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Some expansion theorems and generating relations for the H function of several complex variables*

by

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Summary

The first half of the present work is essetially a sequel to the authors' recent paper [11], and it gives expansion formulas for the multiple H-function in a series of products of several generalized hypergeometric polynomials (cf. Theorems 1 and 2 below); it also demonstrates how these expansion theorems would admit themselves of further generalizations given by Theorems 3 and 4 below. The second half deals with various classes of generating relations involving the H-function of several complex variables. Finally, some possible applications of the results presented here and their relevant connections with a number of known results are indicated briefly.

1. Introduction and definitions.

For the generalized Lauricella function ([6], p. 454; cf. also [8], p. 19 et seq.)

$$F\begin{matrix} A: B'; \cdots; B^{(r)} \\ C: D'; \cdots; D^{(r)} \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}, \tag{1.1}$$

we gave the expansion formula [8, p. 24, Eq. (3. 6)]

$$\begin{aligned} t_1^{\rho_1}\cdots t_r^{\rho_r} F\begin{matrix} A: B'+E'; \cdots; B^{(r)}+E^{(r)} \\ C: D'+G'; \cdots; D^{(r)}+G^{(r)} \end{matrix}\Bigg(&\begin{matrix}[(a): \theta', \cdots, \theta^{(r)}]: \\ [(c): \psi', \cdots, \psi^{(r)}]: \end{matrix} \\ &\begin{matrix}[(e'): 1], [(b'): \phi']; \cdots; [(e^{(r)}): 1], [(b^{(r)}): \phi^{(r)}]; \\ [(g'): 1], [(d'): \delta']; \cdots; [(g^{(r)}): 1], [(d^{(r)}): \delta^{(r)}]; \end{matrix} z_1t_1, \cdots, z_rt_r\Bigg) \\ = \sum_{n=0}^{\infty} F_n[z_1, \cdots, z_r] \prod_{i=1}^{r} \Bigg\{ &\frac{(2n+\gamma_i)[-\rho_i]_n[(e^{(i)})]_{-\rho_i}[(h^{(i)})]_{\rho_i}}{n!\,[n+\gamma_i]_{1+\rho_i}[(g^{(i)})]_{-\rho_i}[(k^{(i)})]_{\rho_i}} B_{n,i}(t_i)\Bigg\}, \end{aligned} \tag{1.2}$$

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together with its confluent form [op. cit., p. 30, Eq. (4.5)]

$$(1.3)\qquad t_1^{\rho_1}\cdots t_r^{\rho_r}F\begin{matrix}A:\ B'+E';\ \cdots;\ B^{(r)}+E^{(r)}\\ C:\ D'+G';\ \cdots;\ D^{(r)}+G^{(r)}\end{matrix}\begin{pmatrix}z_1t_1\\ \vdots\\ z_rt_r\end{pmatrix}$$
$$=\sum_{n=0}^{\infty}G_n[z_1,\ \cdots,\ z_r]\prod_{i=1}^{r}\left\{\frac{[-\rho_i]_n[(e^{(i)})]_{-\rho_i}[(h^{(i)})]_{\rho_i}}{n!\,[(g^{(i)})]_{-\rho_i}[(k^{(i)})]_{\rho_i}}B^*_{n,i}(t_i)\right\},$$

where the generalized Lauricella functions on the left-hand sides of (1.2) and (1.3) are the same,

$$(1.4)\quad F_n[z_1,\ \cdots,\ z_r]=F\begin{matrix}A:\ B'+H'+1;\ \cdots;\ B^{(r)}+H^{(r)}+1\\ C:\ D'+K'+2;\ \cdots;\ D^{(r)}+K^{(r)}+2\end{matrix}\Bigg(\begin{matrix}[(a):\theta',\ \cdots,\ \theta^{(r)}]:\\ [(c):\psi',\ \cdots,\ \psi^{(r)}]:\end{matrix}$$
$$[1+\rho_1:1],\ [(h')+\rho_1:1],\ [(b'):\phi'];\ \cdots;$$
$$[1-n+\rho_1:1],\ [1+n+\gamma_1+\rho_1:1],\ [(k')+\rho_1:1],\ [(d'):\delta'];\ \cdots;$$
$$[1+\rho_r:1],\ [(h^{(r)})+\rho_r:1],\ [(b^{(r)}):\phi^{(r)}];$$
$$[1-n+\rho_r:1],\ [1+n+\gamma_r+\rho_r:1],\ [(k^{(r)})+\rho_r:1],\ [(d^{(r)}):\delta^{(r)}];\ z_1,\ \cdots,\ z_r\Bigg),$$

$$(1.5)\qquad G_n[z_1,\ \cdots,\ z_r]=F\begin{matrix}A:\ B'+H'+1;\ \cdots;\ B^{(r)}+H^{(r)}+1\\ C:\ D'+K'+1;\ \cdots;\ D^{(r)}+K^{(r)}+1\end{matrix}$$
$$\Bigg(\begin{matrix}[(a):\theta',\ \cdots,\ \theta^{(r)}]:\ [1+\rho_1:1],\ [(h')+\rho_1:1],\ [(b'):\phi'];\ \cdots;\\ [(c):\psi',\ \cdots,\ \psi^{(r)}]:\ [1-n+\rho_1:1],\ [(k')+\rho_1:1],\ [(d'):\delta'];\ \cdots;\end{matrix}$$
$$[1+\rho_r:1],\ [(h^{(r)})+\rho_r:1],\ [(b^{(r)}):\phi^{(r)}];$$
$$[1-n+\rho_r:1],\ [(k^{(r)})+\rho_r:1],\ [(d^{(r)}):\delta^{(r)}];\ z_1,\ \cdots,\ z_r\Bigg),$$

$$(1.6)\qquad B_{n,i}(t_i)={}_{E^{(i)}+K^{(i)}+2}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ n+\gamma_i,\ (e^{(i)})-\rho_i,\ (k^{(i)});\\ (g^{(i)})-\rho_i,\ (h^{(i)});\end{matrix}t_i\right]$$

and

$$(1.7)\qquad B^*_{n,i}(t_i)={}_{E^{(i)}+K^{(i)}+1}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ (e^{(i)})-\rho_i,\ (k^{(i)});\\ (g^{(i)})-\rho_i,\ (h^{(i)});\end{matrix}t_i\right],$$
$$(n=0,\ 1,\ 2,\ \cdots;\ i=1,\ \cdots,\ r).$$

{For the sets of sufficient conditions of validity of the expansion formulas (1.2) and (1.3) see Theorem 1, p. 24 and Theorem 3, p. 29, respectively, of reference [8].}

Here as well as throughout this paper we use the abbreviation (a) to denote the sequence of A parameters $a_1,\ \cdots,\ a_A$; for each $i=1,\ \cdots,\ r$, $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters $b_j^{(i)}$, $j=1,\ \cdots,\ B^{(i)}$, with similar interpretations for (c), $(d^{(i)})$, etc., $i=1,\ \cdots,\ r$, it being understood, for example, that $b^{(1)}=b'$, $b^{(2)}=b''$, and so on. Also, for the sake of brevity, we employ the following contracted notations:

$$(1.8)\qquad [(a)]_n=\prod_{j=1}^{A}[a_j]_n,\ [(b^{(i)})]_n=\prod_{j=1}^{B^{(i)}}[b_j^{(i)}]_n,\ i=1,\cdots,r;\ \text{etc.},$$

where $[\alpha]_n$ is the Pochhammer symbol defined by

$$(1.9)\qquad [\alpha]_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=\begin{cases}1, & \text{if } n=0\,.\\ \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n=1,2,3,\cdots\,.\end{cases}$$

In the present paper we first derive extensions of the expansion formulas (1.2) and (1.3) to hold for the H-function of several complex variables $z_1,\cdots,z_r$, which is defined by the multiple contour integral ([10], Eq. (4.1) et seq.; see also [11])

$$(1.10)\qquad H^{0,\,\lambda:\,(\mu',\nu');\cdots;\,(\mu^{(r)},\nu^{(r)})}_{A,\,C:\,[B',D'];\cdots;\,[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:\\ [(c):\psi',\cdots,\psi^{(r)}]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right)$$
$$=\frac{1}{(2\pi\omega)^r}\int_{\mathscr{L}_1}\cdots\int_{\mathscr{L}_r}\Phi_1(\zeta_1)\cdots\Phi_r(\zeta_r)\Psi(\zeta_1,\cdots,\zeta_r)\cdot z_1^{\zeta_1}\cdots z_r^{\zeta_r}d\zeta_1\cdots d\zeta_r\,,\qquad \omega=\sqrt{-1}\,,$$

where

$$(1.11)\qquad \Phi_i(\zeta_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\zeta_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\zeta_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\zeta_i]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}\zeta_i]}\,,$$
$$i=1,\cdots,r\,;$$

$$(1.12)\qquad \Psi(\zeta_1,\cdots,\zeta_r)=\frac{\prod_{j=1}^{\lambda}\Gamma\left[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i\right]}{\prod_{j=\lambda+1}^{A}\Gamma\left[a_j-\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i\right]\prod_{j=1}^{C}\Gamma\left[1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\zeta_i\right]}\,,$$

an empty product is interpreted as 1, the coefficients† $\theta_j^{(i)}$, $j=1,\cdots,A$; $\phi_j^{(i)}$, $j=1,\cdots,B^{(i)}$; $\psi_j^{(i)}$, $j=1,\cdots,C$; $\delta_j^{(i)}$, $j=1,\cdots,D^{(i)}$, $(i=1,\cdots,r)$, are positive numbers, and λ, $\mu^{(i)}$, $\nu^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ are integers such that $0\leqq\lambda\leqq A$, $0\leqq\mu^{(i)}\leqq D^{(i)}$, $C\geqq 0$, and $0\leqq\nu^{(i)}\leqq B^{(i)}$, $i=1,\cdots,r$. The contour $\mathscr{L}_i$ in the complex ζ_i-plane is of the Mellin-Barnes type which runs from $-\omega^{\infty}$ to ω^{∞} with indentations, if necessary, in such a manner that all the poles of $\Gamma[d_j^{(i)}-\delta_j^{(i)}\zeta_i]$, $j=1,\cdots,\mu^{(i)}$, are to the right, and those of $\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\zeta_i]$, $j=1,\cdots,\nu^{(i)}$, and $\Gamma[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i]$, $j=1,\cdots,\lambda$, to the left, of $\mathscr{L}_i$, the various parameters being so restricted that these poles are all simple and none of them coincide; and with

† The special case of (1.10), when these coefficients θ, ϕ, ψ, and δ are all equated to 1, was considered by Khadia and Goyal [2].

the points $z_i=0$, $i=1, \cdots, r$, being tacitly excluded, the multiple integral in (1.10) converges absolutely if

$$|\arg(z_i)| < \frac{1}{2}\pi\Delta_i , \qquad i=1, \cdots, r , \tag{1.13}$$

where

$$\begin{aligned}\Delta_i=&\sum_{j=1}^{\lambda}\theta_j^{(i)}-\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}\\&-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0 , \qquad i=1, \cdots, r .\end{aligned} \tag{1.14}$$

Conditions corresponding to the aforementioned ones will be assumed to hold throughout the present paper, and for convenience, we shall employ a contracted notation and write the first member of (1.10) in the abbreviated form

$$H\begin{matrix}0,\ \lambda:\ (\mu',\ \nu');\ \cdots;\ (\mu^{(r)},\ \nu^{(r)})\\ A,\ C:\ [B',\ D'];\ \cdots;\ [B^{(r)},\ D^{(r)}]\end{matrix}\begin{pmatrix}z_1\\ \vdots\\ z_r\end{pmatrix}, \tag{1.15}$$

wherever no confusion arises.

Thus it is easily verified that

$$\begin{aligned}&H\begin{matrix}0,\ \lambda:\ (\mu',\ \nu');\ \cdots;\ (\mu^{(r)},\ \nu^{(r)})\\ A,\ C:\ [B',\ D'];\ \cdots;\ [B^{(r)},\ D^{(r)}]\end{matrix}\begin{pmatrix}z_1\\ \vdots\\ z_r\end{pmatrix}\\&=\begin{cases}0(|z_1|^{\alpha_1}\cdots|z_r|^{\alpha_r}), & \max\{|z_1|, \cdots, |z_r|\}\to 0 ,\\ 0(|z_1|^{-\beta_1}\cdots|z_r|^{-\beta_r}), & \lambda\equiv 0,\ \min\{|z_1|, \cdots, |z_r|\}\to\infty ,\end{cases}\end{aligned} \tag{1.16}$$

where, with $i=1, \cdots, r$.

$$\begin{cases}\alpha_i=d_j^{(i)}/\delta_j^{(i)}, & j=1, \cdots, \mu^{(i)} .\\ \beta_i=[1-b_j^{(i)}]/\phi_j^{(i)}, & j=1, \cdots, \nu^{(i)} .\end{cases} \tag{1.17}$$

The aforementioned extensions of (1.2) and (1.3), contained in Theorems 1 and 2 below, together with Theorems 3 and 4 of our previous paper [11], are intended to complete the development of expansion theory of the multiple H-function (1.10) in series of various classes of generalized hypergeometric polynomials or their products. Indeed, we also show the main expansions (2.8) and (2.22), given by Theorems 1 and 2 below, admit themselves of further generalizations involving hypergeometric polynomials in several variables.

The remainder of this work is concerned with a considerably large variety of generating relations for the H-function of several complex variables. And we conclude this paper by presenting a brief discussion

on the possible applications of the various results obtained here and on the relevant connections of these results with certain known ones.

2. The main expansion theorems.

By letting $z_1=\cdots=z_r=0$ our expansion formula (1.2) yields

$$(2.1)\qquad t_1^{\rho_1}\cdots t_r^{\rho_r}=\prod_{i=1}^{r}\left\{\frac{\Gamma(1+\rho_i)[(g^{(i)})]_{\rho_i}[(h^{(i)})]_{\rho_i}}{[(e^{(i)})]_{\rho_i}[(k^{(i)})]_{\rho_i}}\right\}$$
$$\cdot\sum_{n=0}^{\infty}\prod_{i=1}^{r}\left\{\frac{(-1)^n(2n+\gamma_i)\Gamma(n+\gamma_i)}{n!\,\Gamma(1-n+\rho_i)\Gamma(1+n+\gamma_i+\rho_i)}\right.$$
$$\left.\cdot\,{}_{E^{(i)}+K^{(i)}+2}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ n+\gamma_i,\ (e^{(i)}),\ (k^{(i)});\\ (g^{(i)}),\ (h^{(i)});\end{matrix}\ t_i\right]\right\},$$

where, for convenience, the contracted notations in (1.8) are used, and the result holds, by the principle of analytic continuation, whenever the series converges to the indicated sum.

As a consequence of (2.1) we shall derive a generalization of our expansion formula (1.2) contained in

THEOREM 1. *With* Δ_i *and* α_i *defined by* (1.14) *and* (1.17), *respectively, let* $\sigma_i>0$, $E^{(i)}+K^{(i)}+1=G^{(i)}+H^{(i)}$,

$$(2.2)\qquad \Delta_i+[E^{(i)}-G^{(i)}]\sigma_i>0,\ |\arg(z_i)|<\frac{1}{2}\pi\{\Delta_i+[E^{(i)}-G^{(i)}]\sigma_i\},$$

and

$$(2.3)\qquad \begin{cases}\mathrm{Re}\,\{1-\rho_i-e_j^{(i)}\}>0,\ j=1,\cdots,E^{(i)},\\ \mathrm{Re}\,\{k_j^{(i)}\}>0,\ j=1,\cdots,K^{(i)};\ i=1,\cdots,r.\end{cases}$$

Also let $0<t_i\leqq 1$, $i=1,\cdots,r$,

$$(2.4)\qquad \mathrm{Re}\left\{\sum_{i=1}^{r}(\rho_i+\sigma_i\alpha_i)\right\}>-\frac{1}{2}(r-1),$$

and

$$(2.5)\qquad \mathrm{Re}\left\{\sum_{i=1}^{r}(\Omega_i-\rho_i-\sigma_i\alpha_i)\right\}<\frac{1}{2}(r-1),\quad \text{if } 0<t_i<1,$$

or

$$(2.6)\qquad \mathrm{Re}\left\{\sum_{i=1}^{r}(\gamma_i-2\rho_i-2\sigma_i\alpha_i+4\Omega_i)\right\}<(r-1),\quad \text{if } t_i=1,$$

where

$$(2.7)\qquad \Omega_i=\frac{1}{2}-\sum_{j=1}^{E^{(i)}}e_j^{(i)}+\sum_{j=1}^{K^{(i)}}k_j^{(i)}+\sum_{j=1}^{G^{(i)}}g_j^{(i)}-\sum_{j=1}^{H^{(i)}}h_j^{(i)}$$
$$+(1-\rho_i)[E^{(i)}-G^{(i)}],\quad i=1,\cdots,r.$$

Then

$$(2.8)\qquad t_1^{\rho_1}\cdots t_r^{\rho_r}H\begin{matrix}0,\ \lambda:(\mu',\nu'+E');\cdots;(\mu^{(r)},\nu^{(r)}+E^{(r)})\\ A,\ C:[B'+E',\ D'+G'];\cdots;[B^{(r)}+E^{(r)},\ D^{(r)}+G^{(r)}]\end{matrix}$$
$$\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(e'):\sigma_1],[(b'):\phi'];\cdots;\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'],(g'):\sigma_1];\cdots;\end{matrix}\right.$$
$$\left.\begin{matrix}[(e^{(r)}):\sigma_r],[(b^{(r)}):\phi^{(r)}];\\ [(d^{(r)}):\delta^{(r)}],[(g^{(r)}):\sigma_r];\end{matrix}\ z_1t_1^{q_1},\cdots,z_rt_r^{q_r}\right)$$
$$=\sum_{n=0}^{\infty}H_n[z_1,\cdots,z_r]\prod_{i=1}^{r}\left\{\Delta_i\frac{(-1)^n(2n+\gamma_i)\Gamma(n+\gamma_i)}{n!}P_{n,i}(t_i)\right\},$$

where, for convenience,

$$(2.9)\qquad \Delta_i=\frac{\prod_{j=1}^{E^{(i)}}\Gamma[1-\rho_i-e_j^{(i)}]\prod_{j=1}^{K^{(i)}}\Gamma[k_j^{(i)}]}{\prod_{j=1}^{G^{(i)}}\Gamma[1-\rho_i-g_j^{(i)}]\prod_{j=1}^{H^{(i)}}\Gamma[h_j^{(i)}]},$$

$$(2.10)\qquad P_{n,i}(t_i)=_{E^{(i)}+K^{(i)}+2}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ n+\gamma_i,\ 1-\rho_i-(e^{(i)}),\ (k^{(i)});\\ 1-\rho_i-(g^{(i)}),\ (h^{(i)});\end{matrix}\ t_i\right],$$
$$(i=1,\cdots,r),$$

and

$$(2.11)\quad H_n[z_1,\cdots,z_r]$$
$$=H\begin{matrix}0,\ \lambda:\ (\mu',1+\nu'+H');\cdots;\ (\mu^{(r)},1+\nu^{(r)}+H^{(r)})\\ A,\ C:[1+B'+H',\ 2+D'+K';\cdots;[1+B^{(r)}+H^{(r)},\ 2+D^{(r)}+K^{(r)}]\end{matrix}$$
$$\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:\\ [(c):\psi',\cdots,\psi^{(r)}]:\end{matrix}\right.$$
$$[-\rho_1:\sigma_1],[1-\rho_1-(h'):\sigma_1],[(b'):\phi'];\cdots;$$
$$[(d'):\delta'],[n-\rho_1:\sigma_1],[-\rho_1-\gamma_1-n:\sigma_1],[1-\rho_1-(k'):\sigma_1];\cdots;$$
$$[-\rho_r:\sigma_r],[1-\rho_r-(h^{(r)}):\sigma_r],[(b^{(r)}):\phi^{(r)}];$$
$$\left.[(d^{(r)}):\delta^{(r)}],[n-\rho_r:\sigma_r],[-\rho_r-\gamma_r-n:\sigma_r],[1-\rho_r-(k^{(r)}):\sigma_r];\ z_1,\cdots,z_r\right),$$
$$n\geqq 0.$$

Proof. In order to establish the expansion formula (2.8) we first replace the H-function on its left-hand side by means of the corresponding multiple Mellin-Barnes integral given by (1.10), apply (2.1) with the $e^{(i)}$ and $g^{(i)}$ parameters replaced by $1-\rho_i-e^{(i)}$ and $1-\rho_i-g^{(i)}$, respectively, $i=1,\cdots,r$, and then interchange the order of summation and integration. On interpreting the resulting multiple contour integral as an H-function of r variables $z_1,\cdots,z_r$, by using (1.10) again, we shall arrive formally at the expansion formula (2.8).

So far we have only shown that formula (2.8) is a formal identity. We now proceed to demonstrate that (2.8) is indeed valid under the hypotheses of Theorem 1 above.

To begin with, notice that since

$$\forall i \in \{1, \cdots, r\},\ \sigma_i > 0 \quad \text{and} \quad E^{(i)} + K^{(i)} + 1 = G^{(i)} + H^{(i)}\ , \tag{2.12}$$

the inequalities in (2.2) are sufficient to insure that the H-functions occurring on both sides of equation (2.8) are well defined.

On the other hand, the precise (sufficient) conditions of convergence of the infinite series in (2.8) may be determined by considering the asymptotic behaviours of the function $H_n[z_1, \cdots, z_r]$ for large n and fixed $|z_1|, \cdots, |z_r|$, and of the product

$$A_n(t_1, \cdots, t_r) = \prod_{i=1}^{r} \left\{ \frac{(-1)^n (2n+\gamma_i)\Gamma(n+\gamma_i)}{n!} P_{n,i}(t_i) \right\}\ , \tag{2.13}$$

when $n \to \infty$ and $0 < t_i \leqq 1$, $i =, \cdots, r$.

Indeed, for bounded $z_1, \cdots, z_r$, it is readily verified by using (1.10), (1.16), (2.11), and the familiar result (cf., e.g., [3], Vol. I, p. 33)

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}[1 + 0(1/n)]\ , \qquad n \to \infty\ , \tag{2.14}$$

that

$$H_n[z_1, \cdots, z_r] \sim n^{-r - \sum_{i=1}^{r} (\gamma_i + 2\rho_i + 2\sigma_i \alpha_i)}\ , \qquad n \to \infty\ , \tag{2.15}$$

where α_i is given by equation (1.17).

Also, since $E^{(i)} + K^{(i)} + 1 = G^{(i)} + H^{(i)}$, $i = 1, \cdots, r$, by appealing to (2.14) and the known result ([3], Vol. I, p. 250, Eq. (8); cf. also Vol. II, p. 3, Eq. (7)), we have

$$\begin{aligned} P_{n,i}(t_i) \sim & \sum_{j=1}^{E^{(i)}} n^{-2(1-\rho_i - e_j^{(i)})} \mathscr{L}^{(1-\rho_i - e_j^{(i)})}_{E^{(i)}+K^{(i)}+2,\ G^{(i)}+H^{(i)}}(t_i) \\ & + \sum_{j=1}^{K^{(i)}} n^{-2k_j^{(i)}} \mathscr{L}^{(k_j^{(i)})}_{E^{(i)}+K^{(i)}+2,\ G^{(i)}+H^{(i)}}(t_i) \\ & + \frac{n^{2\Omega_i}}{\sqrt{\pi}\,\Delta_i} K_{n,i}(t_i), \quad 0 < t_i < 1, \quad n \to \infty\ , \end{aligned} \tag{2.16}$$

where $|K_{n,i}(t_i)|$ is bounded for all n, and Ω_i, Δ_i are given by (2.7) and (2.9), respectively.

The precise nature of the $\mathscr{L}$-terms in (2.16) are available in the literature cited. For our purpose, however, it would be sufficient to know the order estimate

$$\mathscr{L}^{(\varepsilon_j^{(i)})}_{E^{(i)}+K^{(i)}+2,\ G^{(i)}+H^{(i)}}(t_i) = \Xi_i t_i^{-\varepsilon_j^{(i)}}[1 + 0(1/n)]\ , \qquad n \to \infty\ , \tag{2.17}$$

where Ξ_i is independent of n and t_i, $i = 1, \cdots, r$.

These considerations, followed by another appeal to (2.14) and the

known asymptotic estimate [op. cit., p. 259, Eq. (23)] in order to dispose of the situation with $t_1=\cdots=t_r=1$, will evidently lead to the hypotheses (2.3) through (2.6), and our demonstration of Theorem 1 is completed.

In a similar manner, we now apply the formula

$$(2.18)\quad t_1^{\rho_1}\cdots t_r^{\rho_r}=\prod_{i=1}^{r}\left\{\frac{\Gamma(1+\rho_i)[(g^{(i)})]_{\rho_i}[(h^{(i)})]_{\rho_i}}{[(e^{(i)})]_{\rho_i}[(k^{(i)})]_{\rho_i}}\right\}$$
$$\cdot\sum_{n=0}^{\infty}\prod_{i=1}^{r}\left\{\frac{(-1)^n}{n!\,\Gamma(1-n+\rho_i)}\,{}_{E^{(i)}+K^{(i)}+1}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ (e^{(i)}),\ (k^{(i)});\\ (g^{(i)}),\ (h^{(i)});\end{matrix}\ t_i\right]\right\},$$

which obviously would follow either as a special case of our expansion formula (1.3) when $z_1=\cdots=z_r=0$, or as a confluent case of (2.1), when each t_i is replaced by t_i/γ_i and $\gamma_i\to\infty$, $i=1,\cdots,r$, since

$$(2.19)\quad \lim_{\gamma\to\infty}{}_{A+2}F_B\left[\begin{matrix}-n,\ n+\gamma,\ (a);\\ (b);\end{matrix}\ t/\gamma\right]={}_{A+1}F_B\left[\begin{matrix}-n,\ (a);\\ (b);\end{matrix}\ t\right];$$

by making use of the asymptotic estimates [3, Vol. I, p. 264, Eq. (2), and p. 265, Eq. (7); see also Vol. II, p. 10], we are thus led to

THEOREM 2. *With Δ_i, α_i and Ω_i defined by* (1.14), (1.17) *and* (2.7), *respectively, let $\sigma_i>0$, $i=1,\cdots,r$, and the hypotheses* (2.2) *and* (2.3) *of Theorem* 1 *hold.*

Also let $0<t_i<\infty$, and

$$(2.20)\quad \mathrm{Re}\left\{\sum_{i=1}^{r}(\rho_i-\Omega_i+\sigma_i\alpha_i)\right\}>1-r,\ \textit{if}\ E^{(i)}+K^{(i)}+1=G^{(i)}+H^{(i)},$$

or

$$(2.21)\quad \mathrm{Re}\left\{\sum_{i=1}^{r}(\rho_i-2\Omega_i+\sigma_i\alpha_i)\right\}>1-\frac{3}{2}r,\ \textit{if}\ E^{(i)}+K^{(i)}=G^{(i)}+H^{(i)},$$
$$i=1,\cdots,r.$$

Then

$$(2.22)\quad t_1^{\rho_1}\cdots t_r^{\rho_r}H\begin{matrix}0,\ \lambda:\\ A,\ C:\end{matrix}\begin{matrix}(\mu',\ \nu'+E');\cdots;\\ [B'+E',\ D'+G'];\cdots;\end{matrix}\begin{matrix}(\mu^{(r)},\ \nu^{(r)}+E^{(r)})\\ [B^{(r)}+E^{(r)},\ D^{(r)}+G^{(r)}]\end{matrix}\begin{pmatrix}z_1t_1^{\sigma_1}\\ \vdots\\ z_rt_r^{\sigma_r}\end{pmatrix}$$
$$=\sum_{n=0}^{\infty}K_n[z_1,\cdots,z_r]\prod_{i=1}^{r}\left\{\Delta_i\frac{(-1)^n}{n!}P_{n,i}^{*}(t_i)\right\},$$

where the H-functions on the left-hand sides of (2.8) *and* (2.22) *are identical, Δ_i is given by* (2.9),

$$(2.23)\quad P_{n,i}^{*}(t_i)={}_{E^{(i)}+K^{(i)}+1}F_{G^{(i)}+H^{(i)}}\left[\begin{matrix}-n,\ 1-\rho_i-(e^{(i)}),\ (k^{(i)});\\ 1-\rho_i-(g^{(i)}),\ (h^{(i)});\end{matrix}\ t_i\right],$$
$$(i=1,\cdots,r),$$

and

$$(2.24)\quad K_n[z_1, \cdots, z_r]=H\begin{matrix}0,\ \lambda: & (\mu', 1+\nu'+H'); \cdots; \\ A,\ C: & [1+B'+H', 1+D'+K']; \cdots; \\ & (\mu^{(r)}, 1+\nu^{(r)}+H^{(r)}) \\ & [1+B^{(r)}+H^{(r)}, 1+D^{(r)}+K^{(r)}]\end{matrix}$$

$$\left(\begin{matrix}[(a): \theta', \cdots, \theta^{(r)}]: [-\rho_1: \sigma_1], [1-\rho_1-(h'): \sigma_1], [(b'): \phi']; \cdots; \\ [(c): \psi', \cdots, \psi^{(r)}]: [(d'): \delta'], [n-\rho_1: \sigma_1], [1-\rho_1-(k'): \sigma_1]; \cdots; \\ [-\rho_r: \sigma_r], [1-\rho_r-(h^{(r)}): \sigma_r], [(b^{(r)}): \phi^{(r)}]; \\ [(d^{(r)}): \delta^{(r)}], [n-\rho_r: \sigma_r], [1-\rho_r-(k^{(r)}): \sigma_r];\end{matrix} z_1, \cdots, z_r\right), \quad n\geqq 0 .$$

Remark 1. The hypotheses (2.20) and (2.21) stem, among other considerations, from the asymptotic estimates

$$(2.25)\quad \frac{K_n[z_1, \cdots, z_r]}{(n!)^r}\sim n^{-r-\sum_{i=1}^{r}(\rho_i+\sigma_i\alpha_i)}, \quad n\to\infty ,$$

$$\max\{|z_1|, \cdots, |z_r|\}<M_0 , \quad \text{for some } M_0>0 ,$$

which would follow fairly readily from the definition (2.24) in conjunction with (1.10), (1.16) and (2.14)

Remark 2. Since

$$(2.26)\quad \lim_{\gamma_i\to\infty} P_{n,i}(t_i/\gamma_i)=P^*_{n,i}(t_i) , \quad i=1, \cdots, r ,$$

which is an immediate consequence of (2.19), it is not difficult to verify that Theorem 2 is essentially a confluent form of Theorem 1. Free use will be made here of the familiar principle of confluence exhibited, for instance, by

$$(2.27)\quad \lim_{\xi\to\infty}\{(\xi z)^n/[\xi]_n\}=z^n; \ \lim_{\eta\to\infty}\{[\eta]_n(t/\eta)^n\}=t^n ; \qquad (n=0, 1, 2, \cdots) .$$

3. Generalizations of theorems 1 and 2.

The desired generalizations of our expansion formulas (2.8) and (2.22) are contained in the following theorems.

THEOREM 3. *Let the hypotheses* (2.3) *through* (2.6) *of Theorem* 1 *be satisfied. Also, in addition to the various coefficients in* (1.11) *and* (1.12), *let* $\xi_j^{(i)}$, $j=1, \cdots, U$; $\eta_j^{(i)}$, $j=1, \cdots, V$, *and* σ_i, $i=1, \cdots, r$, *be all positive such that*

$$(3.1)\quad \Delta_i+\left[E^{(i)}-G^{(i)}+\sum_{j=1}^{U}\xi_j^{(i)}-\sum_{j=1}^{V}\eta_j^{(i)}\right]\sigma_i>0 ,$$

$$(3.2)\quad U+E^{(i)}+K^{(i)}+1=V+G^{(i)}+H^{(i)} ,$$

and

$$(3.3)\qquad |\arg(z_i)|<\frac{1}{2}\pi\left\{\Delta_i+\left[E^{(i)}-G^{(i)}+\sum_{j=1}^{U}\xi_j^{(i)}-\sum_{j=1}^{V}\eta_j^{(i)}\right]\sigma_i\right\},\qquad (i=1,\cdots,r),$$

where, as before, Δ_i *is given by* (1.14).

Then

$$(3.4)\quad t_1^{\sigma_1}\cdots t_r^{\sigma_r}H\begin{matrix}0,\ \lambda+U: & (\mu',\nu'+E');\cdots; & (\mu^{(r)},\nu^{(r)}+E^{(r)})\\ A+U,\ C+V: & [B'+E',D'+G'];\cdots; & [B^{(r)}+E^{(r)},D^{(r)}+G^{(r)}]\end{matrix}\left(\begin{matrix}[(u):\sigma_1\xi',\cdots,\sigma_r\xi^{(r)}],[(a):\theta',\cdots,\theta^{(r)}]:[(e'):\sigma_1],[(b'):\phi'];\cdots;\\ [(v):\sigma_1\eta',\cdots,\sigma_r\eta^{(r)}],[(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'],[(g'):\sigma_1];\cdots;\\ [(e^{(r)}):\sigma_r],[(b^{(r)}):\phi^{(r)}];\\ [(d^{(r)}):\delta^{(r)}],[(g^{(r)}):\sigma_r];\end{matrix}z_1t_1^{\sigma_1},\cdots,z_rt_r^{\sigma_r}\right)$$

$$=\frac{\prod_{j=1}^{U}\Gamma(1-u_j-M_j)}{\prod_{j=1}^{V}\Gamma(1-v_j-N_j)}\sum_{n=0}^{\infty}H_n[z_1,\cdots,z_r]\prod_{i=1}^{r}\left\{\Lambda_i\frac{(-1)^n(2n+\gamma_i)\Gamma(n+\gamma_i)}{n!}\right\}$$

$$\cdot F\begin{matrix}U:E'+K'+2;\cdots;E^{(r)}+K^{(r)}+2\\ V:G'+H';\cdots;G^{(r)}+H^{(r)}\end{matrix}\left(\begin{matrix}[1-(u)-M:\xi',\cdots,\xi^{(r)}]:\\ [1-(v)-N:\eta',\cdots,\eta^{(r)}]:\\ [-n:1],[n+\gamma_1:1],[1-\rho_1-(e'):1],[(k'):1];\cdots;\\ [1-\rho_1-(g'):1],[(h'):1];\cdots;\\ [-n:1],[n+\gamma_r:1],[1-\rho_r-(e^{(r)}):1],[(k^{(r)}):1];\\ [1-\rho_r-(g^{(r)}):1],[(h^{(r)}):1];\end{matrix}t_1,\cdots,t_r\right),$$

where Λ_i *and* $H_n[z_1,\cdots,z_r]$ *are given by* (2.9) *and* (2.11), *respectively, and*

$$(3.5)\qquad M_j=\sum_{i=1}^{r}\rho_i\xi_j^{(i)},\qquad N_j=\sum_{i=1}^{r}\rho_i\eta_j^{(i)},$$

it being understood that $1-(u)-M$ *stands for the set of* U *parameters*

$$1-u_1-M_1,\cdots,1-u_U-M_U,$$

with similar interpretation for $1-(v)-N$.

THEOREM 4. *Let* $\xi_j^{(i)}>0$, $j=1,\cdots,U$; $\eta_j^{(i)}>0$, $j=1,\cdots,V$; $\sigma_i>0$, $i=1,\cdots,r$, *and let the assumptions* (3.1) *and* (3.3) *of Theorem* 3 *be satisfied. Furthermore, let* $0<t_i<\infty$, $i=1,\cdots,r$, *and suppose that the inequality in* (2.20) *or* (2.21) *holds according as* (3.2) *or*

$$(3.6)\qquad U+E^{(i)}+K^{(i)}=V+G^{(i)}+H^{(i)},\quad i=1,\cdots,r.$$

Then

$$(3.7)\qquad t_1^{\rho_1}\cdots t_r^{\rho_r}$$

$$\cdot H\begin{matrix}0,\ \lambda+U: & (\mu',\ \nu'+E'); \cdots; & (\mu^{(r)},\ \nu^{(r)}+E^{(r)})\\ A+U,\ C+V: & [B'+E',\ D'+G']; \cdots; & [B^{(r)}+E^{(r)},\ D^{(r)}+G^{(r)}]\end{matrix}\begin{pmatrix} z_1 t_1^{\sigma_1} \\ \vdots \\ z_r t_r^{\sigma_r}\end{pmatrix}$$

$$=\frac{\prod_{j=1}^{U}\Gamma(1-u_j-M_j)}{\prod_{j=1}^{V}\Gamma(1-v_j-N_j)}\sum_{n=0}^{\infty}K_n[z_1,\cdots,z_r]\prod_{i=1}^{r}\left\{\Delta_i\frac{(-1)^n}{n!}\right\}$$

$$\cdot F\begin{matrix}U:\ E'+K'+1;\ \cdots;\ E^{(r)}+K^{(r)}+1\\ V:\ G'+H';\ \cdots;\ G^{(r)}+H^{(r)}\end{matrix}\begin{matrix}[1-(u)-M:\ \xi',\ \cdots,\ \xi^{(r)}]:\\ [1-(v)-N:\ \eta',\ \cdots,\ \eta^{(r)}]:\end{matrix}$$

$$[-n:1],\ [1-\rho_1-(e'):1],\ [(k'):1];\ \cdots;$$

$$[1-\rho_1-(g'):1],\ [(h'):1];\ \cdots;$$

$$[-n:1],\ [1-\rho_r-(e^{(r)}):1],\ [(k^{(r)}):1];\ t_1,\cdots,t_r\Big)\ ,$$

$$[1-\rho_r-(g^{(r)}:1],\ [(h^{(r)}):1];$$

where the H-functions on the left-hand sides of equations (3.4) *and* (3.7) *are identical, the M, N are defined by* (3.5) *with the same notational convention as already stated in Theorem* 3, *and* Δ_i, $K_n[z_1,\cdots,z_r]$ *are given by* (2.9) *and* (2.24), *respectively.*

Proofs. Our proofs of Theorems 3 and 4 are by induction on the nonnegative integers U and V. In fact, the special cases of (3.4) and (3.7) with $U=V=0$ are the same as our expansion formulas (2.8) and (2.22), respectively. We content ourselves by giving an outline of our demonstration of Theorem 3 as follows.

Assuming (3.4) to hold true for some fixed positive integers U and V, we replace each t_i by $t_i t^{\xi_i}$, $i=1,\cdots,r$, multiply both sides by $t^{\rho-1}$, and (formally) take their Laplace transforms using the familiar result

$$(3.8)\qquad \int_0^{\infty}e^{-t}t^{z-1}dt=\Gamma(z)\ ,\quad \mathrm{Re}\,(z)>0\ .$$

Now replace ρ on both sides by $1-\rho-P$, where

$$(3.9)\qquad P=\sum_{i=1}^{r}\rho_i\xi_i\ ,$$

and the induction on U is evidently completed.

Next we effect the induction with respect to V by first replacing each t_i in (3.4) by t_i/ζ^{η_i}, $i=1,\cdots,r$, multiplying both of its members by $\zeta^{-\sigma}$, and then (formally) taking their inverse Laplace transforms by means of the known result

$$(3.10)\qquad \frac{1}{2\pi\omega}\int_{\tau-\omega\infty}^{\tau+\omega\infty}e^{\zeta}\zeta^{-z}d\zeta=\frac{1}{\Gamma(z)}\ ,\quad \mathrm{Re}\,(z)>0,\ \omega=\sqrt{-1}\ .$$

If we write $1-\sigma-Q$ for σ, where

$$Q=\sum_{i=1}^{r}\rho_i\eta_i\ , \tag{3.11}$$

we shall thus observe that V is replaced by $V+1$.

This evidently completes a (formal proof of the expansion formula (3.4) by induction, and indeed the final result as stated in Theorem 3 would follow by appealing, as before, to the principle of analytic continuation.

Similar is our demonstration of Theorem 4, and we omit details.

4. Generating function.

In this section we derive a number of generating relations involving the H-function of several complex variables. The first set of such relations would stem from the following consequence of the Lagrange (expansion) theorem [4, p. 349, Problem 216]

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}t^n=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\ , \tag{4.1}$$

where α, β are complex parameters independent of n, and ζ is a function of t defined by

$$\zeta=t(1+\zeta)^{\beta+1}\ ,\quad \zeta(0)=0\ . \tag{4.2}$$

Indeed, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}&H\begin{matrix}0,\ \lambda+1:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})\\ A+1,\ C+1:[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}\\ &\cdot\left(\begin{matrix}[1+\alpha+\beta n:\sigma_1,\cdots,\sigma_r],\ [(a):\theta',\cdots,\theta^{(r)}]:\\ [1+\alpha+(\beta+1)n:\sigma_1,\cdots,\sigma_r],\ [(c):\psi',\cdots,\psi^{(r)}]:\\ [(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right)t^n\\ &=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}H\begin{matrix}0,\ \lambda:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})\\ A,\ C:[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}\begin{pmatrix}z_1/(1+\zeta)^{\sigma_1}\\ \vdots\\ z_r/(1+\zeta)^{\sigma_r}\end{pmatrix},\end{aligned} \tag{4.3}$$

where ζ is given by (4.2), $\sigma_i>0$, $i=1,\cdots,r$, and the inequalities in (1.13) and (1.14) hold;

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}&H\begin{matrix}0,\ \lambda:(\mu',\nu');\cdots;(\mu^{(i)},\nu^{(i)}+1);\cdots;(\mu^{(r)},\nu^{(r)})\\ A,\ C:[B',D'];\cdots;[B^{(i)}+1,D^{(i)}+1];\cdots;[B^{(r)},D^{(r)}]\end{matrix}\\ &\cdot\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(b'):\phi'];\cdots;\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'];\cdots;\end{matrix}\right.\end{aligned} \tag{4.4}$$

$$\left.\begin{matrix}[1+\alpha+\beta n: \sigma], & [(b^{(i)}): \phi^{(i)}]; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [(d^{(i)}): \delta^{(i)}], [1+\alpha+(\beta+1)n: \sigma]; \cdots; & [(d^{(r)}): \delta^{(r)}]; \end{matrix} z_1, \cdots, z_r\right) t^n$$

$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} H \begin{matrix} 0, \lambda: (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)}) \\ A, C: [B', D']; \cdots; [B^{(r)}, D^{(r)}] \end{matrix} \left(\begin{matrix} z_1 \\ \vdots \\ z_i/(1+\zeta)^\sigma \\ \vdots \\ z_r \end{matrix} \right),$$

where $1 \leqq i \leqq r$, ζ is defined by (4.2), $\sigma > 0$, and the conditions given by (1.13) and (1.14) are satisfied.

To prove (4.3) or (4.4), we replace the H-function on the left-hand side by its Mellin-Barnes contour integral given by (1.10), invert the order of summation and integration, and then apply (4.1) above. By interpreting the resulting multiple integral by means of (1.10), and appealing to the principle of analytic continuation, we thus arrive at the generating relations (4.3) and (4.4) under the aforementioned conditions.

By a similar application of Gould's identity [1, p. 196, Eq. (6.1)]

$$\text{(4.5)} \qquad \sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n$$

$$= (1+\zeta)^\alpha \sum_{n=0}^{\infty} (-1)^n \binom{\alpha-\gamma}{n} \binom{n+\gamma/(\beta+1)}{n}^{-1} \left(\frac{\zeta}{1+\zeta}\right)^n,$$

where α, β, γ are arbitrary complex numbers independent of n, and ζ is defined by (4.2), we shall obtain generalizations of (4.3) and (4.4) given by

THEOREM 5. *With* Δ_i *given by* (1.14), *let* $\Delta_i > 0$, $|\arg(z_i)| < (1/2)\pi\Delta_i$, $\sigma_i > 0$, $i=1, \cdots, r$, *and* $\sigma > 0$. *Also let*

$$\text{(4.6)} \qquad L_n[z_1, \cdots, z_r] = H \begin{matrix} 0, \lambda+1: (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)}) \\ A+1, C+1: [B', D']; \cdots; [B^{(r)}, D^{(r)}] \end{matrix}$$

$$\left(\begin{matrix} [1+\alpha-\gamma-n: \sigma_1, \cdots, \sigma_r], [(a): \theta', \cdots, \theta^{(r)}]: [(b'): \phi']; \cdots; \\ [1+\alpha-\gamma: \sigma_1, \cdots, \sigma_r], [(c): \psi', \cdots, \psi^{(r)}]: [(d'): \delta']; \cdots; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(r)}): \phi^{(r)}]; \\ [(d^{(r)}): \delta^{(r)}]; \end{matrix} z_1, \cdots, z_r \right)$$

and

$$\text{(4.7)} \qquad L_{n,i}[z_1, \cdots, z_r] = H \begin{matrix} 0, \lambda: (\mu', \nu'); \cdots; (\mu^{(i)}, \nu^{(i)}+1); \cdots; \\ A, C: [B', D']; \cdots; [B^{(i)}+1, D^{(i)}+1]; \cdots; \end{matrix}$$

$$(\mu^{(r)}, \nu^{(r)})\left([(a): \theta', \cdots, \theta^{(r)}]: [(b'): \phi']; \cdots; [1+\alpha-\gamma-n: \sigma],\right.$$
$$[B^{(r)}, D^{(r)}]\left[(c): \psi', \cdots, \psi^{(r)}\right]: [(d'): \delta']; \cdots; [(d^{(i)}): \delta^{(i)}],$$
$$[(b^{(i)}): \phi^{(i)}]; \cdots; \quad [(b^{(r)}): \phi^{(r)}];$$
$$\left.[1+\alpha-\gamma: \sigma]; \cdots; [(d^{(r)}): \delta^{(r)}]; z_1, \cdots, z_r\right), \quad 1 \leqq i \leqq r .$$

Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\gamma}{\gamma+(\beta+1)n} H_n^{(\alpha,\beta)}(z_1, \cdots, z_r)t^n$$
$$= (1+\zeta)^{\alpha}\Theta(z_1/(1+\zeta)^{\sigma_1}, \cdots, z_r/(1+\zeta)^{\sigma_r}; \zeta/(1+\zeta)) \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\gamma}{\gamma+(\beta+1)n} H_{n,i}^{(\alpha,\beta)}(z_1, \cdots, z_r)t^n$$
$$= (1+\zeta)^{\alpha}\Theta_i(z_1, \cdots, z_i/(1+\zeta)^{\sigma}, \cdots, z_r; \zeta/(1+\zeta)), \quad 1 \leqq i \leqq r , \tag{4.9}$$

where $H_n^{(\alpha,\beta)}(z_1, \cdots, z_r)$ *and* $H_{n,i}^{(\alpha,\beta)}(z_1, \cdots, z_r)$ *are the H-functions occurring on the left-hand sides of equations* (4.3) *and* (4.4), *respectively,* ζ *is given by* (4.2), *and*

$$\Theta[z_1, \cdots, z_r; t] = \sum_{n=0}^{\infty} \binom{n+\gamma/(\beta+1)}{n}^{-1} L_n[z_1, \cdots, z_r]\frac{t^n}{n!} , \tag{4.10}$$

$$\Theta_i[z_1, \cdots, z_r; t]$$
$$= \sum_{n=0}^{\infty} \binom{n+\gamma/(\beta+1)}{n}^{-1} L_{n,i}[z_1, \cdots, z_r]\frac{t^n}{n!} , \quad 1 \leqq i \leqq r . \tag{4.11}$$

Remark 3. The parameters α, β and γ are assumed to take on such values that our equations like (4.3), (4.4), (4.8) and (4.9) make sense.

Remark 4. For $\gamma=\alpha$, the generating function (4.8) and (4.9) would simplify considerably. On the other hand, their limiting cases when $\gamma \to \infty$ correspond formally to our formulas (4.3) and (4.4), respectively. Thus it would seem obvious that, for bounded γ, $\gamma \neq \alpha$, Theorm 5 may be looked upon as being independent of the generating relations (4.3) and (4.4).

5. Multiple series relations.

The generating-function relations of the preceding section would admit themselves of further generalizations involving multiple series. Indeed, if we let $\sigma_i > 0$, $\sigma_i^{(j)} > 0$, $i, j = 1, \cdots, r$, and define

$$(5.1)\quad P^{(\alpha,\beta)}_{n_1,\cdots,n_r}(z_1,\cdots,z_r)=H\begin{matrix}0,\ \lambda+r:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})\\ A+r,\ C+r:[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}$$
$$\left(\begin{matrix}[1+\alpha_1+\beta_1 n_1:\sigma_1',\cdots,\sigma_r'],\cdots,[1+\alpha_r+\beta_r n_r:\sigma_1^{(r)},\cdots,\sigma_r^{(r)}],\\ [1+\alpha_1+(\beta_1+1)n_1:\sigma_1',\cdots,\sigma_r'],\cdots,[1+\alpha_r+(\beta_r+1)n_r:\sigma_1^{(r)},\cdots,\sigma_r^{(r)}],\\ [(a):\theta',\cdots,\theta^{(r)}]:\ [(b'):\phi'];\cdots;\ [(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right)$$

and

$$(5.2)\quad P^{(\alpha,\beta)}_{n_1,\cdots,n_r}(z_1,\cdots,z_r)=H\begin{matrix}0,\ \lambda:(\mu',\nu'+1);\cdots;(\mu^{(r)},\nu^{(r)}+1)\\ A,\ C:[B'+1,D'+1];\cdots;[B^{(r)}+1,D^{(r)}+1]\end{matrix}$$
$$\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[1+\alpha_1+\beta_1 n_1:\sigma_1],\ [(b'):\phi'];\cdots;\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'],\ [1+\alpha_1+(\beta_1+1)n_1:\sigma_1];\cdots;\\ [1+\alpha_r+\beta_r n_r:\sigma_r],\ [(b^{(r)}):\phi^{(r)}];\\ [(d^{(r)}):\delta^{(r)}],\ [1+\alpha_r+(\beta_r+1)n_r:\sigma_r];\end{matrix}\ z_1,\cdots,z_r\right),$$

where, for convenience, $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ stand for the linear arrays $\alpha_1,\cdots,\alpha_r$ and $\beta_1,\cdots,\beta_r$, respectively, we shall readily get the following results.

$$(5.3)\quad \sum_{n_1,\cdots,n_r=0}^{\infty}\frac{(-1)^{n_1+\cdots+n_r}}{n_1!\cdots n_r!}P^{(\alpha,\beta)}_{n_1,\cdots,n_r}(z_1,\cdots,z_r)t_1^{n_1}\cdots t_r^{n_r}$$
$$=\prod_{i=1}^{r}\left\{\frac{(1+\zeta_i)^{\alpha_i+1}}{1-\beta_i\zeta_i}\right\}H\begin{matrix}0,\ \lambda:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})\\ A,\ C:[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}\begin{pmatrix}z_1/[(1+\zeta_1)^{\sigma_1'}\cdots(1+\zeta_r)^{\sigma_1^{(r)}}]\\ \vdots\\ z_r/[(1+\zeta_1)^{\sigma_r'}\cdots(1+\zeta_r)^{\sigma_r^{(r)}}]\end{pmatrix};$$

$$(5.4)\quad \sum_{n_1,\cdots,n_r=0}^{\infty}\frac{(-1)^{n_1+\cdots+n_r}}{n_1!\cdots n_r!}Q^{(\alpha,\beta)}_{n_1,\cdots,n_r}(z_1,\cdots,z_r)t_1^{n_1}\cdots t_r^{n_r}$$
$$=\prod_{i=1}^{r}\left\{\frac{(1+\zeta_i)^{\alpha_i+1}}{1-\beta_i\zeta_i}\right\}H\begin{matrix}0,\ \lambda:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})\\ A,\ C:[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}\begin{pmatrix}z_1/(1+\zeta_1)^{\sigma_1}\\ \vdots\\ z_r/(1+\zeta_r)^{\sigma_r}\end{pmatrix},$$

where ζ_i is a function of t_i defined by

$$(5.5)\quad \zeta_i=t_i(1+\zeta_i)^{\beta_i+1},\ \zeta_i(0)=0,\ i=1,\cdots,r\ .$$

Remark 5. For $n_2=\cdots=n_r=0$, the series relations (5.3) and (5.4) would evidently correspond to our generating functions given by (4.3) and (4.4), respectively. Similar generalizations of the assertions (4.8) and (4.9) of Theorem 5 are contained in

THEOREM 6. *Let α_i, β_i and γ_i $(i=1,\cdots,r)$ be arbitrary complex numbers independent of $n_1,\cdots,n_r$, and let ζ_i be defined by* (5.5). *Also let*

$$(5.6)\qquad I_{n_1,\cdots,n_r}[z_1,\cdots,z_r]=H\begin{matrix}0,\ \lambda+r:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})\\ A+r,\ C+r:[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}$$
$$\left(\begin{matrix}[1+\alpha_1-\gamma_1-n_1:\sigma_1',\cdots,\sigma_r'],\cdots,[1+\alpha_r-\gamma_r-n_r:\sigma_1^{(r)},\cdots,\sigma_r^{(r)}],\\ [1+\alpha_1-\gamma_1:\sigma_1',\cdots,\sigma_r'],\cdots,[1+\alpha_r-\gamma_r:\sigma_1^{(r)},\cdots,\sigma_r^{(r)}],\\ [(a):\theta',\cdots,\theta^{(r)}];[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\cdots,\psi^{(r)}];[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right),$$

$$(5.7)\qquad J_{n_1,\cdots,n_r}[z_1,\cdots,z_r]=H\begin{matrix}0,\ \lambda:\ (\mu',\nu'+1);\cdots;(\mu^{(r)},\nu^{(r)}+1)\\ A,\ C:[B'+1,D'+1];\cdots;[B^{(r)}+1,D^{(r)}+1]\end{matrix}$$
$$\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[1+\alpha_1-\gamma_1-n_1:\sigma_1],[(b'):\phi'];\cdots;\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'],[1+\alpha_1-\gamma_1:\sigma_1];\cdots;\\ [1+\alpha_r-\gamma_r-n_r:\sigma_r],[(b^{(r)}):\phi^{(r)}];\\ [(d^{(r)}):\delta^{(r)}],[1+\alpha_r-\gamma_r:\sigma_r];\end{matrix}\ z_1,\cdots,z_r\right),$$

$\sigma_i>0$, $\sigma_i^{(j)}>0$, $i,j=1,\cdots,r$, *and suppose that the inequalities in* (1.13) *and* (1.14) *hold true.*

Then

$$(5.8)\qquad \sum_{n_1,\cdots,n_r=0}^{\infty}\prod_{i=1}^{r}\left\{\frac{(-1)^{n_i}}{n_i!}\frac{\gamma_i}{\gamma_i+(\beta_i+1)n_i}\right\}P^{(\alpha,\beta)}_{\underset{\sim}{n_1},\cdots,\underset{\sim}{n_r}}(z_1,\cdots,z_r)t_1^{n_1}\cdots t_r^{n_r}$$
$$=(1+\zeta_1)^{\alpha_1}\cdots(1+\zeta_r)^{\alpha_r}\Phi[w_1,\cdots,w_r;\zeta_1/(1+\zeta_1),\cdots,\zeta_r/(1+\zeta_r)]$$

and

$$(5.9)\qquad \sum_{n_1,\cdots,n_r=0}^{\infty}\prod_{i=1}^{r}\left\{\frac{(-1)^{n_i}}{n_i!}\frac{\gamma_i}{\gamma_i+(\beta_i+1)n_i}\right\}Q^{(\alpha,\beta)}_{\underset{\sim}{n_1},\cdots,\underset{\sim}{n_r}}(z_1,\cdots,z_r)t_1^{n_1}\cdots t_r^{n_r}$$
$$=(1+\zeta_1)^{\alpha_1}\cdots(1+\zeta_r)^{\alpha_r}\Psi[\omega_1,\cdots,\omega_r;\zeta_1/(1+\zeta_1),\cdots,\zeta_r/(1+\zeta_r)],$$

where

$$(5.10)\qquad w_i=z_i/[(1+\zeta_1)^{\sigma_i'}\cdots(1+\zeta_r)^{\sigma_i^{(r)}}],\quad \omega_i=z_i/(1+\zeta_i)^{\sigma_i},\quad i=1,\cdots,r,$$

$$(5.11)\qquad \Phi[z_1,\cdots,z_r;t_1,\cdots,t_r]=\sum_{n_1,\cdots,n_r=0}^{\infty}I_{n_1,\cdots,n_r}[z_1,\cdots,z_r]$$
$$\cdot\prod_{i=1}^{r}\left\{\binom{n_i+\gamma_i/(\beta_i+1)}{n_i}^{-1}\frac{t_i^{n_i}}{n_i!}\right\},$$

and

$$(5.12)\qquad \Psi[z_1,\cdots,z_r;t_1,\cdots,t_r]=\sum_{n_1,\cdots,n_r=0}^{\infty}J_{n_1\cdots,n_r}[z_1,\cdots,z_r]$$
$$\cdot\prod_{i=1}^{r}\left\{\binom{n_i+\gamma_i/(\beta_i+1)}{n_i}^{-1}\frac{t_i^{n_i}}{n_i!}\right\}.$$

Remark 6. A number of variations of the generating-function relations (4.4), (4.9), (5.4), and (5.9) may be proven fairly easily. For example, one such variation of our formula (4.4) has the form

$$(5.13)\qquad \sum_{n=0}^{\infty} H\begin{matrix}0,\ \lambda\colon (\mu',\nu');\ \cdots;\ (\mu^{(i)}+1,\nu^{(i)});\ \cdots;\ (\mu^{(r)},\nu^{(r)})\\ A,\ C\colon [B',D'];\ \cdots;\ [B^{(i)}+1,D^{(i)}+1];\ \cdots;\ [B^{(r)},D^{(r)}]\end{matrix}$$
$$\left(\begin{matrix}[(a)\colon \theta',\ \cdots,\ \theta^{(r)}]\colon [(b')\colon \phi'];\ \cdots;\\ [(c)\colon \psi',\ \cdots,\ \psi^{(r)}]\colon [(d')\colon \delta'];\ \cdots;\\ [(b^{(i)})\colon \phi^{(i)}],\ [1+\alpha+(\beta+1)n\colon \sigma];\ \cdots;\ [(b^{(r)})\colon \phi^{(r)}];\\ [1+\alpha+\beta n\colon \sigma],\ [(d^{(i)})\colon \delta^{(i)}];\ \cdots;\ [(d^{(r)})\colon \delta^{(r)}];\end{matrix}\ z_1,\ \cdots,\ z_r\right)\frac{t^n}{n!}$$
$$=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}H\begin{matrix}0,\ \lambda\colon (\mu',\nu');\ \cdots;\ (\mu^{(r)},\nu^{(r)})\\ A,\ C\colon [B',D'];\ \cdots;\ [B^{(r)},D^{(r)}]\end{matrix}\left(\begin{matrix}z_1\\ \vdots\\ z_i/(1+\zeta)^{\sigma}\\ \vdots\\ z_r\end{matrix}\right),$$

where, as before, $1\leqq i\leqq r$, ζ is defined by (4.2), $\sigma>0$, and the inequalities in (1.13) and (1.14) hold.

It is not difficult to verify that this last generating-function relation (5.13) is essentially equivalent to (4.4).

6. Applications.

At the outset we must remark that Theorems 1 through 6 above are very general in character; these and their various special forms considered in the preceding sections can be suitably applied to obtain several classes of expansion formulas or generating-function relations involving a fairly large variety of useful functions (or products of several such functions) that are expressible as the G or H functions in one or more variables. For example, if $\lambda=A=C=0$, the second member of (1.10) would obviously degenerate into the product of r mutually independent (and single) Mellin-Barnes contour integrals, each representing the familiar H-function of argument z_i, $i=1,\cdots,r$. Thus the various results presented here can be first reduced fairly easily to hold for the product of several H-functions (and hence also G-functions, the Wright's generalized or ordinary hypergeometric functions, etc.) of different arguments.

Next we recall the known relationship ([10], Eq. (4.7); see also [6], p. 455, and [11], Eq. (4.11))

$$(6.1)\qquad H\begin{matrix}0,\ A\colon (1,B');\ \cdots;\ (1,B^{(r)})\\ A,\ C\colon [B',D'+1];\ \cdots;\ [B^{(r)},D^{(r)}+1]\end{matrix}\left(\begin{matrix}[(a)\colon \theta',\ \cdots,\ \theta^{(r)}]\colon\\ [(c)\colon \psi',\ \cdots,\ \psi^{(r)}]\colon\end{matrix}\right.$$

$$
\begin{matrix}
[(b'):\phi']; \cdots; [(b^{(r)}):\phi^{(r)}]; \\
[0:1], [(d'):\delta']; \cdots; [0:1], [(d^{(r)}):\delta^{(r)}];
\end{matrix}
z_1, \cdots, z_r\Bigg)
$$

$$
= \frac{\prod_{j=1}^{A}\Gamma[1-a_j]\prod_{j=1}^{B'}\Gamma[1-b'_j]\cdots\prod_{j=1}^{B^{(r)}}\Gamma[1-b_j^{(r)}]}{\prod_{j=1}^{C}\Gamma[1-c_j]\prod_{j=1}^{D'}\Gamma[1-d'_j]\cdots\prod_{j=1}^{D^{(r)}}\Gamma[1-d_j^{(r)}]}
$$

$$
\cdot F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\Bigg(
\begin{matrix}
[1-(a):\theta', \cdots, \theta^{(r)}]: \\
[1-(c):\psi', \cdots, \psi^{(r)}]:
\end{matrix}
$$

$$
\begin{matrix}
[1-(b'):\phi']; \cdots; [1-(b^{(r)}):\phi^{(r)}]; \\
[1-(d'):\delta']; \cdots; [1-(d^{(r)}):\delta^{(r)}];
\end{matrix}
-z_1, \cdots, -z_r\Bigg),
$$

which evidently would enable us to apply our results to derive expansion formulas and generating relations for the generalized Lauricella function of several complex variables or more particularly, for Lauricella's hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ of r variables (cf. [6], p. 454 et seq.). Indeed, Theorems 1, 2, 3 and 4 above would thus reduce, respectively, to Theorems 1, 3, 2 and 4 of our earlier work [8] on expansions in series of products of several generalized hypergeometric polynomials. Notice, in this connection, that the hypergeometric polynomials involved in our expansion formulas (2.8) and (2.22) can be appropriately specialized to yield the corresponding expansions in series of products of several Jacobi or Laguerre polynomials or their various known special or limiting cases (cf. [11], § 3, for details). On the other hand, our Theorems 5 and 6 provide elegant extensions, in terms of the multiple H-function defined by (1.10), of a number of known results on generating functions of one or more complex variables (cf., e.g., [7]).

Finally the special cases $r=1$ and $r=2$ of some of our results are worthy of note. Indeed, when $r=1$, Theorems 1 and 2 would yield generalizations of several results in the theory of special functions including, for instance, Theorems III through VI of Wimp and Luke ([12], pp. 359-361; see also [3], Vol. II, p. 14 et seq.), involving the familiar G-function; while, in the special case $r=2$, our generating relation (4.3) would obviously provide an interesting extension, involving the H-function of two variables, of the main result in a recent paper [5, p. 70, Eq. (3)]. In view of the triviality of the analysis to be applied in order to deduce these and other generalizations of known results, however, we omit details, which may well be left as an exercise to the interested reader.

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