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The existence of indiscernibles which do not spread and their undecidabilities

by

YUKIHIRO NISHIKAWA

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It is well known that *every* $\sum_2^1$ *or* $\prod_2^1$ *predicate* is absolute, which is due to J. R. Shoenfield. Under the hypothesis of the existence of some strong infinite cardinal (e.g. $\exists\kappa(\kappa\rightarrow(\aleph_1)_2^{<\omega})$), R. M. Solovay [3] and J. Silver [2] proved that there exists a Δ_3^1 definable real number *not in* L, where L is Goenel's construtive universe. This real is denoted by $0^\#$ [3].

In this paper we discuss about the cardinal of sets of $(0^\#)^M$ for countable transitive model M of $\exists X(X=0^\#)$.

We always treat countable transitive models of ZFC which satisfy $\exists X(X=0^\#)$. These are denoted by letters M, $N\cdots$.

In what follows $\Gamma(X, \alpha)$ has the same meaning as in [3]. And by $(A)^M$ we shall represent the relativization of A with respect to M.

We define the upper bound of X by the minimum countable ordinal α for which $\Gamma(X, \alpha)$ is not well founded.

In this paper we shall prove the following theorems.

THEOREM 1. *If* $\exists X(X=0^\#)$ *and* $\exists\kappa(\kappa$ *is inaccessible), then for every countable ordinal* α *we have a countable transitive model* M *such that* $\exists X(X=0^\#)$, $(0^\#)^M\neq 0^\#$ *and* $(OR)^M>\alpha$, *where* OR *is the class of all ordinals. And for such* M *there exists the upper bound of* $(0^\#)^M$.

COROLLARY. *If* $\exists X(X=0^\#)$ $\exists\kappa(\kappa$ *is inaccessible), then* Card$\{(0^\#)^M$; M *is a countable transitive model of* $\exists X(X=0^\#)\}\geqq\aleph_1$.

THEOREM 2. *If* $\exists\kappa(\kappa\rightarrow(\aleph_1)_2^{<\omega})$ *and Martin's axiom hold, then* Card$\{(0^\#)^M$; M *is a countable transitive model of* $\exists X(X=0^\#)\}=2^{\aleph_0}$.

THEOREM 3. *If* $\exists\kappa(\kappa\rightarrow(\aleph_1)_2^{<\omega})$ *and Martin's axiom hold, then* Card$\{(0^\#)^M$; M *is a countable transitive model of* $\exists X(X=0^\#)$ *such that* $\alpha<(OR)^M$ *and (the upper bound of* $(0^\#)^M)<(\alpha^+)^L\}=2^{\aleph_0}$, *for* α *such that* $(\aleph_1)^L<\alpha<\aleph_1$, *where* α^+ *is the minimum cadinal greater than* α.

§ 1. Preliminaries

In this and next sections we assume that $\exists X(X=0^\#)$ and $\exists\kappa(\kappa$ is inaccessible). We shall prove some Lemmas.

LEMMA 1. *For every countable ordinal α, there exists a countable transitive model M of $\exists X(X=0^{\#})$ such that $(OR)^{M}>\alpha$.*

Proof. Let κ be an inaccessible cardinal. Then $\langle R(\kappa), \varepsilon\rangle \models \exists X(X=0^{\#})$ holds, where $R(\kappa)$ is the set, the rank of which is less than κ. Let N be a countable elementary substructure of $\langle R(\kappa), \varepsilon\rangle$ such that $\alpha \cup \{\alpha\} \subseteqq |N|$. And M be the transitive model isomorphic to N. Then M is the required.

LEMMA 2 (J. R. Shoenfield). *Every $\sum_2^1$ or $\prod_2^1$ predicate is absolute.*

Proof. See [1].

LEMMA 3. *Let $P=\langle |P|, \langle p\rangle$ be a partially orderd set in L. Then, for every $G(\subseteqq |P|)$ generic over L, we have $0^{\#} \notin L[G]$.*

Proof. Assume that $0^{\#} \in L[G]$. Then for some α $0^{\#} \in L_\alpha[G]$. We take γ such that $\langle L_\gamma, \varepsilon\rangle$ is isomorphic to $\Gamma(0^{\#}, \alpha+\omega)$. Now we have $0^{\#}$, $\alpha+\omega \in L_\gamma[G]$ and, on the other hand $L_\gamma[G] \models ZFC$, because G is generic over L_γ.

Thus we can perfom the construction of $\Gamma(0^{\#}, \alpha+\omega)$ and so construct the transitive model isomorphic to $\Gamma(0^{\#}, \alpha+\omega)$ within $L_\gamma[G]$. So $L_\gamma \in L_\gamma[G]$. This is a contradiction.

LEMMA 4. *Let $P=\langle |P|, \langle p\rangle$ be a partially ordered set in $L_{\aleph_1}$. Then there exists $G(\subseteqq |P|)$ generic over L.*

Proof. The set of dense subsets of $|P|$ in L is countable, because the power set of $|P|$ in L is countable (cf. [2] or [3]).

LEMMA 5. *If $\exists\kappa(\kappa\rightarrow(\aleph_1)_2^{<\omega})$ holds, then* $\mathrm{Card}(P(\alpha)\cap L[X]) \leqq \aleph_0$, *where $X \subseteqq L_\alpha$ and $\alpha<\aleph_1$.*

Proof. See [2] or [3].

LEMMA 6. *If G is generic over L and G' is generic over $L[G]$, then $L[G]\cap L[G']=L$.*

Proof. See Lemma 1.2.5. of [4].

Next we define the following two predicates. $P(M, R) \equiv M= \langle\omega, \varepsilon_M, =_M\rangle \wedge \varepsilon_M \subseteqq \omega^2_\Lambda =_M \subseteqq \omega^2 \wedge M \models ZFC \cup \{\exists X(X=0^{\#})\} \wedge \exists F[(F$ is a total function from ω to $\omega) \wedge \forall x(M \models \mathrm{ord}(x) \rightarrow \exists y(x=F(y))) \wedge \forall xy(\langle xy\rangle \in R \leftrightarrow M \models (\mathrm{ord}(F(x)) \wedge \mathrm{ord}(F(y)) \wedge F(x)<F(x)))]$, which is a predicate. $Q(M, R) \equiv M=\langle\omega, \varepsilon_M, =_M\rangle \wedge \varepsilon_M \subseteqq \omega^2_\Lambda =_M \subseteqq \omega^2 \wedge M \models ZFC \cup \{\exists X(X=0^{\#})\} \wedge (\varepsilon_M$ is a well founded relation) $\wedge \exists F[(F$ is a total function from ω to $\omega) \wedge \forall xy(\langle xy\rangle \in R \leftrightarrow M \models (\mathrm{ord}(F(x)) \wedge \mathrm{ord}(F(y)) \wedge F(x)<F(y))]$, which is a predicate.

And we define a partially orderd set $P_\alpha=\langle |P_\alpha|, \langle p_\alpha\rangle$ for a countable ordinal α by the followings; $|P_\alpha|$ is the set of finite functions $\omega\rightarrow\alpha$ and $<_{P_\alpha}$ means the usual inclusion.

Every generic set $G(\subseteqq|P_\alpha|)$ is a total and surjective mapping [4]. And if α is an ordinal and G is a totol function $\omega\to\alpha$, then we define the binary reration R_G on ω by $\langle xy\rangle\in R_G\equiv G(x)<G(y)$.

§ 2. Proofs of Theorems

Proof of THEOREM 1. Let α be an arbitary countable ordinal. We can take a countable transitive model N of $\exists X(X=0^{\#})$ such that $\alpha<(OR)^N$ by Lemma 1. Let N be a model isomorphic to N such that $|N|=\omega$. And we can take $G(\subseteqq|P_{(OR)^N}|)$ generic over L by Lemma 4.

Now we have $P(N, R_G)$ by the definition of P and construction of N. Then $\exists MP(M, R_G)$. And we have $\exists MP(M, R_G)\equiv L[G]\models\exists MP(M, R_G)$ by Lemma 2. Thus we have a model M in $L[G]$ such that $L[G]\models P(M, R_G)$. For such M we have the transitive model $\boldsymbol{M}$ isomorpic to M.

Then $(0^{\#})^{\boldsymbol{M}}\neq 0^{\#}$ by Lemma 3. And the existence of the upper bound $(0^{\#})^{\boldsymbol{M}}$ is an immediate conclusion of the uniqueness of $0^{\#}$(cf. [3]). Thus proof of Theorem 1 is completed.

And Corollary is an immdiate conclusion of Theorem 1.

Proof of THEOREM 2. Let α be a countable ordinal such that $(\aleph_1)^L<\alpha$ and $\alpha=(OR)^N$, for some countable transitive model N of $\exists X(X=0^{\#})$, the existence of which is assured by Lemma 1.

And we put $X=\{(0^{\#})^{\boldsymbol{M}};\ \boldsymbol{M}$ is a countable transitive model of $\exists X(X=0^{\#})$ such that $\boldsymbol{M}\in L[G]$ for some $G(\subseteqq|P_\alpha|)$ generic over $L\}$.

In order to prove Theorem 2, it is sufficient to show $\overline{\overline{X}}=2^{\aleph_0}$. Now assume $\overline{\overline{X}}<2^{\aleph_0}$.

Let $\{(0^{\#})^{\boldsymbol{M}_\gamma}\}_{\gamma<\beta}$ be an enumeration of X, where $\beta<2^{\aleph_0}$, $\boldsymbol{M}_\gamma\in L[G_\gamma]$ $(\gamma<\beta)$ and $G(\subseteqq|P_\alpha|)$ is generic over $L(\gamma<\beta)$. Let F_γ be the set dense subsets of $|P_\alpha|$ in $L[G_\gamma]$. From $\exists\kappa(\kappa\to(\aleph_1)_2^{<\omega})$ $\overline{\overline{F}}_\gamma\leqq\aleph_0$ by Lemma 5. Then $\bigcup_{\gamma<\beta}\overline{\overline{F}}_\gamma<2^{\aleph_0}$. By Martin's axiom there is $\bigcup_{\gamma<\beta}F_\gamma$-generic set $G(\subseteqq|P_\alpha|)$, which is generic over $L[G_\gamma]$ for all $\gamma(\gamma<\beta)$.

Let N be a model isomorphic to N such that $|N|=\omega$. We have have $P(N, R_G)$ and so $\exists MP(M, R_G)$. We have $\exists MP(M, R_G)\equiv L[G]\models \exists MP(M, R_G)$ by Lemma 2. Then we have $L[G]\models\exists MP(M, R_G)$. Then we have a model M in $L[G|$ such that $L[G]\models P(M, R_G)$. and let $\boldsymbol{M}$ be the model which is transitive and isomorphic to M.

Then $(0^{\#})^{\boldsymbol{M}}=(0^{\#})^{\boldsymbol{M}_\gamma}$ for some $\gamma(\gamma<\beta)$. So $(0^{\#})^{\boldsymbol{M}}\in L[G]\cap L[G_\gamma]$. Hence $(0^{\#})^{\boldsymbol{M}}\in L$ by Lemma 6.

Then the upper bound of $(0^{\#})^{\boldsymbol{M}}$ does not exist from the fact that $L\models\aleph_1<\alpha$ and $L\models\alpha\leqq$(the upper bound of $(0^{\#})^{\boldsymbol{M}}$).

So $(0^{\#})^{\boldsymbol{M}}=0^{\#}$. This contradicts to Lemma 3. Then $\overline{\overline{X}}=2^{\aleph_0}$

Proof of THEOREM 3. Let α be an arbitrary countable ordinal such that $(\aleph_1)^L<\alpha$.

We put $X=\{(0^{\#})^{\boldsymbol{M}};\ \boldsymbol{M}$ is a countable transitive model of $\exists X(X=0^{\#})$

such that $\alpha<(OR)^M$ and (the upper bound of $(0^\#)^M)<(\alpha^+)^L\}$, $Y=\{(0^\#)^M$, M is a countable transitive model of $\exists X(X=0^\#)$ such that $\alpha<(OR)^M$, $M\in L[G]$ for some $G(\subseteqq|P_\alpha|)$ generic over L and (the upper bound of $(0^\#)^M)<(\alpha^+)^L\}$ and $Z=\{(0^\#)^M; M$ is a countable transitive model of $\exists X(X=0^\#)$ such that $\alpha<(OR)^M$ and $M\in L[G]$ for some $G(\subseteqq|P_\alpha|)$ generic over $L\}$.

If $M\in L[G]$ for some $G(\subseteqq|P_\alpha|)$ generic over L, then the upper bound $(0^\#)^M$ exists and (the upper bound of $(0^\#)^M)<(\aleph_1)^{L[G]}$ by Lemma 3 and the definition of $0^\#$. $(\aleph_1)^{L[G]}=(\alpha^+)^{L[G]}=(\alpha^+)^L$ by the definition of P_α.

Then we have $Z=Y\subseteqq X$. Therefore, in order to prove Theorem 3 it is sufficient to show $\overline{\overline{Z}}=2^{\aleph_0}$. Now assume $\overline{\overline{Z}}<2^{\aleph_0}$.

Let $\{(0^\#)^{M_\gamma}\}_{\gamma<\beta}$ be an enumeration of Z where $\beta<2^{\aleph_0}$, $M_\gamma\in L[G_\gamma]$ and $G_\gamma(\subseteqq|P_\alpha|)$ is generic over L. Let F_γ be the set of dense subsets of $|P_\alpha|$ in $L[G_\gamma]$.

From $\exists\kappa(\kappa\rightarrow(\aleph_1)_2^{<\omega})$ $\overline{\overline{F_\gamma}}\leqq\aleph_0$ by Lemma 5. Then $\bigcup_{\gamma<\beta}\overline{\overline{F_\gamma}}<2^{\aleph_0}$ By Martin's axiom there exists $\bigcup_{\gamma<\beta}F_\gamma$-generic set $G(\subseteqq|P_\alpha|)$.

We have a countable transitive model N such that $\alpha<(OR)^N$ by Lemma 1. We have a model N isomorphic to N such that $|N|=\omega$. And $Q(N, R_G)$ by the definition of Q. $\exists MQ(M, R_G)$ is a $\sum_2^1$ predicate. $\exists MQ(M, R_G)\equiv L[G]\models\exists MQ(M, R_G)$ by Lemma 2.

Then we have a model M in $L[G]$ such that $L[G]\models Q(M, R_G)$ and the transitive model M in $L[G]$ which is isomorphic to M.

And $(0^\#)^M=(0^\#)^{M_\gamma}$ for some $\gamma(\gamma<\beta)$. So $(0^\#)^M\in L[G_\gamma]\cap L[G]$ Hence $(0^\#)^M\in L$ by Lemma 6.

Then the upper bound of $(0^\#)^M$ does not exist from the fact that $L\models\aleph_1<\alpha$ and $L\models\alpha\leqq$(the upper bound of $(0^\#)^M$) So, $(0^\#)^M=0^\#$. This contradicts to Lemma 3.

Then $\overline{\overline{Z}}=2^{\aleph_0}$. Q.E.D.

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REFERENCES

[1] SHOENFIELD, J. R.; *Mathematical logic*, Addison Wesley, 1967.

[2] SILVER, J.; Some applications of model thory in set theory, *Annals of Math. Logic* **3** (1971) p 45–110.

[3] SOLOVAY, R. M.; A nonconstructive Δ_3^1 set of integers, *Trans. Amer. Math. Society* **127** (1967) p 50–75.

[4] ———, A model of set-theory in which every set of reals is Lebegue measurable, *Annals of Math.* **92** (1970) p 1–56.

Tokyo University of Education