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On some fixed point theorems in a Banach space

by

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Let X be a Banach space and T be an operator mapping of X into itself. In the present paper we will be concerned with the mapping T which satisfies

$$||T(x)-T(y)||\leqq\frac{1}{3}\{||x-T(x)||+||y-T(y)||+||x-y||\}$$

for $x, y\in X$.

The condition which T satisfies above has been introduced essentially by Reich [2] in connection with some fixed point theorems in a metric space. The theorems obtained in this paper have been motivated by the work of Kannan [1] regarding fixed points in a Banach space for operator mapping which satisfies

$$||T(x)-T(y)||\leqq\frac{1}{2}\{||x-T(x)||+||y-T(y)||\}$$

First we state certain well-known definitions and results quoted in [1].

DEFINITION. A norm $||\cdot||$ in a normed linear space X is *uniformly convex* if $||x_n||=||y_n||=1$ $(n=1, 2, \cdots)$, $\lim_{n\uparrow\infty}||x_n+y_n||=2$ imply $\lim_{n\to\infty}||x_n-y_n||=0$ for $x_n, y_n\in X$

THEOREM A. *Let X be a uniformly convex normed linear space and let ε, M be positive constants. Then there exists a constant δ with $0<\delta<1$ such that*

$$||x||\leqq M,\quad ||y||\leqq M,\quad ||x-y||\geqq\varepsilon$$

imply

$$||x+y||\leqq 2\delta\max(||x||, ||y||)\ .$$

Theorem A is given in [3], page 28. The following two theorems are also quoted in [1].

THEOREM B. *Every uniformly convex Banach space is norm reflexive.*

THEOREM C. *A necessary and sufficient condition that a Banach space X be reflexive is that: Every bounded descending sequence (trans-*

finite) of non-empty closed convex subsets of X has a non-empty intersection.

We prove now the following theorems:

THEOREM 1. *Let X be reflexive Banach space and K be a non-void closed convex bounded subset of X. If T is a mapping of K into itself such that*

(i) $\|T(x)-T(y)\| \leqq 1/3\{\|x-T(x)\|+\|y-T(y)\|+\|x-y\|\}$ *for* $x, y \in K$

(ii) $\sup_{y \in G}\|y-T(y)\| \leqq D(G)/2$

where G is any non-void convex subset of K which is mapped into itself by T and $D(G)$ is the diameter of G, then T has a unique fixed point in K.

If G is a non-void closed convex subset of K then for G we define the following quantities:

$$\rho_x(G)=\sup_{y \in G} \frac{2}{3}\|x-y\|+\sup_{z \in G} \frac{\|z-T(z)\|}{3}, \quad x \in G$$

$$\rho(G)=\inf_{x \in F} \rho_x(G)$$

and

$$G_c=\{x \in G: \rho_x(G)=\rho(G)\}$$

LEMMA 1. *G_c is non-void, closed and convex.*

Proof of Lemma 1 is essentially analogous to the lemma proved on page 170 in [1].

PROOF OF THEOREM 1. Let $\mathscr{F}$ denote the family of all non-void closed and convex subsets of K, each of which is mapped by T into itself. Then by the result of Smulian as in [1] and Zorn's lamma it follows that $\mathscr{F}$ possesses a minimal element, which we denote by G.

Let $x \in G_c$, by Lemma 1 G_c is non-void. Also for $y \in G$,

$$\begin{aligned} \|T(x)-T(y)\| &\leqq \frac{1}{3}\|x-T(x)\|+\frac{1}{3}\|y-T(y)\|+\frac{1}{3}\|x-y\| \\ &\leqq \frac{2}{3} \sup_{y \in G}\|x-y\|+\sup_{y \in G} \frac{1}{3}\|y-T(y)\| \\ &= \rho_x(G)=\rho(G) . \end{aligned}$$

Therefore $T(G)$ is contained in a closed sphere $\bar{S}$ centered at $T(x)$ and radius $\rho(G)$. Hence $T(G \cap \bar{S}) \subset G \cap \bar{S}$ and therefore by minimality of G, we obtain $G \subset \bar{S}$. Hence for $y \in G$, $\|T(x)-y\| \leqq \rho(G)$.

$$\sup_{y \in G}\|T(x)-y\| \leqq \rho(G) \tag{1}$$

Now

$$\rho_{T(x)}(G)=\sup_{y\in G}\frac{2}{3}\|T(x)-y\|+\sup_{z\in G}\frac{1}{3}\|z-T(z)\|$$

therefore

$$\rho_{T(x)}(G)\leqq\frac{2\rho(G)}{3}+\sup\frac{\|z-T(z)\|}{3}\quad\text{(by (1))}\tag{2}$$

Also

$$\begin{aligned}\sup_{z\in G}\frac{\|z-T(z)\|}{3}&=\sup_{z\in G}\frac{2}{9}\|z-T(z)\|+\sup_{z\in G}\frac{\|z-T(z)\|}{9}\\&\leqq\frac{D(G)}{9}+\sup_{z\in G}\frac{\|z-T(z)\|}{9},\ \text{by condition (ii)}\\&=\sup_{z,u\in G}\frac{\|z-x\|}{9}+\sup_{z\in G}\frac{\|z-T(z)\|}{9}\end{aligned}$$

So

$$\begin{aligned}\sup_{z\in G}\frac{\|z-T(z)\|}{3}&\leqq\sup_{z\in G}\frac{\|z-x\|}{9}+\sup_{u\in G}\frac{\|u-x\|}{9}+\sup_{z\in G}\frac{\|z-T(2)\|}{9}\\&=\sup_{z\in G}\frac{2}{9}\|z-x\|+\sup_{z\in G}\frac{\|z-T(z)\|}{9}\\&=\frac{\rho_x(G)}{3}=\frac{\rho(G)}{3}\end{aligned}$$

Therefore from (2) we have $\rho_{T(x)}(G)\leqq\rho(G)$, which implies that $\rho_{T(x)}(G)=\rho(G)$ i.e., $T(x)\in G_c$. Hence T maps G_c into itself.

We now show that if G contains more than one element, then G_c is properly contained in G. Suppose on the contrary $G_c=G$. Then for $x,\ y\in G$,

$$\rho_x(G)=\rho_y(G)=\rho(G)\ .$$

hence

$$\sup_{u\in G}\|x-y\|=\sup_{u\in G}\|y-u\|\ \text{for}\ x,\ y\in G\ .$$

This implies that $\sup_{u\in G}\|x-u\|=M$, a constant, for all $x\in G$. Hence $D(G)=\sup_{x,u\in G}\|x-u\|=M$, where $D(G)$ is the diameter of G. This in turn implies that $\sup_{u\in G}\|T(x)-u\|=D(G)$. Again,

$$\begin{aligned}\|T(x)-T(y)\|&\leqq\frac{\|x-T(x)\|}{3}+\frac{\|y-T(y)\|}{3}+\frac{\|x-y\|}{3}\quad y\in G\\&\leqq\frac{D(G)}{6}+\frac{D(G)}{6}+\frac{D(G)}{3}=\frac{2D(G)}{3}\quad\text{by condition (ii)}\end{aligned}\tag{3}$$

Again by arguments similar to that in obtaining (1), it can be seen that $\sup_{y\in G}\|T(x)-y\|\leqq(2/3)D(G)$ which contradicts (3) because G contains more than one element.

Hence we conclude that if G contains more than one element, then G_c is a proper subset of G. But because of (ii) this contradicts the minimality of G. Hence G contains one element. Since T maps G itself therefore T has fixed a point in K.

The uniqueness follows from the argument given below:

Let $T(x)=x$ and $T(y)=y$ where $x, y \in K$. Then

$$\| T(x)-T(y) \| \leqq \frac{1}{3}\| x-T(x) \| + \frac{1}{3}\| y-T(y) \| + \frac{1}{3}\| x-y \|$$

therefore

$$\frac{2}{3}\| T(x)-T(y) \| \leqq \frac{1}{3}\| x-T(x) \| + \frac{1}{3}\| y-T(y) \| = 0$$

from which we get $T(x)=T(y)=x=y$.

THEOREM 2. *Let K be a non-void, bounded, closed and convex subset of a uniformly convex Banach space X. Let T be a mapping of K into itself such that*

(i) $\| T(x)-T(y) \| \leqq \| x-T(x) \|/3 + \| y-T(y) \|/3 + \| x-y \|/3$ *and* (ii) $\sup_{z \in G} \| z-T(z) \| \leqq D(G)/2$, *where G is any non-void convex subset of K which is mapped into itself by T. Then the sequence $\{x_n\}$, where $x_{n+1}=(x_n+T(x_n))/2$, converges to the fixed point of T in K, where x_0 is any arbitrary point of K.*

PROOF. We know that T has a fixed point in K (by Theorem 1). Consider the sequence $\{x_n - T(x_n)\}$. There are two cases to be considered.

Case 1. $\exists$ an $\varepsilon>0$ such that $\|x_n - T(x_n)\| \geqslant \varepsilon$ for all $n>N$. Let $y \in K$, and $Ty=y$. Now

$$\| (x_n - y) - (T(x_n) - y) \| = \| x_n - T(x) \| \geqq \varepsilon \text{ for } n>N .$$

Since X is uniformly convex and $x_n \in K$ therefore we have

$$\begin{aligned} \| x_{n+1} - y \| &= \left\| \frac{x_n + T(x_n)}{2} - \frac{y+T(y)}{2} \right\| \\ &\leqq \delta \max(\| x_n - y \|, \| T(x_n) - T(y) \|) \quad n>N,\ 0<\delta<1 \end{aligned}$$

Now

$$\| T(x_n) - T(y) \| \leqq \frac{1}{3}[\| x_n - T(x_n) \| + \| y - T(y) \| + \| x_n - y \|]$$

Therefore,

$$\| T(x_n) - T(y) \| \leqq \frac{1}{3}\| x_n - y \| + \frac{1}{3}\| T(y) - T(x_n) \| + \frac{1}{3}\| x_n - y \|$$

from which

(4) $$||T(x_n)-T(y)||\leqq||x_n-y||$$

Hence

$$||x_{n+1}-y||\leqq\delta||x_n-y||,\ n>N,\ 0<\delta<1$$

Therefore $\{||x_n-y||\}$, for $n>N$ is a monotonic decreasing sequence which tends to zero as $n\to\infty$. Hence $\lim_{n\to\infty} x_n=y$, which proves the theorem in this case.

Case 2. $\exists$ a sequence of integers $\{n_k\}$ such that

$$\lim_{k\to\infty}||x_{n_k}-T(x_{n_k})||=0\ .$$

Now

(5) $$||T(x_{n_k})-T(x_{n_e})||\leqq\frac{1}{3}||T(x_{n_k})-x_{n_k}||+\frac{1}{3}||T(x_{n_e})-x_{n_e}||+\frac{1}{3}||x_{n_k}-x_{n_e}||$$

Also

(6) $$\frac{1}{3}||x_{n_k}-x_{n_e}||\leqq\frac{1}{3}||x_{n_k}-T(x_{n_k})||+\frac{1}{3}||T(x_{n_k})-T(x_{n_e})||+\frac{1}{3}||T(x_{n_e})-x_{n_e}||$$

Combining (5) and (6) we get as $n_k\to\infty$ and $n_e\to\infty$

$$\lim_{n_k,n_e\to\infty}||T(x_{n_k})-T(x_{n_e})||=0$$

Therefore $\{T(x_{n_k})\}$ is a Cauchy sequence and hence it converges to u say. So that $\lim_{k\to\infty} x_{n_k}=\lim_{k\to\infty} T(x_{n_k})=u$. Also

$$||u-T(u)||\leqq||u-x_{n_k}||+||x_{n_k}-T(x_{n_k})||+||T(x_{n_k})-T(u)||$$
$$\leqq||u-x_{n_k}||+||x_{n_k}-T(x_{n_k})||+\frac{1}{3}||T(x_{n_k})-x_{n_k}||$$
$$+\frac{1}{3}||T(u)-u||+\frac{1}{3}||x_{n_k}-u||$$

for each $k>0$. From which we infer that $T(u)=u$. Also

$$||x_{n+1}-u||=\left|\left|\frac{x_n+T(x_n)}{2}-\frac{u+T(u)}{2}\right|\right|$$
$$\leqq\frac{1}{2}||x_n-u||+\frac{1}{2}||T(x_n)-T(u)||$$

but $||(x_n)-T(u)||\leqq||x_n-u||$ from the same argument as in getting (4). Therefore $||x_{n+1}-u||\leqq||x_n-u||$. Now because of $\lim_{k\to\infty} x_{n_k}=u$, we have $\lim_{n\to\infty} x_n=u$ which proves the theorem in Case 2.

THEOREM 3. *Let X be a uniformly convex Banach space and let T be a mapping of X into itself such that*

(i) $||T(x)-T(y)||\leqq1/3\,||T(x)-T(x)||+1/3\,||y-T(y)||+1/3\,||x-y||$ *and*

(ii) $\sup_{y \in G} \|y - T(y)\| \leqq D(G)/2$, *where* G *is any non-void convex subset of* X *which is mapped into itself by* T. *Then if* T *has a fixed point* u *in* X, *the sequence defined by* $x_{n+1} = 1/2(x_n + T(x_n))$, *where* x_0 *is in* X, *converges to* u.

PROOF. Consider the set $K = \{x \in X: \|u - x\| \leqq d, \text{ for } d = \|u - x_0\|\}$. If $y \in K$, then we get

$$\|T(y) - u\| \leqq \|T(y) - T(u)\| \leqq \|y - u\|$$

by argument similar to that in obtaining (4). Hence

$$\|T(y) - u\| \leqq \|y - u\| \leqq d$$

In other words $T(y) \in K$. Also K is non-void, bounded, closed and convex. Therefore by Theorem 1, T has a unique fixed point in K and by Theorem 2, the sequence $\{x_n\}$ converges to the limit u, which proves the theorem.

THEOREM 4. *Let* X *be Banach space and* x_0 *an arbitrary point of* X. *Let* T *be a mapping of* X *into itself such that*

$$\|T(x) - T(y)\| \leqq \frac{1}{3}\|x - T(x)\| + \frac{1}{3}\|y - T(y)\| + \frac{1}{3}\|x - y\|$$

for $x, y \in X$.

Then if the sequence $\{x_n\}$, *where* $x_{n+1} = (x_n + T(x_n))/2$ *converges to* u, *then* u *is a unique fixed point of* T *in* X.

PROOF. We define an operator T_1 as follows: Let $T_1(x) = (1/2)x + (1/2)T(x)$. Then T_1 maps X into itself and the sequence $\{x_n\}$ becomes the sequence of iterates of x_0 by T. Now for $x, y \in X$ we have

$$\begin{aligned}\|T_1(x) - T_1(y)\| &\leqq \frac{\|x - y\|}{2} + \frac{1}{2}\|T(x) - T(y)\| \\ &\leqq \frac{2\|x - y\|}{3} + \frac{1}{6}\|x - T(x)\| + \frac{1}{6}\|y - T(y)\|\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1} - T_1(u)\| &\leqq \|T_1(x_n) - T_1(u)\| \\ &\leqq \frac{2\|x_n - u\|}{3} + \frac{1}{6}\|x_n - T_1(x_n)\| + \frac{1}{6}\|u - T_1(u)\| \\ &\leqq \frac{2}{3}\|x_n - u\| + \frac{1}{6}\|x_n - x_{n+1}\| + \frac{1}{6}\|u - x_{n+1}\| + \frac{1}{6}\|x_{n+1} - T_1(u)\|\end{aligned}$$

from which we get

$$\frac{7}{6}\|x_{n+1} - T_1(u)\| \leqq \frac{2}{3}\|x_n - u\| + \frac{1}{6}\|x_n - x_{n+1}\| + \frac{1}{6}\|u - x_{n+1}\|$$

Since $\lim_{n\to\infty} x_n = u$, the above inequalities imply that $u = T_1(u)$, so that $u = (u/2) + (1/2)T(u)$, which in turn implies that $u = T(u)$, and the proof of the theorem is complete.

THEOREM 5. *Let $\{g_n\}$ be sequence of elements in a Banach space X. Let u_n be the unique solution of the equation $u - T(u) = g_n$ where T is a mapping of X into itself and T satisfies the following condition:*

$$\|T(x) - T(y)\| \leqq \frac{\|x - T(x)\|}{3} + \frac{\|y - T(y)\|}{3} + \frac{\|x - y\|}{3} \quad x, y \in X$$

If $\|g_n\| \to 0$ as $n \to \infty$, the sequence $\{u_n\}$ converges to the solution of $u = T(u)$.

PROOF. We have

$$\begin{aligned}\|u_n - u_m\| &\leqq \|u_n - T(u_n)\| + \|T(u_n) - T(u_m)\| + \|T(u_m) - u_m\| \\ &\leqq \|g_n\| + \frac{1}{3}\|u_n - T(u_n)\| + \frac{1}{3}\|u_m - T(u_m)\| + \frac{1}{3}\|u_n - u_m\| + \|g_m\|\end{aligned}$$

Therefore

$$\frac{2}{3}\|u_n - u_m\| \leqq \|g_n\| + \frac{\|g_n\|}{3} + \frac{\|g_m\|}{3} + \|g_m\|$$

and hence $\{u_n\}$ is a Cauchy sequence in X, and therefore it converges to some $u \in X$. Also

$$\begin{aligned}\|u - T(u)\| &\leqq \|u - u_n\| + \|u_n - T(u_n)\| + \|T(u_n) - T(u)\| \\ &\leqq \|u - u_n\| + \|g_n\| + \frac{1}{3}\|u_n - T(u_n)\| + \frac{1}{3}\|u - T(u)\| + \frac{1}{3}\|u_n - u\|\end{aligned}$$

Therefore

$$\frac{2}{3}\|u - T(u)\| \leqq \frac{4}{3}\|u - u_n\| + \frac{4}{3}\|g_n\|,$$

$\forall\, n$, taking limit as $n \to \infty$ we see that $u = T(u)$ which completes the proof of the theorem.

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