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Orbit Structure of the Closure of a Homogeneous Cone

by

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Dedicated to Professor Fumihiro Sato on the occasion of his 65th birthday

Abstract. Let Ω be a homogeneous cone on which a split solvable Lie group H acts linearly and simply transitively. We describe the closure relation between H-orbits in the closure $\overline{\Omega}$ of the cone Ω, and discuss related results in connection with the Riesz distributions on Ω.

1. Introduction

In studying the wave equation, M. Riesz [12] considered a family of tempered distributions $T_\alpha \in \mathcal{S}'(\mathbb{R}^n)$ $(\alpha \in \mathbb{C})$ obtained by the analytic continuation with respect to the parameter α of the function

$$2^{1-\alpha}\pi^{(2-n)/2}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\Gamma\left(\frac{\alpha-n+2}{2}\right)^{-1}(x_1^2-x_2^2-\cdots-x_n^2)^{(\alpha-n)/2}$$

on the Lorentz cone. For a positive integer m, the distribution T_{2m} is a fundamental solution of the differential operator $\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\cdots-\frac{\partial^2}{\partial x_n^2}\right)^m$, and the validity of the Huygens principle of the operator depends on the nature of the support of T_{2m}. Gindikin [2] developed Riesz's idea in full generality in the theory of homogeneous cones. Let Ω be a homogeneous cone on which a split solvable Lie group H acts linearly and simply transitively. The Riesz distribution $\mathcal{R}_{\underline{s}}$ on Ω is defined by the analytic continuation with respect to the parameter $\underline{s}=(s_1,\dots,s_r)\in\mathbb{C}^r$ of the complex measure $\Gamma_\Omega(\underline{s})^{-1}f_1^{s_1}f_2^{s_2}\cdots f_r^{s_r}\,d\mu_\Omega$, where f_k $(k=1,\dots,r)$ is a certain H-relatively invariant rational function, μ_Ω an H-relatively invariant measure on Ω, and $\Gamma_\Omega(\underline{s})$ an appropriate gamma factor. Then $\mathcal{R}_{\underline{m}}$ with a special value $\underline{m}\in\mathbb{Z}^r$ is the fundamental solution of a differential operator $F(\frac{\partial}{\partial x})$, where F is an H-relatively invariant polynomial associated to the dual cone of Ω ([2, 4, 9]). The description of the support of the distributions $\mathcal{R}_{\underline{s}}$ is of fundamental importance, and it is still an open problem. When $\mathcal{R}_{\underline{s}}$ is a positive measure, it plays significant roles in representation theory ([3, 7, 14]) and statistics ([5, 6]). The parameter set $\Xi := \left\{\underline{s}\in\mathbb{C}^r\,;\,\mathcal{R}_{\underline{s}}\text{ is positive}\right\}$ is determined first by Gindikin [3] (see also [1, Theorem VII.3.2] and [13]), while the support of each $\mathcal{R}_{\underline{s}}$ $(\underline{s}\in\Xi)$ is given by [8, Theorem B]. Indeed, all positive $\mathcal{R}_{\underline{s}}$ are described explicitly in [8], related to the H-orbit structure of the closure $\overline{\Omega}$.

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In this paper, we discuss geometric structures of the H-orbits in $\overline{\Omega}$. Our main theorem is a simple criterion for the closure relation $\overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2}$ between H-orbits $\mathcal{O}_1,\ \mathcal{O}_2 \subset \overline{\Omega}$. The result is closely related to the study of Riesz distributions $\mathcal{R}_{\underline{s}}$. We can deduce a part of the main theorem from observation of the support of $\mathcal{R}_{\underline{s}}$. And we expect to utilize the theorem in future research on the distributions $\mathcal{R}_{\underline{s}}$.

Let us explain the present work in more detail. Let Ω be an open convex cone containing no line in a real vector space V. We assume that the cone Ω is homogeneous, that is, the linear automorphism group $GL(\Omega) := \{\, g \in GL(V)\,;\ g\Omega = \Omega \,\}$ acts on Ω transitively. Let H be a maximal connected split solvable Lie subgroup of $GL(\Omega)$. Such H is unique up to inner automorphisms of $GL(\Omega)$, and H acts on Ω simply transitively ([16]). Moreover, if a Lie group $G \subset GL(\Omega)$ acts on Ω transitively and its Lie algebra $\mathrm{Lie}(G)$ is algebraic, then there exists $g_0 \in G$ for which $g_0 H g_0^{-1} \subset G$. In this case, a G-orbit in V is a union of g_0-images of H-orbits. This observation tells us the significance of investigation of the H-action in the theory of homogeneous cones.

Let $\mathfrak{h}$ be the Lie algebra of H. Then the solvable Lie algebra $V \rtimes \mathfrak{h}$ admits a normal j-algebra structure ([8]), and one can make use of its root space decomposition due to Piatetskii-Shapiro [11]. As a result, there exist a commutative Cartan subalgebra $\mathfrak{a}$ of $V \rtimes \mathfrak{h}$ and linear forms $\alpha_k : \mathfrak{a} \to \mathbb{R}$ $(k = 1, \ldots, r,\ r := \dim \mathfrak{a})$ for which

$$\mathfrak{h} = \mathfrak{a} \oplus \sum_{1 \le k < l \le r}^{\oplus} \mathfrak{h}_{(\alpha_l - \alpha_k)/2}\,, \tag{1.1}$$

$$V = \sum_{k=1}^{r}{}^{\oplus}\, V_{\alpha_k} \oplus \sum_{1 \le k < l \le r}^{\oplus} V_{(\alpha_l + \alpha_k)/2}\,, \tag{1.2}$$

where

$$\begin{aligned} \mathfrak{h}_\alpha &:= \{\, T \in \mathfrak{h}\,;\ [C, T] = \alpha(C)T \quad (\forall C \in \mathfrak{a}) \,\}\,, \\ V_\alpha &:= \{\, x \in V\,;\ Cx = \alpha(C)x \quad (\forall C \in \mathfrak{a}) \,\} \end{aligned}$$

for a linear form $\alpha \in \mathfrak{a}^*$. We have $\dim V_{\alpha_k} = 1$ for $k = 1, \ldots r$, while some $n_{lk} := \dim V_{(\alpha_l+\alpha_k)/2}$ $(1 \le k < l \le r)$ may be zero. We can take $E_k \in V_{\alpha_k}$ $(k = 1, \ldots, r)$ so that $\sum_{k=1}^r E_k$ belongs to Ω. For $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0, 1\}^r$, let $E_{\underline{\varepsilon}}$ be the element $\sum_{k=1}^r \varepsilon_k E_k$ of V, and $\mathcal{O}_{\underline{\varepsilon}} \subset V$ the H-orbit through $E_{\underline{\varepsilon}}$. In particular, the cone Ω itself equals the H-orbit $\mathcal{O}_{(1,\ldots,1)}$. By [8, Theorem 3.5], the H-orbit decomposition of the closure of Ω is given by $\overline{\Omega} = \bigsqcup_{\underline{\varepsilon} \in \{0,1\}^r} \mathcal{O}_{\underline{\varepsilon}}$. For $\underline{\varepsilon} \in \{0, 1\}^r$, we define $\underline{\sigma}(\underline{\varepsilon}) = (\sigma_1(\underline{\varepsilon}), \ldots, \sigma_r(\underline{\varepsilon})) \in \mathbb{Z}^r$ by $\sigma_k(\underline{\varepsilon}) := \sum_{i=1}^k \varepsilon_i n_{ki}$ $\quad (k = 1, \ldots, r)$, where we put $n_{kk} := \dim V_{\alpha_k} = 1$ for $k = 1, \ldots, r$. Now we state our result.

MAIN THEOREM . *For $\underline{\varepsilon}^1,\ \underline{\varepsilon}^2 \in \{0, 1\}^r$, one has $\overline{\mathcal{O}_{\underline{\varepsilon}^1}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}^2}}$ if and only if $\sigma_k(\underline{\varepsilon}^1) \le \sigma_k(\underline{\varepsilon}^2)$ for all $k = 1, \ldots, r$.*

In order to study the H-orbits in $\overline{\Omega}$, we may assume that the cone $\Omega \subset V$ and the group H are realized as the set of matrices with specific block decompositions owing to the results in [10] about the symplectic representation of the normal j-algebra $V \rtimes \mathfrak{h}$. In section 2, we explain the detail of the matrix realization of a homogeneous cone, and present some results about geometry of H-orbits (Propositions 3 and 4). Main Theorem is proved in section 3 by rather elementary argument of linear algebra. In section 4, after brief review of results about Riesz distributions, we obtain a refinement of the 'if' part of Main Theorem (Theorem 9 (ii)) by observing the support of positive Riesz distributions. We emphasize that the meaning of the integers $\sigma_k(\underline{\varepsilon})$ becomes clearer when it is related to the theory of Riesz distributions. As a biproduct, we obtain the path-connectedness of the parameter set Ξ (Theorem 10).

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2. Matrix realization of a homogeneous cone

In studying a homogeneous cone, we may discuss only the cone $\Omega_{\mathcal{V}}$ constructed in a specific way explained below without loss of generality because every homogeneous cone is shown to be linearly isomorphic to such a cone ([10, Theorem D]). In what follows, we denote by $\mathrm{Mat}(p, q; \mathbb{R})$ the vector space of $p \times q$ real matrices, by $\mathrm{Mat}(p, \mathbb{R})$ the space $\mathrm{Mat}(p, p; \mathbb{R})$, and by $\mathrm{Sym}(p, \mathbb{R})$ the vector space of $p \times p$ real symmetric matrices. The transpose of a matrix A is denoted by ${}^{\mathrm{t}}A$. Let us take a partition $N = \nu_1+\nu_2+\cdots+\nu_r$ $(\nu_k > 0,\ k = 1, \dots, r)$ of a positive integer N, and $\mathcal{V} = \{V_{lk}\}_{1\le k\le l\le r}$ a system of vector spaces $V_{lk} \subset \mathrm{Mat}(\nu_l, \nu_k; \mathbb{R})$ satisfying

(V1) $A \in V_{lk},\ B \in V_{ki} \Rightarrow AB \in V_{li}$ $(1 \le i \le k \le l \le r)$,

(V2) $A \in V_{li},\ B \in V_{ki} \Rightarrow A\,{}^{\mathrm{t}}B \in V_{lk}$ $(1 \le i \le k \le l \le r)$,

(V3) $A \in V_{lk} \Rightarrow A\,{}^{\mathrm{t}}A \in \mathbb{R}I_{\nu_l}$ $(1 \le k \le l \le r)$,

(V4) $V_{ll} = \mathbb{R}I_{\nu_l}$ $(l = 1, \dots, r)$.

Let $\mathcal{Z}_{\mathcal{V}}$ be the vector space consisting of symmetric matrices $X \in \mathrm{Sym}(N, \mathbb{R})$ of the form

$$X = \begin{pmatrix} X_{11} & {}^{\mathrm{t}}X_{21} & \cdots & {}^{\mathrm{t}}X_{r1} \\ X_{21} & X_{22} & & {}^{\mathrm{t}}X_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{pmatrix} \qquad (X_{lk} \in V_{lk}, \quad 1 \le k \le l \le r)\,,$$

and $\Omega_{\mathcal{V}} \subset \mathcal{Z}_{\mathcal{V}}$ the subset $\{\, X \in \mathcal{Z}_{\mathcal{V}}\,;\ X \text{ is positive definite}\,\}$. Then $\Omega_{\mathcal{V}}$ is an open convex cone in the vector space $\mathcal{Z}_{\mathcal{V}}$. Let $\mathfrak{h}_{\mathcal{V}}$ be the vector space of lower triangular matrices $T \in \mathrm{Mat}(N, \mathbb{R})$ of the form

$$T = \begin{pmatrix} T_{11} & & & \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \cdots & T_{rr} \end{pmatrix} \qquad (T_{lk} \in V_{lk}, \quad 1 \le k \le l \le r)\,.$$

Then $\mathfrak{h}_{\mathcal{V}}$ is an $\mathbb{R}$-subalgebra of the matrix algebra $\mathrm{Mat}(N, \mathbb{R})$ by (V1). Diagonal components T_{kk} $(k = 1, \dots, r)$ of $T \in \mathfrak{h}_{\mathcal{V}}$ are scalar matrices by (V4), that is, we have $T_{kk} = t_{kk} I_{\nu_k}$ with $t_{kk} \in \mathbb{R}$. Regarding $\mathfrak{h}_{\mathcal{V}}$ as a real Lie algebra, we denote by $H_{\mathcal{V}}$ the Lie group $\exp \mathfrak{h}_{\mathcal{V}} \subset GL(N, \mathbb{R})$. Then we have

$$H_{\mathcal{V}} = \{\, T \in \mathfrak{h}_{\mathcal{V}}\,;\ t_{kk} > 0 \quad (k = 1, \dots, r)\,\} \subset \mathfrak{h}_{\mathcal{V}}\,.$$

If $T \in H_{\mathcal{V}}$ and $X \in \mathcal{Z}_{\mathcal{V}}$, then $\rho(T)X := TX\,{}^{t}T$ belongs to $\mathcal{Z}_{\mathcal{V}}$ thanks to (V1)–(V4). Moreover $\rho(H_{\mathcal{V}})$ acts on the cone $\Omega_{\mathcal{V}} \subset \mathcal{Z}_{\mathcal{V}}$ simply transitively (see [10, Proposition 3.2]). Namely, $\Omega_{\mathcal{V}}$ is a homogeneous cone. For instance, if $\nu_k = 1$ $(k = 1, \dots, r)$ and $V_{lk} = \mathbb{R}$ $(1 \le k \le l \le r)$, then we have $\mathcal{Z}_{\mathcal{V}} = \mathrm{Sym}(r, \mathbb{R})$, and $\Omega_{\mathcal{V}}$ is the cone $\mathrm{Sym}^{+}(r, \mathbb{R})$ of positive definite symmetric matrices. See [5, p. 331] for other examples of the homogeneous cone $\Omega_{\mathcal{V}}$.

Let $\mathfrak{a}$ be the subspace of $\mathfrak{h}$ consisting of diagonal matrices, that is,

$$\mathfrak{a} := \left\{ C = \begin{pmatrix} c_1 I_{\nu_1} & & & \\ & c_2 I_{\nu_2} & & \\ & & \ddots & \\ & & & c_r I_{\nu_r} \end{pmatrix} ;\ c_1, c_2, \dots, c_r \in \mathbb{R} \right\},$$

and $\alpha_k \in \mathfrak{a}^*$ $(k = 1, \dots, r)$ the linear form given by $\alpha_k(C) := 2c_k$ $(C \in \mathfrak{a})$. Then the root space decompositions (1.1) and (1.2) coincide with the block decompositions of $\mathfrak{h}_{\mathcal{V}}$ and $\mathcal{Z}_{\mathcal{V}}$ respectively, where $\mathfrak{h}_{(\alpha_l - \alpha_k)/2}$ and $V_{(\alpha_l + \alpha_k)/2}$ naturally correspond to V_{lk} for $1 \le k < l \le r$.

For $\underline{\varepsilon} \in \{0, 1\}^r$, let $E_{\underline{\varepsilon}}$ be the element of $\mathcal{Z}_{\mathcal{V}}$ given by

$$E_{\underline{\varepsilon}} := \begin{pmatrix} \varepsilon_1 I_{\nu_1} & & \\ & \ddots & \\ & & \varepsilon_r I_{\nu_r} \end{pmatrix},$$

and $\mathcal{O}_{\underline{\varepsilon}}$ the $H_{\mathcal{V}}$-orbit $\rho(H_{\mathcal{V}})E_{\underline{\varepsilon}} \subset \mathcal{Z}_{\mathcal{V}}$ through $E_{\underline{\varepsilon}}$. In particular, if we write $\underline{0}$ and $\underline{1}$ for $(0, \dots, 0)$ and $(1, \dots, 1)$ respectively, then $\mathcal{O}_{\underline{0}}$ is the origin $\{0\}$, and $\mathcal{O}_{\underline{1}} = \rho(H_{\mathcal{V}})I_N$ equals the cone $\Omega_{\mathcal{V}}$.

THEOREM 1 ([8, Theorem 3.5]). *The $H_{\mathcal{V}}$-orbit decomposition of the closure $\overline{\Omega_{\mathcal{V}}}$ of the homogeneous cone $\Omega_{\mathcal{V}}$ is given by*

$$\overline{\Omega_{\mathcal{V}}} = \bigsqcup_{\underline{\varepsilon} \in \{0,1\}^r} \mathcal{O}_{\underline{\varepsilon}}.$$

We denote by $H_{\underline{\varepsilon}}$ the stabilizer subgroup $\left\{ T \in H_{\mathcal{V}} ;\ \rho(T)E_{\underline{\varepsilon}} = E_{\underline{\varepsilon}} \right\}$ of $H_{\mathcal{V}}$ at $E_{\underline{\varepsilon}}$. Then we see easily that $H_{\underline{\varepsilon}}$ equals

$$\left\{ T \in H_{\mathcal{V}} ;\ \text{if } \varepsilon_k = 1,\ \text{then } T_{kk} = I_{\nu_k} \text{ and } T_{lk} = 0\ (l > k) \right\}.$$

Let $H(\mathcal{O}_{\underline{\varepsilon}})$ be the group $H_{\underline{1} - \underline{\varepsilon}}$, that is,

$$H(\mathcal{O}_{\underline{\varepsilon}}) := \left\{ T \in H_{\mathcal{V}} ;\ \text{if } \varepsilon_k = 0,\ \text{then } T_{kk} = I_{\nu_k} \text{ and } T_{lk} = 0\ (l > k) \right\}.$$

Then the Lie algebra of $H(\mathcal{O}_{\underline{\varepsilon}})$ is

$$\begin{aligned} \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) &:= \{ T \in \mathfrak{h} ;\ \text{if } \varepsilon_k = 0,\ \text{then } T_{lk} = 0\ (l \ge k) \} \\ &= \left\{ T \in \mathfrak{h} ;\ TE_{\underline{\varepsilon}} = T \right\}. \end{aligned} \tag{2.1}$$

We set

$$\mathfrak{h}^{+}(\mathcal{O}_{\underline{\varepsilon}}) := \left\{ T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) ;\ \text{if } \varepsilon_k = 1,\ \text{then } t_{kk} > 0 \right\} \subset \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}),$$

so that we have a bijection

$$\mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}}) \ni T \overset{\sim}{\mapsto} E_{\underline{1}-\underline{\varepsilon}} + T \in H(\mathcal{O}_{\underline{\varepsilon}})\,.$$

Though the following results are shown in [5] and [8], we give the proofs here for reader's convenience.

LEMMA 2. (i) *One has a diffeomorphism* $\mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}}) \ni T \mapsto T\,{}^{\mathrm{t}}T \in \mathcal{O}_{\underline{\varepsilon}}$.
(ii) *The group* $H(\mathcal{O}_{\underline{\varepsilon}})$ *acts on* $\mathcal{O}_{\underline{\varepsilon}}$ *simply transitively.*
(iii) *The closure of the orbit* $\mathcal{O}_{\underline{\varepsilon}}$ *is described as*

$$\overline{\mathcal{O}_{\underline{\varepsilon}}} = \left\{ T\,{}^{\mathrm{t}}T\,;\ T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) \right\}.$$

Proof. Let X be an element of the $H_{\mathcal{V}}$-orbit of $\mathcal{O}_{\underline{\varepsilon}}$. Then there exists $\tilde{T} \in H_{\mathcal{V}}$ for which $X = \rho(\tilde{T})E_{\underline{\varepsilon}}$. Put $T := \tilde{T}E_{\underline{\varepsilon}} \in \mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}})$. We observe

$$\begin{aligned} X &= \tilde{T}E_{\underline{\varepsilon}}{}^{\mathrm{t}}\tilde{T} = (\tilde{T}E_{\underline{\varepsilon}})\,{}^{\mathrm{t}}(\tilde{T}E_{\underline{\varepsilon}}) = T\,{}^{\mathrm{t}}T \\ &= (E_{\underline{1}-\underline{\varepsilon}} + T)E_{\underline{\varepsilon}}{}^{\mathrm{t}}(E_{\underline{1}-\underline{\varepsilon}} + T) = \rho(E_{\underline{1}-\underline{\varepsilon}} + T)E_{\underline{\varepsilon}}\,. \end{aligned}$$

Since $E_{\underline{1}-\underline{\varepsilon}} + T \in H(\mathcal{O}_{\underline{\varepsilon}})$, the orbit map $H(\mathcal{O}_{\underline{\varepsilon}}) \ni T^0 \mapsto \rho(T^0)E_{\underline{\varepsilon}} \in \mathcal{O}_{\underline{\varepsilon}}$ is surjective, while it is injective because $H(\mathcal{O}_{\underline{\varepsilon}}) \cap H_{\underline{\varepsilon}} = \{I_N\}$. Therefore (i) and (ii) hold. Let us prove (iii). We consider the quadratic map

$$q_{\underline{\varepsilon}} : \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) \ni T \mapsto T\,{}^{\mathrm{t}}T \in \mathcal{Z}_{\mathcal{V}}\,.$$

It follows from (i) that the image $q_{\underline{\varepsilon}}(\mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}))$ is contained in $\overline{\mathcal{O}_{\underline{\varepsilon}}}$. Thus it is sufficient to show that $q_{\underline{\varepsilon}}(\mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}))$ is a closed set in $\mathcal{Z}_{\mathcal{V}}$. In general, for a real vector space W, we denote by $\mathbb{P}(W)$ the projective space $(W \oplus \mathbb{R} \setminus \{(0,0)\})/\mathbb{R}^\times$, and by ι_W the projective imbedding $\iota_W : W \ni w \mapsto [w, 1] \in \mathbb{P}(W)$. We extend the map $q_{\underline{\varepsilon}} : \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) \to \mathcal{Z}_{\mathcal{V}}$ to the continuous map $\tilde{q}_{\underline{\varepsilon}} : \mathbb{P}(\mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}})) \ni [T, c] \mapsto [T\,{}^{\mathrm{t}}T, c^2] \in \mathbb{P}(\mathcal{Z}_{\mathcal{V}})$, noting that $T\,{}^{\mathrm{t}}T \neq 0$ if $T \neq 0$. The image $\tilde{q}_{\underline{\varepsilon}}(\mathbb{P}(\mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}})))$ is compact, so that the set $q_{\underline{\varepsilon}}(\mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}})) = \iota_{\mathcal{Z}_{\mathcal{V}}}^{-1}(\tilde{q}_{\underline{\varepsilon}}(\mathbb{P}(\mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}))))$ is closed in $\mathcal{Z}_{\mathcal{V}}$. $\square$

We see from Lemma 2 (i) that the $H_{\mathcal{V}}$-orbit $\mathcal{O}_{\underline{\varepsilon}}$ is homeomorphic to a vector space. Furthermore, we deduce the following.

PROPOSITION 3 ([8, Proposition 3.6]). *Let* $\underline{\varepsilon}$ *and* $\underline{\varepsilon}'$ *be elements of* $\{0,1\}^r$ *for which* $\underline{\varepsilon} + \underline{\varepsilon}' \in \{0,1\}^r$. *Then one has a bijection*

$$\mathcal{O}_{\underline{\varepsilon}} \times \mathcal{O}_{\underline{\varepsilon}'} \ni (X, X') \mapsto X + X' \in \mathcal{O}_{\underline{\varepsilon}+\underline{\varepsilon}'}\,.$$

Proof. Clearly the map

$$\mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}}) \times \mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}'}) \ni (T, T') \mapsto T + T' \in \mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}+\underline{\varepsilon}'})$$

is bijective, and we have $(T+T')\,{}^{\mathrm{t}}(T+T') = T\,{}^{\mathrm{t}}T + T'\,{}^{\mathrm{t}}T'$. These observations together with Lemma 2 (i) imply the statement. $\square$

Let $\mathcal{C} \subset \mathbb{R}^n$ be a closed convex cone. For $x \in \mathcal{C}$, we say that the half line $\mathbb{R}_+x = \{\,\lambda x\,;\ \lambda > 0\,\}$ is an extremal ray of $\mathcal{C}$ if $x = x' + x''$ with $x', x'' \in \mathcal{C}$ implies $x', x'' \in \mathbb{R}_+x$. We denote by $\mathrm{Ex}(\mathcal{C})$ the set of $x \in \mathcal{C}$ for which $\mathbb{R}_+x$ is an extremal ray of $\mathcal{C}$.

PROPOSITION 4. *Let $\underline{\delta}(k) \in \{0,1\}^r$ be the element of $\{0,1\}^r$ whose k-th component is* 1 *and other components are* 0. *Then one has*

$$\mathrm{Ex}(\overline{\Omega_{\mathcal{V}}}) = \bigsqcup_{k=1}^{r} \mathcal{O}_{\underline{\delta}(k)} .$$

Proof. If $X \in \mathcal{O}_{\underline{\varepsilon}}$ with $\underline{\varepsilon} \neq \underline{\delta}(k)$ for $k = 1, \dots, r$, then $\mathbb{R}_+ X$ is not an extremal ray because of Proposition 3. Thus it is sufficient to show that $\mathbb{R}_+ E_{\underline{\delta}(k)}$ is an extremal ray for $k = 1, \dots, r$. Suppose $E_{\underline{\delta}(k)} = X' + X''$ with $X', X'' \in \overline{\Omega_{\mathcal{V}}}$. By (V4), we have $X'_{ii} = x'_{ii} I_{\nu_i}$ and $X''_{ii} = x''_{ii} I_{\nu_i}$ with $x'_{ii}, x''_{ii} \in \mathbb{R}$ for $i = 1, \dots, r$. Since X' and X'' are positive semi-definite, we have $x'_{ii} \geq 0$ and $x''_{ii} \geq 0$. On the other hand, if $i \neq k$, we have $x'_{ii} + x''_{ii} = 0$. Thus $X'_{ii} = X''_{ii} = 0$, which together with the positive semi-definiteness of X' and X'' implies that the off-diagonal (l, i)-components X'_{li} and X''_{li} equal 0 also for $1 \leq i < l \leq r$. Therefore X' and X'' belong to $\mathbb{R}_+ E_{\underline{\delta}(k)}$, which completes the proof. $\square$

3. Main result

We put $n_{lk} := \dim V_{lk}$ $(1 \leq k \leq l \leq r)$. For $\underline{\varepsilon} \in \{0,1\}^r$, define $\underline{\sigma}(\underline{\varepsilon}) = (\sigma_1(\underline{\varepsilon}), \dots, \sigma_r(\underline{\varepsilon})) \in \mathbb{Z}^r$ by

$$\sigma_k(\underline{\varepsilon}) := \sum_{i=1}^{k} \varepsilon_i n_{ki} \quad (k = 1, \dots, r) . \tag{3.1}$$

THEOREM 5. *For $\underline{\varepsilon}^1, \underline{\varepsilon}^2 \in \{0,1\}^r$, one has $\overline{\mathcal{O}_{\underline{\varepsilon}^1}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}^2}}$ if and only if $\sigma_k(\underline{\varepsilon}^1) \leq \sigma_2(\underline{\varepsilon}^2)$ for all $k = 1, \dots, r$.*

Proof. The key idea of the proof is to introduce the vector space $\mathfrak{r}_k(\underline{\varepsilon})$ $(1 \leq k \leq r,\ \underline{\varepsilon} \in \{0,1\}^r)$ below whose dimension is equal to $\sigma_k(\underline{\varepsilon})$. For $T \in \mathfrak{h}$ and $k = 1, \dots, r$, we set

$$R_k(T) := \begin{pmatrix} T_{k1} & T_{k2} & \cdots & T_{kk} \end{pmatrix} \in \mathrm{Mat}(\nu_k, \nu_1 + \nu_2 + \cdots + \nu_k; \mathbb{R}) .$$

Namely, $R_k(T)$ is the r-th block raw of the triangular matrix T. Put

$$\mathfrak{r}_k(\underline{\varepsilon}) := \left\{ R_k(T) \,;\ T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) \right\} \subset \mathrm{Mat}(\nu_k, \nu_1 + \nu_2 + \cdots + \nu_k; \mathbb{R}) .$$

Then we have $\dim \mathfrak{r}_k(\underline{\varepsilon}) = \sigma_k(\underline{\varepsilon})$. Similarly, we define

$$\check{R}_k(T) := \begin{pmatrix} T_{k1} & T_{k2} & \cdots & T_{k,k-1} \end{pmatrix} \in \mathrm{Mat}(\nu_k, \nu_1 + \nu_2 + \cdots + \nu_{k-1}; \mathbb{R})$$

for $k = 2, \dots, r$ and $T \in \mathfrak{h}$, and

$$\check{\mathfrak{r}}_k(\underline{\varepsilon}) := \left\{ \check{R}_k(T) \,;\ T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}}) \right\} \subset \mathrm{Mat}(\nu_k, \nu_1 + \nu_2 + \cdots + \nu_{k-1}; \mathbb{R}) .$$

Clearly, we have

$$\dim \check{\mathfrak{r}}_k(\underline{\varepsilon}) = \sigma_k(\underline{\varepsilon}) - \varepsilon_k . \tag{3.2}$$

Let us show the 'only if' part of the statement. Assume that $\overline{\mathcal{O}_{\underline{\varepsilon}^1}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}^2}}$. Since $E_{\underline{\varepsilon}^1} \in \overline{\mathcal{O}_{\underline{\varepsilon}^2}}$, we can find $\tilde{T} \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^2})$ for which $E_{\underline{\varepsilon}^1} = \tilde{T}\,{}^{\mathrm{t}}\tilde{T}$ because of Lemma 2 (iii).

Keeping (2.1) in mind, we have a linear map

$$\phi : \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^1}) \ni T \mapsto T\tilde{T} \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^2})\,.$$

We observe that ϕ is injective. Indeed, if $\phi(T) = 0$ for some $T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^1})$, then

$$0 = (T\tilde{T})^{\mathrm{t}}(T\tilde{T}) = TE_{\underline{\varepsilon}^1}{}^{\mathrm{t}}T = T\,{}^{\mathrm{t}}T\,,$$

so that $T = 0$. Let $\tilde{T}^{[k]}$ $(k = 1, \ldots, r)$ be the submatrix of $\tilde{T}$ defined by

$$\tilde{T}^{[k]} := \begin{pmatrix} \tilde{T}_{11} & & \\ \vdots & \ddots & \\ \tilde{T}_{k1} & \cdots & \tilde{T}_{kk} \end{pmatrix} \in \mathrm{Mat}(\nu_1 + \cdots + \nu_k; \mathbb{R})\,.$$

Let us consider a linear map

$$\phi_k : \mathfrak{r}_k(\underline{\varepsilon}^1) \ni R_k(T) \mapsto R_k(T)\tilde{T}^{[k]} = R_k(T\tilde{T}) \in \mathfrak{r}_k(\underline{\varepsilon}^2) \qquad (T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^1}))\,.$$

In a similar way to the case for ϕ, we see that ϕ_k is injective. Therefore we obtain $\dim \mathfrak{r}_k(\underline{\varepsilon}^1) \le \dim \mathfrak{r}_k(\underline{\varepsilon}^2)$, which means that $\sigma_k(\underline{\varepsilon}^1) \le \sigma_k(\underline{\varepsilon}^2)$.

Next we show the 'if' part. Assume that $\sigma_k(\underline{\varepsilon}^1) \le \sigma_k(\underline{\varepsilon}^2)$ for $k = 1, \ldots, r$. In order to show $\overline{\mathcal{O}_{\underline{\varepsilon}^1}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}^2}}$, it is sufficient to find $\tilde{T} \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^2})$ for which $E_{\underline{\varepsilon}^1} = \tilde{T}\,{}^{\mathrm{t}}\tilde{T}$ thanks to Lemma 2 (iii). Since $E_{\underline{\varepsilon}^1} = (E_{\underline{\varepsilon}^1}\tilde{T})^{\mathrm{t}}(E_{\underline{\varepsilon}^1}\tilde{T})$ and $E_{\underline{\varepsilon}^1}\tilde{T} \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^2})$ for such $\tilde{T}$, we can assume further that the matrix $\tilde{T}$ satisfies $\tilde{T} = E_{\underline{\varepsilon}^1}\tilde{T}$ without loss of generality. We shall get $\tilde{T}$ by determining $\tilde{T}^{[k]}$ recursively for $k = 1, \ldots, r$ so that

$$\tilde{T}^{[k]}\,{}^{\mathrm{t}}\tilde{T}^{[k]} = \begin{pmatrix} \varepsilon_1^1 I_{\nu_1} & & \\ & \ddots & \\ & & \varepsilon_k^1 I_{\nu_k} \end{pmatrix} =: E_{\underline{\varepsilon}^1}^{[k]}, \tag{3.3}$$

$$\tilde{T}^{[k]} E_{\underline{\varepsilon}^2}^{[k]} = \tilde{T}^{[k]}\,, \tag{3.4}$$

and

$$E_{\underline{\varepsilon}^1}^{[k]} \tilde{T}^{[k]} = \tilde{T}^{[k]}\,. \tag{3.5}$$

Since $\varepsilon_1^1 = \sigma_1(\underline{\varepsilon}^1) \le \sigma_1(\underline{\varepsilon}^2) = \varepsilon_1^2$, we set $\tilde{T}^{[1]} := \varepsilon_1^1 I_{\nu_1}$ which satisfies (3.3), (3.4) and (3.5) with $k = 1$. Assume that $\tilde{T}^{[k-1]}$ $(2 \le k \le r)$ is determined. If $\varepsilon_k^1 \le \varepsilon_k^2$, then we set

$$\tilde{T}^{[k]} := \begin{pmatrix} \tilde{T}^{[k-1]} & \\ 0 & \varepsilon_k^1 I_{\nu_k} \end{pmatrix}$$

for the required properties. Let us consider the case $\varepsilon_k^1 = 1$ and $\varepsilon_k^2 = 0$. By (3.2), we have $\dim \check{\mathfrak{r}}_k(\underline{\varepsilon}^1) = \sigma_k(\underline{\varepsilon}^1) - 1 < \sigma_k(\underline{\varepsilon}^2) = \dim \check{\mathfrak{r}}_k(\underline{\varepsilon}^2)$. Thus the linear map

$$\psi_k : \check{\mathfrak{r}}_k(\underline{\varepsilon}^1) \ni \check{R}_k(T) \mapsto \check{R}_k(T)\tilde{T}^{[k-1]} \in \check{\mathfrak{r}}_k(\underline{\varepsilon}^2) \qquad (T \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^1}))\,.$$

is not surjective. We take a non-zero element $\tilde{Y}$ of the orthogonal complement $(\mathrm{Image}\,\psi_k)^{\perp} \subset \check{\mathfrak{r}}_k(\underline{\varepsilon}^2)$ with respect to the inner product defined by $(Y_1|Y_2) := \mathrm{tr}\,(Y_1\,{}^{\mathrm{t}}Y_2)$ $(Y_1, Y_2 \in \check{\mathfrak{r}}_k(\underline{\varepsilon}^2))$. Then we have

$$0 = (\tilde{Y}|\psi_k(Y)) = \mathrm{tr}\,(\tilde{Y}\,{}^{\mathrm{t}}\tilde{T}^{[k-1]}\,{}^{\mathrm{t}}Y) \tag{3.6}$$

for any $Y \in \check{\mathfrak{r}}_k(\underline{\varepsilon}^1)$. Since $\tilde{Y}\,{}^t\tilde{Y} \in \mathbb{R}I_{\nu_k}$ by (V3), we can normalize $\tilde{Y}$ so that

$$\tilde{Y}\,{}^t\tilde{Y} = I_{\nu_k}\,.$$

On the other hand, we see from (V2) that $\tilde{Y}\,{}^t\tilde{T}^{[k-1]}$ belongs to the space $\check{\mathfrak{r}}_k = \check{\mathfrak{r}}_k(1,\dots,1)$, while we obtain $\tilde{Y}\,{}^t\tilde{T}^{[k-1]}E_\varepsilon^{[k-1]} = \tilde{Y}\,{}^t\tilde{T}^{[k-1]}$ by (3.5). Thus $\tilde{Y}\,{}^t\tilde{T}^{[k-1]} \in \check{\mathfrak{r}}_k(\underline{\varepsilon}^1)$. Therefore, putting $Y = \tilde{Y}\,{}^t\tilde{T}^{[k-1]}$ in (3.6), we have $\tilde{Y}\,{}^t\tilde{T}^{[k-1]} = 0$. We set

$$\tilde{T}^{[k]} := \begin{pmatrix} \tilde{T}^{[k-1]} & \\ \tilde{Y} & 0 \end{pmatrix},$$

so that (3.3), (3.4) and (3.5) are satisfied. In this way, we obtain $\tilde{T} = \tilde{T}^{[r]} \in \mathfrak{h}(\mathcal{O}_{\underline{\varepsilon}^2})$ for which $\tilde{T}\,{}^t\tilde{T} = E_{\underline{\varepsilon}^1}$. $\square$

4. Riesz distribution

In this section, we investigate the $H_{\mathcal{V}}$-orbits in $\overline{\Omega_{\mathcal{V}}}$ in connection with the theory of Riesz distributions, so that a refinement of the 'if' part of Theorem 5 is obtained (Theorem 9 (ii)). Let us recall the definition of Riesz distributions on the homogeneous cone $\Omega_{\mathcal{V}} \subset \mathcal{Z}_{\mathcal{V}}$ and their basic properties. See [5] and [8] for the details. For $\underline{s} = (s_1,\dots,s_r) \in \mathbb{C}^r$, let $\chi_{\underline{s}} : H_{\mathcal{V}} \to \mathbb{C}^\times$ be the one-dimensional representation of $H_{\mathcal{V}}$ given by $\chi_{\underline{s}}(T) := \prod_{k=1}^r t_{kk}^{2s_k}$ $(T \in H_{\mathcal{V}})$. Define $\Delta_{\underline{s}} : \Omega_{\mathcal{V}} \to \mathbb{C}$ by $\Delta_{\underline{s}}(X) := \chi_{\underline{s}}(T)$ $(X = T\,{}^tT \in \Omega_{\mathcal{V}},\ T \in H_{\mathcal{V}})$. Then $\Delta_{\underline{s}}$ is an $H_{\mathcal{V}}$-relatively invariant function on the cone $\Omega_{\mathcal{V}}$. Moreover, $\Delta_{\underline{s}}$ can be expressed as a product of powers of minors as follows. For $p = 1,\dots,N$ and $S \in \mathrm{Sym}(N,\mathbb{R})$, we denote by $\det^{[p]} S$ the principal minor $\det(S_{\alpha\beta})_{1\le\alpha\le p,\,1\le\beta\le p}$ of degree p. Put $f_1 := \det^{[1]}$ and

$$f_k := \frac{\det^{[\nu_1+\cdots+\nu_{k-1}+1]}}{\det^{[\nu_1+\cdots+\nu_{k-1}]}} \quad (k = 2,\dots,r)\,.$$

Then we have $\Delta_{\underline{s}} = \prod_{k=1}^r f_k^{s_k}$.

Keeping (V3) in mind, we define an inner product on each V_{lk} $(1 \le k \le l \le r)$ by $(A|B) := (\mathrm{tr}\,A\,{}^tB)/\nu_l$ $(A, B \in V_{lk})$, so that

$$X_{lk}\,{}^tX_{lk} = (X_{lk}|X_{lk})I_{\nu_l} \quad (X_{lk} \in V_{lk})\,.$$

We denote by dX the Lebesgue measure $\prod_{1\le k\le l\le r} dX_{lk}$ on $\mathcal{Z}_{\mathcal{V}}$ normalized by the inner product. We define also $\underline{d} := (d_1,\dots,d_r) \in \mathbb{Z}^r/2$ by $d_k := 1 + (\sum_{i<k} n_{ki} + \sum_{l>k} n_{lk})/2$ for $k = 1,\dots,r$. Let $\mu_{\mathcal{V}}$ be a measure on the cone $\Omega_{\mathcal{V}}$ given by $d\mu_{\mathcal{V}}(X) := \Delta_{-\underline{d}}(X)dX$ $(X \in \Omega_{\mathcal{V}})$. Then $\mu_{\mathcal{V}}$ is invariant under the action of $H_{\mathcal{V}}$. For $X \in \mathcal{Z}$, we denote by x_{kk} $(k = 1,\dots,r)$ the real number for which $X_{kk} = x_{kk}I_{\nu_k}$. It is shown in [2, Theorem 2.1] that the integral

$$\Gamma_{\mathcal{V}}(\underline{s}) := \int_{\Omega_{\mathcal{V}}} e^{-\sum_{k=1}^r x_{kk}}\,\Delta_{\underline{s}}(X)\,d\mu_{\mathcal{V}}(X)$$

converges if and only if $2\Re s_k > p_k := \sum_{i<k} n_{ki}$ for $k = 1, \dots, r$, and in this case

$$\Gamma_{\mathcal{V}}(\underline{s}) = \pi^{(n-r)/2} \prod_{k=1}^{r} \Gamma\left(s_k - \frac{p_k}{2}\right),$$

where $n := \dim \mathcal{Z}_{\mathcal{V}} = \sum_{1 \le k \le l \le r} n_{lk}$. Moreover, $\mathcal{R}_{\underline{s}} := \Gamma_{\mathcal{V}}(\underline{s})^{-1} \Delta_{\underline{s}}\, d\mu_{\mathcal{V}}$ defines a complex Radon measure on $\mathcal{Z}_{\mathcal{V}}$ when $\Gamma_{\mathcal{V}}(\underline{s})$ converges, and admits the analytic continuation to whole $\underline{s} \in \mathbb{C}^r$ as a tempered distribution.

For $\underline{\varepsilon} \in \{0, 1\}^r$, we define $p_k(\underline{\varepsilon}) := \sum_{i<k} \varepsilon_i n_{ki}$ $(k = 1, \dots, r)$,

$$\Xi(\underline{\varepsilon}) := \left\{ \underline{s} \in \mathbb{R}^r \,;\, s_k = p_k(\underline{\varepsilon})/2 \text{ (if } \varepsilon_k = 0), \quad s_k > p_k(\underline{\varepsilon})/2 \text{ (if } \varepsilon_k = 1) \right\},$$

and $\Xi := \bigsqcup_{\underline{\varepsilon} \in \{0,1\}^r} \Xi(\underline{\varepsilon})$.

THEOREM 6 ([8, Theorem B]). *The Riesz distribution $\mathcal{R}_{\underline{s}}$ is positive if and only if $\underline{s} \in \Xi$. Moreover, if $\underline{s} \in \Xi(\underline{\varepsilon})$, then $\mathcal{R}_{\underline{s}}$ is a measure on the $H_{\mathcal{V}}$-orbit $\mathcal{O}_{\underline{\varepsilon}}$.*

Using the diffeomorphism $q_{\underline{\varepsilon}} : \mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}}) \ni T \mapsto T\,{}^{\mathrm{t}}T \in \mathcal{O}_{\underline{\varepsilon}}$ in Lemma 2 as a coordinate map of $\mathcal{O}_{\underline{\varepsilon}}$, we can describe the measure $\mathcal{R}_{\underline{s}}$ on $\mathcal{O}_{\underline{\varepsilon}}$ for $\underline{s} \in \Xi(\underline{\varepsilon})$ as follows.

PROPOSITION 7 ([5, Proposition 3.10]). *If $\underline{s} \in \Xi(\underline{\varepsilon})$, one has*

$$d\mathcal{R}_{\underline{s}}(X) = \prod_{\varepsilon_k=1} \left\{ \frac{2(t_{kk})^{2s_k - p_k(\underline{\varepsilon}) - 1}\, dt_{kk}}{\Gamma(s_k - \frac{p_k(\underline{\varepsilon})}{2})} \cdot \prod_{l>k} \frac{dT_{lk}}{\pi^{n_{lk}/2}} \right\},$$

where $X = T\,{}^{\mathrm{t}}T \in \mathcal{O}_{\underline{\varepsilon}}$ with $T \in \mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}})$.

We have a concise algorithm to know whether $\underline{s}$ belongs to Ξ for a given $\underline{s} \in \mathbb{R}^r$.

PROPOSITION 8 ([8, Proposition 6.1]). *For $\underline{s} \in \mathbb{R}^r$, define $u_k^i \in \mathbb{R}$ $(1 \le i \le k \le r)$ by $u_k^1 := s_k$ $(k = 1, \dots, r)$ and*

$$u_k^i := \begin{cases} u_k^{i-1} - n_{ki}/2 & (\textit{if } u_{i-1}^{i-1} > 0), \\ u_k^{i-1} & (\textit{if } u_{i-1}^{i-1} \le 0) \end{cases}$$

for $2 \le i \le k \le r$. Then $\underline{s} \in \Xi$ if and only if $u_k^k \ge 0$ for all $k = 1, \dots, r$. In this case, putting

$$\varepsilon_k := \begin{cases} 1 & (\textit{if } u_k^k > 0), \\ 0 & (\textit{if } u_k^k = 0) \end{cases}$$

for $k = 1, \dots, r$, one has $\underline{s} \in \Xi(\underline{\varepsilon})$.

Let us recall the parameter $\underline{\sigma}(\underline{\varepsilon}) = (\sigma_1(\underline{\varepsilon}), \dots, \sigma_r(\underline{\varepsilon})) \in \mathbb{Z}^r$ defined by (3.1). We note that $\underline{\sigma}(\underline{\varepsilon})/2$ $(\underline{\varepsilon} \in \{0, 1\}^r)$ belongs to $\Xi(\underline{\varepsilon})$. Indeed, Proposition 7 tells us that $\mathcal{R}_{\underline{\sigma}(\underline{\varepsilon})/2}$ equals the image of the Lebesgue measure $\prod_{\varepsilon_k=1} \prod_{l>k} dt_{kk} dT_{lk}$ on $\mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}})$ via the map $q_{\underline{\varepsilon}} : \mathfrak{h}^+(\mathcal{O}_{\underline{\varepsilon}}) \to \mathcal{O}_{\underline{\varepsilon}}$ up to a positive constant multiple.

THEOREM 9. *Let $\underline{\varepsilon}^1$ and $\underline{\varepsilon}^2$ be elements of $\{0,1\}^r$ such that $\sigma_k(\underline{\varepsilon}^1) \le \sigma_k(\underline{\varepsilon}^2)$ for all $k = 1, \dots, r$. For $j = 1, \dots, r$ and $t \in [0,1]$, define $\underline{s}(j,t) \in \mathbb{R}^r$ by*

$$s_k(j,t) := \begin{cases} \sigma_k(\underline{\varepsilon}^1) & (k < j)\,, \\ t\sigma_k(\underline{\varepsilon}^1) + (1-t)\sigma_k(\underline{\varepsilon}^2) & (k = j)\,, \\ \sigma_k(\underline{\varepsilon}^2) & (k > j)\,. \end{cases}$$

(i) *For $j = 1, \dots, r$, there exists $\underline{\varepsilon}(j) \in \{0,1\}^r$ such that $\underline{s}(j,t)/2 \in \Xi(\underline{\varepsilon}(j))$ for all $t \in [0,1)$.*

(ii) *One has $\overline{\mathcal{O}_{\underline{\varepsilon}^1}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}(r)}} \subset \cdots \overline{\mathcal{O}_{\underline{\varepsilon}(2)}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}(1)}} = \overline{\mathcal{O}_{\underline{\varepsilon}^2}}$.*

Proof. We show the assertion (i) by induction on j. Let us consider the case $j = 1$. By the assumption, we have $\varepsilon_1^1 = \sigma_1(\underline{\varepsilon}^1) \le \sigma_1(\underline{\varepsilon}^2) = \varepsilon_1^2$. If $\varepsilon_1^1 = \varepsilon_1^2$, then $\underline{s}(1,t) = \underline{\sigma}(\underline{\varepsilon}^2)$ for $t \in [0,1]$, so that the claim holds with $\underline{\varepsilon}(1) = \underline{\varepsilon}^2$. For the case $\varepsilon_1^1 = 0$ and $\varepsilon_1^2 = 1$, we have $s_1(1,t) = 1 - t > 0$ for $t \in [0,1)$ and $s_k(1,t) = \sigma_k(\underline{\varepsilon}^2)$ $(k = 2, \dots, r)$. Thus $\underline{s}(1,t)/2 \in \Xi(\underline{\varepsilon}(1))$ with $\underline{\varepsilon}(1) = \underline{\varepsilon}^2$ again.

Assume that the claim holds for $j = i < r$. Let φ be any non-negative function on $\mathcal{Z}_{\mathcal{V}}$ with compact support. Since $\mathcal{R}_{\underline{s}(i,t)/2}$ is a positive measure on $\mathcal{O}_{\underline{\varepsilon}(i)}$ for $t \in [0,1)$ by the induction hypothesis and Theorem 6, we have

$$\langle \mathcal{R}_{\underline{s}(i,1)/2}, \varphi \rangle = \lim_{t \to 1-0} \langle \mathcal{R}_{\underline{s}(i,t)/2}, \varphi \rangle \ge 0\,. \tag{4.1}$$

Therefore the distribution $\mathcal{R}_{\underline{s}(i,1)/2}$ is positive, so that Theorem 6 tells us the existence of $\underline{\varepsilon}(i+1) \in \{0,1\}^r$ for which

$$\underline{s}(i,1)/2 \in \Xi(\underline{\varepsilon}(i+1))\,.$$

Furthermore, (4.1) implies that $\overline{\mathcal{O}_{\underline{\varepsilon}(i+1)}} = \operatorname{supp} \mathcal{R}_{\underline{s}(i,1)/2}$ is contained in $\overline{\mathcal{O}_{\underline{\varepsilon}(i)}} = \operatorname{supp} \mathcal{R}_{\underline{s}(i,t)/2}$ $(t \in [0,1))$. We note that $\underline{s}(i+1,0) = \underline{s}(i,1)$ by definition. Let us check that $p_{i+1}(\underline{\varepsilon}(i+1)) = \sum_{h=1}^{i} \varepsilon_h(i+1) n_{i+1,h}$ is equal to $p_{i+1}(\underline{\varepsilon}^1)$. In view of Proposition 8, we see that the h-th component $\varepsilon_h(i+1)$ of $\underline{\varepsilon}(i+1)$ is determined from $s_\alpha(i+1,1)$ $(\alpha = 1, \dots, h)$, which is equal to $\sigma_\alpha(\underline{\varepsilon}^1)$ if $h \le i$. Thus $\varepsilon_h(i+1) = \varepsilon_h^1$ for $h = 1, \dots, i$, so that

$$p_{i+1}(\underline{\varepsilon}(i+1)) = p_{i+1}(\underline{\varepsilon}^1) \le \sigma_{i+1}(\underline{\varepsilon}^1) \le \sigma_{i+1}(\underline{\varepsilon}^2)\,.$$

It follows that

$$p_{i+1}(\underline{\varepsilon}(i+1)) \le s_{i+1}(i+1,t) \le \sigma_{i+1}(\underline{\varepsilon}^2) \quad (0 \le t \le 1)\,.$$

Moreover, $s_{i+1}(i+1,t) > p_{i+1}(\underline{\varepsilon}(i+1))$ for all $t \in [0,1)$ if and only if $\sigma_{i+1}(\underline{\varepsilon}^2) > p_{i+1}(\underline{\varepsilon}(i+1))$. Therefore we see that $\underline{s}(i+1,t)/2 \in \Xi(\underline{\varepsilon}(i+1))$ for all $t \in [0,1)$. The assertion (i) is verified.

For the assertion (ii), it remains to check that $\overline{\mathcal{O}_{\underline{\varepsilon}^1}} \subset \overline{\mathcal{O}_{\underline{\varepsilon}(r)}}$, which follows from $\mathcal{R}_{\underline{\sigma}(\underline{\varepsilon}^1)/2} = \mathcal{R}_{\underline{s}(r,1)/2} = \lim_{t \to 1-0} \mathcal{R}_{\underline{s}(r,t)/2}$. □

Let L_j be the segment $\{\underline{s}(j,t)/2\,;\ t \in [0,1]\} \subset \mathbb{R}^r$ for $j = 1, \dots, r$. Theorem 9 (i) tells us that the union $\bigcup_{j=1}^r L_j$ gives the path from $\underline{\sigma}(\underline{\varepsilon}^2)/2$ to $\underline{\sigma}(\underline{\varepsilon}^1)/2$ in the set $\Xi = \bigsqcup_{\underline{\varepsilon} \in \{0,1\}^r} \Xi(\underline{\varepsilon})$. In particular, putting $\underline{\varepsilon}^1 := \underline{0}$ and $\underline{\varepsilon}^2 := \underline{\varepsilon}$ for any $\underline{\varepsilon} \in \{0,1\}^r$, we see that $\underline{\sigma}(\underline{\varepsilon})/2$ is connected to $\underline{0} = \underline{\sigma}(\underline{0})/2$ by the path, while any $\underline{s} \in \Xi(\underline{\varepsilon})$ is connected to $\underline{\sigma}(\underline{\varepsilon})/2$ by a segment because $\Xi(\underline{\varepsilon})$ is a convex set. As a result, we obtain the final result:

THEOREM 10. *The parameter set $\Xi \subset \mathbb{R}^r$ is path-connected.*

We note that Ξ is not a convex set in general. Indeed, if $\Omega_{\mathcal{V}}$ is an irreducible symmetric cone, the set $\{\alpha \in \mathbb{R}\,;\ (\alpha, \dots, \alpha) \in \Xi\}$ coincides with the so-called Wallach set, which is of the form $\left\{0, \frac{d}{2}, \dots, \frac{(r-1)d}{2}\right\} \cup \left(\frac{(r-1)d}{2}, +\infty\right)$ with some positive integer d ([1, Theorem VII.3.1]).

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