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Certain expansion formulas involving the generalized Lauricella functions*

by

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This paper attempts to unify and extend certain recent works of J. Wimp and Y. L. Luke [10], and V. L. Deshpande [2], on expansion theory of hypergeometric functions of one and two variables in series of similar hypergeometric functions. The main results, given by Theorems 1 through 4 below, involve the generalized Lauricella function of several variables introduced earlier by H. M. Srivastava and M. C. Daoust [8]. Suitable specializations to several classes of expansion formulas associated with various hypergeometric functions, or their products, are discussed briefly.

1. Introduction and notations

Recently, Srivastava and Daoust [8] introduced the generalized Lauricella function of r complex variables $z_1, \cdots, z_r$ in the form [op. cit., p. 454]

$$F\begin{matrix} A: B'; \cdots; B^{(r)} \\ C: D'; \cdots; D^{(r)} \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \equiv F\begin{matrix} A: B'; \cdots; B^{(r)} \\ C: D'; \cdots; D^{(r)} \end{matrix}\left(\begin{matrix} [(a): \theta', \cdots, \theta^{(r)}]: \\ [(c): \psi', \cdots, \psi^{(r)}]: \end{matrix}\right.$$

$$(1.1)\qquad \left.\begin{matrix} [(b'): \phi']; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [(d'): \delta']; \cdots; [(d^{(r)}): \delta^{(r)}]; \end{matrix}\, z_1, \cdots, z_r\right)$$

$$= \sum_{m_1, \cdots, m_r=0}^{\infty} \Omega(m_1, \cdots, m_r)\frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!},$$

where, for convenience,

$$(1.2)\qquad \Omega(m_1, \cdots, m_r) = \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta_j'+\cdots+m_r\theta_j^{(r)}} \prod_{j=1}^{B'} (b_j')_{m_1\phi_j'} \cdots \prod_{j=1}^{B^{(r)}} (b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C} (c_j)_{m_1\psi_j'+\cdots+m_r\psi_j^{(r)}} \prod_{j=1}^{D'} (d_j')_{m_1\delta_j'} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r\delta_j^{(r)}}},$$

the coefficients

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$$\text{(1.3)}\qquad \begin{cases} \theta_j^{(i)},\ j=1,\cdots,A;\ \phi_j^{(i)},\ j=1,\cdots,B^{(i)};\ \psi_j^{(i)},\ j=1,\cdots,C; \\ \qquad\qquad \delta_j^{(i)},\ j=1,\cdots,D^{(i)};\ 1\leqq i\leqq r; \end{cases}$$

are real and positive, and (a) is taken to abbreviate the sequence of A parameters $a_1,\cdots,a_A$; $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters

$$b_1^{(i)},\cdots,b_{B^{(i)}}^{(i)}\ ,\qquad i=1,\cdots,r\ ;$$

with similar interpretations for (c) and $(d^{(i)})$, $i=1,\cdots,r$; etc. Also, for the sake of brevity, we use the following notations throughout this paper:

$$\text{(1.4)}\qquad ((a))_n=\prod_{j=1}^{A}(a_j)_n,\ ((b^{(i)}))_n=\prod_{j=1}^{B^{(i)}}(b_j^{(i)})_n\ ,\quad i=1,\cdots,r;\ \text{etc.}\ ,$$

where, as usual, $(\lambda)_n$ is the Pochhammer symbol defined by

$$\text{(1.5)}\qquad (\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases} 1\ , & \text{if } n=0\ , \\ \lambda(\lambda+1)\cdots(\lambda+n-1)\ , & \text{if } n=1,2,3,\cdots . \end{cases}$$

Indeed it is easy to observe that, when $r=1$ and $r=2$, (1.1) would correspond, respectively, to the generalized hypergeometric function, introduced by Wright ([11] and [12]), and the generalization of Kampé de Fériet's double hypergeometric function, introduced by Srivastava and Daoust (cf., e.g., [8], p. 450).

For a complete set of conditions for convergence of the multiple hypergeometric series in (1.1) see § 5 of a recent paper [9].

In the present paper we first derive certain preliminary results, given by Lemmas 1 and 2 below, on expansions in series of products of several hypergeometric polynomials, and then proceed systematically to appropriately extend these results to hold for the generalized Lauricella functions defined by (1.1).

2. Preliminary results

In this section we first establish

LEMMA 1. *If* $\mu_i\geqq 0$, $0<w_i<1$, $\varepsilon_i>0$ *and* $\lambda_i-\varepsilon_i>-1$, $i=1,\cdots,r$, *then*

$$\text{(2.1)}\qquad \begin{aligned} w_1^{\mu_1}\cdots w_r^{\mu_r}\exp\{w_1z_1+\cdots+w_rz_r\} &=\sum_{n=0}^{\infty}\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n(\varepsilon_i)_{\mu_i}}{n!(n+\lambda_i)_{1+\mu_i}}\right\} \\ &\cdot\prod_{i=1}^{r}{}_2F_2\left[\begin{matrix}1+\mu_i\ ,\ \mu_i+\varepsilon_i\ ; \\ 1-n+\mu_i\ ,\ 1+n+\lambda_i+\mu_i\ ;\end{matrix}\ z_i\right]{}_2F_1\left[\begin{matrix}-n\ ,\ n+\lambda_i\ ; \\ \varepsilon_i\ ;\end{matrix}\ w_i\right]. \end{aligned}$$

Proof. In terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n=0,1,2,\cdots$, defined by (cf., e.g., [3], p. 169 and p. 170)

$$(2.2)\qquad \begin{aligned} P_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^{k} \\ &= (-1)^n \binom{n+\beta}{n} {}_2F_1\left[\begin{matrix} -n, & 1+n+\alpha+\beta; \\ & 1+\beta; \end{matrix} \frac{1+x}{2}\right], \end{aligned}$$

we let

$$(2.3)\qquad \begin{aligned} f(x_1,\cdots,x_r) &= \left(\frac{1+x_1}{2}\right)^{\mu_1}\cdots\left(\frac{1+x_r}{2}\right)^{\mu_r} \exp\left\{\frac{(1+x_1)z_1}{2}\right\}\cdots\exp\left\{\frac{(1+x_r)z_r}{2}\right\} \\ &= \sum_{n=0}^{\infty} C_n P_n^{(\alpha_1,\beta_1)}(x_1)\cdots P_n^{(\alpha_r,\beta_r)}(x_r), \quad -1<x_i<1, \end{aligned}$$

which is valid since $f(x_1, \cdots, x_r)$ is continuous and of bounded variation for all $x_i \in (-1, 1)$ and $\mu_i \geqq 0$, $i=1, \cdots, r$.

Now multiply each member of (2.3) by

$$(2.4)\qquad \prod_{i=1}^{r} (1-x_i)^{\alpha_i}(1+x_i)^{\beta_i} P_m^{(\alpha_i,\beta_i)}(x_i), \quad \alpha_i>-1,\ \beta_i>-1,$$

and integrate with respect to $x_1, \cdots, x_r$ from -1 to 1. Making use of the known integral ([7], Vol. I, p. 277 (27)) and the orthogonality property ([7], Vol. I, p. 276 (22)) of the Jacobi polynomials, we thus obtain

$$(2.5)\qquad C_n = \prod_{i=1}^{r}\left\{\frac{(-1)^n(2n+\lambda_i)(-\mu_i)_n(\varepsilon_i)_{\mu_i}}{(\varepsilon_i)_n(n+\lambda_i)_{1+\mu_i}} {}_2F_2\left[\begin{matrix} 1+\mu_i, & \mu_i+\varepsilon_i; \\ 1-n+\mu_i, & 1+n+\lambda_i+\mu_i; \end{matrix} z_i\right]\right\},$$

for $n=0, 1, 2, \cdots$; where, for the sake of brevity, $\varepsilon_i = 1+\beta_i$ and $\lambda_i = 1+\alpha_i+\beta_i$, $1\leqq i \leqq r$.

If we substitute this value of C_n in (2.3), employ the hypergeometric representations given by (2.2) and then replace $(1+x_i)/2$ by w_i, $i=1, \cdots, r$, we shall arrive at the desired result (2.1) under the conditions stated already.

Next we consider a general function of several variables $z_1, \cdots, z_r$ defined by the multiple series

$$(2.6)\qquad F(z_1,\cdots,z_r) = \sum_{m_1,\cdots,m_r=0}^{\infty} \gamma(m_1,\cdots,m_r)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!},$$

where the coefficients $\gamma(m_1, \cdots, m_r)$ are arbitrary constants, real or complex, depending on the indices $m_1, \cdots, m_r$, and the multiple series converges absolutely when $|z_i|<R_i$, with $0<R_i\leqq 1$, $i=1, \cdots, r$. For the function in (2.6), the above analysis will lead to a straightforward generalization of Lemma 1 given by

LEMMA 2. *With* $F(z_1, \cdots, z_r)$ *defined by* (2.6), *let* $\mu_i \geqq 0$, $0<w_i<1$, $\varepsilon_i>0$, $\lambda_i-\varepsilon_i>-1$, *and* $|z_i|<R_i$, $i=1,\cdots, r$.

Then

$$w_1^{\mu_1}\cdots w_r^{\mu_r}F(w_1z_1,\cdots,w_rz_r)=\sum_{n=0}^{\infty}g_n(z_1,\cdots,z_r) \tag{2.7}$$
$$\cdot\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n(\varepsilon_i)_{\mu_i}}{n!(n+\lambda_i)_{1+\mu_i}}\,{}_2F_1\left[\begin{matrix}-n,\ n+\lambda_i;\\ \varepsilon_i;\end{matrix}\ w_i\right]\right\},$$

where, for convenience,

$$g_n(z_1,\cdots,z_r)=\sum_{m_1,\cdots,m_r=0}^{\infty}\gamma(m_1,\cdots,m_r) \tag{2.8}$$
$$\cdot\prod_{i=1}^{r}\left\{\frac{(1+\mu_i)_{m_i}(\mu_i+\varepsilon_i)_{m_i}}{(1-n+\mu_i)_{m_i}(1+n+\lambda_i+\mu_i)_{m_i}}\frac{z_i^{m_i}}{m_i!}\right\},\quad n\geqq 0\,.$$

In particular, if we set

$$\gamma(m_1,\cdots,m_r)=\Omega(m_1,\cdots,m_r),\quad m_i=0,1,2,\cdots;\ 1\leqq i\leqq r\,; \tag{2.9}$$

where $\Omega(m_1,\cdots,m_r)$ is given by (1.2), Lemma 2 will at once yield an expansion for the generalized Lauricella function of r variables, defined by (1.1), in the form

$$w_1^{\mu_1}\cdots w_r^{\mu_r}F^{A:\,B';\cdots;\,B^{(r)}}_{C:\,D';\cdots;\,D^{(r)}}\begin{pmatrix}w_1z_1\\ \vdots\\ w_rz_r\end{pmatrix}=\sum_{n=0}^{\infty}F_n[z_1,\cdots,z_r] \tag{2.10}$$
$$\cdot\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n(\varepsilon_i)_{\mu_i}}{n!(n+\lambda_i)_{1+\mu_i}}\,{}_2F_1\left[\begin{matrix}-n,\ n+\lambda_i;\\ \varepsilon_i;\end{matrix}\ w_i\right]\right\},$$

where

$$F_n[z_1,\cdots,z_r]=F^{A:\,B'+2;\,\cdots;\,B^{(r)}+2}_{C:\,D'+2;\,\cdots;\,D^{(r)}+2}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:\\ [(c):\psi',\cdots,\psi^{(r)}]:\end{matrix}\right. \tag{2.11}$$
$$\begin{matrix}[1+\mu_1:1],\ [\mu_1+\varepsilon_1:1],\ [(b'):\phi'];\ \cdots;\\ [1-n+\mu_1:1],\ [1+n+\lambda_1+\mu_1:1],\ [(d'):\delta'];\ \cdots;\end{matrix}$$
$$\left.\begin{matrix}[1+\mu_r:1],\ [\mu_r+\varepsilon_r:1],\ [(b^{(r)}):\phi^{(r)}];\\ [1-n+\mu_r:1],\ [1+n+\lambda_r+\mu_r:1],\ [(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right),\quad n\geqq 0\,,$$

and, as before, $\mu_i\geqq 0$, $0<w_i<1$, $\varepsilon_i>0$, $\lambda_i-\varepsilon_i>-1$, and

$$\Delta_i\equiv 1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,\quad i=1,\cdots,r\,, \tag{2.12}$$

or $\Delta_i=0$, provided that $|z_i|<S_i$, where

$$S_i=\min_{\nu_1,\cdots,\nu_r>0}\{E_i\},\quad i=1,\cdots,r\,, \tag{2.13}$$

with [9, p. 157]

$$(2.14)\qquad E_i=(\nu_i)^{1+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}}\frac{\prod_{j=1}^{C}\left(\sum_{i=1}^{r}\nu_i\psi_j^{(i)}\right)^{\psi_j^{(i)}}\prod_{j=1}^{D^{(i)}}(\delta_j^{(i)})^{\delta_j^{(i)}}}{\prod_{j=1}^{A}\left(\sum_{i=1}^{r}\nu_i\theta_j^{(i)}\right)^{\theta_j^{(i)}}\prod_{j=1}^{B^{(i)}}(\phi_j^{(i)})^{\phi_j^{(i)}}}\,.$$

3. Some generalizations

By applying the method of finite mathematical induction, using Laplace's transform and its inverse, it is not difficult to derive a formal extension of formula (2.10) in the form

$$\begin{aligned}
& w_1^{\mu_1}\cdots w_r^{\mu_r}F^{A:\,B'+E';\,\cdots;\,B^{(r)}+E^{(r)}}_{C:\,D'+G';\,\cdots;\,D^{(r)}+G^{(r)}}\left(\begin{matrix}[(a):\theta',\,\cdots,\,\theta^{(r)}]: \\ [(c):\psi',\,\cdots,\,\psi^{(r)}]:\end{matrix}\right. \\
& \left.\begin{matrix}[(e'):1],\,[(b'):\phi'];\,\cdots;\,[(e^{(r)}):1],\,[(b^{(r)}):\phi^{(r)}]; \\ [(g'):1],\,[(d'):\delta'];\,\cdots;\,[(g^{(r)}):1],\,[(d^{(r)}):\delta^{(r)}];\end{matrix}\; w_1z_1,\,\cdots,\,w_rz_r\right) \\
(3.1)\qquad & =\sum_{n=0}^{\infty}F_n[z_1,\,\cdots,\,z_r]\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n(\varepsilon_i)_{\mu_i}((e^{(i)}))_{-\mu_i}}{n!(n+\lambda_i)_{1+\mu_i}((g^{(i)}))_{-\mu_i}}\right\} \\
& \cdot{}_{E'+2}F_{G'+1}\left[\begin{matrix}-n,\,n+\lambda_1,\,(e')-\mu_1; \\ \varepsilon_1,\,(g')-\mu_1;\end{matrix}\;w_1\right] \\
& \cdots{}_{E^{(r)}+2}F_{G^{(r)}+1}\left[\begin{matrix}-n,\,n+\lambda_r,\,(e^{(r)})-\mu_r; \\ \varepsilon_r,\,(g^{(r)})-\mu_r;\end{matrix}\;w_r\right],
\end{aligned}$$

provided that both sides have a meaning. Evidently, (2.10) would follow from the last result (3.1) when $E^{(i)}=G^{(i)}=0$, $i=1,\,\cdots,\,r$.

In (3.1) we put $z_i=0$, $i=1,\,\cdots,\,r$, and replace $E^{(i)}$, $G^{(i)}$ by $E^{(i)}+K^{(i)}$ and $G^{(i)}+H^{(i)}$, respectively, $i=1,\,\cdots,\,r$. Also replace $e_j^{(i)}$ by $e_j^{(i)}+m_i$, $j=1,\,\cdots,\,E^{(i)}$, and $g_j^{(i)}$ by $g_j^{(i)}+m_i$, $j=1,\,\cdots,\,G^{(i)}$, and set

$$(3.2)\qquad \begin{cases} e^{(i)}_{E^{(i)}+j}=k_j^{(i)}+\mu_i\,, & j=1,\,\cdots,\,K^{(i)}\,, \\ g^{(i)}_{G^{(i)}+j}=h_j^{(i)}+\mu_i\,, & j=1,\,\cdots,\,H^{(i)}\,, \end{cases}$$

for each $i=1,\,\cdots,\,r$.

Next we divide both members of the resulting equation by $(\varepsilon_1)_{\mu_1}\cdots(\varepsilon_r)_{\mu_r}$, replace w_i by $w_i\varepsilon_i$ and let $\varepsilon_i\to\infty$, $i=1,\,\cdots,\,r$.

Finally, we replace μ_i by μ_i+m_i, $i=1,\,\cdots,\,r$, and (3.1) is reduced to the identity

$$\begin{aligned}
& w_1^{\mu_1+m_1}\cdots w_r^{\mu_r+m_r}=\prod_{i=1}^{r}\left\{\frac{((e^{(i)}))_{-\mu_i}((g^{(i)}))_{m_i}((h^{(i)}))_{\mu_i+m_i}}{((g^{(i)}))_{-\mu_i}((e^{(i)}))_{m_i}((k^{(i)}))_{\mu_i+m_i}}\right\} \\
(3.3)\qquad & \cdot\sum_{n=0}^{\infty}\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n(1+\mu_i)_{m_i}B_{n,i}(w_i)}{n!(n+\lambda_i)_{1+\mu_i}(1-n+\mu_i)_{m_i}(1+n+\lambda_i+\mu_i)_{m_i}}\right\},
\end{aligned}$$

where, for convenience,

$$B_{n,i}(w_i) = {}_{E^{(i)}+K^{(i)}+2}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix} -n,\ n+\lambda_i,\ (e^{(i)})-\mu_i,\ (k^{(i)}); \\ (g^{(i)})-\mu_i,\ (h^{(i)}); \end{matrix}\ w_i\right], \quad (i=1, \cdots, r)\,. \tag{3.4}$$

If we multiply both members of (3.3) by

$$\Omega(m_1, \cdots, m_r) \prod_{i=1}^{r} \left\{\frac{((e^{(i)}))_{m_i}}{((g^{(i)}))_{m_i}} \frac{z_i^{m_i}}{m_i!}\right\}, \tag{3.5}$$

sum the resulting expressions for each m_i from 0 to ∞, $i=1, \cdots, r$, and appeal to the definitions (1.1) and (1.2), we shall obtain the expansion formula

$$\begin{aligned} &w_1^{\mu_1}\cdots w_r^{\mu_r} F^{A:\,B'+E';\,\cdots;\,B^{(r)}+E^{(r)}}_{C:\,D'+G';\,\cdots;\,D^{(r)}+G^{(r)}}\left(\begin{matrix}[(a):\theta', \cdots, \theta^{(r)}]: \\ [(c):\psi', \cdots, \psi^{(r)}]:\end{matrix}\right. \\ &\left.\begin{matrix}[(e'):1],\ [(b'):\phi'];\ \cdots;\ [(e^{(r)}):1],\ [(b^{(r)}):\phi^{(r)}]; \\ [(g'):1],\ [(d'):\delta'];\ \cdots;\ [(g^{(r)}):1],\ [(d^{(r)}):\delta^{(r)}];\end{matrix}\ w_1z_1, \cdots, w_rz_r\right) \\ &= \sum_{n=0}^{\infty} G_n[z_1, \cdots, z_r] \prod_{i=1}^{r} \left\{\frac{(2n+\lambda_i)(-\mu_i)_n((e^{(i)}))_{-\mu_i}((h^{(i)}))_{\mu_i}}{n!(n+\lambda_i)_{1+\mu_i}((g^{(i)}))_{-\mu_i}((k^{(i)}))_{\mu_i}}\right\} \\ &\cdot \prod_{j=1}^{r} {}_{E^{(j)}+K^{(j)}+2}F_{G^{(j)}+H^{(j)}}\left[\begin{matrix} -n,\ n+\lambda_j,\ (e^{(j)})-\mu_j,\ (k^{(j)}); \\ (g^{(j)})-\mu_j,\ (h^{(j)}); \end{matrix}\ w_j\right], \end{aligned} \tag{3.6}$$

where, for brevity,

$$\begin{aligned} &G_n[z_1, \cdots, z_r] = F^{A:\,B'+H'+1;\,\cdots;\,B^{(r)}+H^{(r)}+1}_{C:\,D'+K'+2;\,\cdots;\,D^{(r)}+K^{(r)}+2}\left(\begin{matrix}[(a):\theta', \cdots, \theta^{(r)}]: \\ [(c):\psi', \cdots, \psi^{(r)}]:\end{matrix}\right. \\ &\begin{matrix}[1+\mu_1:1],\ [(h')+\mu_1:1],\ [(b'):\phi'];\ \cdots; \\ [1-n+\mu_1:1],\ [1+n+\lambda_1+\mu_1:1],\ [(k')+\mu_1:1],\ [(d'):\delta'];\ \cdots;\end{matrix} \\ &\left.\begin{matrix}[1+\mu_r:1],\ [(h^{(r)})+\mu_r:1],\ [(b^{(r)}):\phi^{(r)}]; \\ [1-n+\mu_r:1],\ [1+n+\lambda_r+\mu_r:1],\ [(k^{(r)})+\mu_r:1],\ [(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1, \cdots, z_r\right), \end{aligned} \tag{3.7}$$

with $n=0, 1, 2, \cdots$.

So far we have only shown that formula (3.6) is a formal identity. We now proceed to determine the hypotheses under which (3.6) is valid. We first state

THEOREM 1. *The expansion formula* (3.6) *holds true under the assumptions given below*:

1. *Let none of the following quantities be negative integers*:

$$\lambda_i,\ \mu_i;\ e_j^{(i)}-\mu_i-1,\ j=1, \cdots, E^{(i)};\ h_j^{(i)}+\mu_i-1,\ j=1, \cdots, H^{(i)};$$
$$k_j^{(i)}-1,\ j=1, \cdots, K^{(i)};\ 1 \leqq i \leqq r\,.$$

2. *Let* A, $B^{(i)}$, C, $D^{(i)}$, $E^{(i)}$, $G^{(i)}$ *and* $H^{(i)}$, $i=1, \cdots, r$, *be nonnegative integers, and let the parameters in* (1.3) *be positive such that*

(i) $\Delta_i > E^{(i)} - G^{(i)}$ *and* $E^{(i)} + K^{(i)} + 1 = G^{(i)} + H^{(i)}$, $|z_i| < \infty$, $i = 1, \cdots, r$;

or

(ii) $\Delta_i = E^{(i)} - G^{(i)}$ *and* $E^{(i)} + K^{(i)} + 1 = G^{(i)} + H^{(i)}$, $|z_i| < S_i$, $i = 1, \cdots, r$, *where* Δ_i, S_i *are given by* (2.12) *and* (2.13), *respectively.*

3. *Let* $0 < w_i \leqq 1$, $i = 1, \cdots, r$.

4. *Let the following inequalities be satisfied.*

(i) $\begin{cases} \mathscr{R}\{e_j^{(i)} - \mu_i\} > 0, \ j = 1, \cdots, E^{(i)}; \\ \mathscr{R}\{k_j^{(i)}\} > 0, \ j = 1, \cdots, K^{(i)}; \ i = 1, \cdots, r; \end{cases}$

(ii) $\mathscr{R}\left\{\sum_{i=1}^{r} \mu_i\right\} > -\frac{1}{2}(r-1)$;

and

(iii) $\mathscr{R}\left\{\sum_{i=1}^{r} (\zeta_i - \mu_i)\right\} < \frac{1}{2}(r-1)$, *if* $0 < w_i < 1$, $i = 1, \cdots, r$; *or*

(iv) $\mathscr{R}\left\{\sum_{i=1}^{r} (\lambda_i - 2\mu_i + 4\zeta_i)\right\} < (r-1)$, *if* $w_1 = \cdots = w_r = 1$,

where

$$2\zeta_i = \frac{1}{2} + \sum_{j=1}^{E^{(i)}} e_j^{(i)} + \sum_{j=1}^{K^{(i)}} k_j^{(i)} - \sum_{j=1}^{G^{(i)}} g_j^{(i)} - \sum_{j=1}^{H^{(i)}} h_j^{(i)} + \mu_i[G^{(i)} - E^{(i)}] , \quad \textit{for } i = 1, \cdots, r . \tag{3.8}$$

This expansion is also valid when $w_1 = \cdots = w_r = 0$ *provided* $\mathscr{R}(\mu_i) > 0$, $i = 1, \cdots, r$.

The assumptions 1 are required to insure that the various gamma functions, which appear outside the hypergeometric functions occurring in formula (3.6), are finite. The conditions $\Delta_i \geqq E^{(i)} - G^{(i)}$, with equality when $|z_i| < S_i$, $i = 1, \cdots, r$, are necessary to insure the (absolute) convergence and meaning of the generalized Lauricella functions (cf. [9], p. 158). Note, in this connection, that the conditions under 2, viz.

$$\Delta_i \geqq E^{(i)} - G^{(i)} \text{ and } E^{(i)} + K^{(i)} + 1 = G^{(i)} + H^{(i)} \quad (\text{equality when } |z_i| < S_i) , \tag{3.9}$$

imply

$$1 + \Delta_i \geqq H^{(i)} - K^{(i)} \text{ (equality when } |z_i| < S_i) \quad (i = 1, \cdots, r) \tag{3.10}$$

which indeed are the conditions of convergence of the multiple hypergeometric series of the generalized Lauricella function $G_n[z_1, \cdots, z_r]$ defined by (3.7). Also, since $0 < w_i \leqq 1$, we have

$$|w_i z_i| \leqq |z_i| < S_i , \qquad i = 1, \cdots, r , \tag{3.11}$$

and the left-hand side of the expansion formula (3.6) is well-defined by virtue of 3 and the Δ-conditions in 2(i) and (ii).

The remaining hypotheses arise from the consideration of the convergence of the infinite series in (3.6), and depend on the asymptotic behaviour of the hypergeometric functions $G_n[z_1, \cdots, z_r]$ and $B_{n,i}(w_i)$, $i=1, \cdots, r$, defined by (3.7) and (3.4), respectively, for large n. In view of the assumptions (3.9) the multiple hypergeometric series in (3.7) converges absolutely, as we have just observed, and for $z_1, \cdots, z_r$ fixed and n sufficiently large and positive, we have

$$G_n[z_1, \cdots, z_r]=1+o(1) , \tag{3.12}$$

since (see, for instance, [7], Vol. I, p. 33)

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}=n^{\alpha-\beta}[1+0(n^{-1})] \quad \text{for large} \quad n>0 . \tag{3.13}$$

Next we consider the behaviour of

$$\Gamma_n(w_1, \cdots, w_r)=A_n B_{n,1}(w_1)\cdots B_{n,r}(w_r) , \tag{3.14}$$

where

$$A_n=\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n}{n!(n+\lambda_i)_{1+\mu_i}}\right\} , \tag{3.15}$$

and the $B_{n,i}(w_i)$, $i=1, \cdots, r$, are given by (3.4).

Making use of (3.13) we have, for n large and positive,

$$A_n=\frac{2^r}{\Gamma(-\mu_1)\cdots\Gamma(-\mu_r)}n^{-r-2(\mu_1+\cdots+\mu_r)}[1+0(n^{-1})] . \tag{3.16}$$

On the other hand, the behaviour of $B_{n,i}(w_i)$, $0<w_i\leqq 1$, $i=1, \cdots, r$, for large $n>0$ would follow from references [4], [5], [6] and [7, Vol. I, p. 250 (8) and p. 259 (23)]. Since $E^{(i)}+K^{(i)}+1=G^{(i)}+H^{(i)}$, $i=1, \cdots, r$, by applying (3.13) and the known result (2.5), p. 399 in [4], we have

$$\begin{aligned} B_{n,i}(w_i)\sim &\sum_{j=1}^{E^{(i)}} n^{-2(e_j^{(i)}-\mu_i)}\mathscr{L}^{(e_j^{(i)}-\mu_i)}_{E^{(i)}+K^{(i)}+2,G^{(i)}+H^{(i)}}(w_i) \\ &+\sum_{j=1}^{K^{(i)}} n^{-2k_j^{(i)}}\mathscr{L}^{(k_j^{(i)})}_{E^{(i)}+K^{(i)}+2,G^{(i)}+H^{(i)}}(w_i) \\ &+\frac{\prod_{j=1}^{G^{(i)}}\Gamma(g_j^{(i)}-\mu_i)\prod_{j=1}^{H^{(i)}}\Gamma(h_j^{(i)})}{\Gamma\left(\frac{1}{2}\right)\prod_{j=1}^{E^{(i)}}\Gamma(e_j^{(i)}-\mu_i)\prod_{j=1}^{K^{(i)}}\Gamma(k_j^{(i)})}n^{2\zeta_i}K_{n,i}(w_i) , \end{aligned} \tag{3.17}$$

where $|K_{n,i}(w_i)|$ is bounded for all n if $0<w_i<1$, $i=1, \cdots, r$, and the ζ_i are given by (3.8).

The precise nature of the $\mathscr{L}$-terms in (3.17) are readily available in the literature cited. For our present object, however, it would suffice to know the order estimate

$$\mathscr{L}^{(\alpha_j^{(i)})}_{E^{(i)}+K^{(i)}+2,G^{(i)}+H^{(i)}}(w_i)=\Lambda_i w_i^{-\alpha_j^{(i)}}[1+0(n^{-1})] , \tag{3.18}$$

where the Λ_i are independent of n and w_i, $i=1, \cdots, r$. On substituting from (3.16), (3.17) and (3.18) in (3.14) it follows fairly easily that the right-hand side of the expansion formula (3.6) converges if the hypotheses 4 (i), (ii) and (iii) are satisfied, provided $0<w_i<1$, $i=1, \cdots, r$.

The situation with $w_1=\cdots=w_r=1$ can be disposed of in a similar manner by using (3.13) and the known formula (2.16), p. 597 in reference [6].

Finally, we note that if the assumptions 4 (i), (ii) and (iii) are not satisfied, then an analysis similar to the above, but based upon certain results given in refèrence [5], would show that, in general, the series

$$\sum_{n=0}^{\infty} \Gamma_n(w_1, \cdots, w_r)$$

diverges.

This evidently completes the proof of Theorem 1.

Now we turn to a generalization of Theorem 1 given by

THEOREM 2. *Let the hypotheses* 1, 3 *and* 4 *of Theorem* 1 *be satisfied. Also let* $A, B^{(i)}, C, D^{(i)}, E^{(i)}, G^{(i)}, H^{(i)}, U$ *and* V, $i=1, \cdots, r$, *be nonnegative integers, and let the parameters in* (1.3) *together with* $\xi_j^{(i)}$, $j=1, \cdots, U$; $\eta_j^{(i)}$, $j=1, \cdots, V$; $1\leqq i\leqq r$, *be positive such that*

(i) $\Delta_i>E^{(i)}-G^{(i)}+U-V$ *and* $U+E^{(i)}+K^{(i)}+1=V+G^{(i)}+H^{(i)}$, $|z_i|<\infty$, $i=1, \cdots, r$; *or*

(ii) $\Delta_i=E^{(i)}-G^{(i)}+U-V$ *and* $U+E^{(i)}+K^{(i)}+1=V+G^{(i)}+H^{(i)}$, $|z_i|<T_i$, $i=1, \cdots, r$,

where

$$T_i=\min_{\nu_1,\cdots,\nu_r>0} \{G_i\}, \qquad i=1, \cdots, r, \tag{3.19}$$

with the G_i *defined in terms of the* E_i *in* (2.14) *by*

$$G_i=E_i\frac{\prod_{j=1}^{U}\left(\sum_{i=1}^{r}\nu_i\xi_j^{(i)}\right)^{\xi_j^{(i)}}}{\prod_{j=1}^{V}\left(\sum_{i=1}^{r}\nu_i\eta_j^{(i)}\right)^{\eta_j^{(i)}}}, \qquad i=1, \cdots, r. \tag{3.20}$$

Then

$$w_1^{\mu_1}\cdots w_r^{\mu_r}F^{A+U:\,B'+E';\,\cdots;\,B^{(r)}+E^{(r)}}_{C+V:\,D'+G';\,\cdots;\,D^{(r)}+G^{(r)}}\left(\begin{matrix}[(a):\theta', \cdots, \theta^{(r)}], [(u):\xi', \cdots, \xi^{(r)}]:\\ [(c):\psi', \cdots, \psi^{(r)}], [(v):\eta', \cdots, \eta^{(r)}]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(e'):1], [(b'):\phi']; \cdots; [(e^{(r)}):1], [(b^{(r)}):\phi^{(r)}];\\ [(g'):1], [(d'):\delta']; \cdots; [(g^{(r)}):1], [(d^{(r)}):\delta^{(r)}];\end{matrix}\; w_1z_1, \cdots, w_rz_r\right)$$
$$=\frac{((u))_{-M}}{((v))_{-N}}\sum_{n=0}^{\infty} G_n[z_1, \cdots, z_r]\prod_{i=1}^{r}\left\{\frac{(2n+\lambda_i)(-\mu_i)_n((e^{(i)}))_{-\mu_i}((h^{(i)}))_{\mu_i}}{n!(n+\lambda_i)_{1+\mu_i}((g^{(i)}))_{-\mu_i}((k^{(i)}))_{\mu_i}}\right\}$$

$$(3.21)\qquad \cdot F^{U:\ E'+K'+2;\ \cdots;\ E^{(r)}+K^{(r)}+2}_{V:\quad G'+H';\ \cdots;\ G^{(r)}+H^{(r)}} \left(\begin{matrix}[(u)-M:\xi',\cdots,\xi^{(r)}]:[-n:1],[n+\lambda_1:1],\\ [(v)-N:\eta',\cdots,\eta^{(r)}]:\end{matrix}\right.$$

$$\begin{matrix}[(e')-\mu_1:1],[(k'):1];\cdots;[-n:1],[n+\lambda_r:1],\\ [(g')-\mu_1:1],[(h'):1];\cdots;\end{matrix}$$

$$\left.\begin{matrix}[(e^{(r)})-\mu_r:1],[(k^{(r)}):1];\\ [(g^{(r)})-\mu_r:1],[(h^{(r)}):1];\end{matrix}\ w_1,\cdots,w_r\right),$$

where $G_n[z_1,\cdots,z_r]$ *is given by* (3.7), *and for convenience,*

$$(3.22)\qquad M=\sum_{i=1}^{r}\mu_i\xi^{(i)}\,,\qquad N=\sum_{i=1}^{r}\mu_i\eta^{(i)}\,.$$

Proof. Our proof of Theorem 2 is by induction on the integers U and V. As a matter of fact, the special case $U=V=0$ of (3.21) is the expansion formula (3.6). Assuming (3.21) to hold true for some values of the nonnegative integers U and V, replace each w_i by $w_i t^{\xi_i}$, $i=1,\cdots,r$, multiply both sides by $t^{\rho-1}$, and take their Laplace transforms using the familiar result

$$(3.23)\qquad \int_0^\infty e^{-t}t^{z-1}dt=\Gamma(z)\,,\qquad \mathscr{R}(z)>0\,.$$

Now replace ρ on both sides by $\rho-P$, where

$$(3.24)\qquad P=\sum_{i=1}^{r}\mu_i\xi_i\,,$$

and the induction on U is completed.

To effect the induction with respect to V, replace each w_i in (3.21) by w_i/ζ^{η_i}, $i=1,\cdots,r$, multiply both of its members by $\zeta^{-\sigma}$, and take their inverse Laplace transforms using the known result

$$(3.25)\qquad \frac{1}{2\pi\omega}\int_{\tau-\omega\infty}^{\tau+\omega\infty}e^{\zeta}\zeta^{-z}d\zeta=\frac{1}{\Gamma(z)}\,,\quad \mathscr{R}(z)>0\,,\quad \omega=\sqrt{-1}\,.$$

Replacing σ by $\sigma-Q$, where

$$(3.26)\qquad Q=\sum_{i=1}^{r}\mu_i\eta_i\,,$$

we thus find that V is replaced by $V+1$.

Hence the proof of (3.21) by induction is completed, and indeed the final result stated in Theorem 2 would follow by appealing to the principle of analytic continuation.

4. Specialized and confluent cases

At the outset we remark that Theorems 1 and 2 are very general in character, and can be suitably specialized to obtain several classes of expansion formulas involving hypergeometric functions. For example, if $A=C=0$, Theorem 1 will yield an expansion for the product of several generalized hypergeometric functions introduced by Wright (cf. [11] and [12]). By setting each of the ϕ's and δ's equal to 1, this expansion can be reduced in terms of the product

$$
\tag{4.1}
\begin{aligned}
&{}_{B'+E'}F_{D'+G'}\left[\begin{matrix}(b'),\,(e');\\(d'),\,(g');\end{matrix}\;w_1z_1\right]\\
&\cdots{}_{B^{(r)}+E^{(r)}}F_{D^{(r)}+G^{(r)}}\left[\begin{matrix}(b^{(r)}),\,(e^{(r)});\\(d^{(r)}),\,(g^{(r)});\end{matrix}\;w_rz_r\right]
\end{aligned}
$$

of r (ordinary) hypergeometric functions.

On the other hand, by appropriately specializing the various parameters involved, for instance, in (3.21) we can easily deduce expansion formulas associated with Lauricella's hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ of r variables (cf., e.g., [1], p. 115). In view of the triviality of the analysis to be applied we omit details, which may well be left as an exercise for the interested reader.

With each of the parameters in (1.3) equated to 1, the special cases $r=1$ and $r=2$ of Theorem 1 are substantially the same as the expansions given earlier by Wimp and Luke [10], and Deshpande [2], respectively; the latter [2, p. 41, Eq. (2.1)] also gave an equivalent of the special case $r=2$ of Theorem 2 with the aforementioned choice of the various parameters listed in (1.3) and with

$$
\tag{4.2}
\xi'_j=\xi''_j=1,\; j=1,\cdots,U;\; \eta'_j=\eta''_j=1,\; j=1,\cdots,V\,.
$$

See also Luke [7, Vol. II, § 9.1]. Note that some of the hypotheses of Theorems 1 and 2 can be considerably relaxed in these special cases.

Next we derive two further results similar to Theorems 1 and 2. As a matter of fact, these are confluent forms of (3.6) and (3.21) which would follow if we appeal to the familiar principle of confluence exhibited by

$$
\tag{4.3}
\lim_{\lambda\to\infty}\{(\lambda)_m(z/\lambda)^m\}=z^m;\; \lim_{\mu\to\infty}\{(\mu w)^m/(\mu)_m\}=w^m;\; m=0,1,2,\cdots\,.
$$

If, in (3.6) and (3.21), we replace each w_i, z_i by w_i/λ_i and $z_i\lambda_i$, respectively, $i=1,\cdots,r$, and let $\lambda_1,\cdots,\lambda_r\to\infty$, we shall arrive formally at the following confluent forms of Theorems 1 and 2.

THEOREM 3. *Let the assumptions* 1 (*with the* λ_i *deleted from the* *quantities listed*), 2, *and* 4 (i) *of Theorem* 1 *be satisfied. Also let* $\Delta_i \geqq 1 + H^{(i)} - K^{(i)}$, $0 < w_i < \infty$, $i = 1, \cdots, r$, *and*

$$\mathscr{R}\left\{\sum_{i=1}^{r} \mu_i\right\} > 1 - r\,, \quad \mathscr{R}\left\{\sum_{i=1}^{r} (\zeta_i - \mu_i)\right\} < r - 1\,, \tag{4.4}$$

where the ζ_i *are given by* (3.8).

Then

$$\begin{aligned} & w_1^{\mu_1} \cdots w_r^{\mu_r} F^{A:\, B' + E';\, \cdots;\, B^{(r)} + E^{(r)}}_{C:\, D' + G';\, \cdots;\, D^{(r)} + G^{(r)}} \begin{pmatrix} w_1 z_1 \\ \vdots \\ w_r z_r \end{pmatrix} \\ & = \sum_{n=0}^{\infty} H_n[z_1, \cdots, z_r] \prod_{i=1}^{r} \left\{ \frac{(-\mu_i)_n ((e^{(i)}))_{-\mu_i} ((h^{(i)}))_{\mu_i}}{n! ((g^{(i)}))_{-\mu_i} ((k^{(i)}))_{\mu_i}} \right\} \\ & \cdot \prod_{j=1}^{r} {}_{E^{(j)} + K^{(j)} + 1}F_{G^{(j)} + H^{(j)}} \left[\begin{matrix} -n,\, (e^{(j)}) - \mu_j,\, (k^{(j)}); \\ (g^{(j)}) - \mu_j,\, (h^{(j)}); \end{matrix} \; w_j \right], \end{aligned} \tag{4.5}$$

where the generalized Lauricella function on the left-hand side is the same as in (3.6), *and*

$$\begin{aligned} & H_n[z_1, \cdots, z_r] = F^{A:\, B' + H' + 1;\, \cdots;\, B^{(r)} + H^{(r)} + 1}_{C:\, D' + K' + 1;\, \cdots;\, D^{(r)} + K^{(r)} + 1} \\ & \left(\begin{matrix} [(a): \theta', \cdots, \theta^{(r)}]: \quad [1 + \mu_1: 1], [(h') + \mu_1: 1], [(b'): \phi']; \cdots; \\ [(c): \psi', \cdots, \psi^{(r)}]: [1 - n + \mu_1: 1], [(k') + \mu_1: 1[, [(d'): \delta']; \cdots; \end{matrix} \right. \\ & \qquad \left. \begin{matrix} [1 + \mu_r: 1], [(h^{(r)}) + \mu_r: 1], [(b^{(r)}): \phi^{(r)}]; \\ [1 - n + \mu_r: 1], [(k^{(r)}) + \mu_r: 1], [(d^{(r)}): \delta^{(r)}]; \end{matrix} \; z_1, \cdots, z_r \right), \end{aligned} \tag{4.6}$$

with $n = 0, 1, 2, \cdots$.

THEOREM 4. *Let the hypotheses* 1 (*with the* λ_i *deleted from the quantities listed*) *and* 4 (i) *of Theorem* 1, *and the assumptions* (i) *and* (ii) *of Theorem* 2, *be satisfied. Also let* $\Delta_i \geqq 1 + H^{(i)} - K^{(i)}$, $0 < w_i < \infty$, $i = 1, \cdots, r$, *and the inequalities in* (4.4) *hold true.*

Then

$$\begin{aligned} & w_1^{\mu_1} \cdots w_r^{\mu_r} F^{A + U:\, B' + E';\, \cdots;\, B^{(r)} + E^{(r)}}_{C + V:\, D' + G';\, \cdots;\, D^{(r)} + G^{(r)}} \begin{pmatrix} w_1 z_1 \\ \vdots \\ w_r z_r \end{pmatrix} \\ & = \frac{((u))_{-M}}{((v))_{-N}} \sum_{n=0}^{\infty} H_n[z_1, \cdots, z_r] \prod_{i=1}^{r} \left\{ \frac{(-\mu_i)_n ((e^{(i)}))_{-\mu_i} ((h^{(i)}))_{\mu_i}}{n! ((g^{(i)}))_{-\mu_i} ((k^{(i)}))_{\mu_i}} \right\} \\ & \cdot F^{U:\, E' + K' + 1;\, \cdots;\, E^{(r)} + K^{(r)} + 1}_{V: \quad G' + H';\, \cdots; \quad G^{(r)} + H^{(r)}} \left(\begin{matrix} [(u) - M: \xi', \cdots, \xi^{(r)}]: \\ [(v) - N: \eta', \cdots, \eta^{(r)}]: \end{matrix} \right. \\ & [-n: 1], [(e') - \mu_1: 1], [(k'): 1]; \cdots; \\ & \qquad [(g') - \mu_1: 1[, [(h'): 1]; \cdots; \end{aligned} \tag{4.7}$$

$$\left.\begin{matrix}[-n\colon 1],\ [(e^{(r)})-\mu_r\colon 1],\ [(k^{(r)})\colon 1];\\ [(g^{(r)})-\mu_r\colon 1],\ [(h^{(r)})\colon 1];\end{matrix} w_1, \cdots, w_r\right),$$

where the generalized Lauricella function on the left-hand side is the same as in (3.21), *M, N are defined by* (3.22), *and the* $H_n[z_1, \cdots, z_r]$ *are given by* (4.6).

Proofs of these theorems are very similar to those of Theorems 1 and 2. Indeed, if $\Delta_i \geqq 1+H^{(i)}-K^{(i)}$, $i=1, \cdots, r$, it is readily seen that, for $z_1, \cdots, z_r$ fixed and n sufficiently large and positive,

$$H_n[z_1, \cdots, z_r]=1+o(1). \tag{4.8}$$

Also, the order estimate of

$$B^*_{n,i}(w_i)={}_{E^{(i)}+K^{(i)}+1}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ (e^{(i)})-\mu_i,\ (k^{(i)});\\ (g^{(i)})-\mu_i,\ (h^{(i)});\end{matrix}\ w_i\right] \tag{4.9}$$

for large $n>0$, with $E^{(i)}+K^{(i)}+1=G^{(i)}+H^{(i)}$, $i=1, \cdots, r$, follows fairly easily from references [5, p. 446] and [7, Vol. I, p. 264 (2)]. We omit details.

Our remarks about Theorems 1 and 2, made at the beginning of this section, would apply also to Theorems 3 and 4 which evidently incorporate, as their special cases, a considerably large number of known or new expansion formulas associated with various classes of hypergeometric functions.

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