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Common fixed points theorems of two mappings

By

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Introduction. In this paper, we investigate two self-mappings S, T on a complete metric space (X, d) such that the following condition holds:

(A) $$d(Sx, Ty)\leqslant k(d(x, y)) \max \{d(x, y), d(x, Sx), d(y, Ty)\}$$

where k is upper semicontinuous from the right on $\bar{P}-\{0\}$, $k(t)<1$ for all t in $\bar{P}-\{0\}$, and $P=\{d(x, y); x, y \text{ in } X\}$.

In [1], D. W. Boyd and J. S. W. Wong proved that if X be a complete metric space and let $T: X\rightarrow X$ satisfies

(B) $$d(Tx, Ty)\leqslant \bar{k}(d(x, y)) ,$$

where $\bar{k}: \bar{P}\rightarrow[0\infty)$ is upper semicontinuous from the right on $\bar{P}$ and satisfies $\bar{k}(t)<t$ for all t in $\bar{P}-\{0\}$. Then, T has a unique fixed point y and $T^n(x)\rightarrow y$ for each $x\in X$. We shall show that one need only assume the condition (A).

A number of examples are given to show that the results do in fact improve the results of Boyd and Wong [1].

THEOREM 1: *Let S, T be two self-mappings on a complete metric space (X, d), S or T be continuous on X and S, T satisfy* (A). *Then, S or T has a fixed point.*

Proof: For each $x_0\in X$, we define a sequence $\{x_n\}$ recursively as follows:

$$x_1=x_0 , \ x_2=S(x_1) , \ x_3=T(x_2), \cdots, x_{2n}=S(x_{2n-1}) , \ x_{2n+1}=T(x_{2n}), \cdots .$$

Let $c_m=d(x_m, x_{m+1})$

Case I: There exists a positive integer m such that $c_m=0$. Hence S or T has a fixed point.

Case II: Suppose that $c_m>0$ for all $m\geqslant 1$. Hence we have

$$c_{2n}=d(x_{2n}, x_{2n+1})=d(S(x_{2n-1}), T(x_{2n}))$$
$$\leqslant k(c_{2n-1}) \max \{c_{2n-1}, c_{2n-1}, c_{2n}\}$$

Hence $$c_{2n}\leqslant c_{2n-1} .$$

$$c_{2n-1}=d(S(x_{2n-1}), T(x_{2n-2}))$$
$$\leqslant k(c_{2n-2}) \max \{c_{2n-2}, c_{2n-1}, c_{2n-2}\}$$

Hence $$c_{2n-1}\leqslant c_{2n-2} .$$

Therefore $\{c_n\}$ is a decreasing sequence, and hence has a limit c. But, if $c>0$, we have

$$c_{n+1}<k(c_n)c_n$$

so that

$$c\leqslant \limsup_{t\to c+} k(t)\leqslant k(c)<c$$

which is a contradiction. Hence $c=0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there is an $s>0$, and sequences of integers $\{m(i)\}$, $\{n(i)\}$, with $m(i)>n(i)\geqslant i$, $m(i)$ is an odd integer, $n(i)$ is an even integer and such that

(a) $$d_i=d(x_{m(i)}, x_{n(i)})\geqslant s \quad \text{for} \quad i=1, 2, \cdots .$$

We may assume that

(b) $$d(x_{m(i)-1}, x_{n(i)})<s$$

by choosing $m(i)$ to be the smallest number exceeding $n(i)$ for which (a) holds. Since

$$d_i\leqslant d(x_{m(i)}, x_{m(i)-1})+d(x_{m(i)-1}, x_{n(i)})\leqslant c_{m(i)-1}+s\leqslant c_i+s .$$

Hence $d_i\to s^+$, as $i\to\infty$

But now,

(c) $$\begin{aligned} d_i=d(x_{m(i)}, x_{n(i)})&\leqslant d(x_{m(i)}, x_{m(i)+1})+d(x_{m(i)+1}, x_{n(i)+1})+d(x_{n(i)+1}, x_{n(i)})\\ &\leqslant 2c_i+k(d(x_{m(i)}, x_{n(i)}))\max\{d(x_{m(i)}, x_{n(i)}), d(x_{m(i)}, x_{m(i)+1}), d(x_{n(i)}, x_{n(i)+1})\}\end{aligned}$$

Thus, as $i\to\infty$ in (c), we obtain $s\leqslant k(s)s<s$, which is a contradiction for $s>0$. Hence $\{x_n\}$ is a Cauchy sequence. We may suppose $\{x_n\}$ converges to y. By hypothesis, let S be continuous on X and hence $\{S(x_{2n-1})\}=\{x_{2n}\}$ converges to $S(y)$. Consequently $S(y)=y$. This completes the proof.

Corollary 1.1: *Let S be a continuous self-mapping on a complete metric space (X, d) and S satisfies (C)*

(C) $$d(Sx, Sy)\leqslant k(d(x, y))\max\{d(x, y), d(x, Sx), d(y, Sy)\}$$

for all $x\neq y$ in X, where $k\colon \bar{P}-\{0\}\to[0\ 1)$ is upper semicontinuous from the right on $\bar{P}-\{0\}$. Then, S has a unique fixed point y and $S^n(x)$ converges to y for each x in X.

The continuity condition on S or T cannot be omitted entirely from Theorem 1 as the following example shows:

Example 1: Let $X=[0\ 1]$ and let

$$S(x)=\begin{cases} \dfrac{x}{3} & \text{if } x\in(0\ 1] \\ 1 & \text{if } x=0 \end{cases}$$

$$T(x)=\begin{cases} \dfrac{x}{3} & \text{if } x\in(0\ 1] \\ \dfrac{2}{3} & \text{if } x=0 \end{cases}$$

and

$$k(d)=1-\frac{d}{10} \qquad \text{for all } d \text{ in } [0\ 1]$$

Then S and T satisfy (A) but S and T have no a fixed point. (note that $k(0)=1$).

EXAMPLE 2: Let $X=[0\ 1]$ and let

$$S(x)=\begin{cases} \dfrac{x}{3} & \text{if } x\in(0\ 1] \\ 1 & \text{if } x=0 \end{cases}$$

$$T(x)=\frac{x}{4} \qquad \text{for all } x\in[0\ 1]$$

and

$$k(d)=1-\frac{d}{10} \qquad \text{for all } d \text{ in } [0\ 1]$$

then S and T satisfy (A), $k(0)=1$, $T(0)=0$.

EXAMPLE 3: Let $X=\{1, 2, 3\}$ and let

$$T(1)=2\ , \quad T(2)=2\ , \quad T(3)=1$$

and

$$k(d)<1 \qquad \text{for all } d \text{ in } \{1, 2\} \text{ and } k(d)>\frac{1}{2}$$

then T satisfies (C) and $T^n(x)$ converges to 2 for all $x=1, 2$ or 3 but T does not satisfy (B).

Examples 2 and 3 show that Theorem 1 and Corollary 1.1 do generalize the results of Boyd and Wong [1].

If we omit the continuity on S and T and strengthen the function $k(d)$, we can obtain the following theorems.

THEOREM 2: *Let S, T be two self-mappings on a complete metric space (X, d) and S, T (S, T are not necessary continuous on X.) satisfy* (D)

(D) $$d(Sx, Ty) \leqslant k(d(x, y)) \max \{d(x, y), d(x, Sx), d(y, Ty)\}$$

for all $x \neq y$ *in* X, *where* $k: \bar{P} \to [0\ 1)$ *is upper semicontinuous from the right on* $\bar{P}$. *Then,* S *or* T *has a fixed point in* X.

Proof: We define a sequence $\{x_n\}$ which is the same as in Theorem 1. Let $c_m = d(x_m, x_{m+1})$.

Case I: There exist a positive integer m such that $c_m = 0$. Hence S or T has a fixed point.

Case II: Suppose that $c_m > 0$ for all $m \geqslant 1$ and hence $\{x_n\}$ is a Cauchy sequence. We may assume that $\{x_n\}$ converges to y.

Case 1: There exists a positive integer N such that $x_{2n} = y$ for all $n \geqslant N$. Since $c_n > 0$ for all $n \geqslant 1$ and hence $d(x_{2n+1}, y) > 0$ for all $n \geqslant N$.
If $n \geqslant N$, then

$$\begin{aligned} d(y, Ty) &= d(x_{2n}, Ty) = d(S(x_{2n-1}), Ty) \\ &\leqslant k(d(x_{2n-1}, y)) \max \{d(x_{2n-1}, y), d(x_{2n-1}, x_{2n}), d(y, Ty)\}\,. \end{aligned}$$

Let $n \to \infty$, we have $d(y, Ty) \leqslant k(0) d(y, Ty)$. Therefore $d(y, Ty) = 0$ and $Ty = y$.

Case 2: There exists a subsequence $\{x_{2n_j}\}$ of $\{x_{2n}\}$ such that $x_{2n_j} \neq y$ for all $j = 1, 2, \cdots$.
Then,

$$\begin{aligned} d(y, Sy) &\leqslant d(y, x_{2n_j+1}) + d(x_{2n_j}, Sy) \\ &\leqslant d(y, x_{2n_j+1}) + d(Sy, T(x_{2n_j})) \\ &\leqslant d(y, x_{2n_j+1}) + k(d(x_{2n_j}, y)) \max \{d(y, x_{2n_j}), \\ &\qquad d(y, Sy), d(x_{2n_j}, x_{2n_j+1})\}\,. \end{aligned}$$

Let $j \to \infty$, we have $d(y, Sy) \leqslant k(0) d(y, Sy)$. Therefore $d(y, Sy) = 0$ and $Sy = y$. Combining Cases I and II, this completes the proof.

Corollary 2.1: *Let* T *be a self-mapping on a complete metric space* (X, d) *and* T *satisfies* (E)

(E) $$d(Tx, Ty) \leqslant k(d(x, y)) \max \{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x \neq y$ *in* X, *where* $k: \bar{P} \to [0\ 1)$ *is upper semicontinuous from the right on* $\bar{P}$. *Then,* T *has a unique fixed point* y *and* $T^n(x)$ *converges to* y *for each* x *in* X.

Example 4: Let $X = \{1, 2\}$ and let

$$S(1) = 1 \qquad S(2) = 2\,,$$
$$T(1) = 2 \qquad T(2) = 1\,,$$

and

$$k(d)=\frac{1}{2} \qquad \text{for all } d \text{ in } [0\ 1]$$

then S and T satisfy (D) but S and T have no common fixed point and S and T are continuous on X.

Example 1 shows that the property $k(0)<1$ cannot be omitted entirely form Theorem 2.

THEOREM 3: *Let S, T be two self-mappings on a complete metric space (X, d) and S, T (S and T are not necessarily continuous on X.) satisfy (F)*

(F) $$d(Sx, Ty)\leqslant k(d(x, y)) \max \{d(x, y), d(x, Sx), d(y, Ty)\}$$

for all x, y in X. Then, S and T have a unique and common fixed point y and the sequence $\{x_n\}$ converges to y, where the sequence $\{x_n\}$ is the same as in Theorem 1.

Proof: Case I: Suppose that there exists a positive integer m such that $c_m=0$, by (F), then $c_n=0$ for all $n\geqslant m$. Hence S and T have a unique and common fixed point $y=x_m$ and $\{x_n\}$ converges to y.

Case II: Suppose that $c_m>0$ for all $m\geqslant 1$, then c_m has a limit 0 and $\{x_n\}$ is a Cauchy sequence, we may assume that the sequence converges to y.

Therefore

$$\begin{aligned} d(Sy, y) &\leqslant d(Sy, T(x_{2n}))+d(x_{2n+1}, y) \\ &\leqslant d(x_{2n+1}, y)+k(d(x_{2n}, y)) \max \{d(y, x_{2n}) , \\ &\qquad d(y, Sy), d(x_{2n}, x_{2n+1})\} . \end{aligned}$$

Let $y\longrightarrow\infty$, we have $d(Sy, y)\leqslant k(0)d(y, S(y))$. Hence $d(Sy, y)=0$ and $Sy=y$. Similarly, we have $Ty=y$.

To prove the uniqueness of y. Suppose there is an z in X such that $S(z)=z$ with $z\neq y$ then,

$$\begin{aligned} d(Sz, Ty) &\leqslant k(d(z, y)) \max \{d(z, y), d(z, Sz), d(y, Ty)\} \\ &=k(d(z, y))d(z, y) \end{aligned}$$

that is, $d(z, y)\leqslant k(d(z, y))d(z, y)$ and $d(z, y)=0$. This is a contradiction. Hence $y=z$ and S has a unique fixed point y and the sequence $\{x_n\}$ converges to y.

COROLLARY 3.1: *Let S, T be two self-mapping on a complete metric space (X, d). Suppose that there exist two positive integers p and q such that S^p and T^q satisfy (F). Then, S and T have a unique and common fixed point.*

Proof: By Theorem 3, S^p and T^q have a unique and common fixed point y (to say).

Since

$$S^p(Sy)=S^{p+1}(y)=S(S^p(y))=S(y)$$
$$T^q(Ty)=T^{q+1}(y)=T(T^q(y))=T(y)$$

Consequently $S(y)=T(y)=y$. The uniqueness is clear.

COROLLARY 3.2: *Let T be a self-mapping on a complete metric space (X, d) and T satisfy* (G)

(G) $$d(T(x), T(y))\leqslant k(d(x, y)) \max \{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all x, y in X, where $k: \bar{P}\to[0\ 1)$ is upper semicontinuous from the right on $\bar{P}$, then, T has a unique fixed point y and $\{T^n x\}$ converges to y for each $x\in X$.

EXAMPLE 5: Let $X=[0\ 1]$ and let

$$S(x)=\frac{x}{100} \qquad \text{for all } x \text{ in } [0\ 1]$$

$$T(x)=\frac{x}{50} \qquad \text{for all } x \text{ in } [0\ 1]$$

and

$$k(d)=\frac{9}{10} \qquad \text{for all } d\in[0\ 1]$$

Then S and T satisfy (F) and S and T have a common fixed point 0.

Examples 4 and 5 show that, in general, Theorems 2 and 3 are different.

COROLLARY 3.3: *Let S, T be two self-mappings on a complete metric space (X, d) and S, T satisfy* (H)

(H) $$d(Sx, Ty)\leqslant r \max \{d(x, y), d(x, Sx), d(y, Ty)\}$$

for all x, y in X. Where $0\leqslant r<1$. Then, S and T have a unique and common fixed point.

COROLLARY 3.4: (C. L. Yen [5] Theorem 1): *Let S, T be two self-mappings on a complete metric space (X, d) and S, T satisfy* (I)

(I) $$d(Sx, Ty)\leqslant ad(x, y)+bd(x, Sx)+cd(y, Ty)$$

for all x, y in X. Then S and T have a unique and common fixed point, if $a+b+c<1$, $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$.

COROLLARY 3.5: [R. Kannan [3] Theorem 1]: *Let T be a mapping of a complete metric space (X, d) into itself. Suppose that there exists a number r in $[0\ 1/2)$ such that*

(J) $$d(Tx, Ty)\leqslant r(d(x, Tx)+d(y, Ty))$$

for all x, y in X. Then T has a unique fixed point.

COROLLARY 3.6: [P. Srivastava and V. K. Gupta [4] Theorem 1] *Let S, T be mappings of a complete metric space (X, d) into itself. Suppose that there exists nonnegative real numbers a, b such that* $a+b<1$ *and*

(K) $$d(Sx, Ty) \leqslant ad(x, Sx) + bd(y, Ty)$$

for all x, y in X. Then, S, T have a unique and common fixed point.

C. L. Yen [5], Srivastava and Gupta [4] stated the corollary 3.4 and 3.6 in a more general form with S, T replaced by S^p, T^q for some positive integers p, q.

When $S=T$, Theorems 2 and 3 are coincide.

EXAMPLE 6: Let $X=[0\ 1]$ and let

$$S(x)=\begin{cases} \dfrac{x}{40} & \text{if } x\in[0\ 1) \\ 0 & \text{if } x=1 \end{cases}$$

$$T(x)=\begin{cases} \dfrac{x}{20} & \text{if } x\in[0\ 1) \\ \dfrac{1}{3} & \text{if } x=1 \end{cases}$$

and

$$k(d)=\frac{9}{10} \qquad \text{for all } d \text{ in } [0\ 1]$$

Then S and T satisfy (F), $S\neq T$ and S and T are not continuous on X.

We consider conditions (L) and (M).

(L) $d(Sx, Ty) < \max\{d(x, y), d(x, Sx), d(y, Ty)\}$ if $x\neq y$.

(M) $d(Sx, Ty) \leqslant \max\{d(x, y), d(x, Sx), d(y, Ty)\}$ for all x, y in X.

EXAMPLE 7: Let $X=[0\ 1]$ and let

$$S(x)=\begin{cases} \dfrac{x}{2} & \text{if } x\in(0\ 1] \\ 1 & \text{if } x=0 \end{cases}$$

and

$$T(y)=\begin{cases} \dfrac{x}{2} & \text{if } x\in(0\ 1] \\ \dfrac{2}{3} & \text{if } x=0 \end{cases}$$

Then S and T satisfy (L) and (M) but S and T have no fixed point.

Even X is a compact metric space. However, if S and T satisfy the condition (N).

(N) $$d(Sx, Ty)<d(x, y) \quad \text{if} \quad x\neq y \, .$$

Then we have the following theorem.

THEOREM 4: *Let S, T be two self-mappings on a compact metric space (X, d) and S, T satisfy* (N). *Then, S or T has a fixed point. More generally, we have the following theorem.*

THEOREM 5: *Let (X, d) be a nonempty compact metric space, Let S, T be functions of X into itself, Suppose further that there exist nonnegative real-valued functions* $b_1=b_1(x, y)$, $b_2=b_2(x, y)$, $b_3=b_3(x, y)$, $b_4=b_4(x, y)$, $b_5=b_5(x, y)$ on $X\times X-\Delta$, where $\Delta=\{(x, x): x\in X\}$ *such that*

(a) $b_1+b_2+b_3+b_4+b_5\leqslant 1$

(b) $b_1+b_4<1$ *and* $b_2+b_3<1$

(c) $b_3=b_4$

(d) *for any distinct x, y in X,*

(O) $$d(Sx, Ty)<b_1d(x, Sx)+b_2d(y, Ty)+b_3d(x, Ty)+b_4d(y, Sx)+b_5d(x, y) \, .$$

Then S or T has a fixed point, If both S and T have fixed points, then each of S and T has a unique fixed point and these two fixed points coincid.

We compare Theorem 5 with Theorem 4[9] as follows.

(1) In Theorem 4[9], one of S and T be continuous on X, yet in Theorem 5, S and T are not necessarily continuous on X.

(2) In Theorem 4[9], all α_i are decreasing functions, yet in Theorem 5, all b_i are not decreasing functions.

(3) In Theorem 4[9], all α_i are defined on $(0 \ \infty)$, yet in Theorem 5 all b_i are defined on $X\times X-\Delta$. (It is possible that $b_i(x, y)\neq b_i(y, x)$ may be true but $\alpha_i(d(x, y))=\alpha_i(d(x, y))$.)

(4) In Theorem 4[9], $\alpha_1=\alpha_2$, yet in Theorem 5, b_i does not necessarily equal b_2.

(5) In Theorem 4[9], $\alpha_1+\alpha_4\leqslant 1/2$, $\alpha_2+\alpha_3\leqslant 1/2$, yet in Theorem 5, $b_1+b_4<1$ and $b_2+b_3<1$.

A number of examples are given to show that the results do in fact improve Theorem 4[9].

The proof of theorem 5 is in K. J. Chung [8].

EXAMPLE 8: Let $X=[0 \ 1]$ and let

$$S(x)=\begin{cases} \dfrac{1}{2} & \text{if } x=0 \\[2ex] \dfrac{x}{100} & \text{if } x\in(0 \ 1) \end{cases}$$

$$T(y)=\begin{cases} 0 & \text{if } y=1 \\ \dfrac{y}{50} & \text{if } y\in[0\ 1)\,. \end{cases}$$

Then it is clear that S and T satisfy the condition (0), S and T are not continuous, $T(0)=0$ and S has no fixed point in X.

Example 8 does improve the result of C. S. Wong[9].

In C. S. Wong[9], let $a_3=a_4=0$ then Theorems 1, 2, 3 and 4 in [9] are four special cases by our Theorems 1, 2, 3 and 5 respectively.

THEOREM 6: *Let X be a compact metric space and let S, T be two self-mappings. S or T is continuous on X. S and T satisfy (P)*

(P) $$d(Sx,\ Ty)<\max\{d(x,\ y),\ d(x,\ Sx),\ d(y,\ Ty)\}$$

for all $x\neq y$ in X. Then S or T has a fixed point in X.

Proof: By hypothesis, we may suppose that S is continuous on X. Let

$$s=\inf\{d(x,\ Sx);\ x\in X\}\ ,$$
$$t=\inf\{d(x,\ Tx);\ x\in X\}\ .$$

There exists x in X such that $s=d(x,\ Sx)$.

Case I: If $s=0$, hence $Sx=x$, this theorem has proved.

Case II: If $s>0$, by compactness of X, there exists a sequence $\{x_n\}\subset X$ such that

(Q) $$\lim_{n\to\infty} d\,(x_n,\ Tx_n)=t\ .$$

If $t>0$, we may suppose that $d(x_n,\ Tx_n)>0$ for all $n\geqslant 1$. Therefore

(R) $$\begin{aligned} d(STx_n,\ Tx_n)&<\max\{d(Tx_n,\ x_n),\ d(Tx_n,\ STx_n),\ d(x_n,\ Tx_n)\} \\ &=d(Tx_n,\ x_n)\ . \end{aligned}$$

Let $n\longrightarrow\infty$, we have $s\leqslant t$. Also

$$\begin{aligned} d(Sx,\ TSx)&<\max\{d(x,\ Sx),\ d(x,\ Sx),\ d(Sx,\ TSx)\} \\ &=d(x,\ Sx)=s\ . \end{aligned}$$

Hence we have $t<s$, this is a contradiction. Therefore $t=0$. If T has no a fixed point, from (R), we have $s=0$. This is a contradiction. Consequently T has a fixed point.

Example 7 shows that the continuity on S or T cannot omit entirely from Theorem 6.

COROLLARY 6.1: *Let X be a compact metric space and S be a continuous self-mapping such that (U) holds.*

(U) $$d(Sx,\ Sy)<\max\{d(x,\ y),\ d(x,\ Sx),\ d(y,\ Sy)\}$$

for all $x \neq y$ *in* X. *Then* S *has a unique fixed point in* X.

Corollary 6.1 was showed by V. M. Sehgal[10]. But it is a special case of our Theorem 6.

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