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Certain expansion formulas involving the generalized Lauricella functions, II*

by

H. M. SRIVASTAVA and Rekha PANDA†

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Summary

The present note is a sequel to the authors' paper [6], and it aims at deriving two new classes of expansion formulas for the generalized Lauricella function of several variables introduced earlier by H. M. Srivastava and M. C. Daoust [4]. The main results, given by Theorems 1 and 2 below, are shown to unify and extend a fairly large number of expansions of hypergeometric functions of one and two variables in series of similar hypergeometric functions.

1. Introduction

Following the usual notations (cf. [6]) let

$$F\begin{matrix} A: B'; \cdots; B^{(r)} \\ C: D'; \cdots; D^{(r)} \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \tag{1.1}$$

denote the generalized Lauricella function of several complex variables $z_1, \cdots, z_r$ (see also [4], p. 454). Also let the associated coefficients

$$\begin{cases} \theta_j^{(i)}, j=1, \cdots, A; \phi_j^{(i)}, j=1, \cdots, B^{(i)}; \psi_j^{(i)}, j=1, \cdots, C; \\ \qquad \delta_j^{(i)}, j=1, \cdots, D^{(i)}; 1 \leqq i \leqq r; \end{cases} \tag{1.2}$$

be real and positive. Then it is well known that the multiple hypergeometric series defining the function (1.1) would converge absolutely when

$$\Delta_i \equiv 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} \geqq 0 , \quad (i=1, \cdots, r) , \tag{1.3}$$

where the equality holds provided $|z_i| < \rho_i$, $1 \leqq i \leqq r$, with the ρ_i defined by equation (5.3), p. 157 in [5].

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† On study leave from Ravenshaw College, Cuttack-3, India.

Throughout the present note we use the abbreviation (a) to denote the sequence of A parameters $a_1, \cdots, a_A$; $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters

$$b_1^{(i)}, \cdots, b_{B^{(i)}}^{(i)}, \quad i=1, \cdots, r\ ;$$

with similar interpretations for (c) and $(d^{(i)})$, $i=1, \cdots, r$; etc. Also, for the sake of brevity, we employ the following notations:

$$(1.4) \qquad ((a))_n=\prod_{j=1}^{A} (a_j)_n,\ ((b^{(i)}))_n=\prod_{j=1}^{B^{(i)}} (b_j^{(i)})_n\ , \qquad i=1, \cdots, r;\ \text{etc.}\ ,$$

where $(\lambda)_n$ is the usual Pochhammer symbol defined by

$$(1.5) \qquad (\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases} 1\ , & \text{if } n=0\ , \\ \lambda(\lambda+1)\cdots(\lambda+n-1)\ , & \text{if } n=1, 2, 3, \cdots . \end{cases}$$

For the generalized Lauricella function (1.1) we first give here a new expansion in series of Jacobi polynomials. We then proceed systematically to appropriately extend this Fourier-Jacobi series to expansions in series of certain classes of hypergeometric polynomials. Our main results, given by Theorems 1 and 2 below, and the four expansion theorems, obtained in our previous paper [6], together with the expansion formulas (4.2), p. 455 and (4.3), p. 456 in reference [4], are intended to complete[†] the development of expansion theory for the generalized Lauricella functions in series of various hypergeometric polynomials or their products.

The following known integral formula will be required in our analysis:

$$(1.6) \qquad \int_0^1 t^{\sigma}(1-t)^{\beta}P_n^{(\alpha,\beta)}(1-2t)F\begin{matrix} A: B'; \cdots; B^{(r)} \\ C: D'; \cdots; D^{(r)} \end{matrix}\begin{pmatrix} z_1 t^{\mu_1} \\ \vdots \\ z_r t^{\mu_r} \end{pmatrix} dt$$

$$=\frac{(\alpha-\sigma)_n\Gamma(\sigma+1)\Gamma(\beta+n+1)}{n!\ \Gamma(\beta+\sigma+n+2)}F\begin{matrix} A+2: B'; \cdots; B^{(r)} \\ C+2: D'; \cdots; D^{(r)} \end{matrix}$$

$$\left(\begin{matrix} [\sigma+1: \mu_1,\cdots,\mu_r], & [\sigma-\alpha+1: \mu_1,\cdots,\mu_r], & [(a): \theta',\cdots,\theta^{(r)}]: \\ [\beta+\sigma+n+2: \mu_1,\cdots,\mu_r], & [\sigma-\alpha-n+1: \mu_1,\cdots,\mu_r], & [(c): \psi',\cdots,\psi^{(r)}]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'): \phi']; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [(d'): \delta']; \cdots; [(d^{(r)}): \delta^{(r)}]; \end{matrix}\ z_1, \cdots, z_r\right),$$

which holds true when n is a nonnegative integer, $\mathrm{Re}(\beta)>-1$, $\mathrm{Re}(\sigma)>-1$, $\mu_i>0$, and $\Delta_i>0$, or $\Delta_i=0$ and $|z_i|<\rho_i$, $i=1, \cdots, r$.

Formula (1.6) would follow as an immediate consequence of the integral (2.4) of Panda [2, p. 117], since the Jacobi polynomials

[†] A systematic discussion of the expansion theory of G and H functions of several complex variables will be presented in a forthcoming paper.

$\{P_n^{(\alpha,\beta)}(x)\,|\,n=0,1,2,\cdots\}$ are given by (cf., e.g., [1], Vol. I, p. 274)

$$P_n^{(\alpha,\beta)}(1-2t)=\binom{\alpha+n}{n}{}_2F_1\left[\begin{matrix}-n,\ \alpha+\beta+n+1;\\ \alpha+1;\end{matrix}\ t\right],\quad n\geqq 0\,. \tag{1.7}$$

2. The Fourier-Jacobi series

In terms of the Fourier-Jacobi series we let

$$f(t)\equiv t^{\mu}F\begin{matrix}A:\ B';\ \cdots;\ B^{(r)}\\ C:\ D';\ \cdots;\ D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{\mu_1}\\ \vdots\\ z_rt^{\mu_r}\end{pmatrix}=\sum_{n=0}^{\infty}\Omega_n(z_1,\cdots,z_r)P_n^{(\alpha,\beta)}(1-2t)\,, \tag{2.1}$$

which is valid since $f(t)$ is continuous and of bounded variation on the interval (0, 1) when $\mu\geqq 0$, $\mu_i>0$, and $\Delta_i>0$, or $\Delta_i=0$ and $|z_i|<\rho_i$, $i=1,\cdots,r$.

In order to determine the unknown coefficients $\Omega_n(z_1,\cdots,z_r)$, $n=0,1,2,\cdots$, we multiply both sides of equation (2.1) by $t^{\alpha}(1-t)^{\beta}P_m^{(\alpha,\beta)}(1-2t)$, $\mathrm{Re}(\alpha)>-1$, $\mathrm{Re}(\beta)>-1$, integrate with respect to t from 0 to 1, and then make use of formula (1.6) and the orthogonality property ([1], Vol. I, p. 276 (22)) of the Jacobi polynomials. On substituting in (2.1) the value of $\Omega_n(z_1,\cdots,z_r)$ thus obtained, if we set $\varepsilon=\alpha+1$ and $\lambda=\alpha+\beta+1$, and appeal to the hypergeometric representation given by (1.7), we shall finally arrive at the following

LEMMA. *If* $\mu\geqq 0$, $0<t<1$, $\mathrm{Re}(\varepsilon)>0$, $\mathrm{Re}(\lambda-\varepsilon)>-1$, $\mu_i>0$, *and* $\Delta_i>0$, *or* $\Delta_i=0$ *and* $|z_i|<\rho_i$, $i=1,\cdots,r$, *then*

$$t^{\mu}F\begin{matrix}A:\ B';\ \cdots;\ B^{(r)}\\ C:\ D';\ \cdots;\ D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{\mu_1}\\ \vdots\\ z_rt^{\mu_r}\end{pmatrix}=\sum_{n=0}^{\infty}\frac{(\lambda+2n)(-\mu)_n(\varepsilon)_{\mu}}{n!\,(\lambda+n)_{\mu+1}}\cdot F_n[z_1,\cdots,z_r]\,{}_2F_1\left[\begin{matrix}-n,\ \lambda+n;\\ \varepsilon;\end{matrix}\ t\right], \tag{2.2}$$

where, for convenience,

$$F_n[z_1,\cdots,z_r]=F\begin{matrix}A+2:\ B';\ \cdots;\ B^{(r)}\\ C+2:\ D';\ \cdots;\ D^{(r)}\end{matrix}\left(\begin{matrix}[(a):\ \theta',\cdots,\theta^{(r)}],\ [\mu+\varepsilon:\ \mu_1,\cdots,\mu_r],\ [\mu+1:\ \mu_1,\cdots,\mu_r]:\\ [(c):\ \psi',\cdots,\psi^{(r)}],\ [\lambda+\mu+n+1:\ \mu_1,\cdots,\mu_r],\ [\mu-n+1:\ \mu_1,\cdots,\mu_r]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\ \phi'];\ \cdots;\ [(b^{(r)}):\ \phi^{(r)}];\\ [(d'):\ \delta'];\ \cdots;\ [(d^{(r)}):\ \delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right),\quad n\geqq 0\,. \tag{2.3}$$

3. Further generalizations

The form of our expansion formula (2.2) would suggest the existence of an obvious generalization in which the ${}_2F_1$ polynomials are replaced by a class of generalized hypergeometric polynomials. Indeed if we apply the method of finite mathematical induction, using the Laplace transform and its inverse, we shall formally obtain

$$
\begin{aligned}
(3.1)\quad & t^{\mu}F^{A+E:B';\cdots;B^{(r)}}_{C+G:D';\cdots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}],\ [(e):\mu_1,\cdots,\mu_r]:\\ [(c):\psi',\cdots,\psi^{(r)}],\ [(g):\mu_1,\cdots,\mu_r]:\end{matrix}\right.\\
&\qquad \left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1t^{\mu_1},\cdots,z_rt^{\mu_r}\right)\\
&=\frac{(\varepsilon)_{\mu}((e))_{-\mu}}{((g))_{-\mu}}\sum_{n=0}^{\infty}\frac{(\lambda+2n)(-\mu)_n}{n!\,(\lambda+n)_{\mu+1}}F_n[z_1,\cdots,z_r]\\
&\qquad\cdot{}_{E+2}F_{G+1}\left[\begin{matrix}-n,\ \lambda+n,\ (e)-\mu;\\ \varepsilon,\ (g)-\mu;\end{matrix}\ t\right],
\end{aligned}
$$

provided that each side has a meaning. Evidently, this last result (3.1) with $E=G=0$ would yield our expansion formula (2.2).

Now in equation (3.1) we set $z_i=0$, $i=1,\cdots,r$, and replace E by $E+K$, and G by $G+H$. Also replace the parameters e_j by e_j+M, $j=1,\cdots,E$, and g_j by g_j+M, $j=1,\cdots,G$, where

$$(3.2)\qquad M=\mu_1m_1+\cdots+\mu_rm_r\,,$$

and set

$$(3.3)\qquad \begin{cases} e_{E+j}=k_j+\mu, & j=1,\cdots,K\,,\\ g_{G+j}=h_j+\mu, & j=1,\cdots,H\,.\end{cases}$$

Next we divide both sides of the resulting equation by $(\varepsilon)_\mu$, replace t by εt, and let $\varepsilon\to\infty$.

By replacing μ by $\mu+M$, where M is given by (3.2), we observe that equation (3.1) is finally reduced to the identity

$$
\begin{aligned}
(3.4)\qquad t^{\mu+M}=&\frac{(\mu+1)_M((e))_{-\mu}((g))_M((h))_{\mu+M}}{((g))_{-\mu}((e))_M((k))_{\mu+M}}\\
&\cdot\sum_{n=0}^{\infty}\frac{(\lambda+2n)(-\mu)_nB_n(t)}{n!\,(\lambda+n)_{\mu+1}(\mu-n+1)_M(\lambda+\mu+n+1)_M},
\end{aligned}
$$

where, for convenience,

$$(3.5)\qquad B_n(t)={}_{E+K+2}F_{G+H}\left[\begin{matrix}-n,\ \lambda+n,\ (e)-\mu,\ (k);\\ (g)-\mu,\ (h);\end{matrix}\ t\right].$$

Making use of the identity (3.4) it is easy to derive an expansion formula for the first member of equation (3.1) in a series of the hypergeometric polynomials $B_n(t)$ given by (3.5). The result thus obtained may be stated as

THEOREM 1. *With Δ_i given by* (1.3) *and $\sigma_i(U,V)$ by*

$$(3.6)\qquad \sigma_i(U,V)=\min_{\nu_1,\cdots,\nu_r>0}\left\{\frac{E_i}{(\mu_1\nu_1+\cdots+\mu_r\nu_r)^{(U-V)\mu_i}}\right\},\ i=1,\cdots,r,$$

where the E_i *are defined by equation* (2.14), p. 23 *in* [6], *let* $\Delta_i>(E-G)\mu_i$ *and* $E+K+1=G+H$, $|z_i|<\infty$, $i=1,\cdots,r$; *or* $\Delta_i=(E-G)\mu_i$ *and* $E+K+1=G+H$, $|z_i|<\min\{\sigma_i(E,G),\ \sigma_i(H,K+1)\}$, $i=1,\cdots,r$. *Also let* $\mu_i>0$, $i=1,\cdots,r$; $0<t\leqq 1$; $\mathrm{Re}(e_j)>0$, $j=1,\cdots,E$; $\mathrm{Re}(k_j+\mu)>0$, $j=1,\cdots,K$; *and* $\mathrm{Re}(\zeta-\mu)<0$, *if* $0<t<1$, *or* $\mathrm{Re}(\lambda-2\mu+4\zeta)<0$, *if* $t=1$.

Then

$$(3.7)\qquad t^\mu F\begin{matrix}A+E:B';\cdots;B^{(r)}\\ C+G:D';\cdots;D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{\mu_1}\\ \vdots\\ z_rt^{\mu_r}\end{pmatrix}$$
$$=\frac{((e))_{-\mu}((h))_\mu}{((g))_{-\mu}((k))_\mu}\sum_{n=0}^{\infty}\frac{(\lambda+2n)(-\mu)_n}{n!\,(\lambda+n)_{\mu+1}}G_n[z_1,\cdots,z_r]B_n(t),$$

where the generalized Lauricella function on the left-hand side is the same as in (3.1), *the* $B_n(t)$ *are defined by* (3.5),

$$(3.8)\qquad 2\zeta=\frac{1}{2}+\sum_{j=1}^{E}e_j+\sum_{j=1}^{K}k_j-\sum_{j=1}^{G}g_j-\sum_{j=1}^{H}h_j+\mu(G-E),$$

and, for brevity,

$$(3.9)\qquad G_n[z_1,\cdots,z_r]=F\begin{matrix}A+H+1:B';\cdots;B^{(r)}\\ C+K+2:D';\cdots;D^{(r)}\end{matrix}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}],\\ [(c):\psi',\cdots,\psi^{(r)}],\end{matrix}\right.$$
$$\begin{matrix}[\mu+1:\mu_1,\cdots,\mu_r],[(h)+\mu:\mu_1,\cdots,\mu_r]:\\ [\mu-n+1:\mu_1,\cdots,\mu_r],[\lambda+\mu+n+1:\mu_1,\cdots,\mu_r],[(k)+\mu:\mu_1,\cdots,\mu_r]:\end{matrix}$$
$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}z_1,\cdots,z_r\right),\quad n\geqq 0.$$

This expansion formula is valid also when $t=0$ *provided* $\mathrm{Re}(\mu)>0$.

The various hypotheses stated with the theorem arise from the consideration of convergence of the multiple hypergeometric series of the generalized Lauricella functions occurring on either side of (3.7) and of the infinite series on the right-hand side. The former can be disposed of fairly easily in terms of the aforementioned Δ-conditions (cf. [5], p. 158). In order to handle the latter situation, viz. the convergence of the second member of equation (3.7), we consider the asymptotic behaviour of the hypergeometric functions $G_n[z_1,\cdots,z_r]$ and $B_n(t)$, defined by (3.9) and (3.5), respectively, for large n, provided $0<t\leqq 1$ and $z_1,\cdots,z_r$ are fixed. Note that the given conditions

$$(3.10)\qquad \Delta_i\geqq(E-G)\mu_i\quad\text{and}\quad E+K+1=G+H$$
$$(\text{equality when } |z_i|<\min\{\sigma_i(E,G),\ \sigma_i(H,K+1)\})$$

imply

(3.11) $$\mu_i+\Delta_i \geqq (H-K)\mu_i \text{ (equality when } |z_i|<\sigma_i(H, K+1)) \quad (i=1, \cdots, r);$$

thus the multiple hypergeometric series in (3.9) converges absolutely under the Δ-conditions of the theorem, and for large n and fixed $z_1, \cdots, z_r$ we have

(3.12) $$G_n[z_1, \cdots, z_r]=1+o(1),$$

since it is fairly well known that (cf., e.g., [1], Vol. I, p. 33)

(3.13) $$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}=n^{\alpha-\beta}[1+O(n^{-1})]$$

for large positive n.

On the other hand, the asymptotic behaviour of

(3.14) $$\Gamma_n(t)=A_n B_n(t), \quad (0<t\leqq 1, \ E+K+1=G+H)$$

where

(3.15) $$A_n=\frac{(\lambda+2n)(-\mu)_n}{n!\,(\lambda+n)_{\mu+1}}=\frac{2n^{-1-2\mu}}{\Gamma(-\mu)}[1+O(n^{-1})], \ n\to\infty,$$

would follow readily from certain known results in the literature [1, Vol. I, p. 250 (8) and p. 259 (23)]. We omit details.

Next we turn to the derivation of another result similar to Theorem 1. As a matter of fact, this is a confluent form of the expansion formula (3.7); it would follow if we appeal to the familiar principle of confluence exhibited by

(3.16) $$\lim_{\lambda\to\infty}\{(\lambda z)^m/(\lambda)_m\}=z^m; \ \lim_{\mu\to\infty}\{(\mu)_m(t/\mu)^m\}=t^m; \quad (m=0, 1, 2, \cdots).$$

Indeed it is readily verified that

(3.17) $$\lim_{\lambda\to\infty} B_n(t/\lambda)=B_n^*(t),$$

where $B_n(t)$ is given by (3.5), and

(3.18) $$B_n^*(t)={}_{E+K+1}F_{G+H}\left[\begin{matrix}-n, (e)-\mu, (k); \\ (g)-\mu, (h);\end{matrix} t\right], \ n\geqq 0.$$

In the expansion formula (3.7) we replace z_i and t by $z_i\lambda^{\mu_i}$ and t/λ, respectively, $i=1, \cdots, r$, and let $\lambda\to\infty$. We thus arrive formally at the following confluent form of Theorem 1.

THEOREM 2. *With* Δ_i *defined by* (1.3) *and* $\sigma_i(U, V)$ *by equation* (3.6), *let* $\Delta_i>(E-G)\mu_i$ *and* $E+K+1=G+H$, $|z_i|<\infty$, $i=1, \cdots, r$; *or* $\Delta_i=(E-G)\mu_i$ *and* $E+K+1=G+H$, $|z_i t^{\mu_i}|<\sigma_i(E, G)$, $i=1, \cdots, r$. *Also* *let* $\mu_i>0$, $\Delta_i\geqq(H-K)\mu_i$ *(equality holds when* $|z_i|<\sigma_i(H,K)$*),* $i=1,\cdots,r$; $0<t<\infty$; $\mathrm{Re}(e_j)>0, j=1,\cdots,E$; $\mathrm{Re}(k_j+\mu)>0, j=1,\cdots,K$; *and* $\mathrm{Re}(\zeta-\mu)<0$, *where* ζ *is given by equation* (3.8).

Then

$$\text{(3.19)}\qquad t^{\mu}F\begin{matrix}A+E: B'; \cdots; B^{(r)}\\ C+G: D'; \cdots; D^{(r)}\end{matrix}\begin{pmatrix}z_1t^{\mu_1}\\ \vdots\\ z_rt^{\mu_r}\end{pmatrix}$$
$$=\frac{((e))_{-\mu}((h))_{\mu}}{((g))_{-\mu}((k))_{\mu}}\sum_{n=0}^{\infty}\frac{(-\mu)_n}{n!}H_n[z_1,\cdots,z_r]B_n^*(t)\ ,$$

where the generalized Lauricella function on the left-hand side is the same as in (3.1) *and* (3.7), *the* $B_n^*(t)$ *are defined by* (3.18), *and*

$$\text{(3.20)}\qquad H_n[z_1,\cdots,z_r]$$
$$=F\begin{matrix}A+H+1: B'; \cdots; B^{(r)}\\ C+K+1: D'; \cdots; D^{(r)}\end{matrix}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}], [(h)+\mu:\mu_1,\cdots,\mu_r],\\ [(c):\psi',\cdots,\psi^{(r)}], [(k)+\mu:\mu_1,\cdots,\mu_r],\end{matrix}\right.$$
$$\left.\begin{matrix}[\mu+1:\mu_1,\cdots,\mu_r]: [(b'):\phi']; \cdots; [(b^{(r)}):\phi^{(r)}];\\ [\mu-n+1:\mu_1,\cdots,\mu_r]: [(d'):\delta']; \cdots; [(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right),\quad n\geqq 0\ .$$

The demonstration of this theorem would run parallel to that of Theorem 1. Indeed, if $\Delta_i\geqq(H-K)\mu_i$ (equality when $|z_i|<\sigma_i(H,K)$), $i=1,\cdots,r$, it is easy to observe that, for $z_1,\cdots,z_r$ appropriately fixed and n sufficiently large,

$$\text{(3.21)}\qquad H_n[z_1,\cdots,z_r]=1+o(1)\ ,$$

and the order estimate of $B_n^*(t)$ for large n and $E+K+1=G+H$ follows at once from the known result [1, Vol. I, p. 264 (2)]. The details are omitted.

4. Applications

Evidently, Theorems 1 and 2 of the preceding section are very general in character, and they can be suitably applied to obtain a large number of expansion formulas for various hypergeometric functions, or their products, in series of generalized hypergeometric polynomials such as Jacobi, Laguerre, Rice, and several other familiar polynomials. For instance, if $A=C=E=G=0$, these theorems will yield polynomial expansions for the product of several Wright functions. By further setting each of the ϕ's and δ's equal to 1, these expansions can easily be reduced in terms of the product

$$\text{(4.1)}\qquad t^{\mu}{}_{B'}F_{D'}\left[\begin{matrix}(b');\\ (d');\end{matrix}\ z_1t^{\mu_1}\right]\cdots{}_{B^{(r)}}F_{D^{(r)}}\left[\begin{matrix}(b^{(r)});\\ (d^{(r)});\end{matrix}\ z_rt^{\mu_r}\right]$$

of r generalized hypergeometric functions.

Next we remark that, by appropriately specializing the various parameters involved, these theorems would lead to expansion formulas associated with Lauricella's hypergeometric functions $F_A^{(r)}, F_B^{(r)}, F_C^{(r)}$ and $F_D^{(r)}$ of r variables (cf., e.g., [4], p. 455).

With each of the parameters in (1.2) and $\mu_1, \cdots, \mu_r$ equated to 1, the special cases of Theorems 1 and 2 when $r=1$ are essentially the same as the expansions given earlier by Wimp and Luke ([7], p. 353, Theorem I; p. 357, Theorem II; see also [1], Vol. II, p. 5 (11) and p. 9, Theorem 3). On the other hand, with the same choices of the aforementioned parameters, a special case of the Fourier-Jacobi expansion (2.2) when $r=2$ is equivalent to the main result (3.1), p. 144 of Singh and Sharma [3]. Note, however, that this expansion formula of Singh and Sharma [loc. cit.] is contained in a number of known results, involving G and H functions of two variables, which were given by several earlier writers.

References

[1] LUKE, Y. L.; *The special functions and their approximations*, Vols. I and II, Academic Press, New York and London, 1969.

[2] PANDA, R.; Some integrals associated with the generalized Lauricella functions, *Publ. Inst. Math.* (Beograd) Nouvelle Sér. **16** (**30**) (1973), 115–122.

[3] SINGH, F. and SHARMA, L. K.; Coulomb wave functions in terms of Jacobi polynomial and generalized confluent hypergeometric functions, *Indian J. Pure Appl. Math.* **3** (1972), no. 1, 142–147.

[4] SRIVASTAVA, H. M. and DAOUST, M. C.; Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser.* A **72**=*Indag. Math.* **31** (1969), 449–457.

[5] SRIVASTAVA, H. M. and DAOUST, M. C.; A note on the convergence of Kampé de Fériet's double hypergeometric series, *Math. Nachr.* **53** (1972), 151–159.

[6] SRIVASTAVA, H. M. and PANDA, R.; Certain expansion formulas involving the generalized Lauricella functions, *Comment. Math. Univ. St. Paul.* **22** (1973), fasc. 2, 19–32.

[7] WIMP, J. and LUKE, Y. L.; Expansion formulas for generalized hypergeometric functions, *Rend. Circ. Mat. Palermo Ser.* II **11** (1962), 351–366.

Department of Mathematics
University of Victoria
Victoria, British Columbia, Canada
V8W 2Y2