On the limiting lines in diabatic flow

By M. Z. v. KRZYWOBLOCKI and H. A. HASSAN[1)]

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Summary

The hodograph method is extended to diabatic flows, which can be thought of as flows with heat addition or subtraction by means of sources or sinks distributed in the flow. First, a proof of the nonexistence of limiting lines is given. The behavior of the flow near the transition line in a two dimensional Laval nozzle is discussed next.

Introduction

The problem of one dimensional, steady, diabatic flow, i.e. a flow with heat addition or subtraction, is discussed at length in the literature, however, due to the serious limitations of the one dimensional flow, a more general study is required to gain a better understanding of problems associated with combustion and other problems in aerothermodynamics.

Certain problems of steady two dimensional diabatic flow, which can be thought of as a flow with heat addition or subtraction by means of sources or sinks distributed in the flow field, are discussed for an inviscid, nonheat conducting gas as described by the ordinary hydrodynamic equations and the first law of thermodynamics. Previous attempts have been made to discuss such a flow: In a series of papers, Hicks and his coworkers [24]–[28][2)], formulated the equations using different representations (i.e. introducing Crocco's vector, the N vector, etc.) and discussed a few examples. In this work the emphasis is laid on investigating the hodograph method as a tool of attacking the problem.

The logical starting point of such an investigation is to discuss the problem of limiting lines in diabatic flow. It is well-known, in problems dealing with the potential flow past airfoils or inside nozzles, that the mapping of the hodograph plane into the physical plane breaks down at a certain Mach number: The continuation of the solution in the hodograph plane may be regular but can not be mapped into the physical solution. Tarlor [19], F. and M. Clauser [9], Tollmien [20], Ringleb [45], v. Kármán [30], Tsien [53] and others have suggested that the

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2) Numbers in parenthesis denote references at the end of the paper.

breakdown is due to the appearance of a limiting line: The Jacobian of the transformation from the hodograph plane to the physical plane vanishes along a certain curve in the hodograph plane. The image of this curve in the physical plane has a fold in the supersonic region and it is the edge of this fold that is known as the limiting line. A general discussion of the significance of limiting lines may be found in the works of v. Kármán [30], Tsien [53], Tsien and Kuo [54], Courant and Friedrichs [10] and Craggs [11].

Nikolskii and Taganov [43] have shown that such a limiting line, if it exists at all, would have to start at the sonic curve. But, it was Friedrichs [17] who first furnished a rigorous mathematical proof that limiting lines can not appear anywhere in analytic flows which depend continuously on the Mach number and that, therefore, the breakdown of potential flow must be due to other causes. Manwell [37] has shortened the proof considerably and eliminated the difficult lemma supplied by Flanders [17]. Morawetz and Kolodner [42] presented a proof which dispenses with the condition of analyticity and requires only of existence of the second derivatives of the stream function. v. Krzywoblocki [32], [33] generalized Tollmien's considerations and Friedrich's proof to adiabatic rotational and viscous fluids. A proof of the nonexistence of transonic potential flow was given by Busemann [6]. Some methods of handling this kind of flow were proposed by some of the authors mentioned above, Christianovich [8] and many others.

The previous investigations, except that of v. Krzywoblocki, were restricted to isentropic flows. In the present paper the nonexistence proof is carried out for diabatic flows by applying fundamentally the reasoning of Morawetz and Kolodner.

Starting from the usual hydrodynamic equations for an inviscid, nonheat conducting gas in a steady flow, together with the first law of thermodynamics, the stream function and the potential function equations are derived. Instead of choosing to work with the rate of heat added (or subtracted) per unit mass ($\overline{Q}=DQ/Dt$), as previous investigators did, the total heat added (or subtracted) per unit mass Q is used. This, together with the condition of irrotationality, has the advantage of linearizing the equations in the hodograph plane. After deriving the generalized Cauchy-Riemann conditions, the characteristics equations are derived, which show an interesting feature of the flow, namely, by a special choice of the heat distributions, real characteristics may exist in subsonic flow. Another feature of steady diabatic flows is furnished by considering stream tubes: Sonic conditions do not necessarily occur at the minimum area of the tube. Due to this, regions in the flow are designated by elliptic and hyperbolic instead of the usual subsonic and supersonic respectively and the equivalent of the sonic line in isentropic flows by the parabolic line.

The nonexistence proof is carried out under certain assumptions which are given in Section 2.2. In this and the remaining investigations Q and its derivatives up to the required order are assumed bounded, but the functional dependence of Q on q is not specified for the sake of generality. Specifically, what is proved here is that a limiting line can not appear in a plane continuous flow past an airfoil or in a nozzle of bounded curvature if the flow depends continuously on the velocity (or Mach number) and has a bounded hyperbolic region. However, if the profile has an infinite curvature at some point then a limiting line appears. (Analogous theorems are given in [42].)

The problem of diabatic flow in a two dimensional Laval nozzle is of some practical significance. An investigation of the region near the transition line (parabolic line) is undertaken. The corresponding problem in isentropic flow was discussed by Meyer [39], who sought to obtain the velocity potential in the form of a power series in the position coordinates of the physical plane. The case of the nozzle with a plane surface of transition from subsonic to supersonic velocities was considered by Christianovich [8] and his co-workers. Frankl [13], Lighthill [36] and Tomotika and Tamada [51] applied the hodograph method of Moloenbroek-Chaplygin and undertook a detailed investigation of the character of the flow near the transition line. Falkovich [12] proposed a much simpler method to deal with some characteristic features of transonic flows. By suitable transformations, he was able to show in a simple way that, among other items of interest in the problem in question, the sonic line is a parabolic curve. Continuation of a potential gas flow across the sonic line is discussed by Bers [3] and Germain [19]. The application of Tricomi's equation has been treated by Germain and Bader [20], Weinstein [55] and others, and some characteristic features of the transonic flow were discussed by Görtler [22], Kiebel [33], etc.

Following Falkovich, the generalized Cauchy-Riemann conditions are transformed from the $\{q, \theta\}$ plane to the $\{\varphi, \psi\}$ plane. An accelerated symmetrical flow is assumed in the nozzle, which imposes certain symmetry conditions on $\{q, \theta\}$ in terms of $\{\varphi, \psi\}$. Next, the behavior of the flow near the parabolic line is approximated by a simplified form of the equations, which turned out to be formally similar to the corresponding equations in isentropic flows near the sonic line. Thus the problem is reduced formally to the corresponding problem in isentropic flow and consequently, all formal results derived in Ref. 12 hold for both flows.

An important point which is brought up by this investigation is that the sonic line does not necessarily occur at the throat. However, the throat plays the same role in changing the character of the equation that describes the flow from elliptic to hyperbolic; also, the parabolic line, similar to the sonic line in isentropic flows, is a parabola.

Throughout the paper the function dQ in the first law of thermodynamics $dQ=dE+pd(\rho^{-1})$ is integrable in (x, y) plane. This is easy to show since $E=c_vT(x, y)$, $c_v=$constant, and $P=\int pd(\rho^{-1})$ is a Riemann-Stieltjes integral which has a meaning and exists in an interval under consideration if the function $\rho^{-1}(x, y)$ is a function of bounded variation. Since from the physical aspects of the fluid dynamics the functions ρ and ρ^{-1} are decent, regular bounded functions of bounded variation (their graphs do not have infinite lengths in a finite interval or they are not of non-rectifiable lengths in the intervals in question, etc.) it is obvious that in problems of fluid dynamics in question one does not need to refer to the entropy but may deal with the function $Q(x, y)$ since dQ is integrable. In the case of finite and well-defined discontinuities one can refer to the notion of Lebesque-Stieltjes integral. This can be showed formally in the steady flow conditions from the first law of thermodynamics $dQ=c_vdT+pd(\rho^{-1})$, $T=T(x, y)$, etc., with $h=E+p\rho^{-1}$, $dh=dE+\rho^{-1}dp+pd(\rho^{-1})=c_pdT$, $dh=dQ+\rho^{-1}dp$, and from the equation for the conservation of momentum without the mass and extraneous forces $qdq+\rho^{-1}dp=0$, resulting in $qdq+c_pdT=dQ$, which proves that dQ is integrable. But, of course, it is not assumed that $dQ(T, v)$ is a perfect (exact) differential i.e., $\oint dQ(T, v)\neq 0$, $v=\rho^{-1}$, i.e., from the first law of thermodynamics $\partial^2Q/\partial T\partial v\neq\partial^2Q/\partial v\partial T$, or from $dQ=c_pdT-\rho^{-1}dp$, $\oint dQ(T, p)\neq 0$. But, of course, one may have $\oint dQ(x, y)=0$, i.e., $\partial^2Q/\partial x\partial y=\partial^2Q/\partial y\partial x$.

Besides the references discussed above, the bibliography at the end of the paper contains some papers on the hodograph transformations like those of Bergman, Bers, Craggs, Germain, and others, with a particular reference to the transonic regime.

1. Fundamental equations and transformations

§ 1.1. Basic equations

The diabatic flow equations are based upon the hydrodynamic equations for an inviscid, non-heat conducting compressible perfect fluid and the first law of thermodynamics. With the use of Cartesian tensor notation, these equations in a steady flow have the following form:

The equations of motion;

(1.1.1) $$\rho u_j u_{i,j} + p_{,i} = 0\ , \qquad (i, j = 1, 2)$$

the continuity equation and equation of state:

(1.1.2) $$(\rho u_i)_{,i} = 0; \qquad p = R\rho T\ .$$

The first law of thermodynamics $dQ = dE + dW$, (dQ=heat introduced; E=internal energy, W=work done by the fluid) reduces for a perfect gas to the following form: $dW = pd(\rho^{-1})$, $dE = c_v dT$ and by expressing dp from the equation of state; and using $R = c_p - c_v$,

(1.1.3) $$c_p dT = dQ + \rho^{-1} dp\ .$$

From eq. (1.1.3) the energy equation can be written in the form:

(1.1.4) $$u_i (c_p T)_{,i} = u_i Q_{,i} + \rho^{-1} u_i p_{,i}\ ,$$

Expressing the total differentials in eq. (1.1.3) as $dT = T_{,i} dx_i$, etc. and, since x_i, x_j are independent for $i \neq j$, one finds

(1.1.4a) $$c_p T_{,i} = Q_{,i} + \rho^{-1} p_{,i}\ .$$

Dividing equation (1.1.3) by T and defining the entropy $dS = T^{-1} dQ$, one finds for constant c_p,

(1.1.5) $$c_p T^{-1} dT = dS + R p^{-1} dp\ .$$

Integration of equation (1.1.5) gives:

(1.1.6) $$p p_0^{-1} = (\rho \rho_0^{-1})^{\gamma} \exp [c_v^{-1}(S - S_0)]\ , \qquad \gamma = c_p / c_v\ ,$$

which can be written in the form:

(1.1.7) $$p = c\rho^{\gamma} \exp (c_v^{-1} S); \qquad S = \int T^{-1} dQ + \text{const.}\ .$$

This is the pressure-density-entropy relationship. Following previous authors [52], the local "isentropic" velocity of sound is defined as:

(1.1.8) $$a^2 = (\partial p / \partial \rho)_S = \gamma \rho^{-1} p = \gamma R T\ .$$

§ 1.2. The stream function equation

In a steady, two-dimensional motion, the stream function is defined as:

(1.2.1) $$\rho u = \psi_{,y}; \quad -\rho v = \psi_{,x}; \quad \rho^2 q^2 = \psi_{,x}^2 + \psi_{,y}^2; \quad q^2 = u^2 + v^2\ ,$$

Differentiating the third of equations (1.2.1) first with respect to x and multiplying the result by $(-v)$, then with respect to y and multiplying the result by u, using the first of equations (1.2.1) and adding the products furnish the result:

$$u^2\psi_{,yy}-2uv\psi_{,xy}+v^2\psi_{,xx}=q^2(u\rho_{,y}-v\rho_{,x})+\rho q(uq_{,y}-vq_{,x}) . \tag{1.2.2}$$

Eliminating T in equation (1.1.4a) by means of the equation of state, eliminating $p_{,i}$ in the same equation by means of equation (1.1.1) and, using equation (1.1.8) one finds

$$-(\gamma-1)^{-1}\rho^{-1}\alpha^2\rho_{,i}=Q_{,i}+(\gamma-1)^{-1}u_ju_{i,j} . \tag{1.2.3}$$

Putting $i=x, y$ respectively, in equation (1.2.3), i.e., writing $\cdots\rho_{,x}=Q_{,x}+\cdots$, etc., using the last of equations (1.2.1) and multiplying the first by u, the second by v and subtracting give the formula:

$$u\rho_{,y}-v\rho_{,x}=\rho\alpha^{-2}[(\gamma-1)(vQ_{,x}-uQ_{,y})-q(uq_{,y}-vq_{,x})-q^2\omega] , \tag{1.2.4}$$

where the symbol $\omega=v_{,x}-u_{,y}$, denotes the vorticity. Calculating u and v from the first of equations (1.2.1), differentiating these values with respect to y and x, respectively, and subtracting, one finds

$$u\rho_{,y}-v\rho_{,x}=\rho\omega+\psi_{,xx}+\psi_{,yy} . \tag{1.2.5}$$

Combining equations (1.2.4) and (1.2.5), gives

$$\alpha^2(\rho\omega+\psi_{,xx}+\psi_{,yy})=\rho[(\gamma-1)(vQ_{,x}-uQ_{,y})-q(uq_{,y}-vq_{,x})-q^2\omega] . \tag{1.2.6}$$

Eliminating $(u\rho_{,y}-v\rho_{,x})$ between equations (1.2.2) and (1.2.5) and adding the result to equation (1.2.6), one obtains

$$(\alpha^2-u^2)\psi_{,xx}-2uv\psi_{,xy}+(\alpha^2-v^2)\psi_{,yy}=\rho[(\gamma-1)(vQ_{,x}-uQ_{,y})-\alpha^2\omega] . \tag{1.2.7}$$

This is one of possible forms of the stream function equation.

By using equation (1.1.1) to eliminate $p_{,i}$ from equation (1.1.4) one finds:

$$u_i(c_pT)_{,i}+u_iu_ju_{i,j}=u_iQ_{,i} . \tag{1.2.8}$$

But since:

$$q\partial/\partial s=u_i\partial/\partial x_i , \tag{1.2.9}$$

where $\{s, n\}$ represent the orthogonal coordinates system running along a streamline (s-tangential, n-normal); hence equation (1.2.8) furnishes:

(1.2.10) $$\partial/\partial s(c_pT+\tfrac{1}{2}q^2-Q)=0;$$

(1.2.10a) $$c_pT+\tfrac{1}{2}q^2-Q=c_pT_o-Q_o=\text{const.}$$

where the subscript '*o*' refers to the stagnation conditions. Calculating c_pT from equation (1.2.10a), i.e.,

(1.2.11) $$c_pT=Q-Q_o-\tfrac{1}{2}u_ju_j+c_pT_o\ ,$$

and inserting this value into equation (1.1.4a) one obtains

(1.2.12) $$(c_pT_o)_{,i}-Q_{o,i}=u_j(u_{j,i}-u_{i,j})\ .$$

Let $i=x, y$ in equation (1.2.12), multiply the results by v and $(-u)$, respectively and add the products, thus obtaining:

(1.2.13) $$uQ_{o\ y}-vQ_{o,x}+v(c_pT_o)_{,x}-u(c_pT_o)_{,y}=q^2\omega\ .$$

Denoting the angle of the inclination of the streamline to the horizontal axis by θ furnishes the well-known relation:

(1.2.14) $$\partial/\partial n=-\sin\theta\partial/\partial x+\cos\theta\partial/\partial y\ ,$$

or

(1.2.14a) $$q\partial/\partial n=-v\partial/\partial x+u\partial/\partial y\ .$$

Of course:

(1.2.15) $$d\psi/dn=\rho q\ .$$

With the use of equations (1.2.14a) and (1.2.15), equation (1.2.13) reduces to:

(1.2.16) $$\omega=\rho[dQ_o/d\psi-dh_o/d\psi]\ ,\qquad h_o=c_pT_o\ .$$

Inserting equation (1.2.16) into equation (1.2.7) furnishes another form of stream function equation:

(1.2.17) $$(\alpha^2-u^2)\psi_{,xx}-2uv\psi_{,xy}+(\alpha^2-v^2)\psi_{,yy}$$
$$=\rho^2[\alpha^2(dh_o/d\psi-dQ_o/d\psi)-(\gamma-1)q^2dQ/d\psi]\ ,$$

with

(1.2.17a) $$\rho^2=q^{-2}[(\psi_{,x})^2+(\psi_{,y})^2]\ .$$

§ 1.3. The "generalized" potential equation

The concept of a function analogous to the velocity potential was used by v. Krzywoblocki [32]. The equation $v_{,x}-u_{,y}=\omega$ is fulfilled if one puts:

$$u=\varphi_{,x}+g\,; \quad v=\varphi_{,y}+g\,; \quad g_{,x}-g_{,y}=\omega\;, \tag{1.3.1}$$

where the function $g=g(x, y)$ is defined by the last of equations (1.3.1); the condition is that for an irrotational flow the trivial solution of (1.3. 1), i.e., $g\equiv 0$, must be assumed.

The first two equations (1.3.1) are squared and added, thus giving an expression for q^2, differentiating it with respect to x, and y, respectively; multiplying the first expression so obtained by u and the second by v and adding the products, one obtains

$$q(uq_{,x}+vq_{,y})=u^2\varphi_{,xx}+2uv\varphi_{,xy}+v^2\varphi_{,yy}+(u+v)(ug_{,x}+vg_{,y})\;. \tag{1.3.2}$$

Letting $i=x, y$, respectively, in equation (1.2.3); multiplying the first by u and the second by v and adding, and finally eliminating the expression $\rho^{-1}(u\rho_{,x}+v\rho_{,y})$ by substituting the first two equations (1.3.1) into the continuity equation; one finds

$$\alpha^2(\varphi_{,xx}+\varphi_{,yy}+g_{,x}+g_{,y})=(\gamma-1)(uQ_{,x}+vQ_{,y})+q(uq_{,x}+vq_{,y})\;. \tag{1.3.3}$$

Eliminating the last term in equation (1.3.3) by using (1.3.2), and arranging terms, give the equation:

$$\begin{aligned}(\alpha^2-u^2)\varphi_{,xx}-2uv\varphi_{,xy}+(\alpha^2-v^2)\varphi_{,yy}&=(\gamma-1)(uQ_{,x}+vQ_{,y})\\ &-(\alpha^2-u^2)g_{,x}+uv(g_{,x}+g_{,y})-(\alpha^2-v^2)g_{,y}\;.\end{aligned} \tag{1.3.4}$$

This is one of possible forms of the generalized potential equation of diabatic flow. Other forms are listed below. Using equations (1.3.1) and the new function $uQ_{,x}+vQ_{,y}=DQ/Dt=\overline{Q}$ in equation (1.3.4) furnishes the formula:

$$(\alpha^2-u^2)u_{,x}-uv(v_{,x}+u_{,y})+(\alpha^2-v^2)v_{,y}=(\gamma-1)\overline{Q}\;. \tag{1.3.5}$$

where $\overline{Q}$ is the rate of the convection of the total energy introduced. Consider the function Q as function of u, v, i.e., $Q=Q[u(x, y), v(x, y)]$, then:

$$Q_{,x}=Q_{,u}u_{,x}+Q_{,v}v_{,x}\,; \quad Q_{,y}=Q_{,u}u_{,y}+Q_{,v}v_{,y}\;. \tag{1.3.6}$$

Inserting equations (1.3.1) and (1.3.6) in equation (1.3.4) gives:

$$\begin{aligned}&[\alpha^2-u^2-(\gamma-1)uQ_{,u}]u_{,x}-[uv+(\gamma-1)vQ_{,u}]u_{,y}\\ &\quad-[uv+(\gamma-1)uQ_{,v}]v_{,x}+[\alpha^2-v^2-(\gamma-1)vQ_{,v}]v_{,y}=0\;.\end{aligned} \tag{1.3.7}$$

In the case of an irrotational flow: $u_{,x}=\varphi_{,xx}$; $u_{,y}=v_{,x}=\varphi_{,xy}$; $v_{,y}=\varphi_{,yy}$.

For a rotational flow these derivatives should be derived from equation (1.3.1).

§ 1.4. Hodograph transformation

Letting $W=f(x, y)=F(u, v)$ be an arbitrary function, then:

$$F_{,u}=f_{,x}x_{,u}+f_{,y}y_{,u}; \quad F_{,v}=f_{,x}x_{,v}+f_{,y}y_{,v} , \tag{1.4.1}$$

and

$$f_{,x}=J_1^{-1}(F_{,u}y_{,v}-F_{,v}y_{,u}); \quad f_{,y}=J_1^{-1}(F_{,u}x_{,v}-F_{,v}x_{,u}) , \tag{1.4.2}$$

where the Jacobian of the transformation, J_1, is defined as:

$$J_1=x_{,u}y_{,v}-x_{,v}y_{,u} . \tag{1.4.3}$$

Applying equations (1.4.2) to u, v, respectively, i.e., $u_{,x}=J_1^{-1}y_{,v}$; $v_{,x}=-J_1^{-1}y_{,u}$, etc., and inserting the resulting expressions into equation (1.3.5) and into the equation defining the vorticity, one obtains the following system of equations:

$$(\alpha^2-u^2)y_{,v}+uv(x_{,v}=y_{,u})+(\alpha^2-v^2)x_{,u}=(\gamma-1)J_1\overline{Q}; \tag{1.4.4a}$$

$$x_{,v}-y_{,u}=J_1\omega . \tag{1.4.4b}$$

The expressions for the total differentials $d\varphi$, $d\psi$ are given by

$$d\varphi=\varphi_{,u}du+\varphi_{,v}dv=\varphi_{,x}dx+\varphi_{,y}dy=(u-g)dx+(v-g)dy; \tag{1.4.5a}$$

$$d\psi=\psi_{,u}du+\psi_{,v}dv=\psi_{,x}dx+\psi_{,y}dy=\rho(udy-vdx) . \tag{1.4.5b}$$

Treating equations (1.4.5a, b) as an algebraic system in two unknowns dx and dy and calculating dx and dy one can obtain the partial derivatives $x_{,u}$; $x_{,v}$, etc.:

$$x_{,u}=[\rho u\varphi_{,u}-(v-g)\psi_{,u}]E^{-1}; \quad x_{,v}=[\rho u\varphi_{,v}-(v-g)\psi_{,v}]E^{-1}; \tag{1.4.6a}$$

$$y_{,u}=[\rho v\varphi_{,u}+(u-g)\psi_{,u}]E^{-1}; \quad y_{,v}=[\rho v\varphi_{,v}+(u-g)\psi_{,v}]E^{-1}; \tag{1.4.6b}$$

$$E=\rho[q^2-g(u+v)] . \tag{1.4.6c}$$

Substituting these expressions into the system (1.4.4a, b) furnishes the system:

$$\rho\alpha^2(u\varphi_{,u}+v\varphi_{,v})+(q^2-\alpha^2)(v\psi_{,u}-u\psi_{,v})-g\psi_{,u}(uv+v^2-\alpha^2) +g\psi_{,v}(uv+u^2-\alpha^2)=(\gamma-1)\rho[q^2-g(u+v)]J_1\overline{Q}; \tag{1.4.7a}$$

$$(u-g)\psi_{,u}+(v-g)\psi_{,v}+\rho(v\varphi_{,u}-u\varphi_{,v})=-\rho[q^2-g(u+v)]J_1\omega . \tag{1.4.7b}$$

Now:

$$u=q\cos\theta;\quad v=q\sin\theta;\quad D=\frac{\partial(u,v)}{\partial(q,\theta)}=q\,. \tag{1.4.8}$$

Then:

$$\varphi_{,q}=\varphi_{,u}u_{,q}+\varphi_{,v}v_{,q};\quad \varphi_{,\theta}=\varphi_{,u}u_{,\theta}+\varphi_{,v}v_{,\theta}\,, \tag{1.4.9}$$

calculating the derivatives $u_{,q}$, $u_{,\theta}$, $v_{,\theta}$, etc., by use of equations (1.4.8) and inserting these values into equations (1.4.9); treating the system, so obtained, as an algebraic system one can calculate the quantities $\varphi_{,u}$ and $\varphi_{,v}$. Application of the same procedure to the function $\psi=\psi(u,v)=\psi(q,\theta)$ furnishes the following results

$$\varphi_{,u}=\varphi_{,q}\cos\theta-q^{-1}\varphi_{,\theta}\sin\theta;\quad \varphi_{,v}=\varphi_{,q}\sin\theta+q^{-1}\varphi_{,\theta}\cos\theta; \tag{1.4.10a}$$

$$\psi_{,u}=\psi_{,q}\cos\theta-q^{-1}\psi_{,\theta}\sin\theta;\quad \psi_{,v}=\psi_{,q}\sin\theta+q^{-1}\psi_{,\theta}\cos\theta\,. \tag{1.4.10b}$$

The following discussion will be restricted to an irrotational flow only, i.e., $\omega\equiv g\equiv 0$. Use of equations (1.4.10a, b) in equations (1.4.7a, b) furnishes a system of equations corresponding to Cauchy-Riemann equations in incompressible flow:

$$\varphi_{,q}=(\rho q)^{-1}(q^2\alpha^{-2}-1)\psi_{,\theta}+(\gamma-1)q\alpha^{-2}J_1\overline{Q}; \tag{1.4.11a}$$

$$\varphi_{,\theta}=q\rho^{-1}\psi_{,q}\,. \tag{1.4.11b}$$

Identification of f and F with Q in equations (1.4.2) and the use of equations (1.4.6a, b) together with relations similar to those in equations (1.4.10) for $Q_{,u}$ and $Q_{,v}$, yield the following equalities for $\overline{Q}$:

$$\overline{Q}=(\rho J_1)^{-1}(Q_{,u}\psi_{,v}-Q_{,v}\psi_{,u});\quad \overline{Q}=(\rho q J_1)^{-1}(\psi_{,\theta}Q_{,q}-\psi_{,q}Q_{,\theta})\,. \tag{1.4.12}$$

Inserting the second equation (1.4.12) into the system (1.4.11a, b) furnishes the system:

$$\varphi_{,q}=b_1\psi_{,\theta}+c_1\psi_{,q};\quad \varphi_{,\theta}=a_1\psi_{,q}; \tag{1.4.13a}$$

$$a_1=\rho^{-1}q;\quad b_1=(\rho q)^{-1}[q^2\alpha^{-2}-1+(\gamma-1)q\alpha^{-2}Q_{,q}]; \tag{1.4.13b}$$

$$c_1=-(\gamma-1)(\rho\alpha^2)^{-1}Q_{,\theta}\,. \tag{1.4.13c}$$

Differentiation of the first of equations (1.4.13a) with respect to θ, and the second one with respect to q, gives the stream function equation in the hodograph plane:

$$a_1\psi_{,qq}-c_1\psi_{,q\theta}-b_1\psi_{,\theta\theta}+a_{1,q}\psi_{,q}-b_{1,\theta}\psi_{,\theta}-c_{1,\theta}\psi_{,q}=0\,. \tag{1.4.14}$$

§ 1.5. General considerations

The total differentials given in equation (1.1.3), considered as functions of q, θ, can be written as

$$dT = T_{,q}dq + T_{,\theta}d\theta\ ,\ \cdots,\ \text{etc.} \tag{1.5.1a}$$

hence, equation (1.1.3), after collecting terms, gives

$$(c_pT_{,q} - Q_{,q} - \rho^{-1}p_{,q})dq + (c_pT_{,\theta} - Q_{,\theta} - \rho^{-1}p_{,\theta})d\theta = 0\ . \tag{1.5.1b}$$

Since q, θ are two independent variables, equation (1.5.1b) gives

$$c_pT_{,q} = Q_{,q} + \rho^{-1}p_{,q};\quad c_pT_{,\theta} = Q_{,\theta} + \rho^{-1}p_{,\theta}\ . \tag{1.5.1c}$$

Similar reasoning applied to Bernoulli's equation (1.2.10a) gives

$$q + c_pT_{,q} - Q_{,q} = 0\ ,\qquad c_pT_{,\theta} - Q_{,\theta} = 0\ . \tag{1.5.2}$$

Comparison of equations (1.5.1c) and (1.5.2) yields

$$p_{,q} = dp/dq = -\rho q;\quad p_{,\theta} = 0\ , \tag{1.5.3}$$

hence, $p = p(q)$. From equation (1.5.3) it is clear that $\rho = \rho(q)$, from the equation of state $T = T(q)$, from equation (1.5.2) $Q = Q(q)$ and therefore, it follows from equation (1.1.7) $S = S(q)$. Briefly, all the dependent variables are functions of the velocity only. A more general proof of this result is given in the Appendix.

From equation (1.5.2)

$$dT/dq = c_p^{-1}(dQ/dq - q)\ , \tag{1.5.4}$$

and from the equation of state with the use of equations (1.5.3) and (1.5.4)

$$d\rho/dq = -\rho\alpha^{-2}[q + (\gamma - 1)dQ/dq]\ . \tag{1.5.5}$$

The above results show that, in equations (1.4.13a, b, c) $c_1 \equiv 0$, $a_1 \equiv a_1(q)$ and $b_1 = b_1(q)$. Hence, the stream function equation (1.4.14) reduces to the form

$$a_1(q)\psi_{,qq}(q,\theta) - b_1(q)\psi_{,\theta\theta}(q,\theta) + (da_1/dq)\psi_{,q}(q,\theta) = 0\ . \tag{1.5.6}$$

§ 1.6. Characteristics

The equation of characteristics of equation (1.5.6) in the hodograph plane is:

$$a_1(d\theta)^2 - b_1(dq)^2 = 0\ , \tag{1.6.1}$$

or

$$d\theta = \pm h dq; \quad h^2 = b_1 a_1^{-1} = q^{-2}[q^2 a^{-2} - 1 + (\gamma - 1) q \alpha^{-2} dQ/dq] . \tag{1.6.2}$$

Letting

$$J = \frac{\partial(x, y)}{\partial(q, \theta)} = \frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(q, \theta)} = -J_1 q , \tag{1.6.3}$$

or, using equations (1.4.5a, b):

$$J = \left[\frac{\partial(\varphi, \psi)}{\partial(x, y)}\right]^{-1} \frac{\partial(\varphi, \psi)}{\partial(q, \theta)} = (\rho\alpha)^{-1}[\varphi_{,\theta}\psi_{,q} - \psi_{,\theta}\varphi_{,q}]; \tag{1.6.4}$$

using equations (1.4.13a) in equation (1.6.4) furnishes the result:

$$J = q^{-3}(\varphi_{,\theta}^2 - k^2\psi_{,\theta}^2); \quad k^2 = (\rho\alpha)^{-2}[q^2 - \alpha^2 + (\gamma - 1) q dQ/dq] = a_1 b_1 . \tag{1.6.5}$$

Letting

$$U = \varphi_{,\theta} - k\psi_{,\theta}; \quad V = \varphi_{,\theta} + k\psi_{,\theta} , \tag{1.6.6}$$

then the Jacobian

$$J = q^{-3} U V , \tag{1.6.7}$$

vanishes if and only if $U = 0$, or $V = 0$.

Introducing the characteristic variables α, β by:

$$2d\alpha = \rho k q^{-1} dq - d\theta; \quad 2d\beta = \rho k q^{-1} dq + d\theta . \tag{1.6.8}$$

then equations (1.6.8) give

$$dq = (\rho k)^{-1} q (d\alpha + d\beta); \quad d\theta = d\beta - d\alpha . \tag{1.6.9}$$

If "f" is any function, then:

$$f_{,\alpha} = f_{,q} q_{,\alpha} + f_{,\theta}\theta_{,\alpha}; \quad f_{,\beta} = f_{,q} q_{,\beta} + f_{,\theta}\theta_{,\beta} , \tag{1.6.10}$$

by using equations (1.6.9) and (1.6.10) the operators $\partial/\partial\alpha$ and $\partial/\partial\beta$ are given by:

$$\partial/\partial\alpha = (\rho k)^{-1} q \partial/\partial q - \partial/\partial\theta; \quad \partial/\partial\beta = (\rho k)^{-1} q \partial/\partial q + \partial/\partial\theta . \tag{1.6.11}$$

From equations (1.6.6) with the use of equations (1.6.11) and keeping in mind the results of Section 1.5, as well as equations (1.4.13a) and (1.6.5), one finds:

$$U_{,\beta} = -(\rho k)^{-1} q (dk/dq) \psi_{,\theta} , \tag{1.6.12}$$

or applying equation (1.6.6):

(1.6.13) $$U_{,\beta}=\tfrac{1}{2}(\rho k)^{-1}q(dk/dq)(U-V)\ .$$

Similarly:

(1.6.14) $$V_{,\alpha}=\tfrac{1}{2}(\rho k)^{-1}q(dk/dq)(V-U)\ .$$

Since $k=k(q)$, then from equations (1.6.11) one obtains:

(1.6.15) $$(dk/dq)=\rho kq^{-1}k_{,\alpha}=\rho kq^{-1}k_{,\beta}\ ,$$

which, when inserted into equations (1.6.13), (1.6.14) yields:

(1.6.16a) $$U_{,\beta}-\tfrac{1}{2}k^{-1}k_{,\beta}U=-\tfrac{1}{2}\rho^{-1}k^{-2}q(dk/dq)V;$$

(1.6.16b) $$V_{,\alpha}-\tfrac{1}{2}k^{-1}k_{,\alpha}V=-\tfrac{1}{2}\rho^{-1}k^{-2}q(dk/dq)U\ .$$

Multiplying equation (1.6.16a) by $2k^{-1}U$ and equation (1.6.16b) by $2k^{-1}V$ one finds:

(1.6.17) $$(k^{-1}U^2)_{,\beta}=-BUV;\quad (k^{-1}V^2)_{,\alpha}=-BUV;$$

(1.6.17a) $$B=\rho^{-1}k^{-3}q(dk/dq)\ .$$

2. Discussion of the Jacobian

§ 2.1. Streamtubes in steady diabatic flow

Assuming that the flow in the downstream part of a nozzle is accompanied by a divergent shape of each streamtube of a cross section A. Then the following equations are valid in a steady flow:

(2.1.1) $$\rho qA=\text{const.};\quad d\rho/\rho+dq/q+dA/A=0\ .$$

On eliminating $d\rho/\rho$ by use of equation (1.5.5), equation (2.1.1) reduces to:

(2.1.2) $$q^{-1}[q^2\alpha^{-2}-1+(\gamma-1)q\alpha^{-2}dQ/dq]dq=dA/A\ .$$

When $A=A_{\mathrm{MIN}}$, $dA=0$, and

(2.1.3) $$K=q^2\alpha^{-2}-1+(\gamma-1)q\alpha^{-2}dQ/dq=0\ ,$$

downstream of the throat $dA/A>0$, $dq>0$ (since the flow is assumed to be an accelerated one), $q>0$. The locus of points along which $h^2=K=k^2=0$ will be denoted (equations 1.6.2, 1.6.5, 2.1.3) by the "parabolic line"; on this line $q=q_*$.

In the domain where $dA>0$, $K>0$, and

(2.1.4a) $$q\alpha^{-2}[q+(\gamma-1)dQ/dq]>0\ ,$$

or

$$q+(\gamma-1)dQ/dq>0 \ . \tag{2.1.4b}$$

Clearly, $K(q)$ in the neighborhood of the parabolic line is an increasing function of q. It is assumed that $K(q)$ is an increasing function of q in the vicinity of the parabolic line sufficiently large for the derivation and validity of the proofs presented below.

Inspection of equation (2.1.2) shows that the value of $M=q/\alpha=1$ does not occur at the throat unless $dQ/dq=0$. But as far as the character of the partial differential equation governing the flow is concerned, the throat plays the same role as in the isentropic flow, elliptic upstream, $k^2<0$, parabolic at the throat, $k^2=0$ and hyperbolic downstream, $k^2>0$.

The system of characteristic used here reduces the problems of diabatic flow formally to isentropic flow problems. Thus, the Jacobian will be considered in the elliptic, parabolic and hyperbolic regions instead of the subsonic, sonic and supersonic regions.

§ 2.2. Jacobian does not vanish in the elliptic domain

The assumptions analogous to those in [13] are made:

(a) A solution in the hodograph plane of the equations of a diabatic, irrotational, plane flow is given; i.e., we have a potential function $\varphi(q, \theta)$ and stream function $\psi(q, \theta)$; the flow is assumed to follow convergent-divergent streamtubes in a closed simply connected region D of the q, θ-plane;

(b) $\psi_{,qq}$, $\psi_{,q\theta}$, $\psi_{,\theta\theta}$ exist and are bounded in D;

(c) The boundary C of D intersects the "parabolic" line $h^2=K=k^2=0$ in exactly two points;

(d) $\rho>0$ throughout D.

Q and its derivatives up to the required order are assumed to be bounded, and the flow is continuous.

When the governing equation is of an elliptic type, $k^2<0$, and equation (1.6.5) furnishes:

$$\varphi^2_{,\theta}-k^2\psi^2_{,\theta}\geqq 0; \quad J\geqq 0 \ .$$

The equality $J=0$ may hold at most at a finite set of discrete points which will be the singular points of the hodograph transformation. This is admissable. The equality can not hold along a certain line in D since this would imply that $\varphi_{,\theta}=\psi_{,\theta}=0$ and from equations (1.4.13a) $\varphi_{,q}=\psi_{,q}=0$. This would mean that $\varphi(q, \theta)$, $\psi(q, \theta)$ are constants or that u and v are constant in the subdomain in question and this is excluded.

§ 2.3. Jacobian does not vanish in the remaining domain

In the remaining cases the proof given in [42] can be applied directly; this is done below with several modifications. The lemma proved in [42] is valid in the present case, as well:

Lemma If $J>0$ on C then $J>0$ in D for $q \geqq q_*$.

Proof From equations (1.6.5) and (1.6.17):

$$B=\rho^{-1}k^{-3}q\,dk/dq=\tfrac{1}{2}\rho^{-1}k^{-4}q\,d(k^2)/dq \ . \tag{2.3.1}$$

But $k^2=\rho^{-2}K$, and:

$$d(k^2)/dq=-2\rho^{-3}K\,d\rho/dq+\rho^{-2}dK/dq \ . \tag{2.3.1a}$$

From equations (1.5.5) and (2.1.4b) it is obvious that $d\rho/dq<0$, and from equation (2.1.3) that $dK/dq>0$, since K is an increasing function of q. Hence $B>0$ for $q \geqq q_*$.

Assume that $J \leqq 0$ somewhere in the region of $q>q_*$ of D.

Since J is continuous and D is closed, there is a maximum $q_{\mathrm{MAX}}>q_*$ for which $J=0$. Addition of equations (1.6.8) shows that there exists a maximum value $\tau^*=(\alpha^*+\beta^*)$ of $(\alpha+\beta)$ for which $J=0$, and, for $\tau=(\alpha+\beta)>\tau^*$, $J>0$ since $J>0$ on C by assumption. By equation (1.6.7) either $U=0$ or $V=0$ for $\tau=\tau^*$. Suppose that $U=0$ at this point. From equation (1.6.17), $k^{-1}U^2=0$ at τ^*, but $k^{-1}U^2>0$ for $\tau>\tau^*$. Hence for $\alpha=\alpha^*$, $\beta>\beta^*$, $k^{-1}U^2>0$; hence, by the mean value theorem $k^{-1}U^2$ at $(\alpha^*, \beta>\beta^*)$ is equal to $(\beta-\beta^*)(k^{-1}U^2)_{,\beta}$, with the derivative evaluated at $(\alpha^*, \bar{\beta})$, $\beta^*<\bar{\beta}<\beta$. Since $(k^{-1}U^2)>0$ at (α^*, β), $(k^{-1}U^2)_{,\beta}>0$ at $(\alpha^*, \bar{\beta})$; but then by equation (1.6.17a) since $B>0$, one has $UV<0$ at $(\alpha^*, \bar{\beta})$, that is $J<0$ at $(\alpha^*, \bar{\beta})$ by equation (1.6.7). Since $(\alpha^*+\beta)>\tau^*$, this contradicts $J>0$ for $\tau>\tau^*$ and hence U can not vanish for $q>q_*$. Similarly V can not vanish for $q>q_*$ and thus J can not vanish for $q>q_*$.

Now suppose that $J=0$ at a point P where $q(P)=q_*$; let $\alpha(P)$ and $\beta(P)$ be the coordinates of this point in the characteristic coordinates system. By equation (1.6.17), since $J>0$ and hence $UV>0$ for $q>q_*$, $(k^{-1}U^2)$ is a decreasing function in the direction of increasing β along the line $\alpha=\alpha(P)$. Since $k^{-1}U^2>0$ and continuous for $q>q_*$, $k^{-1}U^2$ must be bounded away from zero for $q \geqq q_*$. Hence $|k^{-1/2}U|$ is also bounded away from zero on $\alpha=\alpha(P)$, $\beta \geqq \beta(P)$.

From the second of equations (1.1.13a) and equation (1.6.7) one obtains:

$$k^{-1/2}\psi_{,q}=\rho q^{-1}(k^{-1/2}U+k^{1/2}\psi_{,\theta}) \tag{2.3.2}$$

since $\psi_{,\theta}$ is bounded and $k\to 0$ for $\alpha=\alpha(P)$, $\beta\to\beta(P)$, from equation (2.3.

2) is seen that $|k^{-1/2}\psi_{,q}|$ is also bounded away from zero for $\alpha=\alpha(P)$, $\beta\geqq\beta(P)$ with β being sufficiently close to $\beta(P)$.

Expanding $k^2(q)$ around $q=q_*$, one finds:

(2.3.3) $$k^2(q)=(q-q_*)F(q_1) ,$$

where $q_*<q_1<q$ and $F(q_1)$ is bounded. This implies, that

(2.3.4) $$k^{-1/2}=(q-q_*)^{-1/4}\overline{F}(q_1) ,$$

with $\overline{F}(q_1)$ being bounded. Consequently, the expression $|(q-q_*)^{-1/4}\psi_{,q}|$, using

(2.3.5) $$|k^{-1/2}\psi_{,q}|\sim|(q-q_*)^{-1/4}\psi_{,q}|>\delta>0 ,$$

is bounded away from zero for $\alpha=\alpha(P)$, $\beta\geqq\beta(P)$ for β sufficiently close to $\beta(P)$.

Since $J=0$ at P, from equation (1.6.5) $k^2=0$, $\varphi_{,\theta}=0$ and from (1.4. 13a) $\psi_{,q}=0$.

Since it was assumed that $\psi_{,qq}$ and $\psi_{,q\theta}$ exist and are bounded in D, then, from the mean value theorem:

(2.3.6) $$\psi_{,q}=(q-q_*)\psi_{,qq}(\overline{q},\theta)+(\theta-\theta[P])\psi_{,q\theta}(q,\overline{\theta}) ,$$

for some $\overline{\theta}$, $\theta[P]<\overline{\theta}<\theta$, and some $\overline{q}$, $q_*<\overline{q}<q$. For every point along the parabolic line one has $\alpha=\alpha(P)$; hence along this characteristic coordinate there is always $\alpha=\alpha(P)$, $d\alpha(P)=0$ and from equation (1.6.8) one gets:

(2.3.7) $$d\theta/_P=\rho kq^{-1}dq/_P .$$

Hence by the mean value theorem:

(2.3.8) $$\theta-\theta(P)=\int_{q_*}^{q}\rho kq^{-1}dq=(q-q_*)f(\overline{\overline{q}}) ,$$

for some $\overline{\overline{q}}$, $q_*<\overline{\overline{q}}<q$, and $f(\overline{\overline{q}})$ is bounded. Hence, for

(2.3.9) $$\psi_{,q}=[\psi_{,qq}(\overline{q},\theta)+f(\overline{\overline{q}})\psi_{,q\theta}(q,\overline{\theta})](q-q_*) ,$$

or

(2.3.10) $$\psi_{,q}=F_2(q,\theta)(q-q_*) ,$$

with F_2 being a bounded function of q and θ. But then $|(q-q_*)^{-1/4}\psi_{,q}|\to0$ for $\alpha=\alpha(P)$, $\beta\to\beta(P)$, and this contradicts (2.3.5). Thus $J>0$ for $q=q_*$, as well as for $q>q_*$, and the lemma is proved.

§ 2.4. The Jacobian does not vanish at the wall

The proofs given in [17] and [42] are directly applicable in the present case with $C_{*}=q_{*}$.

3. On some characteristic properties of the laval nozzle in diabatic flow

§ 3.1. Basic equations

The basic equations are the "generalized" Cauchy-Riemann conditions in the hodograph $\{q\text{-}\theta\}$-plane derived in Section (1.4). These equations can be written in the following form:

$$\varphi_{,\theta}=q\rho^{-1}\psi_{,q};\quad \varphi_{,q}=q\rho^{-1}h^{2}\psi_{,\theta}; \tag{3.1.1}$$

where

$$h^{2}=h^{2}(q)=q^{-2}[q^{2}\alpha^{-2}-1+(\gamma-1)q\alpha^{-2}Q_{,q}], \tag{3.1.1a}$$

From:

$$d\varphi=\varphi_{,q}dq+\varphi_{,\theta}d\theta;\quad d\psi=\psi_{,q}dq+\psi_{,\theta}d\theta, \tag{3.1.2}$$

one obtains:

$$\rho q^{2}Jdq=\varphi_{,\theta}d\psi-\psi_{,\theta}d\varphi;\quad -\rho q^{2}Jd\theta=\varphi_{,q}d\psi-\psi_{,q}d\varphi, \tag{3.1.3}$$

Partial derivatives of the functions $q=q(\varphi,\psi)$ and $\theta=\theta(\varphi,\psi)$; can be calculated by means of eqs. (3.1.3):

$$-\overline{A}q_{,\varphi}=\psi_{,\theta};\quad \overline{A}q_{,\psi}=\varphi_{,\theta};\quad \overline{A}\theta_{,\varphi}=\psi_{,q};\quad -\overline{A}\theta_{,\psi}=\varphi_{,q}, \tag{3.1.4}$$

with $\overline{A}=\rho q^{2}J$. Substitution of eqs. (3.1.4) into eqs. (3.1.1) is $J\neq0$ gives:

$$\theta_{,\varphi}-\rho q^{-1}q_{,\psi}=0;\quad \theta_{,\psi}-q\rho^{-1}h^{2}q_{,\varphi}=0\text{[3)]}. \tag{3.1.5}$$

Eqs. (3.1.5) are of the same type as eqs. (3.1.1), i.e., they are of an elliptic type if $h^{2}<0$ and of a hyperbolic if $h^{2}>0$. Introducing a new independent variable $\eta=\eta(q)$ by

$$\eta=\left[3/2\int_{q}^{q*}(-h^{2})^{1/2}dq\right]^{2/3}, \tag{3.1.6}$$

this gives

$$q_{,\varphi}=\eta_{,\varphi}dq/d\eta;\quad q_{,\psi}=\eta_{,\psi}dq/d\eta. \tag{3.1.7}$$

3) For $Q\equiv0$, the second equation (3.1.5) does not reduce to the form (1.9) in (3:2), which shows that there are obviously some misprints in [12].

Basically, the integral in eq. (3.1.6) is an indefinite one and by means of the elementary formula $\int \cdots d\xi = \text{const.} + \int_\alpha^x \cdots d\xi$, one obtains:

$$(3.1.8) \qquad d\eta/dq = -(-h^2\eta^{-1})^{1/2} .$$

Inserting eqs. (3.1.7) and (3.1.8) into eqs. (3.1.5) furnishes a new system in $\{\varphi, \psi\}$ coordinate system:

$$(3.1.9) \qquad \eta_{,\psi} + b(\eta)\theta_{,\varphi} = 0; \quad \eta\eta_{,\varphi} - b^{-1}(\eta)\theta_{,\psi} = 0 ,$$

where

$$(3.1.9a) \qquad b(\eta) = q\rho^{-1}(-h^2\eta^{-1})^{1/2} .$$

This is a function of η since ρ and h^2 are functions of q only and consequently functions of η only as a result of (3.1.6). It is clear that considering only real values of η, one has:

(3.1.10a) elliptic regime: $h^2 < 0$;

$$\int_q^{q_*} (-h^2)^{1/2} dq = \int_q^{q_*} F(q)dq \geqq F_{\text{MIN}}(q_* - q) \geqq 0 , \quad F_{\text{MIN}}(q) \geqq 0, \ q_* > q; \text{ hence } \eta > 0$$

(3.1.10b) parabolic regime: $q = q_*$, $\eta = 0$.

(3.1.10c) hyperbolic regime: $h^2 > 0$

$$\left[\int_q^{q_*} (-h^2)^{1/2} dq\right]^{2/3} = (\pm i)^{2/3} \left[\int_q^{q_*} (h^2)^{1/2} dq\right]^{2/3} \leqq 0 , \quad \text{with } (\pm i)^{2/3} = -1; \ \eta < 0 .$$

Equations (3.1.9), formally identical to those obtained in [12], are the fundamental equations for the investigation of two-dimensional, diabatic flow of an inviscid, non-heat conducting gas when the magnitude of the velocity increases in such a way that the character of the equation governing the motion changes from elliptic to hyperbolic as the gas passes through the "parabolic" line.

§ 3.2. The variable coefficient and the equation of characteristics

The coefficient $b(\eta)$ is considered in more detail. In particular its value for $\eta = 0$, i.e., on the "parabolic" line is of interest. Since $h^2(\eta = 0) = 0$, one has to use l'Hospital's rule to evaluate $\lim_{\eta \to 0} b(\eta)$. Since $\rho|_{q=q_*}$ and $q = q_*$ are finite, one may consider the limit of the expression

$$\lim_{\eta \to 0}(-q^2h^2\eta^{-1}) \text{ or } \lim_{q \to q_*}(-q^2h^2\eta^{-1}) .$$

It is:

(3.2.1) $\lim_{q\to q*}(-q^2h^2\eta^{-1})=\lim_{q\to q*}(q^2h^2)_{,q}/\lim_{q\to q*}(-d\eta/dq)$,

or with the use of eq. (3.1.8):

(3.2.2) $\lim_{q\to q*}[q^2(-h^2\eta^{-1})]\lim_{q\to q*}[(-h^2\eta^{-1})^{1/2}]=\lim_{q\to q*}(q^2h^2)_{,q}$.

It was established in Section (2.1) that the function $q^2h^2(q)$ is, in the sufficiently large vicinity of the "parabolic" line, an increasing function of q, i.e., $(q^2h^2)_{,q}>0$, which implies that $\lim_{\eta\to 0}b(\eta)$ exists and is different from zero. The equation of characteristics in the hyperbolic regime of the hodograph plane are given by eq. (1.6.2), it can be represented as:

(3.2.3) $$\theta=\pm\int hdq+C_1 .$$

From eq. (3.1.8) one gets

(3.2.4) $\eta(d\eta/dq)^2=-h^2;\quad (-\eta)^{1/2}(d\eta/dq)=h$,

or

(3.2.5) $$\theta=\pm\int hdq+C_1=\pm 2/3(-\eta)^{3/2}+C .$$

This is identical to the result in [12], i.e., the characteristics in the plane are semicubical parabolas with the cusps on the axis of abscissas.

§ 3.3. Differential equations of motion of a gas in the neighborhood of the transition line

Following [12] the flow is assumed to satisfy the following conditions:

(i) The nozzle and the flow are symmetrical with respect to the horizontal axis; a straight line perpendicular to the axis of symmetry and directed away from the axis intersects streamlines with constantly increasing curvatures and therefore encounters particles of the gas having constantly increasing velocities. This is physically sound; it implies that the parabolic (transition) line is convex to the hyperbolic region.

(ii) The point of intersection of the parabolic line with the axis of symmetry is taken as the origin in the $\{\varphi, \psi\}$-plane.

The above assumptions impose certain conditions on the functions $\eta(\varphi, \psi)$ and $\theta(\varphi, \psi)$, namely:

(3.3.1) $\eta(\varphi, \psi)=\eta(\varphi, -\psi);\quad \theta(\varphi, \psi)=-\theta(\varphi, -\psi);\quad \eta(0, 0)=0$.

The functions η and θ are assumed to be represented in power series; since η is an even function of ψ, one has:

(3.3.2) $$\eta=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\varphi^m\psi^{2n}, \qquad a_{00}=0;$$

similarly, θ being an odd function of ψ:

(3.3.3) $$\theta=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}\varphi^m\psi^{2n+1}.$$

From eqs. (3.1.9) one obtains:

(3.3.4) $$\eta\eta_{,\varphi}\eta_{,\psi}=\theta_{,\varphi}\theta_{,\psi}.$$

Inserting eqs. (3.3.2) and (3.3.3) into eq. (3.3.4) and equating the terms of like powers in ψ for $m=n=0$ furnishes the result $b_{00}=0$. In the neighborhood of the origin of coordinates, that is, where φ and ψ are small magnitudes such that the terms of higher order in powers of φ and ψ may be neglected, eqs. (3.3.2) and (3.3.3) give the order of the functions η and θ and their derivatives:

(3.3.5) $$\eta=0(\varphi);\quad \theta=0(\varphi\psi);\quad \eta_{,\varphi}=0(1);$$
$$\eta_{,\psi}=0(\psi);\quad \theta_{,\varphi}=0(\psi);\quad \theta_{,\psi}=0(\varphi).$$

Now, considering eqs. (3.1.9) near the origin, making use of eqs. (3.3.5) one gets $\eta_{,\psi}\sim\theta_{,\varphi}\sim0(\psi)$ and $\eta\eta_{,\varphi}\sim\theta_{,\psi}\sim0(\varphi)$ which implies that $b(\eta)$ must be of the order 0(1) in order to preserve the same "weight" in all the terms of the same equation i.e., $b(\eta)\equiv b(0)$; thus, near the origin, the system (3.1.9) may be approximated by the system:

(3.3.6) $$\eta_{,\psi}+b(0)\theta_{,\varphi}=0;\quad \eta\eta_{,\varphi}-b^{-1}(0)\theta_{,\psi},\quad b(0)\neq0.$$

Letting $b(0)\psi=\overline{\psi}$, eqs. (3.3.6), after dropping the bar, reduce to:

(3.3.7) $$\eta_{,\psi}+\theta_{,\varphi}=0;\quad \eta\eta_{,\varphi}-\theta_{,\psi}=0.$$

The solution of these equations is given by

(3.3.7a) $$\theta=A^2\varphi\psi-(A^3/6)\psi^3;\quad \eta=A\varphi-(A^2/2)\psi^2,$$

where A is a constant. This is again identical to the result in [12], i.e., formally the problem reduces to the one discussed by Falkovich [12]. Similarly, all the subsequent formal derivations and results of Falkovich, for the case of an isentropic flow are valid in the present case. The author found some misprints in N. A. C. A. translation of Falkovich's paper which are given in the Appendix. Thus all the characteristics properties of Laval nozzle remain valid for the case of diabatic flow with the only difference being that the "sonic" line in an isentropic flow has to be substituted by the "parabolic" line in a diabatic flow. Moreover, in a diabatic flow the value of the coefficient A is given by:

(3.3.8) $$A=[-q^{-1}(-h^2\eta^{-1})^{1/2}u_{,x}]_{x=0,y=0} .$$

In Section 1.1. it was shown that $(-h^2\eta^{-1})^{1/2}>0$ at the origin; now, $u_{,x}>0$, hence $A<0$.

Appendix

1. Misprints in [12]

Eq. (1.9) should read: $\theta_{,\psi}-\rho_0\rho^{-1}q^{-1}(q^2\alpha^{-2}-1)=0$;
Eq. (4.1) should read: $\theta=A^2\varphi\psi-1/6A^3\psi^3$;
Eq. (5.4) should read: $\theta=1/12A_1^3\psi^3$; $\eta=-1/4A_1^2\psi^3$;
Eq. (5.5) should read: $\theta=-2/3A_2^3\psi^3$; $\eta=-A_2^2\psi^2$;
Eq. (5.7) should read: $2f-2tf'+g'=0$; $ff'-3g+2tg'=0$;
Eq. (5.8) should read: $g=1/3[(f+4t^2)f'-4tf]$;
Eq. (5.10) should read: $f=-A_2^2$ for $t=-\frac{1}{2}A_2$;
Eq. below (5.11) should read: $A_1\leqq A_2\leqq 1/4A_1$.

2. Discussion of Q

Concentrating the attention on a thermodynamic system which consists of a streamline and letting q' represent the heat added per unit mass and per unit length along the streamline, then the First Law of Thermodynamics (see equation 1.1.3) applied to this system can be written as

(a) $$dQ=q'ds=c_p dT-\rho^{-1}dp ,$$

where 's' is the running coordinate along the streamline. Hence

(b) $$Q=\int dQ=\int_0^s q'ds+Q_0 .$$

Q is therefore the total heat added above a certain reference point. Choosing this reference point at stagnation then, $Q=0$ for $q=0$, $(s=0)$ and therefore $Q_0=0$.

The generalized Bernoulli's equation (equation 1.2.10a) was given as

(c) $$1/2q^2+c_pT-Q=\text{const.} .$$

For $q=0$, $Q=0$ and therefore, equation (c) reduces to

(d) $$1/2q^2+c_pT-Q=c_pT_0 .$$

The quantity q' may be related to the rate of heat added or sub tracted per unit mass, $\overline{Q}$. Equation (a) gives

(e) $$\partial Q/\partial s = q' ,$$

but, $\overline{Q}=DQ/Dt=q\partial Q/\partial s$ and therefore,

(f) $$q'=\overline{Q}/q$$

Isentropic flow is defined as $dQ=q'ds\equiv 0$ and therefore $Q\equiv 0$. In this case equation (d) reduces to the familiar Bernoulli's equation in isentropic flow.

An interesting situation arises on considering the physical significance of $Q=c$ (const.). A function q' such that, $q'=0$ for all $s\neq s_0$, $0\leqslant s_0\leqslant s$, and $Q=\int_0^s q'ds=c$ does not exist. However, $Q=c$ is justified mathematically if one considers $Q=\int dQ$ as a Lebesgue-Stieltjes integral. This situation may simulate the presence of a flame front in the flow domain.

3. A remark on the results of section 1.5

The results given in Section 1.5 are, in fact, a consequence of Kelvin's Circulation Theorem. A proof of this statement is given below.

The circulation Γ is given by

(a) $$\oint u_i dx_i$$

hence,

(b) $$\frac{D\Gamma}{Dt}=\oint (Du_i/Dt)dx_i+\oint u_i D(dx_i)/Dt$$
$$=-\oint \rho^{-1}(\partial p/\partial x_i)dx_i+\oint u_i du_i \quad \text{(by using eq. 1.1.1)}$$
$$=-\oint dp/\rho+1/2\oint d(q^2) .$$

$\Gamma=0$ implies that $\omega=0$ by applying Stokes theorem to equation (a). Hence, equation (b) gives, for $\omega=0$, $\oint dp/\rho=0$ since $d(q^2)$ is an exact differential. The vanishing of $\oint dp/\rho$ implies that dp/ρ is an exact differential and therefore, $p=p(\rho)$. Hence, equation (1.1.6) gives $S=S(\rho)$, the equation of state gives $T=T(\rho)$, the First Law of Thermodynamics gives $Q=Q(\rho)$ and finally, Bernoulli's equation gives $\rho=\rho(q)$. This shows that $Q=Q(\rho(q))=F(q)$ or $q=q(Q)$.

Equations (1.2.16) and (1.2.10a) show that the condition of irrotationality implies that the constant of integration in Bernoulli's equation (eq. 1.2.10a) is independent of the stream function. Since the form of the functional relationship $Q=Q(x, y)$ was not used in the proof given above, it follows that any function $Q=Q(x, y)$ transforms into $Q=Q(q)$ in the hodograph plane. Hence $Q(x, y)$ is arbitrary.

$p=p(\rho)$ also follows from Silberstein's theorem which states that "an irrotational flow of an inviscid non-heat conducting fluid subject to conservative forces is either baratropic, isochoric or isobaric". This theorem was given in his paper published in the Bulletin Internationale de l'Academic des Sciences de Cracovie Comptes Rendus in 1896.

The above outlined relation $Q=Q(q)$ may be subject to a radical change when the boundary value problem is considered. In this case one has to satisfy the usual geometric boundary conditions and the "a priori" given heat distribution $Q=Q(x, y)$, which due to $\omega=0$, must result in $Q=Q(q)$ only. The question whether this can be always satisfied seems to be a difficult one and is beyond the scope of the present work.

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University of Illinois