Exponential analogues of Feld's series

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1. Introduction. The Feld series

$$\sum_{n=1}^{\infty} a_n b_n \frac{z^n}{1-a_n z^n},$$

where $\{a_n\}$ and $\{b_n\}$ are arbitrary sequences of complex valued constants, has as its exponential analogue the general B-series

$$B(z)=\sum_{n=1}^{\infty} a_n b_n \frac{e^{-\lambda_n z}}{1-a_n e^{-\lambda_n z}}$$

obtained by replacing z^n by $e^{-\lambda_n z}$, where $\{\lambda_n\}$ is a monotone increasing unbounded sequence of real numbers.

It is the purpose of this paper to consider the regions of convergence of the B-series, the representation of a B-series as a Dirichlet series and conversely, and finally certain restrictions on the a_n, b_n and λ_n which will insure that the function represented by the B-series will have a natural boundary.

Since all but a finite number of the λ_n will be positive, we shall consider λ_n to be positive for all n. Further, in order to avoid the singular points of the various terms of the B-series, we shall restrict the sequence $\{a_n\}$ so that for some constants L and M, $L \leq (\ln|a_n|)/\lambda_n \leq M$ for all n. The following notation will be standard:

M = least upper bound $\{(\ln|a_n|)/\lambda_n\}$

L = greatest lower bound $\{(\ln|a_n|)/\lambda_n\}$

$\mathfrak{M}=\{z\colon \mathrm{R}(z)>M\}$

$\mathfrak{L}=\{z\colon \mathrm{R}(z)<L\}$

$\mathfrak{P}=\mathfrak{M}\cup\mathfrak{L}=\{z\colon \mathrm{R}(z)>M \text{ or } R(z)<L\}$

$\mathfrak{O}=\mathfrak{M}\cap\mathfrak{R}$, where $\mathfrak{R}=\{z\colon \mathrm{R}(z)>0\}$.

Unless otherwise indicated all summations will range from $n=1$ to ∞.

2. Convergence of the B-series. The following are given without proof.

THEOREM 1. (A) *If the series* $\sum b_n$ *converges,*

(i) *and the sequence* $\{a_n\}$ *is bounded, the B-series converges for all* z *in* $\mathfrak{P}$ *for which* $\mathrm{R}(z)>0$;

(ii) *and the sequence* $\{a_n\}$ *is bounded such that for all* n, $0<H\leq|a_n|\leq K$, *the B-series converges for all* z *in* $\mathfrak{P}$ *for which* $\mathrm{R}(z)<0$;

(iii) *the B-series converges for all* z *in* $\mathfrak{P}$ *for which* $\mathrm{R}(z)=0$ *and for which the series* $\sum |a_n e^{-\lambda_n z}|$ *converges.*

(B) *Whether or not the series* $\sum b_n$ *converges,*

(i) *if the sequence* $\{a_n\}$ *is bounded, the B-series converges and diverges with the associated Dirichlet series* $\sum a_n b_n e^{-\lambda_n z}$ *for all* z *in* $\mathfrak{P}$ *for which* $\mathrm{R}(z)>0$;

(ii) *the B-series converges and diverges with the associated Dirichlet series* $\sum a_n b_n e^{-\lambda_n z}$ *for all values of* z *in* $\mathfrak{P}$ *for which* $\mathrm{R}(z)\leq 0$ *and for which the series* $\sum |a_n e^{-\lambda_n z}|$ *converges.*

Theorem 1 remains true if ordinary convergence and divergence is replaced throughout by absolute convergence and divergence.

By the angular region $T=T(z', \alpha)$ will be meant the set of all points z in the angular region with vertex at $z'=x'+iy'$ and defined by $|\arg(z-z')|\leq\alpha<\pi/2$. The $T^*=T^*(z', \alpha)$ region is the set of points symmetric to the set T where z' is the center of symmetry.

THEOREM 2. (A) *If the sequence* $\{a_n\}$ *is bounded and if the B-series converges at* z' *where* $\mathrm{R}(z')>0$, *the B-series converges uniformly throughout* $\mathfrak{P}\cap T$, *where* $T=T(z', \alpha)$.

(B) *If both the B-series and the series* $\sum |a_n e^{-\lambda_n z}|$ *converge at* z' *where* $R(z')<0$, *the B-series converges uniformly throughout* $\mathfrak{P}\cap T^*$, *where* $T^*=T^*(z', \alpha)$.

THEOREM 3. (A) *If the sequence* $\{a_n\}$ *is bounded and the B-series converges absolutely at* z' *where* $\mathrm{R}(z')>0$, *then the B-series converges absolutely and uniformly throughout* $\mathfrak{P}\cap\mathfrak{U}$, *where* $\mathfrak{U}=\{z\colon \mathrm{R}(z)\geq\mathrm{R}(z')\}$.

(B) *If both the B-series and the series* $\sum a_n e^{-\lambda_n z}$ *converge absolutely at* z' *where* $\mathrm{R}(z')\leq 0$, *the B-series converges absolutely and uniformly throughout the region* $\mathfrak{P}\cap\mathfrak{U}^*$, *where* $\mathfrak{U}^*=\{z\colon \mathrm{R}(z)\leq\mathrm{R}(z')\}$.

3. Relationship between the B-series and general Dirichlet series. Performing the indicated division of the n-th term of the B-series gives, at least formally,

$$a_n b_n \frac{e^{-\lambda_n z}}{1-a_n e^{-\lambda_n z}}=\sum_{m=1}^{\infty} b_n a_n^m e^{-m\lambda_n z}$$

and hence

$$\sum a_n b_n \frac{e^{-\lambda_n z}}{1-a_n e^{-\lambda_n z}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n a_n^m e^{-m\lambda_n z}\,. \tag{1}$$

Summing the terms of the double series (1) according to increasing values of $m\lambda_n$ results in a general Dirichlet series

$$\sum_{k=1}^{\infty} h_k e^{-t_k z}\,. \tag{2}$$

In order to establish the convergence of this last series, consider the double series in (1) taking the absolute value of the various terms. The n-th row converges for all $z=x+iy$ in $\mathfrak{D}$ to

$$\frac{|a_n b_n| e^{-\lambda_n x}}{1-|a_n| e^{-\lambda_n x}}.$$

The sum of the "row-sums" will then be

$$\sum |a_n b_n| \frac{e^{-\lambda_n x}}{1-|a_n| e^{-\lambda_n x}}$$

which converges for those z in $\mathfrak{D}$ for which $B(z)$ converges absolutely, provided the sequence $\{a_n\}$ is bounded; hence for such values of z the double series in (1) is absolutely convergent and its terms can be deranged in any manner without affecting convergence. The Dirichlet series is one such derangement. We have therefore

THEOREM 4. *In its region of absolute convergence in the half plane $\mathfrak{D}$ a B-series can be expressed as a Dirichlet series which converges absolutely and represents the same analytic function in that region, provided the sequence $\{a_n\}$ is bounded.*

Conversely we have

THEOREM 5. *A given Dirichlet series can be formally expressed as a B-series. In its region of absolute convergence in the half plane $\mathfrak{D}$ the resulting B-series represents the same analytic function as does the given Dirichlet series, provided the sequence $\{a_n\}$ of the B-series is bounded.*

Verification of this last theorem is again by use of the double series in (1).

4. A special B-series; its relation to ordinary Dirichlet series. If the sequence $\{\lambda_n\}$ of the B-series is taken to be $\{\ln n\}$ there results the ordinary B-series

$$\sum a_n b_n \frac{n^{-z}}{1-a_n n^{-z}}.$$

We shall develop a relationship between this series and ordinary Dirichlet series which is analogous to that developed in the last section between general B-series and general Dirichlet series; however in the present case we shall obtain explicit formulae for relating the coefficients of the two series.

As in the preceding section, continued division of each term of the ordinary B-series results in the double series

$$(3)\qquad \frac{b_1a_1}{1-a_1}+\sum_{m=1}^{\infty}\sum_{n=2}^{\infty}b_na_n^mn^{-mz}\,.$$

Summing the terms of (3) according to increasing values of n^m, $n=2, 3, 4, \cdots$; $m=1, 2, 3, \cdots$, we obtain an ordinary Dirichlet series

$$(4)\qquad \frac{b_1a_1}{1-a_1}+\sum_{k=2}^{\infty}h_kk^{-z}\,.$$

For a given $k>1$, $b_na_n^mn^{-mz}$ enters into the sum h_kk^{-z} if and only if $n^m=k$. Consequently the coefficient of k^{-z} for $k>1$ will be

$$h_k=\sum_{n^m=k}b_na_n^m\,,$$

where the summation is taken over all n such that for some positive integer m, $n^m=k$.

The problem of convergence of (4) will be the same as that of (2). Hence we have

THEOREM 6. *An ordinary B-series*

$$\sum a_nb_n\frac{n^{-z}}{1-a_nn^{-z}}$$

can be expressed as an ordinary Dirichlet series $\sum_{k=1}^{\infty}h_kk^{-z}$ *where* $h_1=a_1b_1/(1-a_1)$ *and for* $k>1$, $h_k=\sum_{n^m=k}b_na_n^m$. *In the region of absolute convergence of the B-series in the half plane* $\mathfrak{D}$ *the resulting Dirichlet series converges absolutely and represents the same analytic function in that region, provided the sequence* $\{a_n\}$ *is bounded.*

Before considering the converse problem, we shall introduce the following modification of Doyle's inversion function [1]. For positive integers t and n, the function $S(t, n)$ is defined recursively as follows:

$$S(1, n)=1, \quad \text{for all} \quad n$$
$$\sum{}' S(t, n)[a(n, v)]^{(v/t-1)}=0, \quad \text{for} \quad v>1$$

for each positive integer v such that $n=s^v$ for some positive integer s, where $\sum'$ indicates that the summation is to be taken over all positive integral divisors t of v, and $a(n, v)=a_{\sqrt[v]{n}}$

Suppose now that we are given an ordinary Dirichlet series $\sum_{k=1}^{\infty}h_kk^{-z}$ and that we are to determine its ordinary B-series representation. Again using the double array (3) as an intermediate step, we shall show that the coefficients for $n>1$ are given by

$$(5)\qquad b_na_n=\sum_{s^t=n}S(t, n)h_s\,,$$

where for a fixed n the summation extends over all positive integers s such that for some integer t, $s^t=n$.

In (5) replace h_s by its equivalent from Theorem 6; the right side of (5) becomes

$$\sum_{s^t=n} S(t, n) \sum_{k^d=s} b_k a_k^d \,. \tag{6}$$

For a fixed $n>1$, $b_m a_m$ appears in this last expression if and only if for some positive integers t and d, $m^d=s$ and $s^t=n$. The total coefficient of each such $a_m b_m$ that does appear will be

$$\sum_{t\mid v} S(t, n)[a(n, v)]^{(v/t-1)}$$

so that (6) can be rewritten

$$\sum_{m^v=n} a_m b_m \sum_{t\mid v} S(t, n)[a(n, v)]^{(v/t-1)} \,.$$

By definition of the S function, this inner sum is zero unless $v=1$ when it has the value one, in which case $m=n$.

THEOREM 7. *A given Dirichlet series* $\sum_{k=1}^{\infty} h_k k^{-z}$ *can be expressed as a B-series of the form*

$$\sum a_n b_n \frac{n^{-z}}{1-a_n n^{-z}} \,,$$

where $b_1 a_1/(1-a_1)=h_1$ *and for* $n>1$, $b_n a_n=\sum_{s^t=n} S(t, n)h_s$. *In its region of absolute convergence in the half plane* $\mathfrak{D}$, *the resulting B-series represents the same analytic function as does the given Dirichlet series, provided the sequence* $\{a_n\}$ *is bounded.*

As with Theorem 4, convergence is verified by use of the double series (3). This procedure does not yield a unique B-series, for only the sequence $\{a_n b_n\}$ is determined. If in a particular case one first chooses $\{a_n\}$, the above procedure determines the sequence $\{b_n\}$; or because of the recursive nature of $S(t, n)$, if $\{b_n\}$ is first chosen, the a_n are uniquely determined.

For example, if $\zeta(z)$ denotes the Riemann zeta function, then $1/\zeta(z)=\sum \mu(n)n^{-z}$ has as its equivalent B-series, where b_n is identically one,

$$1+\sum_{n=2}^{\infty} \frac{\{S(r, n)\mu(m)\}n^{-z}}{1-\{S(r, n)\mu(m)\}n^{-z}}$$
$$=1-\frac{2^{-z}}{1+2^{-z}}-\frac{3^{-z}}{1+3^{-z}}-\frac{4^{-z}}{1+4^{-z}}+\cdots$$

where μ denotes the Möbius function and for each n, m is the least positive integer such that $m^r=n$ for integral r.

5. Natural boundaries of the B-series. This last section will consider the existence of a natural boundary of the function represented by an ordinary *B*-series. An interesting special case of the ordinary *B*-series occurs when the sequence $\{a_n\}$ is determined as follows:

$$a_n = \begin{cases} 0, & \text{for } n=1 \\ n^k, & \text{for } n>1, \text{ where } k \text{ is some real constant.} \end{cases}$$

There results

$$\sum_{n=2}^{\infty} b_n \frac{n^{-z+k}}{1-n^{-z+k}}, \tag{7}$$

so that the natural boundary problem reduces to that of Kennedy's *R*-series [3]. Accordingly, the function represented by (7) will, under the requisite restrictions on the sequence $\{b_n\}$ as determined by Kennedy, have as a natural boundary the line $\mathrm{R}(z)=k$.

For the general case of the ordinary *B*-series, we shall adopt the method of Rust and Regan [4] to reduce the question of a natural boundary to that of Kennedy's *R*-series by expressing both as equivalent Dirichlet series.

THEOREM 8. *If the sequence $\{a_n\}$ is bounded, an ordinary B-series*

$$\sum_{n=2}^{\infty} a_n b_n \frac{n^{-z}}{1-a_n n^{-z}}$$

can, in its region of absolute convergence in the half plane $\mathfrak{D}$, be written as an ordinary R-series

$$\sum_{m=2}^{\infty} c_m \frac{m^{-z}}{1-m^{-z}}$$

where

$$c_m = \sum_{k^t=m} \mu(t) \sum_{n^v=k} b_n a_n^v$$

in which μ denotes the Möbius function.

Proof. By Theorem 6, in its region of absolute convergence in the half plane $\mathfrak{D}$, the series

$$\sum_{n=2}^{\infty} a_n b_n \frac{n^{-z}}{1-a_n n^{-z}}$$

can be expressed as a Dirichlet series $\sum_{k=2}^{\infty} h_k k^{-z}$ where $h_k = \sum_{n^v=k} b_n a_n^v$. If

$$\sum_{k=2}^{\infty} h_k k^{-z} = \sum_{k=2}^{\infty} \Big(\sum_{n^v=k} b_n a_n^v \Big) k^{-z}$$

be the series $\sum_{n=2}^{\infty} b_n n^{-z}$ in Kennedy's representation theorem [3; 448], then at least formally

$$\sum_{n=2}^{\infty} a_n b_n \frac{n^{-z}}{1-a_n n^{-z}} = \sum_{k=2}^{\infty} \Big(\sum_{n^v=k} b_n a_n^v \Big) k^{-z} = \sum_{m=2}^{\infty} c_m \frac{m^{-z}}{1-m^{-z}} \tag{8}$$

where

$$c_m = \sum_{k^t=m} \mu(t) \sum_{n^v=k} b_n a_n^v \,.$$

The proof of the convergence of the series on the right in (8) follows as in the proof of Theorem 4.

We conclude therefore that if the sequence $\{a_n\}$ is bounded and if the region of absolute convergence of the B-series in $\mathfrak{D}$ includes those values of z for which $0<\mathrm{R}(z)<c$ for some constant c, the problem of a natural boundary at the axis of imaginaries for the function represented by the ordinary B-series can be determined by means of the equivalent R-series.

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References

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