mathcal{K}_\Gamma = \mathcal{K}_\Gamma(x)$ is the mean curvature of Γ at $x \in \Gamma$ in the direction of the outer unit normal vector field **n** *of ∂D.*

Proof. We here demonstrate only the subsolution case since the proof of supersolution case is parallel.

By the definition of N-dimensional submanifold with boundary Γ is smooth $N-1$-dimensional submanifold, i.e., smooth hypersurface in $\mathbb{R}^N$. We here choose the orthonormal basis τ_j $(j = 1,\ldots,N-1)$ of tangential space $T_{\hat{x}}\Gamma$ at $\hat{x}$. Then the outer unit normal vector field **n** of Γ is defined, and there exists $\delta_0 > 0$ such that

$$\hat{x} + \delta\hat{\mathbf{n}} \in \mathbb{R}^N \setminus \overline{D}, \quad \hat{x} - \delta\hat{\mathbf{n}} \in \mathrm{Int}D,$$

for $\delta \in (0, \delta_0)$, where $\hat{\mathbf{n}} = \mathbf{n}(\hat{x})$. We next choose a smooth curve ζ_j on Γ, which is defined in a neighborhood of $0 \in \mathbb{R}$, satisfying

$$\zeta_j(0) = \hat{x}, \quad \zeta_j'(0) = \tau_j$$

for $j = 1, \ldots, N-1$.

We now define functions $\Phi_j(\sigma)$, $\Psi(\sigma)$, which are defined in a neighborhood of $0 \in \mathbb{R}$, given as

$$\Phi_j(\sigma) = (\chi_D)^*(\zeta_j(\sigma)) - \varphi(\zeta_j(\sigma)),$$
$$\Psi(\sigma) = (\chi_D)^*(\hat{x} + \sigma\hat{\mathbf{n}}) - \varphi(\hat{x} + \sigma\hat{\mathbf{n}}).$$

Then, both Φ_j and Ψ take their maximum at $\sigma = 0$. Moreover, since $(\chi_D)^* = \chi_{\overline{D}}$, Φ_j is smooth in a neighborhood of $\sigma = 0$ for $j = 1, \ldots, N-1$, and Ψ is smooth in $(-\delta_0, 0)$. Thus, we first obtain

$$\Phi_j'(0) = -\langle \nabla\varphi(\hat{x}), \tau_j \rangle = 0$$

for $j = 1, \ldots, N-1$, which and $|\nabla\varphi(\hat{x})| \neq 0$ yield that

$$\frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|} = \hat{\mathbf{n}} \text{ or } -\hat{\mathbf{n}}.$$

We next obtain

$$\begin{aligned} 0 &\geq \Psi(-\sigma) - \Psi(0) \\ &= 1 - \varphi(\hat{x} - \sigma\hat{\mathbf{n}}) - (1 - \varphi(\hat{x})) \\ &= -\varphi(\hat{x} - \sigma\hat{\mathbf{n}}) + \varphi(\hat{x}) = \sigma\langle \nabla\varphi(\hat{x}), \hat{\mathbf{n}} \rangle + O(\sigma^2) \quad \text{as } \sigma \to 0, \end{aligned}$$

which implies $\langle \nabla\varphi(\hat{x}), \hat{\mathbf{n}} \rangle \leq 0$ and thus

$$\frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|} = -\hat{\mathbf{n}}.$$

Moreover, we obtain $\Phi_j''(0) \leq 0$ for $j = 1, \ldots, N-1$ since $\Phi_j(\sigma)$ attains its maximum at $\sigma = 0$, which implies

$$\begin{aligned} 0 &\geq \sum_{j=1}^{N-1} (-\langle \nabla^2\varphi(\hat{x})\tau_j, \tau_j \rangle - \langle \varphi(\hat{x}), \zeta_j''(0) \rangle) \\ &= \sum_{j=1}^{N-1} \left[-\text{trace} \left\{ (\tau_j \otimes \tau_j)\nabla^2\varphi(\hat{x}) \right\} + |\nabla\varphi(\hat{x})| \langle \hat{\mathbf{n}}, \zeta_j''(0) \rangle \right]. \end{aligned}$$

Note that

$$I_N - \sum_{j=1}^{N-1} \tau_j \otimes \tau_j = \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} = \frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|} \otimes \frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|},$$

and

$$\sum_{j=1}^{N-1} \langle \hat{\mathbf{n}}, \zeta_j''(0) \rangle = -\sum_{j=1}^{N-1} \langle D_{\tau_j}\hat{\mathbf{n}}, \tau_j \rangle = \mathcal{K}_\Gamma(\hat{x}),$$

where $D_{\tau_j}\hat{\mathbf{n}}$ is the tangential derivative of $\mathbf{n}$ in the direction of τ_j at $\hat{x}$. Then the above yields

$$-\mathrm{trace}\left\{\left(I_N - \frac{\nabla\varphi\otimes\nabla\varphi}{|\nabla\varphi|^2}\right)\nabla^2\varphi\right\} + \mathcal{K}_\Gamma|\nabla\varphi| \le 0 \quad \text{at } \hat{x}\,.$$

Since

$$\mathrm{trace}\left\{\left(I_N - \frac{\nabla\varphi\otimes\nabla\varphi}{|\nabla\varphi|^2}\right)\nabla^2\varphi\right\} = |\nabla\varphi|\mathrm{div}\frac{\nabla\varphi}{|\nabla\varphi|}\,,$$

we obtain the conclusion of Theorem 2.8. □

The following corollary looks trivial; however, we need Theorem 2.8 and Lemma 2.2 to prove.

COROLLARY 2.9. *A domain with suitable constant mean curvature boundary is a stationary set theoretic solution to* (1.1). *For example*,

(i) *Ball*: $\{x \in \mathbb{R}^N;\ |x - x_0| < (N-1)/C\}$ *for* $x_0 \in \mathbb{R}^N$ *provided that* $C \neq 0$,

(ii) *Generalized cylindrical surface*: $\{(x', x_{k+1}, \ldots, x_N) \in \mathbb{R}^k \times \mathbb{R}^{N-k};\ |x'| < (N-1-k)/C\}$ *for* $k \in [2, N-1] \cap \mathbb{Z}$ *and its rotation provided that* $C \neq 0$,

(iii) *For* $e \in S^{N-1}$ *and* $x_0 \in \mathbb{R}^N$, *the set under a hyperplane* $\{x \in \mathbb{R}^N;\ \langle e, x - x_0\rangle \le 0\}$ *provided that* $C = 0$.

REMARK 2.10. We remark that the mean curvature of $\partial\{x \in \mathbb{R}^N;\ |x - x_0| < (N-1)/C\}$ is $-C$ since the curvature is defined with the outer unit normal vector field $\mathbf{n}$ and we do not take the average of principal curvature as the mean curvature.

3. Eikonal-curvature flow

In this section we consider a stationary ball for (1.1) in $\mathbb{R}^N$ with a level set formulation. The level set equation of (1.1) is of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{3.1}$$

with (2.3) and (2.4). To consider stability of a stationary ball we consider evolution of compact interface with a level set formulation, so that we consider spatially profile of initial data or solutions to (3.1) in

$$\mathfrak{X}_\alpha := \{f : \mathbb{R}^N \to \mathbb{R};\ \mathrm{supp}(f - \alpha) \text{ is compact}\}$$

for $\alpha \in \mathbb{R}$.

3.1. Barrier solutions for the uniqueness.

The comparison principle is established by [7] or [15]. Note that we now consider $\Omega = \mathbb{R}^N$ thus Ω is unbounded. However, if we assume (2.9) for a sub- and supersolution u, v to (3.1), then we obtain (CP) in this problem with the argument as in [7]. Note that if we assume (2.7) then we observe $\alpha \le \beta$ for $\alpha,\ \beta$ in (2.9). Theorem 2.7 is also available in this problem.

The existence of viscosity solution with initial data $u_0 \in BUC(\mathbb{R}^N)$, where $BUC(\Omega)$ denotes the space of bounded uniformly continuous functions on Ω, is also established in [7] with Perron's method. For the uniqueness of solutions we have to verify that

$$u(t,x) = \alpha \quad \text{if } |x| > R(t)$$

with at least locally bounded $R(t)$. For this purpose we now construct barrier sub- and supersolutions. We here construct only the supersolution since the construction of subsolution is parallel.

We now assume that there exists $R_0 > 0$ such that

$$u_0(x) = u(0,x) = \alpha \quad \text{if } |x| \ge R_0 .$$

We now set

$$b(t,x) = Bt + A(-|x| + R_0 + 1)$$

for constants $A, B > 0$ determined later. Then, for $x \ne 0$ we have

$$\nabla b = -\frac{Ax}{|x|}, \quad \nabla^2 b = -\frac{A}{|x|} I + \frac{Ax \otimes x}{|x|^3} .$$

Thus we obtain

$$F(\nabla b, \nabla^2 b) = F\left(-\frac{A}{|x|}x, -\frac{A}{|x|}I\right) = A\left(\frac{(N-1)}{|x|} - C\right)$$

for $x \ne 0$ by (F4). We now set $B = AC$. Then

$$b_t + F(\nabla b, \nabla^2 b) = \frac{A(N-1)}{|x|} > 0$$

for $x \ne 0$.

We now set $A = \|u_0\|_\infty - \alpha$ and define

$$b^+(t,x) = \min\{\theta_\alpha^+(b(t,x) + \alpha), \|u_0\|_\infty\}$$

with a cut-off function

$$\theta_\alpha^+(\sigma) = \begin{cases} \alpha & \text{if } \sigma < \alpha , \\ \sigma & \text{otherwise} , \end{cases} \tag{3.2}$$

where $\|u_0\|_\infty = \sup_{\mathbb{R}^N} |u_0|$. Note that a constant function is a solution to (3.1) and then b^+ is a supersolution. Then, we observe that b^+ is a supersolution to (3.1) in $(0,\infty) \times \mathbb{R}^N$ from Lemma 2.6 and Proposition 2.3, and satisfies

$$b^+(0,x) = \|u_0\|_\infty \text{ if } |x| \le R_0 , \tag{3.3}$$

$$b^+ \ge \alpha \text{ in } [0,T] \times \mathbb{R}^N \text{, and } b^+(t,x) = \alpha \text{ if } |x| \ge R_0 + 1 + Bt/A . \tag{3.4}$$

The property (3.3) and (3.4) implies that $b^+(0,x) \geq u_0(x)$. Then, the solution u constructed by Perron's method implies $u(t,x) \leq b^+(t,x)$ by taking infimum of the constructed supersolutions and b^+.

By the parallel argument of the above with $A = \|u_0\|_\infty + \alpha$ we obtain a barrier subsolution $b^-(t,x) = \max\{\theta_\alpha^-(-b(t,x)+\alpha), -\|u_0\|_\infty\}$ to (3.1) in $(0,\infty)\times\mathbb{R}^N$ satisfying

$$b^-(0,x) = -\|u_0\|_\infty \text{ if } |x| \leq R_0\,,$$

$$b^- \leq \alpha \text{ in } [0,T]\times\mathbb{R}^N \;\;, \text{ and } \; b^-(t,x) = \alpha \;\text{ if }\; |x| \geq R_0 + 1 + Bt/A\,,$$

where

$$\theta_\alpha^-(\sigma) = \begin{cases} \alpha & \text{if } \sigma > \alpha\,, \\ \sigma & \text{otherwise}\,. \end{cases} \tag{3.5}$$

The above implies that the solution $u(t,x)$ to (3.1) with $u(0,x) = u_0(x)$ satisfies

$$u(t,x) = \alpha \quad \text{if} \quad |x| \geq R(t) := R_0 + 1 + Ct\,,$$

and then the solution u is unique with respect to $u_0 \in UC(\mathbb{R}^N) \cap \mathfrak{X}_\alpha$ in $[0,T)\times\mathbb{R}^N$ for arbitrary $T > 0$, where $UC(\Omega)$ denotes the space of uniformly continuous functions on Ω.

3.2. Instability of stationary ball.

By Corollary 2.9

$$\Xi = \left\{x \in \mathbb{R}^N;\; |x - x_0| < \frac{N-1}{C}\right\}$$

is a set-theoretic solution to (1.1) with a constant $C \neq 0$. It is also easy to find that the above stationary solution is unstable by considering a ball with the different radius from $(N-1)/C$; if the radius is less than $(N-1)/C$ then the ball will vanish, and if the radius is larger than that then the ball will spread whole domain. The following result expresses the above phenomena from a view point with a level set formulation.

THEOREM 3.1. *Let $u \in C([0,\infty)\times\mathbb{R}^N)$ be a solution to* (3.1) *with initial data $u(0,\cdot) = u_0 \in UC(\mathbb{R}^N) \cap \mathfrak{X}_\alpha$ for $\alpha \in \mathbb{R}$ satisfying*

$$\Xi = \{x \in \mathbb{R}^N;\; u(0,x) > c\}\,, \quad \Gamma = \partial\Xi = \{x \in \mathbb{R}^N; u(0,x) = c\}$$

for fixed $c \in \mathbb{R}$. Then

$$\lim_{t\to\infty} u(t,x) = c \quad \text{for } x \in \mathbb{R}^N.$$

Proof. We may assume that $c = 0$ and $x_0 = 0$ without loss of generality by considering $\tilde{u}(t,x) = u(t,x+x_0) - c$ instead of u.

We first note that, for arbitrary $T > 0$ we have

$$u \geq 0 \quad \text{on } [0,T)\times\overline{\Xi}\,, \qquad u \leq 0 \quad \text{on } [0,T)\times\Xi^c \tag{3.6}$$

by (CP), which implies that

$$u \equiv 0 \quad \text{on } [0,\infty)\times\Gamma\,. \tag{3.7}$$

In fact, $\chi_-(x) = -\|u_0\|_\infty \chi_{\Xi^c}(x)$ is a solution to (3.1) by Corollary 2.9 and satisfies

$$\chi_-^*(x) \le u(0,x) \quad \text{for } x \in \mathbb{R}^N \ .$$

Note that $u^* = u_* = u$ since $u \in C([0,\infty) \times \mathbb{R}^N)$. This implies

$$\chi_-^*(x) \le u(t,x) \quad \text{for } (t,x) \in (0,T) \times \mathbb{R}^N$$

with arbitrary $T > 0$ by (CP). Note that $\chi_-^*(x) = \|u_0\|_\infty(\chi_{\overline{\Xi}}(x) - 1)$ and then we obtain the first inequality of (3.6). The second inequality of (3.6) is obtained by the parallel argument of the above with $\chi^+(x) = \|u_0\|_\infty \chi_\Xi(x)$.

Let $\varepsilon > 0$. Then, there exists $\delta > 0$ such that

$$\{x \in \mathbb{R}^N;\ u(0,x) > \varepsilon\} \subset \Xi_\delta := \left\{x \in \mathbb{R}^N;\ |x| < \frac{N-1}{C} - \delta\right\}$$

since u is uniformly continuous in $\mathbb{R}^N$. We may assume that $\delta \to 0$ as $\varepsilon \to 0$ by taking $\delta' = \min\{\delta, \varepsilon\}$ instead of δ.

We now define

$$v_\delta^+(t,x) = -A_1 t - |x|^2 + \left(\frac{N-1}{C} - \delta\right)^2 ,$$

where $A_1 > 0$ is a constant determined later. Then,

$$\Xi_\delta = \{x \in \mathbb{R}^N;\ v_\delta^+(0,x) > 0\}$$

which implies

$$\{x \in \mathbb{R}^N;\ u(0,x) > \varepsilon\} \subset \{x \in \mathbb{R}^N;\ v_\delta^+(0,x) > 0\} . \tag{3.8}$$

By straightforward calculation we obtain

$$(v_\delta^+)_t = -A_1 , \quad \nabla v_\delta^+ = -2x , \quad \nabla^2 v_\delta^+ = -2I$$

and then

$$(v_\delta^+)_t + F^*(\nabla v_\delta^+, \nabla^2 v_\delta^+) = -A_1 + 2(N-1) - 2C|x| .$$

We here have used the fact that

$$F_0^*(0,X) = -\sum_{j=1}^{N-1} \lambda_j ,$$

where λ_j is an eigenvalue of X satisfying $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Then, if $x \in \Xi_{\delta/4}$, then

$$\begin{aligned}(v_\delta^+)_t &+ F^*(\nabla v_\delta^+, \nabla^2 v_\delta^+) \\ &\ge -A_1 + 2(N-1) - 2(N-1) + \frac{C\delta}{2} = -A_1 + \frac{C\delta}{2} .\end{aligned}$$

Hence, we obtain v_δ^+ is a supersolution to (3.1) in $(0,\infty) \times \Xi_{\delta/4}$ provided that $A_1 \in (0, C\delta/2)$.

We extend v_δ^+ to the whole domain. Fix $A_1 \in (0, C\delta/2)$ and let $G^+ \colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$G^+(s) = \begin{cases} -1 & \text{if } s < -M_1\,, \\ \dfrac{s}{M_1} & \text{if } -M_1 \le s < 0\,, \\ s & \text{if } s \ge 0\,, \end{cases}$$

where

$$M_1 = \left(\frac{N-1}{C} - \frac{\delta}{2}\right)^2 - \left(\frac{N-1}{C} - \delta\right)^2 .$$

Then, G^+ is continuous and monotone nondecreasing function and thus

$$\tilde{v}_\delta^+(t,x) = G^+(v_\delta^+(t,x))$$

is still a supersolution to (3.1) in $(0,T) \times \Xi_{\delta/4}$. Moreover, if $|x| \ge C^{-1}(N-1) - \delta/2$, then $\tilde{v}_\delta^+(t,x) = -1$ for $t > 0$ since

$$v_\delta^+(t,x) \le -\left(\frac{N-1}{C} - \frac{\delta}{2}\right)^2 + \left(\frac{N-1}{C} - \delta\right)^2 = -M_1\,.$$

We now define

$$w_\delta^+(t,x) = \begin{cases} \tilde{v}_\delta^+(t,x) & \text{if } x \in \left\{ y \in \mathbb{R}^N;\ |x| < \dfrac{N-1}{C} - \dfrac{\delta}{4} \right\}, \\ -1 & \text{otherwise.} \end{cases}$$

Then, w_δ^+ is a supersolution to (3.1) in $(0,\infty) \times \mathbb{R}^N$. In fact, $w_\delta^+ \equiv -1$ in $(0,\infty) \times \{x \in \mathbb{R}^N;\ |x| > (N-1)/C - \delta/2\}$, which implies

$$(w_\delta^+)_t + F^*(\nabla w_\delta^+, \nabla^2 w_\delta^+) = 0 + F^*(0, O) = 0$$

in $(0,\infty) \times \{x \in \mathbb{R}^N;\ |x| > (N-1)/C - \delta/2\}$.

Theorem 2.7 and (3.8) yield

$$\{x \in \mathbb{R}^N;\ u(t,x) > \varepsilon\} \subset \{x \in \mathbb{R}^N;\ w_\delta^+(t,x) > 0\}$$

for $t \in (0,T)$ with arbitrary fixed $T > 0$. Note that

$$\{x \in \mathbb{R}^N;\ w^+(t,x) > 0\} = \emptyset$$

provided that $t > T_\delta := ((N-1)/C - \delta)^2/A_1$. Hence, we obtain

$$\{x \in \mathbb{R}^N;\ u(t,x) > \varepsilon\} = \emptyset \quad \text{provided that } t > T_\delta\,,$$

which implies

$$\varlimsup_{t\to\infty} u(t,x) \le \varepsilon \quad \text{for } x \in \Xi_\delta\,.$$

Tending $\varepsilon \to 0$ yields that

$$\varlimsup_{t\to\infty} u(t,x) \le 0 \quad \text{for } x \in \Xi\,,$$

which and the second inequality of (3.6) implies

$$\varlimsup_{t\to\infty} u(t,x) \le 0 \quad \text{for } x \in \mathbb{R}^N. \tag{3.9}$$

The lower estimate of u, i.e.,

$$\varliminf_{t\to\infty} u(t,x) \geq 0 \quad \text{for } x \in \mathbb{R}^N$$

is derived with similar way. For $\varepsilon > 0$ we first choose $\delta > 0$ such that $\lim_{\varepsilon\to 0} \delta = 0$ and

$$\{x \in \mathbb{R}^N;\ u(0,x) < -\varepsilon\} \subset \Xi^{\delta} := \left\{x \in \mathbb{R}^N;\ |x| > \frac{N-1}{C} + \delta\right\}. \tag{3.10}$$

We introduce the function v_δ^- of the form

$$v_\delta^-(t,x) = A_2 t - |x|^2 + \left(\frac{N-1}{C} + \delta\right)^2.$$

Then we can find suitable $A_2 > 0$ such that is a subsolution to (3.1) in $(0,\infty) \times \Xi^{\delta/4}$ by the parallel way of the case of v_δ^+. Then, we extend v_δ^- into $(0,T) \times \mathbb{R}^N$ by similar way with

$$G^-(s) = \begin{cases} -1 & \text{if } s \leq -1\,, \\ s & \text{if } -1 < s \leq 0\,, \\ \dfrac{s}{M_2} & \text{if } 0 < s \leq M_2\,, \\ 1 & \text{otherwise,} \end{cases}$$

where

$$M_2 = \left(\frac{N-1}{C} + \delta\right)^2 - \left(\frac{N-1}{C} + \frac{\delta}{2}\right)^2,$$

i.e.,

$$w_\delta^-(t,x) = \begin{cases} G^-(v_\delta^-(t,x)) & \text{if } x \in \Xi^{\delta/4}, \\ 1 & \text{otherwise.} \end{cases}$$

Then we obtain

$$\{x \in \mathbb{R}^N;\ u(t,x) < -\varepsilon\} \subset \{x \in \mathbb{R}^N;\ w_\delta^-(t,x) < 0\}$$

for $t \in [0,T)$ with arbitrary fixed $T > 0$ from (3.10) and $\Xi^\delta = \{x \in \mathbb{R}^N;\ w_\delta^-(0,x) < 0\}$. The above implies

$$u(t,x) \geq -\varepsilon \quad \text{if } |x| \leq \sqrt{A_2 t + \left(\frac{N-1}{C} + \delta\right)^2}$$

and then $\varliminf_{t\to\infty} u(t,x) \geq -\varepsilon$. By tending $\varepsilon \to 0$ we obtain

$$\varliminf_{t\to\infty} u(t,x) \geq 0 \quad \text{for } x \in \mathbb{R}^N\,. \tag{3.11}$$

We obtain the conclusion in Theorem 3.1 by (3.9), (3.11) and (3.7). □

Note that the estimate (3.11) is not uniform with respect to $x \in \mathbb{R}^N$ and $\lim_{t\to\infty} \|u(t,\cdot) - c\|_\infty \neq 0$.

Theorem 3.1 means that every continuous stationary solution to (3.1) at least has no strict local maximum and minimum. This is generalized as follows.

THEOREM 3.2. *There are no continuous stationary solutions* $u \in \mathfrak{X}_\alpha$ *for* $\alpha \in \mathbb{R}$ *to* (3.1) *such that* u *has a connected component* $U \neq \emptyset$ *of super- or sublevel set whose closure is included in the ball* Ξ *whose center and radius, respectively, is* $x_0 \in \mathbb{R}^N$ *and* $(N-1)/C$, *i.e.,* $\Xi \supset \overline{\{x \in \Xi;\ u(x) > c\}} \neq \emptyset$ *or* $\Xi \supset \overline{\{x \in \Xi;\ u(x) < c\}} \neq \emptyset$ *for a constant* $c \in \mathbb{R}$.

Proof. We may assume that $c = 0$ and $x_0 = 0$ without loss of generality. We also assume that $\overline{\{x \in \Xi;\ u(x) > 0\}} \subset \Xi$ since the proof of the other case is parallel.

Fix $\varepsilon \in (0, \max_{\overline{\Xi}} u(x))$. Since $\max_{\partial\Xi} u(x) \leq 0$, the function $\tilde{u}$ defined as

$$\tilde{u} = \begin{cases} \max\{\varepsilon/2, u(x)\} & \text{if } x \in \Xi\,, \\ \varepsilon/2 & \text{otherwise} \end{cases}$$

is still a subsolution to (3.1) by Proposition 2.3 and $\tilde{u} \in \mathfrak{X}_{\varepsilon/2}$. However, the superlevel set $\{x \in \Xi;\ \tilde{u}(x) > \varepsilon\}$ must vanish by the similar argument in the proof of Theorem 3.1, which is the contradiction. □

REMARK 3.3. (i) The proof of Theorem 3.1 is naturally extended to the solution $u(t,x) \in C([0,\infty) \times \mathbb{R}^k \times (\mathbb{R}/\mathbb{Z})^{N-k})$ describing the generalized cylindrical surface

$$\Gamma = \left\{ (x', x_{k+1}, \ldots, x_N) \in \mathbb{R}^k \times \mathbb{R}^{N-k};\ |x'| = \frac{N-1-k}{C} \right\}$$

for $k \in [2, N-1] \cap \mathbb{Z}$ with the comparison principle for unbounded domain as in [15]. Note that the assumption (2.9) for the above problems should be revised as follows.

(2.9)′ There exists $R > 0$ such that $u(t,x) = \alpha$ and $v(t,x) = \beta$ if $|x'| > R$ for $t \in [0,T)$ and $x = (x', x_{k+1}, \ldots, x_N) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

(ii) We also obtain the nonexistence result like as Theorem 3.2 corresponding to the stationary cylindrical surface in (i).

4. Curvature flow equation on axis-symmetric domain

Let $\Omega \subset \mathbb{R}^N$ be an axis-symmetric domain

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R};\ |x'| < r(x_N)\}$$

with some smooth positive function r satisfying (Ω1)–(Ω2) (see §2.2). In this section we consider evolving hypersurface $\Gamma_t \subset \overline{\Omega}$ by (1.1) with $C = 0$ and the right angle condition, i.e.,

$$V = \mathcal{K} \quad \text{on } \Gamma_t\,, \tag{4.1}$$

$$\Gamma_t \perp \partial\Omega\,. \tag{4.2}$$

The level set equation is of the form

$$u_t + F_0(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0,T) \times \Omega\,, \tag{4.3}$$

$$\langle \nabla u, \vec{\nu} \rangle = 0 \quad \text{on } (0.T) \times \partial\Omega \tag{4.4}$$

with (2.4), where $\vec{\nu}$ is the outer unit normal vector field of $\partial\Omega$.

Note that Ω is non-convex. In this section we consider the situation such that (x', x_N) is in interior or exterior if $x_N \to -\infty$ or $x_N \to \infty$, respectively. To describe this situation with the level set method we consider a spatially profile of initial data or solutions in

$$\mathfrak{Y}_{\alpha,\beta} = \left\{ f : \overline{\Omega} \to \mathbb{R};\ \begin{array}{c} \text{there exists } R > 0 \text{ such that} \\ f(x', x_N) = \alpha \text{ if } x_N < -R\,, \\ f(x', x_N) = \beta \text{ if } x_N > R \end{array} \right\}$$

for $\alpha, \beta \in \mathbb{R}$.

4.1. Barrier solutions for the uniqueness.

We here remark on the uniqueness and existence of solutions to (4.3)–(4.4). The comparison principle for (4.3)–(4.4) for a nonconvex and bounded Ω is established by [18]. Although Ω in our problem is unbounded, the comparison principle for our problem is also derived by applying the proof of [18] for sub- and supersolution u and v satisfying (2.10).

The existence of solution u to (4.3)–(4.4) with initial data $u_0 \in UC(\overline{\Omega}) \cap \mathfrak{Y}_{\alpha,\beta}$ is also derived by the Perron's method; see [31]. To show the uniqueness of solution we have to see $u(t,\cdot) \in \mathfrak{Y}_{\alpha,\beta}$ with at least locally bounded $R = R(t) > 0$. For this purpose we make a barrier sub- and a supersolution as well as in §3 and demonstrate that the above R is independent of time $t \geq 0$.

We now assume there exists $R_0 > 0$ such that

$$u_0(x) = \alpha \ \text{ if } \ x_N < -R_0\,, \quad u_0(x) = \beta \ \text{ if } \ x_N > R_0$$

for $x = (x', x_N)$ and $u_0 = u(0,\cdot)$. Then,

$$b(x) = \begin{cases} -A(x_N + R_0)^4 + \|u_0\|_\infty & \text{if } \ x_N < -R_0\,, \\ \|u_0\|_\infty & \text{if } \ |x_N| \leq R_0\,, \\ -A(x_N - R_0)^4 + \|u_0\|_\infty & \text{otherwise} \end{cases}$$

is a C^2 supersolution to (4.3)–(4.4) in $(0,\infty) \times \mathbb{R}^N$. In fact, by straightforward calculation we have $\nabla^2 b = c(x)\nabla b \otimes \nabla b$ for $x \in \overline{\Omega}$, which implies

$$b_t + F_0^*(\nabla b, \nabla^2 b) = 0 + F_0^*(\nabla b, O) = 0 \quad \text{in } (0,T) \times \Omega$$

by (F4). For (4.4) we have

$$\langle \nabla b, \vec{\nu} \rangle = \begin{cases} \dfrac{4A(x_N + R_0)^3 r'(x_N)}{\sqrt{1 + r'(x_N)^2}} & \text{if } \ x_N < -R_0\,, \\ 0 & \text{if } \ |x_N| \leq R_0, \\ \dfrac{4A(x_N - R_0)^3 r'(x_N)}{\sqrt{1 + r'(x_N)^2}} & \text{otherwise,} \end{cases}$$

which implies $\langle \nabla b, \vec{\nu} \rangle \geq 0$ on $(0,T) \times \partial\Omega$ from $(\Omega 1)$.

We now define

$$b^+(x) := \begin{cases} \theta_\alpha^+(b(x)) & \text{if } \ x_N < 0\,, \\ \theta_\beta^+(b(x)) & \text{if } \ x_N \geq 0\,, \end{cases}$$

where θ_α^+ or θ_β^+ is defined in (3.2). Then, $b^+(x)$ is still a supersolution to (4.3)–(4.4) in $(0,\infty)\times\overline{\Omega}$ satisfying $b^+ \ge u_0$. Then, we observe that the solution u by Perron's method satisfies $b^+(x) \ge u(t,x)$ for $t>0$ and $x\in\overline{\Omega}$. By the parallel argument of the above with $-b$ and cut-off functions $\theta_\alpha^-, \theta_\beta^-$ as in (3.5) we also obtain the subsolution

$$b^-(x) := \begin{cases} \theta_\alpha^-(-b(x)) & \text{if } x_N < 0\,, \\ \theta_\beta^-(-b(x)) & \text{if } x_N \ge 0 \end{cases}$$

satisfying $b_-(x) \le u(t,x)$ for $(t,x)\in(0,\infty)\times\overline{\Omega}$. By the definitions of $b^\pm$ there exists $R = R_0 + O(1/A) > R_0$ such that

$$b^-(x) = b^+(x) = \alpha \quad \text{if } x_N < -R\,,$$

$$b^-(x) = b^+(x) = \beta \quad \text{if } x_N > R$$

and thus

$$u(t,x) = \begin{cases} \alpha & \text{if } x_N < -R\,, \\ \beta & \text{if } x_N > R\,, \end{cases}$$

for $t \ge 0$: note that R is independent of t. Hence, we observe that the solution $u \in C([0,\infty)\times\overline{\Omega})$ for $u_0 \in UC(\overline{\Omega})\cap\mathfrak{Y}_{\alpha,\beta}$ is unique.

Moreover, the idea of constructing barrier solutions $b^\pm$ yields the following fundamental property.

LEMMA 4.1. *Let $u\in C([0,\infty)\times\overline{\Omega})$ be a solution to* (4.3)–(4.4) *in* $(0,\infty)\times\overline{\Omega}$ *with $u(0,\cdot)\in\mathfrak{Y}_{\alpha,\beta}$ for $\alpha,\beta\in\mathbb{R}$. If*

$$\{x\in\overline{\Omega};\ u(t_0,x)=c\} \subset \overline{\Omega}_I := \overline{\Omega}\cap(\mathbb{R}^{N-1}\times I)$$

for a connected interval $I\subset\mathbb{R}$ and $t_0\in[0,\infty)$, then,

$$\{x\in\overline{\Omega};\ u(t,x)=c\} \subset \overline{\Omega}_{\bar I}$$

for $t > t_0$.

Proof. Let $p = \inf I$, $q = \sup I$, and

$$\overline{\Omega}_I^{c-} = \{(x',x_N)\in\overline{\Omega};\ x_N < p\}\,, \quad \overline{\Omega}_I^{c+} = \{(x',x_N)\in\overline{\Omega};\ x_N > q\}\,.$$

Note that $\overline{\Omega}_I^{c-}$ or $\overline{\Omega}_I^{c+}$ is empty if $p=-\infty$ or $q=\infty$, respectively. By assumption we have that each $\overline{\Omega}_I^{c\pm}$ is included in $\{x\in\overline{\Omega};\ u(t_0,x)>c\}$ or $\{x\in\overline{\Omega};\ u(t_0,x)<c\}$.

We now demonstrate that

(4.5) $\qquad \{x\in\overline{\Omega};\ u(t,x)=c\}\subset\overline{\Omega}_{[p,\infty)}$ for $t\ge t_0$ provided that $p>-\infty$.

We now assume $\overline{\Omega}_I^{c-} \subset \{x\in\overline{\Omega};\ u(t_0,x)>c\}$. Then, the subsolution

$$\bar b(x) = \begin{cases} \theta_1^-(|x_N-(p+1)|^4-1) & \text{if } x_N < p+1\,, \\ -1 & \text{otherwise} \end{cases}$$

satisfies $\bar b\in\mathfrak{Y}_{1,-1}$ and

$$\{x\in\overline{\Omega};\ \bar b(x)>0\} = \overline{\Omega}_{(-\infty,p)} \subset \{x\in\overline{\Omega};\ u(t_0,x)>c\}\,.$$

Then, by Theorem 2.7 we obtain

$$\overline{\Omega}_{(-\infty,p)} \subset \{x \in \overline{\Omega};\ u(t,x) > c\}$$

for $t > t_0$, which implies (4.5). If $\overline{\Omega}_I^{c-} \subset \{x \in \overline{\Omega};\ u(t,x) < c\}$, then we deduce (4.5) with similar argument as the above with the supersolution

$$\tilde{b}(x) = \begin{cases} \theta_{-1}^{+}(-|x_N - (p+1)|^4 + 1) & \text{if } x_N < p+1\ , \\ 1 & \text{otherwise.} \end{cases}$$

We also deduce

$$\{x \in \overline{\Omega};\ u(t,x) = c\} \subset \overline{\Omega}_{(-\infty,q]} \quad \text{for } t \geq t_0 \text{ provided that } q < \infty$$

with the parallel argument. □

4.2. Stability of plane at the neck.

By Corollary 2.9 we obtain that the plane at $x_N = 0$, i.e.,

$$\Xi = \{(x', x_N) \in \overline{\Omega};\ x_N < 0\}\,, \quad \Gamma = \partial\Xi = \{(x', 0) \in \overline{\Omega};\ x' \in \mathbb{R}^{N-1}\} \tag{4.6}$$

is a set theoretic solution to (4.1)–(4.2). The goal of this section is to show the discontinuous solution is stable in the following sense.

THEOREM 4.2. *Let $u \in C([0,\infty) \times \overline{\Omega})$ be a solution to* (4.3)–(4.4) *in* $(0,\infty) \times \overline{\Omega}$ *with initial data $u_0 \in \mathfrak{Y}_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{R}$. Then,*

$$\lim_{t\to\infty} u(t,x) = \begin{cases} \alpha & \text{if } x_N < 0\,, \\ \beta & \text{if } x_N > 0\,. \end{cases}$$

In other words, for every $c \in \mathbb{R}$ between α and β the level set $\{x \in \overline{\Omega};\ u(t,x) = c\}$ converges to Γ as $t \to \infty$.

Proof. We may assume that $\alpha > \beta$ without loss of generality. Let $R_0 > 0$ be such that

$$u_0(x) = \begin{cases} \alpha & \text{if } x_N < -R_0\,, \\ \beta & \text{if } x_N > R_0 \end{cases}$$

for $x = (x', x_N) \in \overline{\Omega}$. Then, for $\mu > 0$

$$\{x \in \overline{\Omega};\ u_0(x) > \alpha + \mu \ \text{ or } \ u_0(x) < \beta - \mu\} \subset \overline{\Omega}_{(-R_0,R_0)}\,,$$

$$\{x \in \overline{\Omega};\ u_0(x) < \alpha - \mu\} \subset \overline{\Omega}_{(-R_0,+\infty)}\,,$$

$$\{x \in \overline{\Omega};\ u_0(x) > \beta + \mu\} \subset \overline{\Omega}_{(-\infty,R_0)}\ .$$

We now prove that, for $\varepsilon \in (0, \min\{1, R_0\})$, there exists $T_\varepsilon > 0$ such that

$$\{x \in \overline{\Omega};\ u(t,x) > \alpha + \mu \text{ or } u(t,x) < \beta - \mu\} \subset \overline{\Omega}_{(-\varepsilon,\varepsilon)}\,,$$

$$\{x \in \overline{\Omega};\ u(t,x) < \alpha - \mu\} \subset \overline{\Omega}_{(-\varepsilon,+\infty)}\,,$$

$$\{x \in \overline{\Omega};\ u(t,x) > \beta + \mu\} \subset \overline{\Omega}_{(-\infty,\varepsilon)} \tag{4.7}$$

for $t > T_\varepsilon$. We here demonstrate only (4.7) since the proof of the others are parallel.

Let

$$v(x) = -|x-y|^2 + z = -|x'|^2 - |x_N + y_N|^2 + z$$

for $y = (0', -y_N)$, $y_N > 0$ and $z > 0$. We choose y_N and z so that v satisfies

(4.8) $\quad v(x', R_0) = 0 \quad \text{if } (x', R_0) \in \partial\Omega\,,$

(4.9) $\quad v(0', R_0 + 1) \le -1\,,$

(4.10) $\quad v(x', \varepsilon/2) \ge c_\varepsilon + 1 \quad \text{if } (x', \varepsilon/2) \in \partial\Omega\,, \text{ where } c_\varepsilon = v(0', \varepsilon)\,,$

(4.11) $\quad \langle \nabla v, \vec{\nu} \rangle \ge 0 \quad \text{for } (x', x_N) \in \partial\Omega \quad \text{if } x_N \in I_\varepsilon = [\varepsilon/2, R_0 + 1]\,.$

From (4.8) we set

$$z = r(R_0)^2 + (R_0 + y_N)^2.$$

For (4.9), (4.10) and (4.11) it suffices to choose y_N satisfying

$$y_N \ge \max\left\{ \frac{r(R_0)^2}{2} - R_0,\, \frac{r(R_0)^2}{\varepsilon} + \frac{1}{\varepsilon} - \frac{3}{4}\varepsilon,\, \frac{r(R_0+1)}{m_\varepsilon} \right\}\,,$$

where $m_\varepsilon = \inf_{I_\varepsilon} r' > 0$, since

$$\langle \nabla v, \vec{\nu} \rangle = -\frac{2(|x'| - r'(x_N)(x_N + y_N))}{\sqrt{1 + r'(x_N)^2}} \ge -\frac{2(r(R_0+1) - m_\varepsilon y_N)}{\sqrt{1 + r'(x_N)^2}}$$

for $(x', x_N) \in \partial\Omega$ satisfying $x_N > 0$ by (Ω1), and $\vec{\nu} = (1 + r'(x_N)^2)^{-1/2}(x'/|x'|, -r'(x_N))$.

We now introduce

$$w(t, x) = -Bt + Av(x)$$

for $A, B > 0$ chosen later. Then we have

$$\nabla w = -2A(x', x_N + y_N)\,, \quad \nabla^2 w = -2AI\,,$$

which implies

$$F_0^*(\nabla w, \nabla^2 w) = 2A(N-1)\,.$$

Thus, we set $B = 2A(N-1) > 0$ to get

$$w_t + F_0^*(\nabla w, \nabla^2 w) = 0 \quad \text{in } (0,\infty) \times \Omega\,.$$

Moreover we have

$$\langle \nabla w, \vec{\nu} \rangle \ge 0 \quad \text{on } (0,\infty) \times (\partial\Omega \cap \overline{\Omega}_{I_\varepsilon})\,.$$

Hence, we obtain $w(t, x)$ is a supersolution to (4.3)–(4.4) on $(0,\infty) \times \overline{\Omega}_{I_\varepsilon}$.

Let $T_\varepsilon > 0$ be such that

$$w(T_\varepsilon, (0', \varepsilon)) = -BT_\varepsilon + Ac_\varepsilon = 0\,, \quad \text{i.e., } T_\varepsilon = \frac{Ac_\varepsilon}{B} = \frac{c_\varepsilon}{2(N-1)}\,.$$

Then, we have

(4.12) $\quad w(t, (x', R_0 + 1)) \le -A \quad \text{for } (t, (x', R_0 + 1)) \in [0, T_\varepsilon] \times \overline{\Omega}$

(4.13) $\quad w(t, (x', \varepsilon/2)) \ge A \quad \text{for } (t, (x', \varepsilon/2)) \in [0, T_\varepsilon] \times \overline{\Omega}$

from (4.9) and (4.10), respectively. Fix $A > 1$. Then (4.12) and (4.13) imply that there exists $\varepsilon_A > 0$ such that

$$w(t, (x', R_0 + 1)) < -1 \quad \text{for } (t, (x', R_0 + 1)) \in [0, T_\varepsilon + \varepsilon_A] \times \overline{\Omega}\,,$$

$$w(t,(x',\varepsilon/2)) > 1 \quad \text{for } (t,(x',\varepsilon/2)) \in [0, T_\varepsilon + \varepsilon_A] \times \overline{\Omega} .$$

We now define

$$\bar{w}(t,x) = \begin{cases} \min\{1, \max\{-1, w(t,x)\}\} = \bar{\theta}(w(t,x)) & \text{in } [0,\infty) \times \overline{\Omega}_{(\varepsilon/2,\infty)} , \\ 1 & \text{otherwise,} \end{cases}$$

where

$$\bar{\theta}(\sigma) = \begin{cases} -1 & \text{if } \sigma < -1 , \\ \sigma & \text{if } |\sigma| \le 1 , \\ 1 & \text{otherwise.} \end{cases}$$

Then, $\bar{w}$ is still a supersolution to (4.3)–(4.4) in $(0, T_\varepsilon + \varepsilon_A) \times \overline{\Omega}$. Moreover, (4.8) implies

$$\{x \in \overline{\Omega};\ \bar{w}(0,x) > 0\} \supset \overline{\Omega}_{(-\infty,R_0)} \supset \{x \in \overline{\Omega};\ u_0(x) > \beta + \mu\} .$$

Thus we obtain

$$\{x \in \overline{\Omega};\ \bar{w}(T_\varepsilon,x) > 0\} \supset \{x \in \overline{\Omega};\ u(T_\varepsilon,x) > \beta + \mu\}$$

by Theorem 2.7. The definition of T_ε implies

$$\bar{w}(T_\varepsilon,(x',x_N)) \le \bar{w}(T_\varepsilon,(x',\varepsilon)) \le \bar{w}(T_\varepsilon,(0',\varepsilon)) = 0$$

if $x_N \ge \varepsilon$, which and Lemma 4.1 yield

$$\overline{\Omega}_{(-\infty,\varepsilon)} \supset \{x \in \overline{\Omega};\ u(t,x) > \beta + \mu\} \quad \text{for } t \ge T_\varepsilon .$$

Hence, we obtain

$$\overline{\lim_{t\to\infty}}\, u(t,x) \le \beta + \mu \quad \text{if } x_N \ge \varepsilon .$$

Tending $\varepsilon \to 0$, and next $\mu \to 0$ yield that

$$\overline{\lim_{t\to\infty}}\, u(t,x) \le \beta \quad \text{if } x_N > 0 .$$

By the parallel arguments using a subsolution with $Bt - Av(x)$ we obtain

$$\underline{\lim}_{t\to\infty}\, u(t,x) \ge \beta \quad \text{if } x_N > 0 .$$

Hence, we obtain $\lim_{t\to\infty} u(t,x) = \beta$ if $x_N > 0$.

We also deduce $\lim_{t\to\infty} u(t,x) = \alpha$ if $x_N < 0$ by the parallel argument constructing a sub- and supersolution by v with $y = (0, y_N)$, $y_N > 0$. □

One can easily deduce the asymptotic stability of Γ from the proof of Theorem 4.2.

COROLLARY 4.3. *There is no continuous stationary solution* $u \in \mathfrak{Y}_{\alpha,\beta}$ *to* (4.3)–(4.4) *satisfying* $\Gamma = \{x \in \overline{W};\ u(x) = c\}$ *for* $c \in \mathbb{R}$ *between* α *and* β *and* Γ *given as in* (4.6).

5. Concluding remarks

The following problems are still open:

(i) Are there continuous stationary solutions to (3.1) describing a constant mean curvature interface?

(ii) Are there nonconstant stationary solutions to (3.1)?

However, at least every continuous stationary solution to (3.1) has neither strict local maximum nor minimum by Theorem 3.2. It is very strong restriction to continuous stationary solutions except constant functions.

Although we prove the nonexistence of continuous stationary solutions describing stationary interface Γ considered in §3 or §4, there exist continuous and exactly nonstationary solution describing Γ by [7] or [31]. It means that we have to choose the nonstationary solution as the center of linearization if we consider the stability of stationary interface to (1.1) with a level set method from a view point of eigenvalue problem.

We conclude this section to mention anisotropic curvature equation, which we do not treat in this paper. The exact solution to the level set equation of an anisotropic mean curvature equation is presented in [16, §1.7.2]. Thus, one can easily find a vanishing or spreading self similar solution to an anisotropic curvature flow with constant driving force by the parallel argument of the proof of Theorem 3.1.

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