On models of algorithms and flow-charts

By Nobuo ZAMA

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Introduction

The purpose of this paper is a trial to discuss the theory of algorithms in terms of models. The author assumes that the algorithms are the procedures represented by flow-charts.

A number of mathematicians have given various sorts of definitions to algorithms, and many proofs of equivalence between those definitions have been given. The author proposes that such equivalency-proofs are equivalent to give models in a certain sense. The notion of the flow-charts plays an important part in accomplishing his aim, and he owed this notion to Kalznin's paper. However, he attempts a generalization of it, and makes a slight modification.

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1. Graphs

1.1. The definition of Graphs

Definition 1. Let M and N be two arbitrary sets. If at most two elements of N are associated with every element of M, we call such a mapping 2-*valued*. Moreover, if associated elements range over all elements of N, we call it a *total* mapping.

Definition 2. Let S be a set of finite arbitrary symbols that contains at least two elements. If i and o, two elements of S, are different from each other, a total 2-valued mapping from $S-\{o\}$ on $S-\{i\}$ is called a *graph whose ground is* S. If o_1, o_2 and i, three elements of S, are different from each other, a total 2-valued mapping from $S-\{o_1, o_2\}$ on $S-\{i\}$ is called a *logical graph* with the ground S. In both cases, i is called the *input*, o_1, o_2 and o are called *outputs*. Then we write $\Gamma\{S\}$ to mean that the ground of Γ is S. By a *general graph* Γ we shall mean that Γ is a graph or a logical graph.

Definition 3. Let $\Gamma\{S\}$, $\Gamma'\{S'\}$ be two general graphs. Γ and Γ' are called *isomorphic* if there is a one-to-one correspondence ψ from S on S', and if $\Gamma'(\psi(s))=\psi(\Gamma(s))$ for all $s\in S$.

Definition 4. We write $s\overset{\Gamma}{\longrightarrow}t$, or $s\longrightarrow t$ simply, to mean that $\Gamma(s)=(t)$

for a mapping Γ. Such an expression is called a *formula* of Γ. The *left member* of $s \to t$ is s, and the *right member* is t.

If a symbol s is contained in the ground of Γ, it is called a point of Γ and we use the expression s^Γ. Especially the input of Γ is written i^Γ, the output o^Γ.

1.2. Insertions and combinations

Let $\Gamma_1\{S_1\}$, $\Gamma_2\{S_2\}$ be two general graphs. If $S_1 \cap S_2$ is not empty, we take an arbitrary set S_2' which contains as many elements as S_2, and satisfies that $S_1 \cap S_2'$ is empty. Then we take a one-to-one correspondence ψ from S_2 on S_2'. The definition $\psi(\Gamma_2(s)) = \Gamma_2'(\psi(s))$ $(s \in S_2)$ gives us a general graph $\Gamma_2'\{S_2'\}$ which is isomorphic to Γ_2, and we shall take it in place of Γ_2'.

When we consider many graphs at the same time in the Definition 5, 6, 7, 8, we shall assume that their grounds are disjoint after the procedure in the preceding paragraph. As the proof of the Lemma 1, 2, 3, 4 can be obtained by investigations of their formulas, they are omitted.

Definition 5. Let Γ be a graph, and Δ a general graph. The mapping, whose formula consists of

$$\Gamma\text{'s formulas, } \Delta\text{'s formulas and } o^\Gamma \to i^\Delta$$

is called the *combination* of Γ and Δ, and is written $\Gamma \cdot \Delta$.

Lemma 1. *$\Gamma \cdot \Delta$ is a graph when Γ and Δ are graphs, but is a logical graph when Δ is a logical graph.*

Definition 6. Let $\Gamma\{S\}$ be a general graph, and $s \in S$. If there is at most one formula whose left member is s, s is called a *mathematical* point. If otherwise, s is called logical.

Definition 7. Let Δ be a graph. When Γ is a general graph, and when s is a mathematical point of Γ, we suppose that the formulas which contains s as its member, are

$$h_1 \to s, \cdots, h_r \to s, s \to k\,. \tag{2}$$

Then the *insertion of Δ to Γ at s* is the graph whose formulas are

$$\text{the formulas of } \Gamma \text{ except (2), } \Delta\text{'s formulas,}$$
$$h_1 \to i^\Delta, \cdots, h_r \to i^\Delta, o^\Delta \to k\,.$$

The insertion is written $\Gamma\binom{s}{\Delta}$.

Lemma 2. *Under the same assumption as Definition* 7, $\Gamma\binom{s}{\Delta}$ *is* *a graph if Γ is a graph, and it is a logical graph if Γ is a logical graph.*

Definition 8. Let Γ be a general graph, and s^Γ be logical. Suppose that

$$h_1 \to s, \cdots, h_r \to s, s \to k_1, s \to k_2 \tag{4}$$

are the formulas which contain s as their members.

When Δ is a logical graph, we define that *the insertion* $\Gamma\binom{s;\ k_1,\ k_2}{\Delta}$ is the graph whose formulas consist of

$$\Gamma\text{'s formulas except (4), } \Delta\text{'s formula,}$$
$$h_1 \to i^\Delta, \cdots, h_r \to i^\Delta, o_1^\Delta \to k_1, o_2^\Delta \to k_2 .$$

If no confusion is expected, we simply denote $\Gamma\binom{s}{\Delta}$.

Lemma 3. *Under the same assumption as in Definition* 8, $\Gamma\binom{s}{\Delta}$ *is a graph if Γ is a graph.* $\Gamma\binom{s}{\Delta}$ *is a logical graph if Γ is a logical graph.*

The points of $\Gamma\binom{s^\Gamma}{\Delta}$ consist of the points of Γ except s and the points of Δ. Then we shall understand that the points of the insertion are written s^Γ, or s^Δ according as they belong to Γ or Δ.

In the following Lemma 4 and 5, we shall suppose that Γ and Θ_h $(h=1, \cdots, r)$ general graphs. When $s_1, \cdots, s_r$ are some points of Γ, we correspond Θ_h to s_Γ^h $(h=1, \cdots, r)$, supposing that Θ_h is a graph or a logical graph according as s_h is mathematical or logical.

Lemma 4. $\Gamma\binom{s_1}{\Theta_1}\binom{s_2}{\Theta_2}$, *that is, the insertion of Θ_2 at s_2 to* $\Gamma\binom{s_1}{\Theta_1}$ *is a graph or a logical graph according as Γ is a graph or a logical graph. Its formulas coincide with those of* $\Gamma\binom{s_2}{\Theta_2}\binom{s_1}{\Theta_1}$, *and do not contain any formula $t^{\Theta_1} \to u^{\Theta_2}$, where t and u are neither o nor i.*

Lemma 5. $\Xi = \Gamma\binom{s_1}{\Theta_1}\binom{s_2}{\Theta_2}\cdots\binom{s_n}{\Theta_n}$ *is a graph, and any change of the order of insertions doesn't alter the formulas of Ξ. Moreover, Ξ has no formula $t^{\Theta_k} \to u^{\Theta_m}$ $(k \neq m)$ where t and u are neither o nor i.*

The proof of Lemma 5 is easily done by the induction on n. We can use the symbol $\Gamma\begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ \Theta_1 & \Theta_2 & & \Theta_n \end{pmatrix}$ without ambiguity.

2. Flow-charts

2.1. Mathematical Structures

Let $M = \{m_\lambda\}$ be a set of arbitrary subjects. A function with one variable, whose variables and values range over $\boldsymbol{M}$, is called a function in $\boldsymbol{M}$. A predicate with one variable, whose argument a range over $\boldsymbol{M}$, is called a predicate in $\boldsymbol{M}$. Suppose that $F=\{f_\mu\}$ be an arbitrary set of functions in $\boldsymbol{M}$ containing the identity function, and $\boldsymbol{P}=\{p_\nu\}$ be an arbitrary set of predicate in $\boldsymbol{M}$.

Definition 9. A *mathematical structure* $\mathfrak{F}$ is the union of such $\boldsymbol{M}$, $\boldsymbol{F}$ and $\boldsymbol{P}$ as above, and $\mathfrak{F}$ is written $\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M}\rangle$.

2.2. Flow-charts.

Let $\mathfrak{F}$ be a mathematical structure $\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M}\rangle$, and $\Gamma\{S\}$ be a graph.

Definition 10. If $\boldsymbol{s}$ is a logical point, there are two formulas each of which has $\boldsymbol{s}$ as its left member. We indicate arbitrarily that one of them is the *true-formula* (or+-formula), the other is the *false formula* (or—-formula). Then, we proceeds the same indications to all logical points of Γ, and the set of all formulas, +-formulas and—-formulas is called a *chart whose ground is* $\boldsymbol{S}$. Different indications of+and—-formulas give us different charts. From now on, we shall properly consider charts, and not graphs.

Definition 11. Let F be a chart. We associate an element of $\boldsymbol{F}$ with a mathematical point of F, and an element of $\boldsymbol{P}$ with a logical point, one by one. When $\mathfrak{A}$ is such a correspondence, any pair $(\boldsymbol{s}^F, \mathfrak{A}(\boldsymbol{s}^F))$ is called *a block*. $\boldsymbol{s}^F$ is called the point of the block. Hereafter $\bar{\boldsymbol{s}}^F$ means $\mathfrak{A}(\boldsymbol{s}^F)$. To indicate a block, we shall write only its points.

Definition 12. A *flow-chart over a mathematical structure* $\mathfrak{F}$ is a chart whose ground is the set of blocks, and we call them a $FC(\mathfrak{F})$ (or $FC(\boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M})$) for the sake of brevity.

Definition 13. We can similarly define *logical charts* and *logical flow-charts* (abbreviated LFC) by making use of logical graphs. But in addition to the definition of charts, we must indicate that one output is the true output (+-output), and the other the false output (—-output).

2.3. Computations.

Let F be a FC($\mathfrak{F}$). We can define a function by F according to the following procedure.

1) We take an element m of $\boldsymbol{M}$, and substitute it to the variable of $\bar{\boldsymbol{i}}^F$.

2) If $\bar{\boldsymbol{i}}^F$ is a function, and if $\boldsymbol{i}^F \rightarrow \boldsymbol{s}^F$, $\bar{\boldsymbol{i}}^F(m)$ is substituted to the variable[1] of $\bar{\boldsymbol{s}}^F$. When there is no $\bar{\boldsymbol{i}}^F(m)$, the procedure stops.

1) Here, "variable" is replaced to "argument," if $\bar{\boldsymbol{s}}^F$ is logical.

3) If $\bar{i}^F$ is a predicate, there are the+-formula $\bar{i}^F \to \bar{s}^F$ and the formula $\bar{i}^F \to \bar{t}^F$. We substitute m to the argument of $\bar{i}^F$. If $\bar{i}^F(m)$ is true, $\bar{m}$ is substituted to the variable of $\bar{s}^F$. If $\bar{i}^F(m)$ is false, m is substituted to the variable of $\bar{t}^F$. When we can decide whether $\bar{i}^F(m)$ is true or not, the procedure stops.

4) We follow the same procedure to $\bar{s}^F(m), \bar{t}^F(m)$ as $\bar{i}^F(m)$ in 2) and 3).

5) If we can reach $\bar{o}^F$[2] after some steps, and if the value y of $\bar{o}^F(m)$[2] is decided, y is defined as the value gained by F from given m. But if we can not reach $\bar{o}^F$, or we have no value of $\bar{o}^F(m)$, we cannot gain any value by F from m.

By a LFC we can define a predicate when we adopt 1), 2), 3), 4) and 5′). After some steps, if we can reach $\bar{o}^F_+$, and $\bar{o}^F_+(m)$ is m, we assume that the predicate is true for given m. If we can reach $\bar{o}^F_-$, and $\bar{o}^F_-(m)$ is m, we assume that the predicate is false. If we have neither of the previous cases, we don't know whether true or not.

Definition 14. The computation from a given element m by a FC (or a LFC) F is the succession of the procedures 1), 2), 3), 4), 5) or 5′) as above. By Comp (m, F), we shall understand the computation from m by F.

Definition 15. The route of the computation is the sequence of blocks through which the computation is carried out. Its end is the block in whose next block the computation is impossible any more. En (m, F) is the number of blocks from the input to the end of Comp (m, F). By $\mathrm{Res}_n(m, F)$ we shall mean the value gained in the n-th block of the route of Comp (m, F). $\mathrm{Bl}_n(m, F)$ is the n-th block of the route of Comp (m, F).

Definition 16. The value gained by Comp (m, F) is written $F(m)$. The function or the predicate defined by a FC or by a LFC F is also written $F(x)$.

2.4. Insertions and Combinations of FC's.

The insertions and the combinations of a chart are defined similarly as the cases for graphs. But, when we define the insertion of a logical chart Δ to Γ at a logical point s^Γ, we have to modify the definition slightly. That is, if $s^\Gamma \to t^\Gamma$ is the+-formula, and $s^\Gamma \to u^\Gamma$ is the−-formula, we must take $o^\Delta_+ \to t^\Gamma$ and $o^\Delta_- \to u^\Gamma$ in place of $o^\Delta_1 \to k_1$, $o^\Delta_2 \to k_2$ in the Definition 8.

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[2] As $\bar{o}^F$ is necessarily mathematical, $\bar{o}^F$ is a function.

3. Models

3.1. The definition of models.

Definition 17. Let $\mathcal{F}$ be a mathematical structure $\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle$. By $\langle\!\langle \mathcal{F} \rangle\!\rangle$, we shall understand the set of the functions and predicates defined by FC($\mathcal{F}$).

We shall take another mathematical structure $\mathcal{G} = \langle \boldsymbol{G}, \boldsymbol{Q}, \boldsymbol{N} \rangle$. Given a one-to-one mapping φ from $\boldsymbol{M}$ in $\boldsymbol{N}$, we can define the function φf and the predicate by the predicate φp by the stipulation that

$$(\varphi f)(\varphi(m)) \doteq \varphi(f(m)) \quad \text{for all } m \in \boldsymbol{M}\,.$$

$$(\varphi p)(\varphi(m)) \Leftrightarrow \varphi p(m) \quad \text{for all } m \in \boldsymbol{M}\,.$$

Where f is a function in $\boldsymbol{M}$, and p is a predicate in $\boldsymbol{M}$.

Definition 18. $\langle\!\langle \mathcal{G} \rangle\!\rangle$ is a *model* of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ by φ, if the following conditions hold:

(1) for each function $f \in \langle\!\langle \mathcal{F} \rangle\!\rangle$, there exists a function $g(x) \in \langle\!\langle \mathcal{G} \rangle\!\rangle$ such that

$$g(\varphi(m)) = (\varphi f)(\varphi(m)) \quad \text{for all } m \in \boldsymbol{M}\,.$$

(2) for each predicate $p \in \langle\!\langle \mathcal{F} \rangle\!\rangle$, there exists a predicate $q(x) \in \langle\!\langle \mathcal{G} \rangle\!\rangle$ such that

$$q(\varphi(m)) \Leftrightarrow (\varphi p)(\varphi(m)) \quad \text{for all } m \in \boldsymbol{M}\,.$$

The symbol $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteq_{\varphi} \langle\!\langle \mathcal{G} \rangle\!\rangle$ is used when $\langle\!\langle \mathcal{G} \rangle\!\rangle$ is a model of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ by φ.

3.2. A Special Case of Models.

We shall consider the models in the special case where $\mathcal{F}$ is the identical mapping, before we proceed to consider the general case.

In this section 3.2., we shall understand $\mathcal{F}$ is $\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle$, and $\mathcal{G}$ is $\langle \boldsymbol{G}, \boldsymbol{Q}, \boldsymbol{M} \rangle$.

Lemma 6. *$\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteq_{\varphi} \langle\!\langle \mathcal{G} \rangle\!\rangle$ if and only if $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{G} \rangle\!\rangle$, where the mapping φ is identical.*

Proof. (1) Let $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteq_{\varphi} \langle\!\langle \mathcal{G} \rangle\!\rangle$. Then, for each function $f \in \langle\!\langle \mathcal{F} \rangle\!\rangle$, there is a function $g(m) \in \langle\!\langle \mathcal{G} \rangle\!\rangle$ such that $g(m) = f(m)$ for all $m \in \boldsymbol{M}$. Hence $f \in \langle\!\langle \mathcal{G} \rangle\!\rangle$.

Similarly we can prove that every predicate $p \in \langle\!\langle \mathcal{F} \rangle\!\rangle$ is also contained in $\langle\!\langle \mathcal{G} \rangle\!\rangle$.

(2) Let $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{G} \rangle\!\rangle$. Then, every $f \in \langle\!\langle \mathcal{F} \rangle\!\rangle$ is contained in $\langle\!\langle \mathcal{G} \rangle\!\rangle$. Hence,

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3) $f(x) = g(x)$ means "That there exists $f(x)$ is equivalent to the fact there exists $g(x)$; and $g(x) = f(x)$, if there exist $f(x)$ and $g(x)$. $p(x) \Leftrightarrow q(x)$ means "if $p(x)$ is true, $q(x)$ is true; and if $q(x)$ is false, $q(x)$ is false."

4) $\subseteq$ means set-theoretic inclusion.

(1) in Definition 1 is satisfied by taking f as g. Similarly, we have (2) in Definition 1. q.e.d.

In the following, we assume that all structures contain the identity function.

Theorem 1. *$\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$ if and only if $\boldsymbol{F} \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$, $\boldsymbol{P} \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$.*

Proof. (1) Let $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$. It is easily seen that $f \in \langle\!\langle \mathcal{F} \rangle\!\rangle$ and $p \in \langle\!\langle \mathcal{G} \rangle\!\rangle$ for every $f \in \boldsymbol{F}$ and $p \in \boldsymbol{P}$. So $f \in \langle\!\langle \mathcal{G} \rangle\!\rangle$, $p \in \langle\!\langle \mathcal{G} \rangle\!\rangle$. Hence, $\boldsymbol{F} \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$, $\boldsymbol{P} \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$.

(2) Let $\boldsymbol{F} \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$, $\boldsymbol{P} \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$. That is, all $f_\lambda \in \boldsymbol{F}$ and $p_\mu \in \boldsymbol{P}$ are defined by FC($\mathcal{G}$)'s. By $[f_\lambda]$, $[p_\mu]$ we shall understand FC($\mathcal{G}$)'s defining f_λ and p_μ respectively. Therefore the assumption is as follows:

$$\begin{aligned} &f_\lambda(m) \doteq [f_\lambda](m) \quad \text{for all } f_\lambda \in \boldsymbol{F} \text{ and all } m \in \boldsymbol{M}. \\ &p_\mu(m) \Leftrightarrow [p_\mu](m) \quad \text{for all } p_\mu \in \boldsymbol{P} \text{ and all } m \in \boldsymbol{M}. \end{aligned} \qquad \text{(A)}$$

Let F be an arbitrary FC($\mathcal{F}$) and its blocks be $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n$ (Assume that $\boldsymbol{b}_1 = \boldsymbol{i}^F$, $\boldsymbol{b}_n = \boldsymbol{o}^F$.) Then we construct

$$K = F\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \\ [\overline{\boldsymbol{b}_1}] & [\overline{\boldsymbol{b}_2}] & \cdots & [\overline{\boldsymbol{b}_n}] \end{pmatrix}.$$

In order to prove $K(m) = F(m)$ for all $m \in \boldsymbol{M}$, we carry out Comp(m, F) and Comp(m, K) for an arbitrary $m \in \boldsymbol{M}$.

If the substitution of m to $\boldsymbol{i}^F$ gives us no result, Comp$(m, [\overline{\boldsymbol{i}^F}])$ also gives us no result by (A). By Lemma 5, there is not a formula $\boldsymbol{b}^{[\overline{i^F}]} \to \boldsymbol{b}^{[\overline{b_q}]} (q \neq 1)$ where these blocks are neither $\boldsymbol{o}$ nor $\boldsymbol{i}$. Then Comp(m, K) doesn't reach $\boldsymbol{o}^{[\overline{b_1}]}$, and so doesn't reach $\boldsymbol{o}^K$. Hence there is no value of $K(m)$.

Next we assume that $\overline{\boldsymbol{i}}^F(m)$ gives us any result. Then we shall prove the following (1) and (2) for all k less than En(m, F):

(1) If $\mathrm{Bl}_k(m, F) = \boldsymbol{b}$, $\mathrm{Bl}_p(m, K) = \boldsymbol{o}^{[\overline{b}]}$ for some p.

(2) And $\mathrm{Res}_k(m, F) = \mathrm{Res}_p(m, K)$.

The proof can be done by induction on k less than En(m, F). For $k=1$,

$$\mathrm{Bl}_1(m, F) = \boldsymbol{i}^F, \quad \text{and} \quad \mathrm{Res}_n(m, F) = \overline{\boldsymbol{i}}^F(m).$$

Then, $\overline{\boldsymbol{o}}^{[\overline{i^F}]}(m) = \overline{\boldsymbol{i}}^F(m)$ by (A). Hence Comp(m, K) reach $\boldsymbol{o}^{[\overline{i^F}]}$, and $\mathrm{Bl}_p(m, K) = \boldsymbol{o}^{[\overline{i^F}]}$ for some p. Therefore we have (1) and (2) for $k=1$.

We assume that (1) and (2) can be proved for some k less than En(m, F). If $\mathrm{Bl}_{k+1}(m, F) = \boldsymbol{c}$.

$$\begin{aligned} \mathrm{Res}_{k+1}(m, F) &= \overline{\boldsymbol{c}}(\mathrm{Res}_k(m, F)) \\ &= \boldsymbol{o}^{[\overline{c}]}(\mathrm{Res}_k(m, F)) && \text{(by (A))} \\ &= \boldsymbol{o}^{[\overline{c}]}(\mathrm{Res}_p(m, K)). && \text{(by (2))} \end{aligned}$$

Then, when $\boldsymbol{o}^{[\overline{c}]}$ is q-th block of the computation, we have

(2) If $\mathrm{Bl}_{k+1}(m, F)=\boldsymbol{c}$, $\mathrm{Bl}_q(m, K)=o^{[c]}$

(2) $\mathrm{Res}_{k+1}(m, F)=\mathrm{Res}_q(m, K)$.

Hence (1) and (2) are proved.

By (1) and (2), we can conclude "If $r=\mathrm{E}(m, F)$ and $\mathrm{Bl}_r(m, F)=\boldsymbol{e}$, then $\mathrm{Bl}_s(m, K)=\boldsymbol{o}^{[e]}$ for some s." Therefore $\mathrm{Comp}(m, K)$ can reach $\boldsymbol{o}^K=\boldsymbol{o}^{[\bar{F}]}$ if $\mathrm{Bl}_r(m, F)=\boldsymbol{o}^F$. But $\mathrm{Comp}(m, K)$ cannot reach $\boldsymbol{o}^K$, if $\mathrm{Bl}_r(m, F)\neq\boldsymbol{o}^F$.

On the contrary, if $\mathrm{Comp}(m, F)$ has no end, $\mathrm{Comp}(m, K)$ has no end either, as (1) and (2) are true for all integer n. q.e.d.

3.3. General Cases.

Let us suppose that $\mathcal{F}=\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M}\rangle$, $\mathcal{G}=\langle \boldsymbol{G}, \boldsymbol{Q}, \boldsymbol{N}\rangle$, and that a one-to-one mapping φ from $\boldsymbol{M}$ in $\boldsymbol{N}$ which is not identical.

If F is a FC($\mathcal{F}$), we replace each function f in F to φf, and each predicate p to φp. Let φF be the so-gained FC. Then we have

Theorem 2. *$(\varphi F)(\varphi(m))=\varphi(F(m))$ for all $m\in\boldsymbol{M}$.*

Proof. The points of F's blocks are equals to those of φF's blocks. If $\bar{\boldsymbol{i}}^F(m)$ gives us no result, $\bar{\boldsymbol{i}}^{\varphi F}(\varphi(m))=\varphi(\boldsymbol{i}^F(m))$ gives us no result either. If otherwise, it is sufficient to show for all $m\in\boldsymbol{M}$

(1) $\mathrm{Bl}_k(m, F)=\mathrm{Bl}_k(\varphi m, \varphi F)$

(2) $\mathrm{Res}_k(\varphi m, \varphi F)=\varphi(\mathrm{Res}(m, F))$

where k is an integer less than $\mathrm{En}(m, F)$.

It is easily done by the indication on k. For $n=1$,

$$\mathrm{Bl}_1(m, F)=\boldsymbol{i}\ , \qquad \bar{\boldsymbol{i}}^{\varphi F}=\varphi(\bar{\boldsymbol{i}}^F)\ .$$

Then we have

$$\begin{aligned}\mathrm{Res}_1(\varphi(m), \varphi(F))&=\varphi(\bar{\boldsymbol{i}}^F)(\varphi(m))\\ &=\varphi(\bar{\boldsymbol{i}}^F(m))=\varphi(\mathrm{Res}_1(F, m))\ .\end{aligned}$$

If (1) and (2) are true for some k less than $\mathrm{En}(m, F)$, then

$$\mathrm{Bl}_{k+1}(m, F)=\mathrm{Bl}_k(\varphi_m, \varphi F)=\boldsymbol{c}$$

is trivial by the definition of φF. Hence

$$\begin{aligned}\mathrm{Res}_{k+1}(\varphi(m), \varphi F)&=\bar{\boldsymbol{c}}(\mathrm{Res}_k(\varphi(m), \varphi F))\\ &=\bar{\boldsymbol{c}}(\varphi(\mathrm{Res}_k(m, F))\\ &=\varphi\bar{\boldsymbol{c}}(\mathrm{Res}_k(m, F))\\ &=\varphi(\mathrm{Res}_k(m, F))\ .\end{aligned}$$

Hence we have proved (1) and (2). Hereafter we can discuss as in Theorem 1. q.e.d.

By the similary way as Theorem 2, we can prove

Theorem 3. *$(\varphi L)(\varphi(m)) \Leftrightarrow \varphi(L(m))$ for every* LFC$(\mathcal{F})L$.

Lemma 7. $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq_\varphi \langle\!\langle \varphi\mathcal{F} \rangle\!\rangle$
Here, by $\langle\!\langle \varphi\mathcal{F} \rangle\!\rangle$, we mean $\langle\!\langle \varphi\boldsymbol{F}, \varphi\boldsymbol{P}, \varphi\boldsymbol{M} \rangle\!\rangle$ where $\varphi\boldsymbol{F}=\{\varphi f; f \in \boldsymbol{F}\}$, $\varphi\boldsymbol{P}=\{\varphi p; p \in \boldsymbol{P}\}$, $\varphi\boldsymbol{M}=\{\varphi(x); x \in \boldsymbol{M}\}$.

Proof. Le us suppose that f can be defined by a FC$(\mathcal{F})F$. Then, φf can be defined by a FC$(\varphi\mathcal{F})$ by Theorem 2. So we can take φf as g, and φp as q in the definition 18, where a predicate $p \in \langle\!\langle \mathcal{F} \rangle\!\rangle$. q.e.d.

By Definition 18, we can prove easily;

Lemma 8. *$\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq_\varphi \langle\!\langle \mathcal{G} \rangle\!\rangle$ if and only if $\langle\!\langle \varphi\mathcal{F} \rangle\!\rangle \subseteqq \langle\!\langle \mathcal{G} \rangle\!\rangle$.*

Then we have,

Theorem 4. *$\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq_\varphi \langle\!\langle \mathcal{G} \rangle\!\rangle$ if and only if $\varphi\boldsymbol{F} \subseteqq \langle\!\langle \boldsymbol{G}, \boldsymbol{Q}, \varphi\boldsymbol{M} \rangle\!\rangle$ and $\varphi\boldsymbol{P} \subseteqq \langle\!\langle \boldsymbol{G}, \boldsymbol{Q}, \varphi\boldsymbol{M} \rangle\!\rangle$.*

4. Expansion of Models.

4.1. Models by an Equivalence Class.

Let $\sim$ be an equivalence relation in a set $\boldsymbol{M}$. By $\bar{x}$, we shall understand the class containing x.

Definition 19. Let f be a function in $\boldsymbol{M}$. When there exists a function g in $\boldsymbol{M}/\sim$ such that

$$y=f(x) \Leftrightarrow \bar{y}=g(\bar{x}) \qquad \text{for all} \quad x \in \boldsymbol{M},$$

we call f a *class-function* by $\sim$, and g is written $\bar{f}$.

Similarly, if p is a predicate in $\boldsymbol{M}$, and if there exists a predicate q in $\boldsymbol{M}/\sim$ such that

$$p(x) \Leftrightarrow q(\bar{x}) \qquad \text{for all} \quad x \in \boldsymbol{M},$$

we call p a *class-predicate* by $\sim$, and q is written $\bar{p}$.

Let $\mathcal{F}$ be a mathematical structure $\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle$. If every element f_λ of F is a class function, and if every element p_μ of $\boldsymbol{P}$ is a class predicate, $\{\bar{f}_\lambda\}$ is written $\bar{\boldsymbol{F}}$, and $\{\bar{p}_\mu\}$ is written $\bar{\boldsymbol{P}}$.

Lemma 9. *Under the assumption above, every function f of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ is a class function, and there exists a function $\bar{f}$ in $\langle\!\langle \bar{\boldsymbol{F}}, \bar{\boldsymbol{P}}, \bar{\boldsymbol{M}}/\sim \rangle\!\rangle$ such that*

$$y=f(x) \Leftrightarrow \bar{y}=\bar{f}(\bar{x}) \qquad \textit{for every} \quad x \in \boldsymbol{M}. \tag{1}$$

Similarly every predicate p of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ is a class predicate, and there exists a predicate p in $\langle\!\langle \bar{\boldsymbol{F}}, \bar{\boldsymbol{P}}, \bar{\boldsymbol{M}}/\sim \rangle\!\rangle$ such that

$$p(x) \Leftrightarrow \bar{p}(\bar{x}) \qquad \textit{for every} \quad x \in \boldsymbol{M}. \tag{2}$$

Conversely for every element in $\langle\!\langle \bar{\boldsymbol{F}}, \bar{\boldsymbol{P}}, \boldsymbol{M}/\sim \rangle\!\rangle$ there exists a function *f or a predicate p such that* (1) *or* (2) *holds.*

Proof. When a function $g \in \langle\!\langle \mathcal{F} \rangle\!\rangle$ is a defined by a FC F, we replace all f_λ in F to $\bar{f}_\lambda$, and all p_μ in F to $\bar{p}_\mu$. We write the so-gained FC $\bar{F}$. If $\bar{g}$ is a function defined by $\bar{F}$, we can prove

$$y = g(x) \Leftrightarrow \bar{y} = \bar{g}(\bar{x}) \qquad \text{for all} \quad x \in M ,$$

by a similar way as Theorem 2.

A predicate $p \in \langle\!\langle \mathcal{F} \rangle\!\rangle$, we can discuss similarly.

The converse is shown by taking a $f(x)$ for $\bar{f}$ a $p(x)$ for $\bar{p}$. q.e.d.

4.2. Models by a Normal Mapping.

Definition 20. A many-valued mapping φ from $\boldsymbol{M}$ on $\boldsymbol{N}$ is said *normal* when: $\varphi(m_1) \neq \varphi(m_2)$ for $m_1 \neq m_2$.

Then the element x of $\boldsymbol{M}$, for which $\varphi(x) = n$ holds, is determined uniquely for every element in $\boldsymbol{N}$. In other words, there exists the inverse mapping φ^{-1} of φ.

If we define $\sim_\varphi$ by the rule that

$$n_1 \sim_\varphi n_2 \Leftrightarrow \varphi^{-1}(n_1) = \varphi^{-1}(n_2) ,$$

$\sim_\varphi$ is an equivalence relation in $\boldsymbol{N}$. When we suppose that $\bar{n}$ is the class containing n, the mapping $\bar{\varphi}$, defined by the stipulation $\bar{\varphi}(\bar{m}) = \overline{\varphi(m)}$, is a one-valued mapping from $\boldsymbol{M}$ on $\boldsymbol{N}/\sim\varphi$. If $\bar{\varphi}$ is one-to-one,

$$\langle\!\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle\!\rangle \subseteqq_{\bar{\varphi}} \langle\!\langle \bar{\varphi}\boldsymbol{F}, \bar{\varphi}\boldsymbol{P}, \boldsymbol{N}/\sim_\varphi \rangle\!\rangle .$$

Definition 21. Let us suppose that $\mathcal{F} = \langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle$, $\mathcal{G} = \langle \boldsymbol{G}, \boldsymbol{Q}, \boldsymbol{N} \rangle$, φ, $\bar{\varphi}$ are the same as in the preceding section. $\langle\!\langle \mathcal{G} \rangle\!\rangle$ is called a *model* of $\langle\!\langle \mathcal{F} \rangle\!\rangle$, when $\boldsymbol{G}$ and $\boldsymbol{Q}$ contain class functions by $\sim_\varphi$, and class predicates by $\sim_\varphi$ respectively, and when

$$\langle\!\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle\!\rangle \subseteqq_{\bar{\varphi}} \langle\!\langle \bar{\boldsymbol{G}}, \bar{\boldsymbol{Q}}, \boldsymbol{N}/\sim_\varphi \rangle\!\rangle .$$

In this case, we use the symbol $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq_\varphi \langle\!\langle \mathcal{G} \rangle\!\rangle$.

Theorem 5. *Let φ be a normal mapping from $\boldsymbol{M}$ on $\boldsymbol{N}$, and $\bar{\varphi}$ be one-to-one. Then, $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq_\varphi \langle\!\langle \mathcal{G} \rangle\!\rangle$ if and only if every $\bar{f}_\lambda \in \bar{\varphi}\boldsymbol{F}$ and every $\bar{p}_\mu \in \bar{\varphi}\boldsymbol{P}$ is contained in $\langle\!\langle \bar{\boldsymbol{G}}, \bar{\boldsymbol{Q}}, \boldsymbol{N}/\sim_\varphi \rangle\!\rangle$.*

Proof. $\langle\!\langle \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{M} \rangle\!\rangle \subseteqq_{\bar{\varphi}} \langle\!\langle \bar{\boldsymbol{G}}, \bar{\boldsymbol{Q}}, \boldsymbol{N}/\sim_\varphi \rangle\!\rangle$ if and only if $\langle\!\langle \bar{\varphi}\boldsymbol{F}, \bar{\varphi}\boldsymbol{P}, \boldsymbol{N}/\sim_\varphi \rangle\!\rangle \subseteqq \langle\!\langle \bar{\boldsymbol{G}}, \bar{\boldsymbol{Q}}, \boldsymbol{N}/\sim_\varphi \rangle\!\rangle$ by Lemma 8. Hence we have the conclusion by Theorem 4. q.e.d.

Theorem 6. *Let us suppose that φ is a normal mapping from $\boldsymbol{M}$ on $\boldsymbol{N}$, and $\langle\!\langle \mathcal{F} \rangle\!\rangle \subseteqq_\varphi \langle\!\langle \mathcal{G} \rangle\!\rangle$.*

If $y = g(x)$ for a $g \in \langle\!\langle \mathcal{G} \rangle\!\rangle$, there exists a function $f \in \langle\!\langle \mathcal{F} \rangle\!\rangle$ *such that* $\varphi^{-1}(y) = f(\varphi^{-1}(x))$.

As the proof is easily seen from the Definition 21, it is omitted. The theorem points out the common character between two models in 3.1. and 4.2.

5. An Application.

5.1. Turing Machines

By an alphabet A we shall mean a set of finite symbols. We consider a tape of infinite length divided into squares, and we write finite symbols of A on it according to the rule that one symbol is written in one square. A sequence of symols (containing blank spaces) is called a tape description. We define the coordinates of the type as follows:

1) We choose a square as the origin arbitrarily and let its coordinate be 0,

2) the coordinates of the right squares to the origin are 1, 2, 3, $\cdots$ respectively from left to right,

3) the coordinates of the left squares to the origin are 1, 2, 3, $\cdots$ respectively from right to left.

By a description we shall understand a pair of a tape description Y and a integer r, $\langle Y; r\rangle$.

A Turing machine (abbreviated to TM) is a FC($\boldsymbol{F}$, $\boldsymbol{P}$, $\boldsymbol{M}$), where $\boldsymbol{F}$, $\boldsymbol{P}$ and $\boldsymbol{M}$ are as follows:

1) $\boldsymbol{M}$ is the set of descriptions.

2) $\boldsymbol{F}$ is a finite set of functions with the following types.

2.1.) aN; (where a is an symbol in A)

$$\mathrm{aN}(\langle\cdots b_{r-1}b_r b_{r+1}\cdots; r\rangle)=\langle\cdots b_{r-1}ab_{r+1}\cdots; r\rangle$$

2.2.) NR, NL:

$$\mathrm{NR}(\langle Y, r\rangle)=\langle Y, r+1\rangle\,; \qquad \mathrm{NL}(\langle Y, r\rangle)=\langle Y, r-1\rangle\,.$$

2.3.) NN

$$\mathrm{NN}(\langle Y, r\rangle)=\langle Y, r\rangle\,.$$

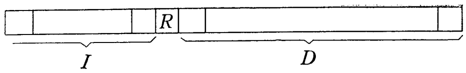
In the above, b_i is the symbol of coordinate i in Y, and Y is a tape description, and r is an arbitrary integer.

3) $\boldsymbol{P}$ is a finite set of predicates a $(\langle Y, r\rangle)$. This means a symbol a in A is written in the square of Y with the coordinate r.

5.2. We can give the answer, "Yes," to the problem whether any digital-computer M with the inner program can be done the same behaviour as a TM or not.

First, we give the outlone of the structure of M. It has an inner memory S divided to sections of the same bits. Each section is numbered with an integer called an address. S is separated to parts, one is the instruction part I (including working spaces), the other the data part D.

The author's idea is summerised as follows:



1) D is considered as the tape of the TM, and each symbol in A is coded to a symbol in M, for example a 2-adic integer. Then, they are written in D, a symbol for an address. The end of the description and the address R between D and I are marked with a special symbol.

2) But a special bit of each address in D is preserved to write the mark of the working order. That is, only a symbol of a particular co-ordinate is concerned with every function or every predicate. The bit is used to indicate it.

3) We give a FC in terms of M's instructions which is performing the same behaviour as every function or as every predicate, respectively. But in the case when we consider NL, we check whether there is a space of D to the left of the working address. These FC's gives us the desired FC by Theorem 4.

4) The FC given 4) is written in the form of the program of M.

Summing up, if the instructions of M enable us to check what a symbol is, and to change a symbol to another, we have a programm of M. But the weakest point of M is its limitation of the memory-capacity for I and D. To cancell the weak point, we have to use the tapes of M for either the tape of TM or the instructions of M, otherwise for both. We can also give a programm according to that idea.

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