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Lommel type integrals involving products of three Bessel functions

by

P. L. BHATNAGAR

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1. Introduction. In a recent investigation* relating to the flow of a compressible fluid with a swirl in a circular cylindrical tube, the author had to evaluate a number of indefinite and difinite integrals involving the products of two or three Bessel functions. The evaluation of some of the indefinite integrals involving products of three Bessel functions is presented here in the form which is much more general than required in the afore-said problem for a ready reference. Earlier, Fettis [1] has evaluated the integral $I=\int_0^x tJ_0(\alpha t)J_0(\beta t)J_0(\gamma t)dt$ in terms of an infinite series of products of two Bessel functions of the same order, where α, β, γ are arbitrary constants. He has also deduced expressions for the integrals of the type $\int_0^x tJ_0(\alpha t)J_1(\beta t)J_1(\gamma t)dt$ and $\int_0^x J_1(\alpha t)J_1(\beta t)J_1(\gamma t)dt$ in terms of the above integral.

In § 2 we evaluate the integral $I_{nnn}(\alpha, \beta, \gamma; -n+1; x)$ and in § 3, the integrals (a) $I_{n-1nn}(\alpha, \beta, \gamma; -n+2; x)$, (b) $I_{n-1n-1n}(\alpha, \beta, \gamma; -n+3; x)$, (c) $I_{n-1n-1n-1}(\alpha, \beta, \gamma; -n+2; x)$ and (d) $I_{nnn}(\alpha, \beta, \gamma; -n+3; x)$ in terms of first integral, where

$$I_{pqr}(\alpha, \beta, \gamma; s, x)=\int_0^x t^s J_p(\alpha t)J_q(\beta t)J_r(\gamma t)dt .$$

In passing we mention that our restrictions on α, β, γ are such that we can obtain the values of the integrals involving the products of modified Bessel functions of first kind by simply substituting $i\alpha, i\beta, i\gamma$ for α, β, γ in the formulae obtained here.

In § 4 we evaluate the integral $\int_{a>0}^x t^{-n+1}J_n(\alpha t)J_n(\beta t)\,Y_n(\gamma t)dt$ and also some important integrals associated with it.

2. Evaluation of $I=I_{nnn}(\alpha, \beta, \gamma; -n+1; x)$. We use the Gegenbauer addition formula [2]

$$(2.1)\quad t^n R^{-n} J_n(Rt)=2^n(\alpha\beta)^{-n}\Gamma(n)\sum_{p=0}^{\infty}(n+p)C_p^n(\cos\theta)J_{n+p}(\alpha t)J_{n+p}(\beta t) ,$$

where

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$R^2=\alpha^2-2\alpha\beta\cos\theta+\beta^2$ and $C_p^n(\cos\theta)$ are Gegenbauer polynomials satisfying the following integral relation:

$$\int_0^\pi C_p^n(\cos\theta)(\sin\theta)^{2n}d\theta=\begin{cases}\dfrac{\pi}{2^{2n}}\dfrac{\Gamma(2n+1)}{[\Gamma(n+1)]^2}, & p=0\\ 0, & p=1,2,\cdots.\end{cases} \tag{2.2}$$

In view of (2.1) and (2.2), we have

$$t^{-n+1}J_n(\alpha t)J_n(\beta t)=\frac{(2\alpha\beta)^n}{\pi}\frac{\Gamma(n+1)}{\Gamma(2n+1)}t\int_0^\pi R^{-n}J_n(Rt)(\sin\theta)^{2n}d\theta \tag{2.3}$$

so that

$$I=\frac{(2\alpha\beta)^n}{\pi}\frac{\Gamma(n+1)}{\Gamma(2n+1)}\int_0^\pi R^{-n}(\sin\theta)^{2n}d\theta\int_0^x tJ_n(\gamma t)J_n(Rt)dt\,. \tag{2.4}$$

Using the Lommel integral [3]

$$(\lambda^2-\mu^2)\int_0^x tJ_n(\lambda t)J_n(\mu t)dt=\mu xJ_n'(\mu x)J_n(\lambda x)-\lambda xJ_n(\mu x)J_n'(\lambda x) \tag{2.5}$$

we can write

$$I=-x\gamma J_n'(\gamma x)G_1(x)+xJ_n(\gamma x)H_1(x)\,, \tag{2.6}$$

where

$$G_1(x)=\frac{(2\alpha\beta)^n}{\pi}\frac{\Gamma(n+1)}{\Gamma(2n+1)}\int_0^\pi\frac{(\sin\theta)^{2n}J_n(Rx)}{(\gamma^2-R^2)R^n}d\theta \tag{2.7}$$

and

$$H_1(x)=\frac{\partial G_1(x)}{\partial x} \tag{2.8}$$

provided that $\gamma^2\neq R^2$ in the range of integration $(0,\pi)$, i.e. if

$$\left|\frac{\gamma^2-\alpha^2-\beta^2}{2\alpha\beta}\right|>1 \qquad \text{(Case (i))}\,.$$

When

$$\left|\frac{\gamma^2-\alpha^2-\beta^2}{2\alpha\beta}\right|\leqq 1 \qquad \text{(Case (ii))}\,,$$

we modify the expression (2.6) and write

$$I=-\gamma xJ_n'(\gamma x)G_2(x)+xJ_n(\gamma x)H_2(x)\,, \tag{2.9}$$

where

$$G_2(x)=\frac{(2\alpha\beta)^n}{\pi}\frac{\Gamma(n+1)}{\Gamma(2n+1)}\int_0^\pi\frac{(\sin\theta)^{2n}}{R^n}\frac{J_n(Rx)-J_n(\gamma x)}{\gamma^2-R^2}d\theta \tag{2.10}$$

and

(2.11) $$H_2(x)=\frac{\partial G_2}{\partial x}\ .$$

Using (2.1) and the following expression [4] for $C_p^n(\cos\theta)$:

$$C_p^n(\cos\theta)=2\sum_{k=0}^{[p/2]}\frac{\Gamma(n+k)\Gamma(n+p-k)}{[\Gamma(n)]^2k!\,(p-k)!}\cos(p-2k)\theta\ ,$$

(2.12) $$G_1(x)=\frac{2^{2n}nx^{-n}}{\pi\Gamma(2n+1)}\sum_{p=0}^{\infty}(n+p)J_{n+p}(\alpha x)J_{n+p}(\beta x)\times\sum_{k=0}^{p}\frac{\Gamma(n+k)\Gamma(n+p-k)}{k!\,(p-k)!}I_k\ ,$$

where

(2.13) $$I_k=\int_0^{2\pi}\frac{(\sin\theta)^{2n}\cos(p-2k)\theta}{\gamma^2-R^2}d\theta=\frac{(-1)^n}{i\alpha\beta 2^{2n+1}}\int_{|z|=1}\frac{(z-z^{-1})^{2n}(z^{p-2k}+z^{-p+2k})}{(z-z_1)(z-z_2)}dz$$

where

$$z_1=-e^{-\phi},\ z^2=-e^{\phi},\ \cosh\phi=\frac{\gamma^2-\alpha^2-\beta^2}{2\alpha\beta},\quad \text{if}\quad |\gamma^2|>|\alpha^2+\beta^2|$$

$$z_1=e^{-\phi'},\ z_2=e^{\phi'},\ \cosh\phi'=\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta},\quad \text{if}\quad |\gamma^2|<|\alpha^2+\beta^2|$$

assuming that either α,β,γ are real or have the same argument so that ϕ and ϕ' are real.

The integrand in (2.13) has simple poles at z_1 and z_2 and a multiple pole at 0 of order $2n+p-2k$. Out of the two simple poles only z_1 lies within the contour $|z|=1$.

We can easily check that the residue R_1 at z_1 and R_2+R_3 at 0 are given by

$$R_1=(-1)^p2^{2n}(\sinh\phi)^{2n-1}\cosh(p-2k)\phi,\quad \text{when}\quad |\gamma^2|>|\alpha^2+\beta^2|$$
$$=(-1)2^{2n}(\sinh\phi')^{2n-1}\cosh(p-2k)\phi',\quad \text{when}\quad |\gamma^2|<|\alpha^2+\beta^2|$$

$$R_2=0,\ 1,\ C_0B_{2n-(p-2k+1)}+C_1B_{2n-(p-2k+3)}+\cdots+C_{q_1}B_1\quad \text{or}\quad C_{q_2}B_0\ ,$$

according as $2n<,=,>p-2k+1$, with

$$C_q=(-1)^q\binom{2n}{q},\ B_q=(-1)^q\frac{\sinh(q+1)\phi}{\sinh\phi}\quad \text{or}\quad \frac{\sinh(q+1)\phi'}{\sinh\phi'}$$

and

$$q_1=n+k-\frac{1}{2}(p+2),\ q_2=n+k-\frac{1}{2}(p+1)\ ,$$

according as p is even or odd,

$$R_3=0,\ 1,\ C_0B_{2n+p-2k-1}+C_1B_{2n+p-2k-3}+\cdots+C_{q_3}B_1 \text{ or } C_{q_4}B_0\ ,$$

according as $2n+p-2k$ less than, equal to, or greater than 1 with

$$q_3=n-k+\frac{1}{2}(p-2),\ q_4=n-k+\frac{1}{2}(p-1)$$

according as p is even or odd. Hence

$$I_k=\frac{(-1)^n\pi}{\alpha\beta 2^{2n}}(R_1+R_2+R_3)$$

and

$$G_1(x)=\frac{(-1)^n n x^{-n}}{\alpha\beta\Gamma(2n+1)}\sum_{p=0}^{\infty}(n+p)J_{n+p}(\alpha x)J_{n+p}(\beta x)\times\sum_{k=0}^{[p/2]}\frac{\Gamma(n+k)\Gamma(n+p-k)}{k!\,(p-k)!}(R_1+R_2+R_3)\ . \tag{2.14}$$

From (2.14) we can get the value of $H_1(x)$ according to the definition (2.8). We shall now evaluate $G_2(x)$ and $H_2(x)$. When $|(\alpha^2+\beta^2-\gamma^2)/2\alpha\beta|\leqq 1$, we write

$$\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}=\cos\phi$$

so that

$$\gamma^2=\alpha^2+\beta^2-2\alpha\beta\cos\phi\ ,$$

and $J_n(\gamma x)$ can also be expanded according to Gegenbauer addition formula. Thus

$$\mathrm{G}_2(x)=\frac{2^{2n-1}\Gamma(n)\Gamma(n+1)x^{-n}}{\pi\alpha\beta\Gamma(2n+1)}\sum_{p=0}^{\infty}(n+p)J_{n+p}(\alpha x)J_{n+p}(\beta x)I'_p\ , \tag{2.15}$$

where

$$I'_p=\int_0^{\pi}(\sin\theta)^{2n}\frac{C_p^n(\cos\theta)-C_p^n(\cos\phi)}{\cos\theta-\cos\phi}d\theta$$

$$=2\sum_{k=0}^{[p/2]}\frac{\Gamma(n+k)\Gamma(n+p-k)}{[\Gamma(n)]^2k!\,(p-k)!}I(p-2k)$$

so that finally

$$G_2(x)=\frac{2^{2n}nx^{-n}}{\pi\alpha\beta\Gamma(2n+1)}\sum_{p=0}^{\infty}(n+p)J_{n+p}(\alpha x)J_{n+p}(\beta x)\times\sum_{k=0}^{[p/2]}\frac{\Gamma(n+k)\Gamma(n+p-k)}{k!\,(p-k)!}I(p-2k) \tag{2.16}$$

where

(2.17)
$$I(k)=\int_0^{\pi}(\sin\theta)^{2n}\frac{\cos k\theta-\cos k\phi}{\cos\theta-\cos\phi}d\theta$$

We shall now evaluate $I(k)$ using the following expression

$$\cos k\alpha=\sum_{s=0}^{S}(-1)^s\frac{k(k-s-1)!}{s!\,(k-2s)!}2^{k-2s-1}(\cos\alpha)^{k-2s}\,,$$

where

$$S=\left[\frac{k}{2}\right].$$

If we substitute

$$\frac{\cos k\theta-\cos k\phi}{\cos\theta-\cos\phi}=\sum_{s=0}^{S}(-1)^s\frac{k(k-s-1)!}{s!\,(k-2s)!}2^{k-2s-1}\sum_{p=0}^{k-2s-1}(\cos\phi)^{k-2s-p-1}(\cos\theta)^p$$

and use the following formula

$$\int_0^{\pi}(\sin\theta)^{2n}(\cos\theta)^p d\theta=\begin{cases}0, & \text{if } p \text{ is odd}\\ \dfrac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{2n+2+p}{2}\right)}, & \text{if } p \text{ is even}\end{cases}$$

in (2.17), we can show that

$$I(k)=A_n\sum_{s=0}^{S}\frac{(-1)^s k(k-s-1)!}{s!\,(k-2s)!}(2\cos\phi)^{k-2s-1}\sum_{p=0}^{P}(2\cos\phi)^{-2p}\frac{(2p)!}{p!\,(n+1)_p}\,,$$

where

$$A_n=\frac{\pi^{1/2}\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}$$

and

$$P=\left[\frac{k-1}{2}\right]-s\,.$$

Using the relation

$$\frac{\sin kx}{\sin x}=\sum_{q=0}^{Q}(-1)^q(2\cos x)^{k-2q-1}\binom{k-q-1}{q}\,,$$

where

$$Q=\left[\frac{k-1}{2}\right]$$

and simplifying considerably, we have

$$I(k)=\frac{\pi^{1/2}\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}\left[\frac{\sin k\phi}{\sin\phi}\right. \tag{2.18}$$

$$\left.+\sum_{q=1}^{Q}\left\{d_q-(-1)^q\binom{k-q-1}{q}\right\}(2\cos\phi)^{k-2q-1}\right],$$

where

$$d_q=\frac{(2q)!}{q!\,(n+1)_q}+(-1)\frac{k(k-2)!}{1!\,(k-2)!}\,\frac{\{2(q-1)\}!}{(q-1)!\,(n+1)_{q-1}}$$

$$+(-)1^2\frac{k(k-3)!}{2!\,(k-4)!}\,\frac{\{2(q-2)\}!}{(q-2)!\,(n+1)_{q-2}}+\cdots$$

$$+(-1)^{q-1}\frac{k(k-q)!}{(q-1)!\,(k-2q-2)!}\,\frac{2!}{1!\,(n+1)}+(-1)^q\frac{k(k-q-1)!}{q!\,(k-2q)!}\,.$$

For ready reference we note below some particular cases of (2.18):

$$\left\{\begin{aligned}
&I(2)=A_n\frac{\sin 2\phi}{\sin\phi}\,,\quad I(3)=A_n\left[\frac{\sin 3\phi}{\sin\phi}-\frac{2n}{n+1}\right]\\
&I(4)=A_n\left[\frac{\sin 4\phi}{\sin\phi}-\frac{4n}{n+1}\cos\phi\right],\\
&I(5)=A_n\left[\frac{\sin 5\phi}{\sin\phi}-\frac{8n}{n+1}\cos^2\phi+\frac{2n(2n+1)}{(n+1)(n+2)}\right],\\
&I(6)=A_n\left[\frac{\sin 6\phi}{\sin\phi}-\frac{16n}{n+1}\cos^3\phi+\frac{12n}{n+2}\cos\phi\right]\\
&I(7)=A_n\left[\frac{\sin 7\phi}{\sin\phi}-\frac{32n}{n+1}\cos^4\phi+\frac{8n(4n+5)}{(n+1)(n+2)}\cos^2\phi\right.\\
&\qquad\left.-\frac{2n(3n^2+4n+5)}{(n+1)(n+2)(n+3)}\right]
\end{aligned}\right. \tag{2.19}$$

and

$$I(k)=\pi\frac{\sin k\phi}{\sin\phi}\,,\quad \text{when}\quad n=0\,.$$

The last integral in (2.19) is previously recorded [5].

When $n=0$, we obtain as a particular case the result established by Fettis [1]. This completes the evaluation of $I_{nnn}(\alpha,\beta,\gamma;-n+1;x)$.

3. Evaluation of some related integrals.

(a) We write

$$I_{n-1nn}(\alpha,\beta,\gamma,-n+2;x)=\int_0^x[t^{-(n-1)}J_{n-1}(\alpha t)][tJ_n(\beta t)J_n(\gamma t)]dt \tag{3.1}$$

and integrate by parts with the help of (2.5) according to the scheme indicated by the square brackets. Then, on simplifying with the help of the recurrence relations, we get

(3.2)
$$(\beta^2-\gamma^2)I_{n-1nn}+\alpha\beta I_{nn-1n}-\alpha\gamma I_{nnn-1}$$
$$=x^{-n+2}J_{n-1}(\alpha x)[\gamma J_n(\beta x)J_{n-1}(\gamma x)-\beta J_{n-1}(\beta x)J_n(\gamma x)]\ .$$

There are two more similar relations obtained by interchanging α, γ keeping β fixed and interchanging α, β keeping γ fixed, but these three relations are not linearly independent. Therefore, to evaluate the three integrals occuring on the left hand side of (3.2) we must obtain at least one more relation between them. We now write

$$I_{n-1n-1n-1}(\alpha,\ \beta,\ \gamma;\ -n+2;\ x)=\int_0^x[t^{-(n-1)}J_{n-1}(\beta t)][tJ_{n-1}(\alpha t)J_{n-1}(\gamma t)]dt$$

and integrate it with the help of (2.5) by parts according to the scheme indicated by the square brackets. We thus get

(3.3)
$$\alpha\beta I_{nnn-1}-\gamma\beta I_{n-1nn}=x^{-n+2}J_{n-1}(\beta x)[\gamma J_{n-1}(\alpha x)J_n(\gamma x)$$
$$-\alpha J_n(\alpha x)J_{n-1}(\gamma x)]+(\alpha^2-\gamma^2)I_{n-1n-1n-1}\ .$$

We note that in writing the integrals in (3.2) and (3.3) we have ignored the arguments as they are the same for all of them. Out of two more similar relations that can be symmetrically written we shall use the one that is obtained by interchanging β, γ keeping α fixed, namely

(3.4)
$$\alpha\gamma I_{nn-1n}-\beta\gamma I_{n-1nn}=x^{-n+2}J_{n-1}(\gamma x)[\beta J_{n-1}(\alpha x)J_n(\beta x)$$
$$-\alpha J_n(\alpha x)J_{n-1}(\beta x)]+(\alpha^2-\beta^2)I_{n-1n-1n-1}\ .$$

We can easily check by writing the determinant of the coefficients of the unknown integrals that (3.2), (3.3), and (3.4) form a linearly independent system of equation. Solving these equations we get

(3.5)
$$2\beta\gamma I_{n-1nn}=x^{-n+2}[\alpha J_n(\alpha x)J_{n-1}(\beta x)J_{n-1}(\gamma x)$$
$$-\beta J_{n-1}(\alpha x)J_n(\beta x)J_{n-1}(\gamma x)$$
$$-\gamma J_{n-1}(\alpha x)J_{n-1}(\beta x)J_n(\gamma x)]+(\beta^2+\gamma^2-\alpha^2)I_{n-1n-1n-1}$$

together with similar expressions for the remaining two integrals of this type.

(b) We shall now evaluate $I_{n-1n-1n-1}$ in terms of I.

We integrate

$$I=\int_0^x[t^{-n+1}][J_n(\alpha t)J_n(\beta t)J_n(\gamma t)]dt$$

by parts to get

(3.6) $$(4n-2)I=-x^{-n+2}J_n(\alpha x)J_n(\beta x)J_n(\gamma x)+\alpha I_{n-1nn}+\beta I_{nn-1n}+\gamma I_{nnn-1}$$

Substituting the values of the three integrals on the right hand side of (3.6) from (3.5), we have

(3.7)
$$(2\alpha^2\beta^2+2\beta^2\gamma^2+2\gamma^2\alpha^2-\alpha^4-\beta^4-\gamma^4)I_{n-1n-1n-1}$$
$$=2(4n-2)\alpha\beta\gamma I+x^{-n+2}[2\alpha\beta\gamma J_n(\alpha x)J_n(\beta x)J_n(\gamma x)$$
$$-\alpha(\alpha^2-\beta^2-\gamma^2)J_n(\alpha x)J_{n-1}(\beta x)J_{n-1}(\gamma x)$$
$$-\beta(\beta^2-\alpha^2-\gamma^2)J_{n-1}(\alpha x)J_n(\beta x)J_{n-1}(\gamma x)$$
$$-\gamma(\gamma^2-\beta^2-\alpha^2)J_{n-1}(\alpha x)J_{n-1}(\beta x)J_n(\gamma x)]\ .$$

(c) On integrating by parts

$$I_{n-1nn-1}(\alpha,\ \beta,\ \gamma;\ -n+3;\ x)=\int_0^x[t^{-n+2}J_n(\beta t)][tJ_{n-1}(\alpha t)J_{n-1}(\gamma t)]dt$$

we have

(3.8)
$$(\gamma^2-\alpha^2)I_{n-1nn-1}+\beta\alpha I_{nn-1n-1}-\beta\gamma I_{n-1n-1n}$$
$$=x^{-n+3}J_n(\beta x)[\gamma J_n(\gamma x)J_{n-1}(\alpha x)-\alpha J_{n-1}(\gamma x)J_n(\alpha x)]$$
$$+(-2n+2)\alpha I_{nnn-1}-(-2n+2)\gamma I_{n-1nn}\ ,$$

where in writing the integrals we have suppressed the arguments as they are the same as indicated in (a), (b), and (c). Interchanging β,γ keeping α fixed in (3.8)

(3.9)
$$(\beta^2-\alpha^2)I_{n-1n-1n}+\gamma\alpha I_{nn-1n-1}-\beta\gamma I_{n-1nn-1}$$
$$=x^{-n+3}J_n(\gamma x)[\beta J_n(\beta x)J_{n-1}(\alpha x)-\alpha J_{n-1}(\beta x)J_n(\alpha x)]$$
$$+(-2n+2)\alpha I_{nn-1n}-(-2n+2)\beta I_{n-1nn}\ .$$

To obtain another relation between the three integrals occuring on the left hand side of (3.8) or (3.9), we integrate by parts

$$I_{n-1nn}=\int_0^x[t^{-n+2}][J_{n-1}(\alpha t)J_n(\beta t)J_n(\gamma t)]dt$$

to get

(3.10)
$$(-2n+2)I_{n-1nn}=x^{-n+3}J_{n-1}(\alpha x)J_n(\beta x)J_n(\gamma x)-\beta I_{n-1n-1n}-\gamma I_{n-1nn-1}$$
$$+\alpha I_{nnn}(\alpha,\ \beta,\ \gamma;\ -n+3;\ x)$$

By symmetry we also have the following relation

(3.11)
$$(-2n+2)I_{nn-1n}=x^{-n+3}J_n(\alpha x)J_{n-1}(\beta x)J_n(\gamma x)-\alpha I_{n-1n-1n}-\gamma I_{nn-1n-1}$$
$$+\beta I_{nnn}(\alpha,\ \beta,\ \gamma;\ -n+3;\ x)\ .$$

Eliminating I_{nnn} between (3.10) and (3.11), we have

(3.12)
$$\alpha\gamma I_{nn-1n-1}-\beta\gamma I_{n-1nn-1}+(\alpha^2-\beta^2)I_{n-1n-1n}$$
$$=(-2n+2)\beta I_{n-1nn}-(-2n+2)\alpha I_{nn-1n}$$
$$+x^{-n+3}J_n(\gamma x)[\alpha J_n(\alpha x)J_{n-1}(\beta x)-\beta J_{n-1}(\alpha x)J_n(\beta x)]\ .$$

By symmetry the following relation should also hold

(3.13)
$$\alpha\beta I_{nn-1n-1}-\beta\gamma I_{n-1n-1n}+(\alpha^2-\gamma^2)I_{n-1nn-1}$$
$$=(-2n+2)\gamma I_{n-1nn}-(-2n+2)\alpha I_{nnn-1}$$
$$+x^{-n+3}J_n(\beta x)[\alpha J_n(\alpha x)J_{n-1}(\gamma x)-\gamma J_{n-1}(\alpha x)J_n(\gamma x)]\ .$$

From (3.8), (3.9), (3.12), and (3.13) we have

$$\alpha I_{nn-1n-1}=\beta I_{n-1nn-1}=\gamma I_{n-1n-1n}=\lambda(\text{say}) . \tag{3.14}$$

Substituting from (3.14) for the integrals in terms λ in (3.12), and (3.13) and the remaining third equation similar to them we get the values of $I_{n-1n-1n}$, $I_{n-1nn-1}$ and $I_{nn-1n-1}$ respectively from which we construct the following value of λ which remains invariant to operations of interchanging any two of α, β, γ:

$$\begin{aligned}\lambda=\frac{x^{-n+3}}{2}\Big[&\frac{1}{\gamma^2-\alpha^2}\{\beta\gamma J_{n-1}(\alpha x)J_{n-2}(\beta x)J_n(\gamma x)\\ &-\beta\alpha J_n(\alpha x)J_{n-2}(\beta x)J_{n-1}(\gamma x)\}+\frac{1}{\alpha^2-\beta^2}\{\alpha\gamma J_n(\alpha x)J_{n-1}(\beta x)J_{n-2}(\gamma x)\\ &-\gamma\beta J_{n-1}(\alpha x)J_n(\beta x)J_{n-2}(\gamma x)\}+\frac{1}{\beta^2-\gamma^2}\{\beta\alpha J_{n-2}(\alpha x)J_n(\beta x)J_{n-1}(\gamma x)\\ &-\gamma\alpha J_{n-2}(\alpha x)J_{n-1}(\beta x)J_n(\gamma x)\}\Big]-(-2n+2)I_{n-1n-1n-1} ,\end{aligned} \tag{3.15}$$

and then (3.14) gives the values of the integrals $I_{nn-1n-1}$, $I_{n-1nn-1}$, and $I_{n-1n-1n}$.

(d) To obtain a symmetrical expression for $I_{nnn}(\alpha, \beta, \gamma; -n+3; x)$ we write all three relations similar to (3.10) and add them. Thus

$$\begin{aligned}&3\alpha\beta\gamma I_{nnn}(\alpha, \beta, \gamma; -n+3; x)\\ &=2(\alpha^2+\beta^2+\gamma^2)\lambda-x^{-n+3}[\beta\gamma J_{n-1}(\alpha x)J_n(\beta x)J_n(\gamma x)\\ &+\gamma\alpha J_n(\alpha x)J_{n-1}(\beta x)J_n(\gamma x)+\alpha\beta J_n(\alpha x)J_n(\beta x)J_{n-1}(\gamma x)]\\ &+(-2n+2)[\beta\gamma I_{n-1nn}+\gamma\alpha I_{nn-1n}+\alpha\beta I_{nnn-1}] ,\end{aligned} \tag{3.16}$$

where from (3.5) the value of the expression within the last square brackets is

$$\begin{aligned}-\frac{1}{2}x^{-n+2}[&\alpha J_n(\alpha x)J_{n-1}(\beta x)J_{n-1}(\gamma x)+\beta J_{n-1}(\alpha x)J_n(\beta x)J_{n-1}(\gamma x)\\ &+\gamma J_{n-1}(\alpha x)J_{n-1}(\beta x)J_n(\gamma x)]+\frac{1}{2}(\alpha^2+\beta^2+\gamma^2)I_{n-1n-1n-1} .\end{aligned} \tag{3.17}$$

4. Evaluation of $\bar{I}=\int_{a>0}^{x} t^{-n+1}J_n(\alpha t)J_n(\beta t)Y_n(\gamma t)dt$.

From (2.3), we have

$$\begin{aligned}\bar{I}&=\frac{(2\alpha\beta)^n}{\pi}\frac{\Gamma(n+1)}{\Gamma(2n+1)}\int_0^{\pi}\frac{d\theta(\sin\theta)^{2n}}{R^n}\int_a^x tJ_n(Rt)Y_n(\gamma t)dt\\ &=[-\gamma tY_n'(\gamma t)G_1(t)+tY_n(\gamma t)H_1(t)]_a^x ,\end{aligned} \tag{4.1}$$

where $G_1(t)$ and $H_1(t)$ are defined in (2.7) and (2.8) respectively. In deducing (4.1) we have used the well-known Lommel integral

$$(4.2)\quad (\gamma^2-R^2)\int_a^x tJ_n(Rt)\,Y_n(\gamma t)dt=[\gamma tJ_n'(Rt)\,Y_{n+1}(\gamma t)-RtJ_{n+1}(Rt)\,Y_n(\gamma t)]_a^x$$
$$=[tR\,Y_n(\gamma t)J_n'(Rt)-t\gamma J_n(Rt)\,Y_n'(\gamma t)]_a^x$$

and assumed that $\gamma^2\neq R^2$ in the range of integration, i.e., $|(\gamma^2-\alpha^2-\beta^2)/2\alpha\beta|>1$. We have already evaluated the values of $G_1(t)$ and $H_1(t)$ in § 2.

When $|(\gamma^2-\alpha^2-\beta^2)/2\alpha\beta|\leqq 1$, we modify (4.1) using the relation

$$(4.3)\quad R\,Y_n(\gamma t)J_n'(Rt)-\gamma\,Y_n'(\gamma t)J_n(Rt)$$
$$=Y_n(\gamma t)[RJ_n'(Rt)-\gamma J_n'(\gamma t)]-\gamma\,Y_n'(\gamma t)[J_n(Rt)-J_n(\gamma t)]-\frac{2}{\pi t}$$

obtained from the Wronskian relation

$$J_{\nu+1}(z)\,Y_\nu(z)-J_\nu(z)\,Y_{\nu+1}(z)=\frac{2}{\pi z}$$

written in the form

$$(4.4)\quad J_\nu(z)\,Y_\nu'(z)-J_\nu'(z)\,Y_\nu(z)=\frac{2}{\pi z}\,.$$

Thus when $|(\gamma^2-\alpha^2-\beta^2)/2\alpha\beta|\leqq 1$

$$(4.5)\quad \bar{I}=\left[-\gamma t\,Y_n'(\gamma t)G_2(t)+t\,Y_n(\gamma t)H_2(t)-\frac{2}{\pi}K\right]_a^x,$$

where $G_2(t)$ and $H_2(t)$ are defined in (2.10) and (2.11) respectively and evaluated in § 2, and

$$(4.6)\quad K=K(\alpha,\beta,\gamma,n)=\frac{(2\alpha\beta)^n}{\pi}\,\frac{\Gamma(n+1)}{\Gamma(2n+1)}\int_0^\pi\frac{(\sin\theta)^{2n}d\theta}{R^n(\gamma^2-R^2)}$$

is a constant depending on α,β,γ and n. When we substitute the limits $t=a$ and $t=x$ in (4.5), the term containing K does not make any contribution, hence it is not necessary to evaluate it. In fact we can write

$$(4.7)\quad \bar{I}=[-\gamma t\,Y_n'(\gamma t)G_2(t)+t\,Y_n(\gamma t)H_2(t)]_a^x\,.$$

This completes the evaluation of $\bar{I}$.

4.1. We shall now record below the expressions for some of the integrals which can be evaluted in terms of $\bar{I}$ without proof which proceeds on the lines indicated in § 3.

$$(a)\quad 2\beta\gamma\int_a^x t^{-n+1}J_n(\alpha t)J_{n+1}(\beta t)\,Y_{n+1}(\gamma t)dt$$
$$=[t^{-n+1}\{\alpha J_{n+1}(\alpha t)J_n(\beta t)\,Y_n(\gamma t)-\beta J_n(\alpha t)J_{n+1}(\beta t)\,Y_n(\gamma t)$$
$$-\gamma J_n(\alpha t)J_n(\beta t)\,Y_{n+1}(\gamma t)\}]_a^x+(\gamma^2+\beta^2-\alpha^2)\bar{I}\,,$$

along with two similar results obtained by interchanging α, β keeping γ fixed and interchanging α, γ keeping β fixed.

(b)
$$
\begin{aligned}
&4(2n+1)\alpha\beta\gamma\int_a^x t^{-n}J_{n+1}(\alpha t)J_{n+1}(\beta t)Y_{n+1}(\gamma t)dt \\
&= [t^{-n+1}\{\alpha(\alpha^2-\beta^2-\gamma^2)J_{n+1}(\alpha t)J_n(\beta t)Y_n(\gamma t) \\
&\quad +\beta(\beta^2-\gamma^2-\alpha^2)J_n(\alpha t)J_{n+1}(\beta t)Y_n(\gamma t) \\
&\quad +\gamma(\gamma^2-\alpha^2-\beta^2)J_n(\alpha t)J_n(\beta t)Y_{n+1}(\gamma t)\}]_a^x \\
&\quad +(2\alpha^2\beta^2+2\beta^2\gamma^2+2\gamma^2\alpha^2-\alpha^4-\beta^4-\gamma^4)\bar{I}\ .
\end{aligned}
$$

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Department of Applied Mathematics
University of Waterloo*

* On leave from Himachal Pradesh University, Simla, India.