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Fundamental sequences of ordinal diagrams

by

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Let S be a set with a primitive recursive ordering $<$. We say that fundamental sequences are defined for S, or fundamental sequences of elements of S exist, if there is a uniform method according to which we can construct for any limit element of S, says, a $<$-increasing sequence of elements of S, say $\{s_m\}_m$, such that $\{s_m\}_m$ is primitive recursive in m and converges to s (from below) with respect to $<$.

We are going to show that, given a system of ordinal diagrams, $O(I, A)$, fundamental sequences can be defined for $O(I, A)$ with respect to each ordering $<_i$, where i is an element of I.

The ultimate objective of the study of fundamental sequences for ordinal diagrams is its use in a constructive (in some sense) proof of the well-foundedness of ordinal diagrams. (We do not specify here the concept of "constructive".) An accessibility proof for ordinal diagrams, in which the theory of fundamental sequences is essentially used, is our task of the future.

A detailed bibliography concerning the theory of ordinal diagrams and its applications is seen in [3], hence we have avoided repetition; only [4] has been added.

§ 0. Preliminary—a summary of the theory of approximations.

In developing the theory of fundamental sequences, the theory of approximations plays an essential role. Although this theory is explained in detail in [5], we shall here present a summary of the theory for the reader's convenience.

Consider the system of ordinal diagrams based on the primitive recursive sets I and A which have primitive recursive well-orderings. We assume that fundamental sequences are defined for I and A; namely there is a uniform method such that for any limit element of I (A), we can construct according to the method a sequence of elements of I (A) converging to it from below.

DEFINITION 0.1. The definition of the *system of ordinal diagrams*

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based on I and A, $O(I, A)$, and some related notions. "Ordinal diagram" will be abbreviated to "o.d.". We shall omit the reference to I and A most of the time. Let 0 be a special symbol.

(1) The o.d.'s of $O(I, A)$.

1) 0 is a connected o.d. (of $O(I, A)$).

2) Let i be an element of I, a be an element of A and α be an o.d. already defined. Then (i, a, α) is a connected o.d. (of $O(I, A)$).

3) Let $n \geq 2$ and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be connected o.d.'s. Then $\alpha_1 \# \alpha_2 \# \cdots \# \alpha_n$ is a nonconnected o.d. (of $O(I, A)$). Each of $\alpha_1, \alpha_2, \cdots \alpha_n$ is called a component of the o.d. thus defined.

(2) A *sub-o.d. of* α is a part of α which is itself an o.d.

(3) Let (i, a, γ) be a sub-o.d. of α and let X be an expression which is a part of γ. Then we say that the i and the a explicitly written are connected to X (in α). We also say that (i, a) is *connected to* X (in α).

(4) Let X be a part of α and j be an element of I. If every l an element of I which is connected to X (in α) is $\geq j$, then X is said to be *j-active* (in α). A connected, j-active sub-o.d. of α is called a *j-subsection* of α.

(5) Let (i, a, γ) be a j-subsection of α for some $j > i$. Then γ is called an *i-section* of α. If there is an i-section of α, then we say that i is a *index of* α.

(6) If β is a sub-o.d. of α, we say that α *contains* β.

Note that the various notions we have defined above concern occurrences of expressions, not just expressions themselves. We shall not, however, employ a particular notation for an occurrence of an expression in order to distinguish it from the expression itself. we believe that the distinction can be made from the context. Let us also note that a sub-o.d. of α may be α itself. When it is not α, we may say it is a *proper sub-o.d. of* α and α contains it properly. A rigorous treatment of the entire matter is seen in [5].

(7) $\alpha = \beta$ when α and β are identical up to the order of components at each stage of the definitions of α and β.

(8) (i, a) is called a *value* when i is an element of I and a is an element of A. The values are ordered lexicographically.

(9) Let α be (i, a, γ). (i, a) is called the *outermost value of* α.

(10) An element of I is called an *indicator*.

DEFINITION 0.2. Let ∞ be a new symbol and let $\tilde{I}$ be $I \cup \{\infty\}$. For each i in $\tilde{I}$ we define an ordering $<_i$ of o.d.'s of $O(I, A)$. $\alpha \leq_i \beta$ will be an abbreviation of "$\alpha <_i \beta$ or $\alpha = \beta$".

(1) $0 <_i \alpha$ for any i and any α which is not 0.

(2) Let $l + m > 2$, let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be all the components of α

and let $\beta_1, \beta_2, \cdots, \beta_l$ be all the components of β, where $\alpha_1 {}_i{\geq} \alpha_2 {}_i{\geq} \cdots {}_i{\geq} \alpha_m$ and $\beta_1 {}_i{\geq} \beta_2 {}_i{\geq} \cdots {}_i{\geq} \beta_l$. Then $\alpha <_i \beta$ if one of the following conditions holds.

(2.1) $l > m$ and $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_m = \beta_m$.

(2.2) There is an n, $1 \leq n \leq m, l$, such that $\alpha_1 = \beta_1, \cdots, \alpha_{n-1} = \beta_{n-1}$ and $\alpha_n <_i \beta_n$.

(3) α and β are connected, neither is 0 and i is an element of $\boldsymbol{I}$. $\alpha <_i \beta$ if one of the following holds.

(3.1) There is an i-section of β, say δ, such that $\alpha \leq_i \delta$.

(3.2) Let j_0 be the least element j of $\boldsymbol{I}$ satisfying that $i < j$ and j is an index of α or β, if there is such j; let j_0 be ∞ when there is no such j. For this j_0, $\alpha <_{j_0} \beta$ and, for any σ an i-section of α, $\sigma <_i \beta$.

(4) Let α be (j, a, α') and β be (k, b, β'). $\alpha <_\infty \beta$ if one of the following holds.

(4.1) $$j < k\,.$$

(4.2) $$j = k \quad \text{and} \quad a < b\,.$$

(4.3) $$(j, a) = (k, b) \quad \text{and} \quad \alpha' <_j \beta'\,.$$

THEOREM. *For each i in $\tilde{\boldsymbol{I}}$, $<_i$ is a linear ordering of $O(\boldsymbol{I}, \boldsymbol{A})$.*

LEMMA. *Let α be a connected ordinal diagram and let β be an i-active sub-o.d. of α which is distinct from α. Then $\beta <_j \alpha$ for every $j \leq i$.*

We shall now develop the theory of approximations and valuations for a connected o.d. α which is not 0 and an element j of $\boldsymbol{I}$.

DEFINITION 0.3. The maximum among the j-active values of α is called the *0^{th} j-valuation of α* and is denoted by $v_0(j, \alpha)$. Note that any o.d. has the 0^{th} j-valuation (since α is not 0).

PROPOSITION 0.1. *Let α and β be connected (non zero) o.d.'s and j be an element of $\boldsymbol{I}$. If $v_0(j, \beta) < v_0(j, \alpha)$, then $\beta <_j \alpha$.*

DEFINITION 0.4. (1) Let $v_0(j, \alpha) = (i, a)$ and α_0 be the greatest, with respect to $<_i$, among the j-subsections of α whose outermost values are (i, a). Then α_0 is called the *0^{th} j-approximation of α* and is denoted by apr $(0, j, \alpha)$.

More precisely *apr* $(0, j, \alpha)$ denotes any occurrence of a sub-o.d. of α which satisfies the condition stated above. So the single notation α_0 (or apr $(0, j, \alpha)$) represents both an o.d. and its j-active occurrences in α.

(2) If a j-subsection of α, say η, does not contain any (occurrence of) α_0 and is not contained by α_0, then we say that η *j-omits* α_0 (in α).

PROPOSITION 0.2. (1) *If a j-subsection of α, say η, j-omits α_0, then $\eta <_l \alpha_0$ for all $l \geq j$.*

(2) *Let α and β be connected o.d.'s where $v_0(j, \alpha) = v_0(j, \beta) = (i, a)$ and* $\mathrm{apr}(0, j, \beta) <_i \mathrm{apr}(0, j, \alpha)$. *Then $\beta <_j \alpha$.*

DEFINITION 0.5. Let $v_0(j, \alpha) = (i, a)$ and $\mathrm{apr}(0, j, \alpha) = \alpha_0$. Suppose α_0 is a proper sub-o.d. of α. Let ρ be an (j-active) occurrence of α_0 in α. Consider the maximal (i.e., the complexity as an o.d. is the greatest) connected sub-o.d. of α which contains ρ, say $\alpha(\rho)$, such that every element of I in it connected to ρ is i. Consider $\alpha(\rho)$ for every ρ (i.e., every occurrence of α_0) and let α_1 be the greatest, with respect to $<_i$, among the $\alpha(\rho)$'s. Then α_1 is called the *first j-approximation of a* and is denoted by *apr* $(1, j, \alpha)$.

Here again, α_1 denotes an o.d. as well as some of its occurrences in α.

Note that $\alpha_1 = \alpha_0$ is possible.

PROPOSITION 0.3. (1) *If α_1 properly contains α_0, then $j \leq i$.*

(2) *If b is an element of A in α_1 which is connected to α_0, then $b < a$.*

DEFINITION 0.6. If a j-subsection of α, say η, neither contains nor is contained by α_1, and is not properly contained by α_0, then η is said to j-omit α_1.

PROPOSITION 0.4. (1) *Let η be a j-subsection of α which satisfies that η is not contained in α_0 and, for each occurrence of α_0 in η, say ρ, there is an element of I connected to ρ which is $< i$. Let $\rho_1, \cdots, \rho_m$ be all the occurrences of α_0 in η and q_k be the least such element of I connected to ρ_k, $k = 1, \cdots, m$. Let $q = \max(q_1, \cdots, q_k)$. (If η j-omits α_0, then let $q = j$.) Then $\eta <_l \alpha_0$ for any l such that $q < l \leq i$.*

(2) *If a j-subsection of α, say η, j-omits α_1, then $\eta <_l \alpha_1$ for any l such that $j \leq l \leq i$.*

(3) *Suppose $v_0(j, \alpha) = v_0(j, \beta) = (i, a)$ and $\alpha_0 =$* $\mathrm{apr}(0, j, \alpha) = \mathrm{apr}(0, j, \beta) = \beta_0$. *If* $\mathrm{apr}(1, j, \beta) <_i \mathrm{apr}(1, j, \alpha)$, *then $\beta <_j \alpha$. (This includes the case where $\beta_0 =$* $\mathrm{apr}(1, j, \beta)$ *while* $\mathrm{apr}(1, j, \alpha)$ *properly contains α_0.)*

DEFINITION 0.7. Let α be an o.d., j be an element of I, $\alpha_0 = \mathrm{apr}(0, j, \alpha)$, and $\alpha_1 = \mathrm{apr}(1, j, \alpha)$ $(= (i, b, \alpha'))$. We name the i in $v_0(j, \alpha) = (i, a)$ as i_0 and the i in α_1 as i_1.

Suppose we have defined the pairs of sub-o.d.'s of α and indicators occurring in α, say (α_0, i_0), (α_1, i_1), $\cdots$, (α_n, i_n), in a manner that they satisfy the following conditions in (*).

(*) (i) For every l, $1 \leq l < n$, $j \leq i_{l+1} < i_l$.

(ii) Let $l \geq 1$ and (k, b, γ) be a j-subsection of α such that there is an occurrence of α_l as a component of γ. Consider all such j-subsections. Then i_{l+1} is the maximum among those k's.

(iii) Consider an i_{l+1}-active occurrence of α_l, say ρ. Let $\eta(\rho)$ be the maximal (in regards to the complexity as an o.d.), connected sub-o.d. of α which contains ρ such that every element of $\boldsymbol{I}$ in $\eta(\rho)$ connected to ρ is $\geq i_{l+1}$. Consider all such ρ's. Then α_{l+1} is the greatest, with respect to i_{l+1}, among those $\eta(\rho)$'s.

Now we define (α_{n+1}, i_{n+1}), provided that α is not α_n.

i_{n+1} is defined from α_n as i_{l+1} is determined from α_l in (ii). α_{n+1} is defined from α_n and i_{n+1} as α_{l+1} is determined from α_l and i_{l+1} in (iii). $j \leq i_{n+1} < i_n$ is obvious from the definition.

Define $v_n(j, \alpha) = i_n$ for every $n \geq 1$. i_n is called the n^{th} *j-valuation of* α. α_n is called the n^{th} *j-approximation of* α and is denoted by *apr* (n, j, α). α_n represents an o.d. together with some of its occurrences in α.

If $\alpha_n = \alpha$, then (α_{n+1}, i_{n+1}) needs not be defined. We may, however, use the expression $v_n(j, \beta) < v_n(j, \alpha)$ to mean that $v_n(j, \beta)$ is empty, while $v_n(j, \alpha)$ is not (even if the value is 0).

DEFINITION 0.8. Suppose α_n is defined for α and j. Let η be a j-subsection of α, say η. We say that η *j-omits* α_n if η is not contained by α_n, η does not contain α_n and η is not contained in any of $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$.

PROPOSITION 0.5. *Suppose $n \geq 1$. The following three statements are proved simultaneously by induction on n.*

(1) *Let η be a j-subsection of α which satisfies that η is not contained in α_n, and for each occurrence of α_n in η, say ρ, these is an element of $\boldsymbol{I}$ connected to ρ which is $< i_{n+1}$. Let $\rho_1, \cdots, \rho_m$ be all the occurrences of α_n in η and q_k be the least such element of $\boldsymbol{I}$ connected to ρ_k, $k = 1, \cdots, m$. Let $q = \max(q_1, \cdots, q_k)$. If η j-omits α_n, then let $q = j$.) Then $\eta <_l \alpha_n$, for every l such that $q \leq l \leq i_n$.*

(2) *If η j-omits α_{n+1}, then $\eta <_l \alpha_{n+1}$, for every l such that $j \leq l \leq i_{n+1}$.*

(3) *Suppose $\alpha_n = \mathrm{apr}\,(n, j, \alpha) = \mathrm{apr}\,(n, j, \beta) = \beta_n$ (hence $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1 =, \cdots, \alpha_{n-1} = \beta_{n-1}$).*

(3.1) $$\text{If } v_{n+1}(j, \beta) < v_{n+1}(j, \alpha), \text{ then } \beta <_j \alpha\,.$$

(3.2) $$\text{If } v_{n+1}(j, \beta) = v_{n+1}(j, \alpha) = i_{n+1} \text{ and } \beta_{n+1} = \mathrm{apr}\,(n+1, j, \beta) <_{i_{n+1}} \mathrm{apr}\,(n+1, j, \alpha) = \alpha_{n+1}, \text{ then } \beta <_j \alpha\,.$$

DEFINITION 0.9. Here we restrict our attention to j-subsections of α_{n+1} which contain α_n and define refinements of α_{n+1}, $\alpha_{(n,k)}$. $k=$

0, 1, $\cdots$.

(0) $\alpha_{(n,0)}$ is any i_{n+1}-active occurrence of α_n in α_{n+1}.

($k+1$) Suppose $\alpha_{(n,k)}$ has been defined and α_{n+1} is not $\alpha_{(n,k)}$. $\alpha_{(n,k+1)}$ is any occurrence of the greatest, with respect to $<_{i(n+1)+1}$ of the (i_{n+1}-active) sub-o.d.'s of α_{n+1} which contain occurrences of $\alpha_{(n,k)}$.

From the definition of i_{n+1} and α_{n+1}, it is obvious that every $\alpha_{(n,k)}$ is i_{n+1}-active in α_{n+1} and contains an i_{n+1}-active occurrence of α_n. As before, $\alpha_{(n,k)}$ represents an o.d. together with some of its occurrences in α_{n+1}. $\alpha_{(n,k)}$ is called the $(n, k)^{\text{th}}$ *j-approximation of* α and is denoted by $apr((n, k), j, \alpha)$.

PROPOSITION 0.6. *The above definition of* $(n, k)^{\text{th}}$ *j-approximation of* α *is equivalent to the following.*

$\alpha_{(n,0)}$ *is defined as in Definition* 0.9.

Suppose the $(n, k)^{\text{th}}$ j-approximation in the second sense, $\alpha_{(n,k)}$, has been defined. Consider a (i_{n+1}-active) sub-o.d. of α_{n+1} of the form (i_{n+1}, c, δ), where a component of δ, say η, contains an occurrence of $\alpha_{(n,k)}$. $\alpha_{(n,k+1)}$ is defined to be the greatest, with respect to $<_{i(n+1)+1}$, of those η's and α_{n+1}.

DEFINITION 0.10. An i_{n+1}-subsection of α_{n+1}, say η, *j-omits* $\alpha_{(n,k)}$ if it does not contain any $\alpha_{(n,k)}$, is not contained by $\alpha_{(n,k)}$ and is not contained by any of $\alpha_{(m,l)}$ if $(m, l)<(n, k)$.

Note that the notion of α_n and that of $\alpha_{(n,0)}$ are not the same. There may be an occurrence of α_n which is not $\alpha_{(n,0)}$. So it is possible that η j-omits $\alpha_{(n,0)}$ but not α_n.

PROPOSITION 0.7. (1) *Let* η *be an* i_{n+1}*-subsection of* α_{n+1} *which j-omits* $\alpha_{(n,k)}$. *Then* $\eta <_{i_{n+1}} \alpha_{(n,k)}$.

(2) *Suppose* $\text{apr}((n, k-1), j, \beta)=\text{apr}((n, k-1), j, \alpha)$ *and* $v_{n+1}(j, \beta)= v_{n+1}(j, \alpha)=i_{n+1}$, *where* $n\geq 1$ *is assumed. If* $\text{apr}((n, k), j, \beta)<_{i(n+1)+1} \text{apr}((n, k), j, \alpha)$, *then* $\beta_{n+1}<_{i_{n+1}} \alpha_{n+1}$, *hence* $\beta<_j\alpha$.

PROPOSITION 0.8. *If* $n=0$, *then the following definition of* $\alpha_{(0,k)}$ *is equivalent to the the given one: Let* $\alpha_{(0,0)}=\alpha_0$ *and* $a_{00}=a$. *Suppose we have defined* $a_{00}, a_{01}, \cdots, a_{0k}$, *and* $\alpha_{(0,0)}, \alpha_{(0,1)}, \cdots, \alpha_{(0,k)}$ *in a manner that* $a_{00}>a_{01}> \cdots >a_{0k}$ *and have not exhausted* α_1. *Then define* a_{0k+1} *as the greatest among the elements of* A *in* α_1 *which are connected to some occurrences of* $\alpha_{(0,k)}$. *Consider all the i-subsections of* α_1 *which contain* $\alpha_{(0,k)}$ *properly and are of the form* (i, a_{0k+1}, γ). *Let* $\alpha_{(0,k+1)}$ *be the greatest, with respect to* $<_i$, *among these.*

In the subsequent sections, we will be using some of the definitions and propositions here quite frequently, but without quoting them every time.

We also employ the following abbreviated notations. Let i be an element of $\boldsymbol{I}$ and $\{i_m\}_m$ be a sequence of elements of $\boldsymbol{I}$. $i_m \uparrow i$ means that $\{i_m\}_m$ converges to i from below (with the order of $\boldsymbol{I}$). The same notation applies to the elements of $\boldsymbol{A}$. If the order type of $\boldsymbol{I}$ is a limit ordinal (in which case we say that $\boldsymbol{I}$ is limit), then $i_m \uparrow \boldsymbol{I}$ means the order type of i_m converges to that of $\boldsymbol{I}$. The same with $\boldsymbol{A}$.

§ 1. Reduction sequences.

Given a system of o.d.'s, $O(\boldsymbol{I}, \boldsymbol{A})$, we shall first define the notion of reduction sequences (for o.d.'s of $O(\boldsymbol{I}, \boldsymbol{A})$ with respect to each indicator, an element of $\boldsymbol{I}$), proving subsequently that those reduction sequences serve as the fundamental sequences for $O(\boldsymbol{I}, \boldsymbol{A})$.

The definition of the reduction sequences consists of three parts, which we shall present successively in the following subsections § 1.1, § 1.2 and § 1.3. § 1.1 concerns the scanned pairs, scanned o.d.'s marked places and reduction places. Given $\tilde{\alpha}$ a connected o.d. and j_0 an indicator, we define in § 1.1 scanned pairs (sp), which are pairs of indicators and sub-o.d.'s of $\tilde{\alpha}$, and scanned o.d.s (sod), which are the sub-o.d.'s occurring in the sp's, determining marked places, which are sub-o.d.'s of $\tilde{\alpha}$, at each stage, marking the marked places with underlines and locating intermediate reduction pairs when they arise. The meaning of all this will become clear later. These notions are defined relative to $(j_0, \tilde{\alpha})$. The construction of sp's stops and the last sp of this process is called the last reduction pair (of $(j_0, \tilde{\alpha})$). The sod in the last sp will be called the last reduction place.

The reduction of the last reduction place (of $(j_0, \tilde{\alpha})$) is prescribed in § 1.2. Let ν be the last reduction place. Then a sequence of o.d.'s, say $\{\nu_m\}_m$, is defined corresponding to ν according to the prescription; such a sequence will be called the reduction sequence for ν (relative to $(j_0, \tilde{\alpha})$).

Let (j, ν) be a sp of $(j_0, \tilde{\alpha})$. A $<_j$-increasing sequence of o.d.'s, say $\{\gamma_m\}_m$, is defined corresponding to γ; such a sequence is called the reduction sequence for γ (relative to $(j_0, \tilde{\alpha})$). This is done in § 1.3, by induction on the number of stages (sp's) between (j, γ) and the last reduction pair; the case where (j, ν) is an intermediate reduction pair needs a special care. As a special case we obtain the reduction sequence for $\tilde{\alpha}$ with respect to j_0, say $\{\tilde{\alpha}_m\}_m$. It turns out that $\{\tilde{\alpha}_m\}_m$ is primitive recursive, is $<_{j_0}$-increasing and converges to $\tilde{\alpha}$ (from below) with respect to j_0 (i.e. $<_{j_0}$). This shows that fundamental sequences can be defined for $O(\boldsymbol{I}, \boldsymbol{A})$ (with respect to every ordering).

We wish to note a notational convention here: $\alpha+1$ will express an o.d. which has 0 as a component, and it will be called a successor o.d. If an o.d. has no 0 as a component, then it is called a limit o.d.

Let us outline the idea of the reduction before we begin formal definitions.

Given $\tilde{\alpha}$ an o.d. and j_0 an indicator, we would like to define a fundamental sequence of $\tilde{\alpha}$ with respect to j_0; it is a sequence of o.d.'s, say $\{\tilde{\alpha}_m\}_m$, which converges to $\tilde{\alpha}$ from below with respect to j_0. More precisely $\{\tilde{\alpha}_m\}_m$ satisfies that

(*) $\{\tilde{\alpha}_m\}_m$ is $<_{j_0}$-increasing and $\forall m(\tilde{\alpha}_m <_l \tilde{\alpha})$ for every $l \leq j_0$,

and

(**) $$\forall \tilde{\beta} <_{j_0} \tilde{\alpha} \exists m(\tilde{\beta} <_{j_0} \tilde{\alpha}_m) .$$

Let us note here that in establishing the property (**) for $\{\tilde{\alpha}_m\}_m$, it suffices to consider only large β's; viz. those $\tilde{\beta}$'s which are smaller than $\tilde{\alpha}$ (with respect to j_0) but very close to $\tilde{\alpha}$.

Our purpose is to establish a primitive recursive method to construct such sequences uniformly in $\tilde{\alpha}$ and j_0. The key point for this objective is how to locate the last reduction place and how to transform it to obtain $\tilde{\alpha}_m$ for each m.

In an attempt to locate the last reduction place, we define sp's. We let $(j_0, \tilde{\alpha})$ be the first sp. Suppose we have defined a sp. (j, γ). We wish to construct a sequence $\{\gamma_m\}_m$ satisfying (*) and (**) for (j, γ), $\{\gamma_m\}_m$ will be called the reduction sequence for γ. There are several cases. 1°, (j, γ) satisfies that $\gamma = (i, a, \delta+1)$ (for some i, a and δ). Then stop. (j, γ) is the last reduction pair.

Suppose 1° is not the case. In order to simplify our discussion, let us consider the case $\gamma = \mathrm{apr}((n, k+1), j, \gamma)$ where $n > 0$. Consider a connected β for which $\beta <_j \gamma$ holds. As was remarked above, it suffices to consider large β's, viz. those which are close to γ with respect to j. Mathematically we may assume that

(***) $\mathrm{apr}((n, k), j, \beta) = \mathrm{apr}((n, k), j, \gamma)$ and $v_{n+1}(j, \beta) = v_{n+1}(j, \gamma) (= i_{n+1})$.

Therefore we are forced to preserve $\mathrm{apr}((n, k), j, \gamma)$ in $\{\gamma_m\}_m$; $\mathrm{apr}((n, k), j, \gamma_m) = \mathrm{apr}((n, k), j, \gamma)$. Therefore if there is a unique occurrence of $\mathrm{apr}((n, k), j, \gamma)$ in γ, then we must not break it, hence wish to mark it so it will be preserved in the reduction process. This is a marked place.

Under the assumption (***), $\beta <_j \gamma$ is equivalent to

$$\beta_{(n,k+1)} = \mathrm{apr}((n, k+1), j, \beta)$$
$$<_{i_{(n+1)}+1} \mathrm{apr}((n, k+1), j, \gamma) = \gamma_{(n,k+1)} = \gamma .$$

2°. Consider the case where $\gamma = (i_{n+1}, a, \gamma')$. Then $\beta_{(n,k+1)} <_{i_{(n+1)}+1} \gamma_{(n,k+1)} = \gamma$ implies that $\beta_{(n,k+1)}$ is of the form (i_{n+1}, a, β'), hence $\beta_{(n,k+1)} <_{i_{(n+1)}+1} \gamma$ is reduced to $\beta' <_{i_{n+1}} \gamma'$. It suffices to consider the case where $<_{i_{n+1}}$ holds due to a least component of γ': If $\gamma' = \gamma_1 \# \cdots \# \gamma_p$ where $\gamma_1 {}_\iota\!\geq \gamma_2 {}_\iota\!\geq$

$\cdots{}_\iota\geq\gamma_p$ and $\iota=i_{n+1}$, then $\beta'=\gamma_1\# \cdots \#\gamma_{p-1}\#\delta_p\# \cdots$, where the components are written in the $<_\iota$-decreasing order and $\delta_p<_\iota\gamma_p$. Let η denote γ_p. Since 1° is not the case, $\eta\neq 0$.

From the way η was chosen, it is expected that if we can construct a sequence $\{\eta_m\}_m$ converging to η from below with respect to i_{n+1} (within a set of o.d.'s satisfying a certain condition), then by replacing η by $\{\eta_m\}_m$ we obtain $\{\gamma_m\}_m$ from γ satisfying the desired property. Here a "certain condition" is, roughly speaking, that the j-approximations do not exceed (with respect to the appropriate orderings) those of γ which are contained in η. This condition is necessary in order to secure $\gamma_m<_j\gamma$.

We now realize that next we should examine (i_{n+1}, η); namely this is the next sp. Note that η is a proper sub-o.d. of γ.

3° Consider the case where $\gamma=(i, a, \gamma')$ and $i>i_{n+1}$. Let $h=i_{(n+1)}+1$. In order to see why $\beta_{(n,k+1)}<_h\gamma$, examine the h-approximations of γ. If 3° is not the case for (h, γ), then proceed to an appropriate step. If 3° is the case for (h, γ), repeat the same speculation, thus obtaining an increasing sequence of indicators, say $i_{n+1}=h_1, h_2, h_3, \cdots$, occurring in γ. It must eventually reach i, hence 3° can repeat only finitely many times. When out of 3° one can proceed to a different step.

4°. There are several other conditions than 1° which make a sp (j, γ) the last one. The detail is omitted for now.

If a condition in 1° or 4° applies to (j, γ), then (j, γ) is the last reduction pair.

5° If none of 1°~4° applies to (j, γ), then we define the next sp with a similar speculation as in 2°. Consider, for example, the case where $\gamma=\text{apr}(0, j, \gamma)=(i, a, \delta)$ where $a\neq 0$ and δ is a marked place. Let (i, δ) be the next sp. (j, γ) is called an intermediate reduction pair and a special care is taken in defining the reduction sequence for γ so that δ, a marked place, will be preserved.

It is obvious from the definition that the construction of sp's stops, for a sod is a proper sub-o.d. of a preceding one. The last sp satisfies either 1° or 4° and there the reduction is taken place.

We have assumed that $\tilde{\alpha}$ is connected; it is obvious that fundamental sequences for non-connected o.d.'s are naturally induced from the ones for connected o.d.'s.

§ 1.1. Scanned pairs, marked places and reduction pairs.

DEFINITION 1.1. Given $\tilde{\alpha}$ a connected o.d. and j_0 an indicator (an element of I), we shall define *scanned pairs* (*sp*), *scanned* o.d.'s (*sod*), *marked places*, *intermediate reduction pairs*, *intermediate reduction places*, the *last reduction pair* and the *last reduction place of* $\tilde{\alpha}$ *with respect to* j_0 (or of $(j_0, \tilde{\alpha})$). A *pair* in any of the above terms is a pair of an indicator and a connected sub-o.d. of $\tilde{\alpha}$, say (j, γ), and the *place* corresponding to (j, γ) is γ as a sub-o.d. of $\tilde{\alpha}$. Therefore it suffices to define only the respective pairs (except the marked places). What is important here is that a scanned o.d. (sod) is a specified occurrence of a (connected) sub-o.d. of $\tilde{\alpha}$.

The first scanned pair (sp) of $\tilde{\alpha}$ with respect to j_0 is $(j_0, \tilde{\alpha})$. For the inductive stages, we first list all the case conditions. Suppose a sp (of $(j_0, \tilde{\alpha})$), say (j, γ), has been defined, where j is an indicator and γ is a connected sub-o.d. of $\tilde{\alpha}$. The following exhaust all of the cases which can apply to (j, γ).

(1) γ is of the form $(i, a, 0)$.

(2) γ is of the from $(i, a, \alpha+1)$ and it is not the case that $a=0$, $j<i$, $\gamma=\text{apr}((n, k+1), j, \gamma)$ where $n>0$ and $v_{n+1}(j, \gamma)<i$. (For the excluded case, see (5, 3) below. We shall name the excluded case as (2').)

If $\gamma=\text{apr}((n, k+1), j, \gamma)$ for some n and k, and if there is a unique occurrence of the $(n, k)^{\text{th}}$ j-approximation of γ, then mark (underline) it as a marked place.

(2.1) $a=0$, $j\leq i$, $i=i_0+1$ (namely i is a successor indicator) and there is at least one component of α whose outermost value is greater than $(i_0, 0)$.

(2.2) (2.1) is not the case.

For (2.1) we define the notion of $(i_0, 0)$-dominants; a component of α whose outermost value is greater than $(i_0, 0)$ (if such exists) is called an $(i_0, 0)$-dominant (of γ, or of α).

In the subsequent cases, we assume that γ is of the form (i, a, α), where α is a limit o.d., viz. no component of α is 0, except for (2') (cf. (5.3) below).

(3) γ is the 0^{th} j-approximation of itself, or $\gamma=\text{apr}(0, j, \gamma)$.

(3.1) All the components of α are marked and $a\neq 0$.

(3.2) All the components of α are marked and $a=0$.

(3.3) Not all the components of α are marked.

(4) γ is the $(0, k+1)^{\text{th}}$ j-approximation of itself, or $\gamma=\text{apr}((0, k+1), j, \gamma)$, for some k.

If there is a unique occurrence of the $(0, k)^{\text{th}}$ j-approximation of γ, then mark (underline) it as a marked place.

(4.1) All the components of α are marked, $a=0$, $j\leq i$ and i is a successor element.

(4.2) All the components of α are marked and (4.1) is not the case.

(4.3) Not all the components of α are marked.

(5) γ is the $(n, k+1)^{\text{th}}$ j-approximation of itself, or $\gamma=\text{apr}((n, k+1), j, \gamma)$, for some (n, k) where $n>0$.

Note. If γ is the $(n, 0)^{\text{th}}$ j-approximation of itself, then we may regard γ as thə $((n-1), l)^{\text{th}}$ j-approximation of itself for some $l>0$, hence the sufficiency of considering $k+1$.

If there is a unique occurrence of the $(n, k)^{\text{th}}$ j-approximation of γ, then mark it as a marked place. Let $i_{n+1}=v_{n+1}(j, \gamma)$.

(5.1) $i=i_{n+1}$ and all the components of α are marked.

(5.2) $i=i_{n+1}$ and not all the components of α are marked.

(5.3) and (2′) $i_{n+1}<i$ under the conditions of (5) stated immediately above and of (2′), namely, $\gamma=(i, a, \alpha+1)$, $a=0$, $j<i$, $\gamma=\text{apr}((n, k+1), j, \gamma)$ for some $n>0$ and $v_{n+1}(j, \gamma)=i_{n+1}<i$.

Note. 1) (3.3), (4.3) and (5.2) include the case where no components of α are marked.

2) We later learn that in (3.1), (3.2), (4.1), (4.2) and (5.1), the clause "all the components of α are marked" can be replaced by "α is connected and marked."

3) For (2′), we need not mark a new marked place.

The transition form (j, γ) to the next sp is determined according to the cases listed above and an additional condition which is stated below. Namely (3.3), (4.3) and (5.2) will be classified into two subcases, [1°] and [2°].

[1°] $\gamma=(t, b, \lambda\#\nu)$ where ν is a least, with respect to $<_t$, (a t-least) componet of $\lambda\#\nu$ and is of the form $(i, 0, \alpha)$ and (3.2) applies to the pairs (t, ν) (i.e. $\nu=\text{apr}(0, t, \nu)$ and all the components of α are marked).

[2°] [1°] is not the case.

Note. 1) Obviously there is a notational confusion here. In any of (3.3), (4.3) and (5.2), γ is supposed to be of the form (i, a, α), whereas in [1°] $(t, b, \gamma\#\nu)$ is used for γ, and i and α are preserved for ν. This is due to the fact that the condition in [1°] concerns (t, ν), to which (3.2) applies. We shall stick to this notation throughout whenever [1°] comes under consideration. In [2°], we return to the original notation; γ is of the form (i, a, α).

2) In [1°] we are not mentioning anything about λ. λ may or may not be empty.

3) In [1°] and any subsequent discussion, "a least component of α" means that we choose one of the least components of α (with respect to an appropriate indicator). We may assume that the components of α are ordered in the non-increasing order with respect to an appropriate indicator and that we choose the last one as a "least component". This specification of a "least component" is important, especially for (2.1), and we shall observe it throughout.

Now we are comming to the induction steps (of defining sp's). In order to make the situation clear, we shall repeat the case conditions. Let (j, γ) be the sp (of $(j_0, \tilde{\alpha})$) which has just been defined.

Now we are coming to the induction steps (of defining sp's). In order to make the situation clear, we shall repeat the case conditions. Let (j, γ) be the sp (of $(j_0, \tilde{\alpha})$) which has just been defined.

(1) $\gamma=(i, a, 0)$. Stop. (j, γ) is the last reduction pair.

(2) $\gamma=(i, a, \alpha+1)$ and (2′) is not the case. If $\gamma=\text{apr}((n, k+1), j, \gamma)$ and there is a unique occurrence of $\text{apr}((n, k), j, \gamma)$, then mark (underline) it as a marked place.

(2.1) $a=0$, $j\leq i$, $i=i_0+1$ and there is an $(i_0, 0)$-dominant of γ. (j, γ) is an intermediate reduction pair. Let γ' be an i-least $(i_0, 0)$-dominant of γ. (i, γ') is the next sp.

Let us point out that here by an "i-least" $(i_0, 0)$-dominant we mean the last occurrence of an i-least such in the i-decreasing arrangement of the components of α.

(2.2) (2.1) is not the case. Stop. (j, γ) is the last recduction pair.

Now assume that in $\gamma=(i, a, \alpha)$ α is a limit o.d. except for (2′).

(3) $\gamma=\text{apr}(0, j, \gamma)$.

(3.1) All the components of α are marked, and $a\neq 0$. (j, γ) is an intermediate reduction pair. Let γ' be an i-least component of α. (i, γ') is the next sp.

(3.2) All the components of α are marked and $a=0$. Let γ' be an i-least component of α. (i, γ') is the next sp.

(3.3) [1°] Here we assume that $\gamma=(t, b, \lambda\#\nu)$. Not all the components of $\lambda\#\nu$ are marked and the condition [1°] stated above applies. (j, γ) is an intermediate reduction pair and (t, ν) is the next sp.

[2°] Here we are back to the common notation: $\gamma=(i, a, \alpha)$. Not all the components of α are marked and [1°] is not the case. Let γ' be an i-least component of α. (i, γ') is the next sp.

(4) $\gamma=\text{apr}((0, k+1), j, \gamma)$. If there is a unique occurrence of $\text{apr}((0, k), j, \gamma)$, then mark (underline) it as a marked place.

(4.1) All the components of α are marked, $a=0$, $j\leq i$, and i is a successor element. (j, γ) is an intermediate reduction pair. Let γ' be an i-least component of α. (i, γ') is the next sp.

(4.2) All the components of α are marked and (4.1) is not the case. Stop. (j, γ) is the last reduction pair.

(4.3) [1°] $\gamma=(t, b, \lambda\#\nu)$, not all the components of $\lambda\#\nu$ are marked and the condition in [1°] is satisfied. (j, γ) is an intermediate reduction pair and (t, ν) is the next sp.

[2°] $\gamma=(i, a, \alpha)$, not all the components of α are marked and [1°] is not the case. Let γ' be an i-least component of α. (i, γ') is the next sp.

(5) $\gamma=\text{apr}((n, k+1), j, \gamma)$ for an $n>0$. If there is a unique occurrence of $\text{apr}((n, k), j, \gamma)$, then mark it as a marked place. Let $i_{n+1}=v_{n+1}(j, \gamma)$.

(5.1) $i=i_{n+1}$ and all the components of α are marked.

(5.1.1) $a=0$, $j\leq i$ and i is a successor element. (j, γ) is an intermediate reduction pair. Let γ' be an i-least component of α. (i, γ') is the next sp.

(5.1.2) (5.1.1) is not the case. Stop. (j, γ) is the last reduction pair.

(5.2) $i=i_{n+1}$ and not all the components of α are marked.

[1°] $\gamma=(t, b, \lambda\#\nu)$, $i_{n+1}=t$, not all the components of $\lambda\#\nu$ are marked and [1°] applies to (j, γ). (j, γ) is an intermediate reduction pair and (t, ν) is the next sp.

[2°] $\gamma=(i, a, \alpha)$, $i_{n+1}=i$, not all the components of α are marked and [1°] is not the case. Let γ' be an i-least component of α. (i, γ') is the next sp.

(5.3) and (2′) $i_{n+1}<i$. We shall define a sequence of indicators $l_1, l_2, \cdots, l_m, \cdots$ such that $i_{n+1}=l_1<l_2<\cdots<i$ and, for each m, l_m occurs in γ. Evidently the sequence is finite. Let l_u be the last indicator in the sequence. Then the next sp is defined to be (l_u+1, γ).

Put $l_1=i_{n+1}$ and $j_1=l_1+1$.

(*) Consider the j_1-approximations of γ. Suppose $\gamma=\text{apr}((r,s), j_1, \gamma)$ for some (r, s). If there is a unique occurrence of $\text{apr}((r, s-1), j_1, \gamma)$, then mark it as a marked place. If (j_1, γ) satisfies the condition in one of the preceding cases, viz. (1)$\sim$(5.2), then stop. l_1 is the last indicator of the desired desired sequence. Otherwise (5.3) must be the case again; $r>0$, $s>0$ and $v_{r+1}(j_1, \gamma)<i$. Then let $l_2=v_{r+1}(j_1, \gamma)$ and let $j_2=l_2+1$.

Consider the j_2-approximations of γ and go over the speculation described in (*) above, replacing j_1 by j_2. If one of the preceding cases applies to (j_2, γ), then stop; l_2 is the last entry of the sequence. Otherwise define l_3 and j_3 and repeat (*).

From the definition of l_m, $l_m=v_{r+1}(j_{m-1}, \gamma)$ for some r and $l_{(m-1)}+1=j_{m-1}\leq l_m<i$, hence $l_{m-1}<l_m<i$. This completes Definition 1.1.

Remark. When the process (2′) stops, only (2) can apply, since γ remains unchanged.

DEFINITION 1.2. 1) Let $j_1(=l_1+1)$, $j_2(=l_2+1)$, $\cdots$, $j_u(=l_u+1)$ be the indicators defined for (2′) and (5.3) in Definition 1.1. Then each of (j_1, γ), (j_2, γ), $\cdots$, (j_u, γ) is called a *transitory scanned pair* (*tsp*) (for (j, γ)). A tsp is normally distinguished from sp's, although many propositions are stated for both sp's and tsp's. Note that (j_u, γ), the last tsp, is a sp which succeeds (j, γ).

2) Let (j, γ) be a sp and (i, γ') be the next sp. Then (j, γ) is called the *immediate predecessor of* (i, γ') and (i, γ') is called the *immediate successor of* (j, γ). The notion that one sp is the predecessor or a successor of another, not necessarily an immediate one, is defined

naturally.

The same terms will be used for tsp's.

3) A sp, say (j, γ), is called static if (2′) or (5.3) applies to (j, γ); otherwise it is called non-static.

It is obvious that the case conditions in Definition 1.1 exhaust all the possibilities. The soundness of Definition 1.1 will then be established by Proposition 1.1 below.

The following are some consequences of the definitions. Let us remember that all the notions have been defined relative to $(j_0, \tilde{\alpha})$, where j_0 is an indicator and $\tilde{\alpha}$ is a connected o.d.

PROPOSITION 1.1. 1) *For* (j, γ) *a sp its immediate predecessor and its immediate successor (if existent) are unique.*

2) *All the sod's and transitory sod's are connected.*

3) *Let* (j, γ) *be a non-static sp and let* (k, δ) *be its immediate successor, where* $\gamma=(i, a, \alpha)$. *Then* $k=i$ *and* δ *is an* i*-least component of* α *except for* (2.1). *For* (2.1) $\gamma=(i, a, \alpha+1)$ *and* δ *is an* i*-least* $(i_0, 0)$*-dominant of* γ *(of* α*).*

Let (k, δ) *be a sp whose immediate predecessor is non-static and where* δ *is a component of* α *in* $\gamma=(i, a, \alpha)$. *Then* (j, γ) *is the immediate predecessor of* (k, δ) *for some* j, *hence* $k=i$.

4) *In the sequence of sp's, there can never be two consecutive static pairs.*

5) *Let* (k, δ) *be a successor of* (j, γ). *If* (j, γ) *is static and* (k, δ) *is its immediate successor, then* $j<k$ *and* $\delta=\gamma$. *Otherwise* δ *is a proper sub-o.d. of* γ.

PROPOSITION 1.2. *The sequence of sp's is finite; hence the existence of the last reduction pair for any given* $(j_0, \tilde{\alpha})$. *The last reduction pair is uniquely determined for* $(j_0, \tilde{\alpha})$.

Proof. From 2), 4) and 5) of Proposition 1.1.

PROPOSITION 1.3. *If* (j, γ) *is the immediate successor of a static pair, then the case* (1) *in Definition* 1.1 *does not apply to* (j, γ).

PROPOSITION 1.4. 1) *Let* (j, γ) *be a sp and let* $\eta=(l, b, \xi)$ *be a sub-o.d. of* γ *such that a component of* ξ, *say* λ, *contains a sod* δ. *(So* (k, δ) *is a successor of* (j, γ) *for some* k.) *Then* (l, λ) *is a sp which is a successor of* (j, γ) *and is a predecessor of* (k, δ). *(The cases where* $(l, \lambda)=(k, \delta)$ *is included.)*

We say in such a case that (l, λ) *is between* (j, γ) *and* (k, δ).

2) *Let* γ *be a sod (of* $(j_0, \tilde{\alpha})$*) and* (p, δ) *be a successor of* γ *where* δ *is a proper sub-o.d. of* γ. *Then there is a sub-o.d. of* γ, *say* (q, b, β), *such that* $q\leq p$ *and* δ *is a component of* β.

Proof. 1) By induction on the number of sp's between (j, γ) and (k, δ). If (k, δ) is the immediate successor of (j, γ), or (j, γ) is static and (k, δ) is the second successor of (j, γ) (cf. 4) of Proposition 1.1), then $\eta=\gamma$ and the proposition is obvious from 3), 4) and 5) of Proposition 1.1. If the above is not the case, let (p, κ) be the first sp after (j, γ) such that κ is a proper sub-o.d. of γ. (The existence of such pair is guaranteed by 3), 4) and 5) of Proposition 1.1.) For every η which satisfies the condition of the proposition and which is contained in κ, the induction hypothesis applies to (p, k), η and δ. If η is κ, then the basis of induction applies.

2) If the immediate predecessor of (p, δ) is non-static, then $p=q$ by 3) of Proposition 1.1. If it is static, then it is (k, δ) where $k<q$ by 5) of Proposition 1.1. Since no two consecutive sp's are static (4) of Proposition 1.1), the immediate predecessor of (k, δ) is not static. Applying the first part of this proof to (κ, δ), we obtain a (q, b, β) where $q \leq k$, hence $q<p$.

PROPOSITION 1.5. *Let (i, a, α) be any sub-o.d. of $\tilde{\alpha}$. Then there is at most one marked component of α. Furthermore a marked component of α (if there is one) is the i-greatest (the greatest with respect to $<_i$) component of α.*

This leads us to the following important conclusion:

COROLLARY. *The clause in Definition* 1.1 *"all the components of α are marked"* (*cf.* (3.1), (3.2), (4.1), (4.2) *and* (5.1)) *can be equivalently stated as "α is connected and is marked".*

Proof of Proposition 1.5. Suppose that a component of α is marked at the stage of (j, γ) (hence γ contains (i, a, α); by the definition of marked places). Let η be a component of α. Due to the uniqueness condition for a marked place, there is a unique occurrence of η among the components of α. Let δ be another component of α. Since $\gamma=$ apr $((n, k+1), j, \gamma)$ and $\eta=$apr $((n, k), j, \gamma)$ for some (n, k), $j \leq i$ and $i=v_{n+1}(j, \gamma)$. So in particular δ is j-active in γ. If we suppose δ contains apr $((n, k), j, \gamma)$ (as a subsection), then the η in α would not have been marked. So δ j-omits apr $((n, k), j, \gamma)$, hence $\delta <_i \eta$. So a marked component of α (if any) must be greater than all other components of α (with respect to $<_i$).

PROPOSITION 1.6. *In* (3.3), (4.3) *and* (5.2) *in Definition* 1.1, *γ' is not marked.* (*In* [1°] *γ' is denoted by ν.*) *Namely if (j, γ) is a sp, $\gamma=(i, a, \alpha)$ and the next sp is (i, γ'), where γ' is an i-least component of α, then γ' is not marked.*

Proof. This is a corollary of Proposition 1.5 and the condition that not all the components of α are marked.

PROPOSITION 1.7. 1) *In the cases* (3.1) *and* (3.2) *of Definition* 1.1, *the immediate predecessor of* (j, γ) *is non-static and* $j>i$.

2) *If* (3.2) *applies to a sp* (t, ν), *then it cannot be the initial pair. Therefore the following situation must arise; there is a sp of* $\tilde{\alpha}$ *say* (j, γ) *such that* $\gamma=(t, b, \lambda\#\nu)$ *and* [1°] *applies to* (j, γ).

3) *If* [1°] *applies to* (j, γ) *where* $\gamma=(t, b, \lambda\#\nu)$, *then* $t>0$.

Proof. 1) In these cases α is connected and marked (cf. Corollary of Proposition 1.5), hence there is a sp, say (l, δ), which precedes (j, γ) such that $\delta=\text{apr}((n, k+1), l, \delta)$, $\alpha=\text{apr}((n, k), l, \delta)$ for some (n, k) and this occurrence of α is the only l-active one in δ. This implies that $i=v_{n+1}(l, \delta)$.

Suppose that the immediate predecessor of (j, γ), say (k, γ), is static. Then $k<j\leq i$ (cf. (5.3) of Definition 1.1 and 5) of Proposition 1.1). The immediate predecessor of (k, γ) is non-static (cf. 4) of Proposition 1.1). So by 3) of Proposition 1.1 k occurs in δ and is connected to γ (hence to α). But then $i=v_{n+1}(l, \delta)$ implies $k\geq i$, contradicting $k<i$, which was claimed above. Thus, the immediate predecessor of (j, γ) is non-static.

By 3) of Proposition 1.1 applied to (j, γ), j occurs in δ and is connected to γ, hence to α; so $v_{n+1}(l, \delta)=i$ implies $j\geq i$. Suppose $j=i$. Then α is j-active in γ, so the equations $\gamma=(i, a, \alpha)$ and $\gamma=\text{apr}(0, j, \gamma)$ imply $\alpha<_{i+1}\gamma$. Furthermore $j=i$ and $\gamma=\text{apr}(0, j, \gamma)$ imply that the outermost indicator of $\alpha\leq i$. So $\alpha=\text{apr}((n, k), 0, \delta)$, hence $i=v_{n+1}(l, \delta)$, implies that $n=0$. But then $\gamma<_{i+1}\alpha$, yielding a contradiction. Therefore $j>i$.

2) The first statement is obvious, since otherwise α could not be marked in $\nu=(i, 0, \alpha)$, ν being the 0^{th} t-approximation of itself. By 1) here, (t, ν) is not the immediate successor of a static sp. Therefore ν occurs in a sod γ: $\gamma=(t, b, \lambda\#\nu)$. (cf. 3) of Proposition 1.1.)

3) Immediate from the condition of [1°] and 1) of this proposition.

PROPOSITION 1.8. *If* (j, γ) *is the last reduction pair, then it is either of the form* $(i, a, 0)$ *or* $(i, a, \beta+1)$, *or of the form* (i, a, α) *where* α *is (connected and) marked. If* (j, γ) *is an intermediate reduction pair, then either* $\gamma=(i, a, \alpha)$ *where* α *is marked,* γ *is of the form* $(i, a, \beta+1)$ *or* [1°] *applies to* (j, γ).

PROPOSITION 1.9. *Let* (j, γ) *be a sp of* $\tilde{\alpha}$ *with respect to* j_0. *Consider the situation where we also construct the sequence of sp's of* γ *with respect to* j. *Suppose also that the last reduction place of* $\tilde{\alpha}$ *is of the form* $(l, b, 0)$ *or* $(l, b, \beta+1)$ *(i.e. either* (1) *or* (2) *of Definition* 1.1). *Then a sp of* $(j_0, \tilde{\alpha})$ *which is a successor of* (j, γ) *is also a sp* *of* (j, γ), *hence the reduction place of* (j, γ) *is the same as that of* $(j_0, \tilde{\alpha})$.

Proof. We prove the proposition together with the following statement:

(*) Any connected sub-o.d. of γ which is marked for (j, γ) is marked for $(j_0, \tilde{\alpha})$ also.

The statements are true for (j, γ). Let (k, δ) be a sp of $(j_0, \tilde{\alpha})$ which is a successor of (j, γ) and suppose the statements hold for it. Mark an appropriate sub-o.d. of δ with respect to k (if there is any). This process is the same for $\tilde{\alpha}$ and for γ. By our hypothesis, unless $\delta=(l, b, 0)$ or $\delta=(l, b, \beta+1)$, there is a further sp for $\tilde{\alpha}$. If (k, δ) is the last reduction pair of (j, γ), then δ is of the form (i, a, α) where α is connected and marked and certain other conditions are satisfied (cf. Proposition 1.8). But for (*) α is marked also for $\tilde{\alpha}$, and "other conditions" hold for $\tilde{\alpha}$ as well. (Examine the conditions in (4.2) and (5.1.2) of Definition 1.1.) Therefore δ must be the last reduction place of $\tilde{\alpha}$, contradicting the assumption. Therefore (k, δ) has a further sp of (j, γ) also. Then the construction of its immediate successor is the same for $(j_0, \tilde{\alpha})$ (cf. the conditions in (3)$\sim$(5) of Definition 1.1 except (4.2) and (5.1.2)). If δ is the last reduction place of $(j_0, \tilde{\alpha})$ (hence is of the form $(l, b, 0)$ or $(l, b, \beta+1)$), then the same is true for (j, γ).

The following two propositions are important and useful.

PROPOSITION 1.10. *In* (4.1) *and* (4.2), *namely* $\gamma=\text{apr}((0, k+1), j, \gamma)=(i, a, \alpha)$ *and* α *is* (*connected and*) *marked* (*cf. Corollary of Proposition* 1.5), $\alpha=\text{apr}((0, k), j, \gamma)$.

Proof. $\gamma=\text{apr}((0, k+1), j, \gamma)=(i, a, \alpha)$ implies that $j\leq i$. Since α is marked, there is a sp of $\tilde{\alpha}$, say (l, δ), such that $\alpha=\text{apr}((r, s), l, \delta)$ and $\delta=\text{apr}((r, s+1), l, \delta)$ for some (r, s). So $i=v_{r+1}(l, \delta)$. Suppose $r\neq 0$. If we let $\eta=\text{apr}(0, l, \delta)$, η is a connected sub-o.d. of γ such that the outermost indicator of η is greater than i (since $i=v_{r+1}(l, \delta)$ and $r>0$) and all the indices connected to η are $\geq i$; but this would mean that η is j-active in γ, and hence the outermost indicator of η is j-active in γ and is greater than $v_1(j, \gamma)$ $(=i)$, which is impossible. So $r=0$. Therefore $\alpha=\text{apr}((0, s), l, \delta)$ and $v_1(l, \delta)=i$.

Since $i=v_1(l, \delta)$ and $\alpha=\text{apr}((0, s), l, \delta)$, the outermost indicator of α is $\geq i$. Therefore $\alpha=(i, b, \kappa)$ for some b and κ, which means $a<b$ (Proposition 0.8). Suppose $\alpha\neq\text{apr}((0, k), j, \gamma)$. Then α contains $\text{apr}((0, k), j, \gamma)$, so $a\geq b$ (Proposition 0.8), yielding a contradiction. So $\alpha=\text{apr}((0, k), j, \gamma)$.

PROPOSITION 1.11. *In* (5.1) *of Definition* 1.1 *where* $\gamma=(i, a, \alpha)=\text{apr}((n, k+1), j, \gamma)$, $n>0$ *and* α *is marked*, $\alpha=\text{apr}((n, k), j, \gamma)$.

Proof. Since α is marked, $\alpha=\text{apr}((r, s), l, \delta)$ and $\delta=\text{apr}((r, s+1),$

$l, \delta)$ for some l, δ, r and s. This implies $i=v_{r+1}(l, \delta)$; hence $i=i_{n+1}=v_{r+1}(l, \delta)$. Suppose α is not apr $((n, k), j, \gamma)$. Then apr $((n, k), j, \gamma)$ is contained in α properly. So $\alpha <_{i+1} \gamma$ (since γ is the $(n, k+1)^{\text{th}}$ j-approximation of itself). On the other hand $\gamma <_{i+1} \alpha$ since $\alpha=$ apr $((r, s), l, \delta)$ and δ contains γ. But this contradicts the result obtained above. Therefore $\alpha=$apr $((n, k), j, \gamma)$.

Proposition 1.12. *If a sp* (j, γ) *satisfies the condition* [1°], *then only one of the cases* (3.3), (4.3) *and* (5.2) *applies to it; hence the sufficiency of considering two cases* [1°] *and* [2°] *only for* (3.3), (4.3) *and* (5.2) *in Definition* 1.1.

Proof. By virtue of 1) of Proposition 1.7 (applied to (t, ν)), it is obvious that [1°] does not apply to (2) or (5.3). We shall assume that one of the other excluded cases applies to (j, γ) and derive a contradiction.

Let $\gamma=(t, b, \lambda \# \nu)$. The corollary of Proposition 1.5 and the conditions (3.1), (3.2), (4.1), (4.2) and (5.1) imply that λ is empty and ν is marked. So $\gamma=(t, b, \nu)$, $\nu=(i, 0, \alpha)$ and $t>i$ by 1) of Proposition 1.7.

Suppose (3.1) or (3.2) applies to (j, γ). Then $\nu=$apr $((r, s), h, \xi)$ for some ξ, h, r and s, and $t=v_{r+1}(h, \xi)$. So $t \leq i$, contradicting $t>i$, which was claimed above. So neither (3.1) nor (3.2) applies.

Suppose (4.1) or (4.2) applies to (j, γ). Then by Proposition 1.10 $\nu=$apr $((0, k), j, \gamma)$. So $t=i=v_1(j, \gamma)$, which contradicts $t>i$.

Suppose (5.1) applies to (j, γ). Then by Proposition 1.11 $\nu=$ apr $((n, k), j, \gamma)$, so $i \geq v_{n+1}(j, \gamma)=t$, which contradicts $t>i$.

§ 1.2. Reduction; non-critical case

Definition 1.3. Reduction of the last reduction place. Let (t, ν) be the last reduction pair of $(j_0, \tilde{\alpha})$; hence ν is the last reduction place. We shall define a *sequence of* o.d.'s, say $\{\nu_m\}_m$, *corresponding to* (t, ν). Recall that [1°] yields an intermediate reduction pair (cf. Definition 1.1), hence [1°] is not involved here. We refer to the case numbers and the notation in § 1.1.

(1) ν is of the form $(i, a, 0)$.

a) $a \neq 0$.

a.1) a is a limit element. Let $\{a_m\}_m$ be an increasing sequence of elements of A which converges to a; viz. $a_m \uparrow a$. Define $\nu_m=(i, a_m, 0)$.

a.2) $a=b+1$ and $t \leq i$. Define $\{\nu_m\}_m$ as follows. $\nu_0=(i, b, 0)$; $\nu_{m+1}=(i, b, \nu_m)$.

a.3) $a=b+1$ and $t>i$. Let h be the least indicator among those which occur in the sp's of $\tilde{\alpha}$.

a.3.1) $h>i$.

1°. I is limit. Let $i_m \uparrow I$. Define $\nu_m=(i, b, (i_m, 0, 0))$.

2°. I has the maximal indicator ν and A is limit. Let $a_m \uparrow A$. Define $\nu_m=(i, b, (\iota, a_m, 0))$.

3°. I has the maximal indicator ι and A nas the maximal element e. Define $\{\nu_m\}_m$ as follows. $\kappa_0=(\iota, e, 0)$; $\kappa_{m+1}=(\iota, e, \kappa_m)$ and $\nu_m=(i, b, \kappa_m)$.

a.3.2) $h\leq i$. This case is called a "critical" case and requires a special care in the subsequent discussion. We shall deal with this case in § 4; for the time being we leave the reduction sequence undefined. This does not affect other cases, as there is only one last reduction pair for any $(j_0, \tilde{\alpha})$.

b) $a=0$.

b.1) i is limit, $i_m \uparrow i$. Let $\nu_m=(i_m, 0, 0)$.

b.2) $i=i_0+1$.

b.2.1) A is limit; $a_m \uparrow A$. Let $\nu_m=(i_0, a_m, 0)$.

b.2.2) A has the maximal element, say e.

b.2.2;1) $t\geq i$ and $h\geq i$. (See a.3) for h.)

1°. $i_m \uparrow I$. Define ν_m by $\nu_m=(i_0, e, (i_m, 0, 0))$.

2°. I has the maximal element ι. $\nu_m=(i_0, e, (\iota, e, \cdots, (\iota, e, 0)\cdots))$.

b.2.2;2) $t\geq i$ and $h<i$ (or $h\leq i_0$). This is another critical case and will be dealt with in § 4.

b.2.2;3) $t<i$, or $t\leq i_0$. $\nu_m=(i_0, e, \cdots, (i_0, e, 0)\cdots)$.

b.3) $i=0$. Let ν_m be $0\#0\#\cdots\#0$ where 0 repeats $m+1$ times.

(2.2) ν is of the form $(i, a, \alpha+1)$ and neither (2.1) nor (2′) is the case. We name this case as c).

c.1) $i<t$. Let $\nu_m=(i, a, \alpha)\#\cdots\#(i, a, \alpha)$, where (i, a, α) repeats $m+1$ times.

c.2) $t\leq i$.

c.2.1) a is a limit element; $a_m \uparrow a$. $\nu_m=(i, a_m,(i, a, \alpha))$.

c.2.2) $a=b+1$. $\nu_0=(i, b, (i, a, \alpha))$; $\nu_{m+1}=(i, b, \nu_m)$.

c.2.3) $a=0$ and $t=i$. $\nu_m=(i, a, \alpha)\#\cdots\#(i, a, \alpha)\#\rho_m$, where (i, a, α) repeats $m+1$ times and ρ_m is defined as follows.

1°. $i=0$ or i is limit. ρ_m is empty.

2°. $i=i_0+1$ and $a_m \uparrow A$. $\rho_m=(i_0, a_m, (i, a, \alpha))$.

3°. $i=i_0+1$ and e is the maximum element of A.
$\rho_m=(i_0, e, \cdots, (i_0, e, (i, a, \alpha)\#\cdots\#(i, a, \alpha))\cdots)$, where (i_0, e) and (i, a, α) repeat $m+1$ times.

c.2.4) $a=0$, $t<i$ and $i_m \uparrow i$. We may assume that $i_m>t$. Define $\nu_m=(i_m, 0, (i, a, \alpha))$.

c.2.5) $a=0$, $t<i$ and $i=i_0+1$.

1°. $a_m \uparrow A$. $\nu_m=(i_0, a_m, (i, a, \alpha))$.

2°. e is the maximum element of A.
$\nu_m=(i_0, e, \cdots, (i_0, e, (i, a, \alpha)\#\cdots\#(i, a, \alpha))\cdots)$ where (i_0, e) and (i, a, α) repeat $m+1$ times.

We often denote the (i, a, α) in those definitions by μ.

In the following ν is of the form (i, a, α), where α is a limit o.d.

(4.2) $\nu=\mathrm{apr}((0, k+1), t, \nu)$, α is connected and marked (cf. the corollary of Proposition 1.5), and a and i satisfy one of the following conditions; $a\neq 0$, $a=0$ and $t=i=0$, or $a=0$, $t\leq i$ and i is a limit element. (Note that $t\leq i$ must hold in this case.) We name this case as d).

d.1) $a_m \uparrow a$. $\nu_m=(i, a_m, \alpha)$.

d.2) $a=b+1$. $\nu_m=(i, b, \cdots, (i, b, \alpha\# \cdots \#\alpha)\cdots)$ where (i, b) and α repeat $m+1$ times.

d.3) $a=0$, $t<i$ and $i_m \uparrow i$, where $i_m>t$. $\nu_m=(i_m, 0, \alpha)$.

d.4) $a=0$ and $t=i$.

d.4.1) $i_m \uparrow i$. $\nu_m=\alpha\# \cdots \#\alpha\#\rho_m$, where α repeats $m+1$ times and $\rho_m=(i_m, 0, \alpha)$.

d.4.2) $i=0$. $\nu_m=\alpha\# \cdots \#\alpha$.

(5.1.2) $\nu=\mathrm{apr}((n, k+1), t, \nu)$, where $n>0$. $a\neq 0$, $a=0$ and $t=i=0$ or $a=0$, $t\leq i$ and i is limit. α is connected and marked. $\alpha=\mathrm{apr}((n, k), t, \nu)$ by Proposition 1.11; so $t\leq i=v_{n+1}(t, \nu)=i_{n+1}$. We name this case as e).

e.1) $a_m \uparrow a$. $\nu_m=(i, a_m, \alpha)$.

e.2) $a=b+1$. $\nu_m=(i, b, \cdots, (i, b, \alpha\# \cdots \#\alpha)\cdots)$.

e.3) $a=0$ and $t<i$. Let $i_m \uparrow i$, where $t<i_m$ is assumed for every m. Define $\nu_m=(i_m, 0, \alpha)$.

e.4) $a=0$ and $t=i$.

e.4.1) $i_m \uparrow i$. $\nu_m=\alpha\# \cdots \#\alpha\#\rho_m$ where $\rho_m=(i_m, 0, \alpha)$.

e.4.2) $i=0$. $\nu_m=\alpha\# \cdots \#\alpha$.

Note. In c.2.4), d.3) and e.3), the condition that $i_m>t$ is not necessary; it helps when making some statements uniform in m.

§ 1.3. Reduction sequences: non-critical case.

For each (j, γ) a sp (of $(j_0, \tilde{\alpha})$), we define a sequence of o.d.'s, say $\{\gamma_m\}_m$, which is called the *reduction sequence for* (j, γ) (or γ). In most cases γ_m is obtained from γ by replacing the last reduction place, ν, by the corresponding ν_m, viz. $\gamma_m=\gamma(\nu_m)=\gamma\binom{\nu}{\nu_m}$. Precisely the reduction is defined by induction on the number of sp's between (j, γ) and the last reduction pair. Here we assume that the last reduction pair is not a critical case (cf. a.3.2) of Definition 1.3).

DEFINITION 1.4. Let (t, ν) be the last reduction pair. Then the reduction sequence for (t, ν) is the $\{\nu_m\}_m$ defined in Definition 1.3.

Let (j, γ) be a sp and (l, δ) be its immediate successor. If the transition from (j, γ) to (l, δ) is not by $[1°]$ and not by (2.1), (3.1), (4.1), (5.1.1) of (5.3), then $\gamma_m=\gamma\binom{\delta}{\delta_m}$, namely γ_m is obtained from γ by replacing δ by δ_m, which is assumed to have been defined.

For [1°] ((3.3), (4.3) or (5.2)), we define the reduction sequence for γ as follows. Let $\gamma=(t, b, \lambda\#\nu)$. Recall that $\nu=(i, 0, \alpha)$ where $t>i$ (cf. Proposition 1.7). By the induction hypothesis, $\{\nu_m\}_m$ has been defined for ν. Put $\mu_m=(t, b, \lambda\#\nu_m)$.

[1°.1] $b_m \uparrow b$. $\gamma_m=(t, b_m, \lambda\#\nu)\#\mu_m$.

[1°.2] $b=c+1$. $\kappa_0=(t, c, \lambda\#\nu)$; $\kappa_{m+1}=(t, c, \kappa_m)$; $\gamma_m=\kappa_m\#\mu_m$.

[1°.3] $b=0$ and $t_m \uparrow t$. We may assume that $t_m>i$ for all m. $\gamma_m=(t_m b, \lambda\#\nu)\#\mu_m$.

]1°.4] $b=0$, $t=q+1$ and $a_m \uparrow A$. $\gamma_m=(q, a_m, \lambda\#\nu)\#\mu_m$.

[1°.5] $b=0$, $t=q+1$ and A has the maximal element e. $\kappa_1=(q, e, \lambda\#\nu)$; $\kappa_{m+1}=(q, e, \kappa_m)$; $\gamma_m=\kappa_m\#\mu_m$.

Recall that $t=0$ is not possible for this case.

For (2.1), we define the reduction sequence for γ as follows. Put $\mu=(i, 0, \alpha)$. Let α_m be obtained from α by replacing δ (the next sod) by δ_m.

(2.1;1) $j<i$. Define $\gamma_m=(i, 0, \alpha_m\#(i_0, 0, \mu))$.

(2.1;2) $j=i$. $\gamma_m=\mu\#\cdots\#\mu\#(i, 0, \alpha_m\#(i_0, 0, \mu))$ where μ repeats $m+1$ times.

For (3.1), we define the reduction sequence for γ as follows.

Suppose $\gamma=(i, a, \alpha)$. Then the next sp is (i, α) (cf. the condition in (3.1) and the Corollary of Proposition 1.5). By the induction hypothesis, $\{\alpha_m\}_m$ has been defined for α. Recall that $a\neq 0$.

(3.1;1) $a_m \uparrow a$. $\gamma_m=(i, a, \alpha_m)\#(i, a_m, \alpha)$.

(3.1;2) $a=b+1$. $\gamma_m=(i, a, \alpha_m)\#(i, b, \cdots, (i, b, \alpha)\cdots)$.

For (4.1), we define the reduction sequence for γ as follows. Recall that $\gamma=(i, 0, \alpha)$ where $\alpha=\text{apr}((0, k), j, \gamma)=\text{apr}((0, k), i, \gamma)$ (cf. Proposition 1.10) and that the next sp is (i, α) (cf. Corollary of Proposition 1.5). Here $i=i_0+1$. By the induction hypothesis the reduction sequence for (i, α), say $\{\alpha_m\}_m$, is defined.

(4.1;1) $j<i$. $\gamma_m=(i, 0, \alpha_m\#(i_0, 0, \alpha))$.

(4.1;2) $j=i$. $\gamma_m=\alpha\#\cdots\#\alpha\#(i, 0, \alpha_m\#(i_0, 0, \alpha))$.

The reduction sequence for the case (5.1.1) is defined as follows. Recall that $\gamma=(i, 0, \alpha)$, where $i=i_{n+1}=v_{n+1}(j, \gamma)$, $\alpha=\text{apr}((n, k), j, \gamma)=\text{apr}((n, k), i, \gamma)$ (cf. Proposition 1.11), $i=i_0+1$ and the next sp is (i, α). By the induction hypothesis the reduction sequence for (i, α), say $\{\alpha_m\}_m$ is defined.

(5.1.1;1) $j<i$. $\gamma_m=(i, 0, \alpha_m\#(i_0, 0, \alpha))$.

(5.1.1;2) $j=i$. $\gamma_m=\alpha\#\cdots\#\alpha\#(i, 0, \alpha_m\#(i_0, 0, \alpha))$.

Note that the definition is completely parallel to that for (4.1).

If the transition is by (5.3) or (2′), then put $\gamma_m=\delta_m$.

As was remarked after Definition 1.1, γ_m for (2′) assumes one of the forms defined for c.2.3), c.2.4) and (2.1).

This completes the reduction of $\tilde{\alpha}$ with respect to j_0. The reduction sequence for $\tilde{\alpha}$ will be denoted by $\{\tilde{\alpha}_m\}_m$. Note that (3.2) is not an exceptional case; if (t, ν) satisfies the condition in (3.2) and $\nu=(i, 0, \alpha)$, then $\nu_m=(i, 0, \alpha_m)$.

It turns out that the reduction sequences serve as the fundamental sequence of ordinal diagrams.

PROPOSITION 1.13. *Let γ be a sod of $(j_0, \tilde{\alpha})$, and let $\{\gamma_m\}_m$ be the reduction sequence for γ. If γ is of the form (i, a, α), γ is not the last reduction place and none of* (2.1), (4.1) *and* (5.1.1) *is the case, then γ_m contains a component of the form (i, a, α_m), where α_m is obtained from α by replacing its least component by an o.d. of the corresponding sequence.*

Proof. If the transition from γ to thc next stage is not by one of [1°], (2.1), (3.1), (4.1) and (5.1.1), then $\gamma_m=(i, a, \alpha_m)$ has the form described above. If it is by [1°] or (3.1), then (i, a, α_m) is one component of γ_m (cf. Definition 1.4).

For the title of the next section, we define one expression which is not very rigorous but is rather convenient.

DEFINITION 1.5. An o.d. α *dominates another* o.d. β (with respect to j an indicator) (when j is given and fixed) if $\beta<_j\alpha$. α is said to *dominate a sequence* $\{\alpha_m\}_m$ (with respect to j) if $\alpha_m<_j\alpha$ for every m.

§ 2. An o.d. dominates its reduction sequence; non-critical case

In this section we shall show that $\tilde{\alpha}_m<_{j_0}\tilde{\alpha}_{m+1}$ and $\tilde{\alpha}_m<_{j_0}\tilde{\alpha}$ for every m (cf. §1.3 for the notation) except for the case where the last reduction place is critical (i.e. a.3.2)). For the purpose of this section, we prove the following proposition.

PROPOSITION 2.1. *Let (j, γ) be a sp of $(j_0, \tilde{\alpha})$ and let $\{\gamma_m\}_m$ be the reduction sequence for γ. Let h^* be an indicator which is either j or which occurs in a sp which precedes (j, γ). Let h be the least of such h^*'s. If (j, γ) happens to be $(j_0, \tilde{\alpha})$ the initial sp, then $h=j_0=j$. When we wish to emphasize that h depends on γ, we write $h(\gamma)$ for h.*

Now the following hold for every m and every l an indicator satisfying $h\leq l\leq j$.

1. $\gamma_m<_l\gamma$.
2. The maximum, h-active value of $\gamma_m\leq$ the corresponding value of γ.
3. Let σ be an l-section of γ_m. Then $\sigma<_l\gamma$.
4. $\gamma_m<_l\gamma_{m+1}$.
5. Let σ be an l-section of γ_m. Then $\sigma<_l\gamma_{m+1}$.

As a special case of the proposition, we obtain our first result.

THEOREM 1. *Let $\tilde{\alpha}$ be a connected o.d. and j_0 be an indicater.*

Then there is a j_0-increasing sequence of o.d.'s which is dominated by $\tilde{\alpha}$ (with respect to j_0). Furthermore there is a uniform method to construct such a sequence for every $\tilde{\alpha}$ and j_0.

Uniformity of the construction is obvious from Definition 1.3 and 1.4.

We prove Proposition 2.1 in two parts: for the last reduction pairs (§ 2.1) and for the induction stages (§ 2.2).

In the course of the proof, the following definition will become necessary.

DEFINITION. Let γ be a non-connected o.d. Then we define $apr((n, k), j, \gamma)$ to be the $(n, k)^{\text{th}}$ j-approximation of the j-greatest components of γ.

§ 2.1. Proposition 2.1; Basis.

The basis of the proof is carried out according to the cases in Definition 1.3. We quote the numbering there. Since many cases are trivial, we shall pick up a few points which need some explanation.

(1) $\nu=(i, a, 0)$.

a) $a\neq 0$.

a.1) $\nu_m=(i, a_m, 0)<_l(i, a, 0)$ and $(i, a_m, 0)<_l(i, a_{m+1}, 0)$ are obvious for all l. The only section of ν_m is 0. The value $(i, a_m)<(i, a)$.

a.2) $\nu_0=(i, b, 0)<_l(i, b+1, 0)=\nu$ for every l. Suppose $\nu_m<_l\nu$ for all l. Then $\nu_{m+1}=(i, b, \nu_m)<_l\nu$ is obvious if $l>i$. The i-section of ν_{m+l} is ν_m and $\nu_m<_i\nu$ by the hypothesis, hence $\nu_{m+1}<_i\nu$. There is no other section. Therefore $\nu_{m+1}<_l\nu$ follow for all l. $v_0(h, \nu_m)=(i, b)<(i, b+1)=v_0(h, \nu)$. $\nu_m<_l\nu_{m+1}$ for all l and ν_{m-1}, the i-section of ν_m, $<_i\nu_{m+1}$.

a.3) $\nu=(i, b+1, 0)$ and $t<i$.

a.3.1) $h>i$. $\nu_m=(i, b, \nu')$ for some ν' (depending on m) for any of the three cases. So $\nu_m<_l\nu$ if $l>i$. Since $h>i$ this is sufficient. There is no section of ν_m for such l. $\nu_m<_l\nu_{m+1}$ is also easily proved. Since $h>i$ the large values which occur in ν' do not matter when considering h-active values.

b.2.2;1) $h>i_0$. So $\nu_m<_l\nu$ is $l>i_0$. Since $h\geq i>i_0$, this is sufficient.

(2.2) $\nu=(i, a, \alpha+1)$.

c.2) $t\leq i$.

c.2.1) $\nu_m=(i, a_m, (i, a, \alpha))<_l(i, a, \alpha+1)=\nu$ is obvious if $l>i$. $(i, a, \alpha)<_i\nu$. Suppose $l<i$. Any l-section of ν_m is an l-section of α, hence is an l-section of ν. Thus follows $\nu_m<_l\nu$ for all l. The values do not increase.

d.4.1) $\nu=\text{apr}((0, k+1), t, \nu)=(i, a, \alpha)$, where α is connected and marked, $a=0$, $t=i$ and $i_m\uparrow i$. $\nu_m=\alpha\#\cdots\#\alpha\#\rho_m$. $\alpha<_l(i, 0, \alpha)$ if $l\leq i=t$. $\rho_m<_l(i, 0, \alpha)$ is also obvious.

Note that a.3.1) and b.2.2;2) are the only cases where the lower bound h is necessary for l and d.4) and e.4) are the only cases where the upper bound is necessary.

§ 2.2. Proposition 2.1: Induction step.

Assume that the proposition has been established for the successors of a sp (j, γ). Then we prove the proposition for (j, γ). Let $\gamma=(i, a, \alpha)=(i, a, \alpha' \# \gamma')$, where (i, γ') is the next sp. α' may be empty.

[1°] of (3.3), (4.3) or (5.2). Here we adhere to the notation of [1°]: $\gamma=(t, b, \lambda \# \nu)$ and ν is a t-least component of $\lambda \# \nu$. We write ρ for $\lambda \# \nu$.

[1°.1] $(t, b_m, \rho)<_j(t, b, \rho)$ and $(t, b_m, \rho)<_j(t, b_{m+1}, \rho)$ are obvious. For any l, an l-section of (t, b_m, ρ) is an l-section of γ and of γ_{m+1}. Let ρ_m be $\lambda \# \nu_m$. Then by the induction hypothesis for any l, $h'=h(\nu_m)\leq l\leq t$,

$$\rho_m<_l\rho \quad \text{and} \quad \rho_m<_l\rho_{m+1} .$$

So if $l\geq t$, then

$$\mu_m=(t, b, \rho_m)<_l(t, b, \rho)$$

and

$$(t, b, \rho_m)<_l(t, b, \rho_{m+1}) .$$

Suppose $h=h(\gamma)\leq l<t$. $h'\leq h$. Any σ an l-section of (t, b, ρ_m) is an l-section of ρ_m, hence by the induction hypothesis $\sigma<_l\rho$ and $\sigma<_l\rho_{m+1}$. Therefore

$$(t, b, \rho_m)<_l(t, b, \rho) \quad \text{and} \quad <_l(t, b, \rho_{m+1})$$

for all $l\geq h$. This includes the case $l=j$. h-active values do not increase.

[1°.2] For κ_m, see a.2). For μ_m, see [1°.1].

(2.1;1) $\gamma=(i, 0, \alpha+1)$ where $j<i$ and $i=i_0+1$. Let (i, δ) be the immediate successor. The proposition holds for it by the induction hypothesis. In particular $\delta_m<_l\delta$ for every l, $h\leq l\leq i$. l includes j.

Case 1. $\gamma=\text{apr}(0, j, \gamma)$. Then it can be easily shown that $\mu=(i, 0, \alpha)=\text{apr}(0, j, \mu)$. We need a lemma for this case.

LEMMA 1. $\text{apr}(0, j, \gamma_m)=\mu$, $v_2(j, \gamma_m)=i_0$ *and* $\gamma_m=\text{apr}((1, 1), j, \gamma_m)$.

Proof. By the induction hypothesis $v_0(j, \delta_m)\leq v_0(j, \delta)\leq(i, 0)$ and $\delta_m<_i\delta<_i\mu$. If $v_0(j, \delta_m)=(i, 0)$, then $v_0(j, \delta)=(i, 0)$. Let $\sigma=\text{apr}(0, j, \delta_m)$. Then $\sigma\leq_i \text{apr}(0, j, \delta)<_i\mu$ (since $\mu=\text{apr}(0, j, \mu)$). Also, $\delta_m<_i\delta\leq_i\alpha<_i\mu$, hence $\alpha_m<_i\mu$. $(i_0, 0, \mu)<_i\delta\leq_i\alpha$, since $i>i_0$ and δ is an $(i_0, 0)$-dominant of α. Therefore $\alpha_m \# (i_0, 0, \mu)<_i\alpha$, and so $\gamma_m=(i, 0, \alpha_m \# (i_0, 0, \mu))<_i(i, 0, \alpha)=\mu$. Thus $\mu=\text{apr}(0, j, \gamma_m)$.

From this follows the latter equation immediately.

Now the proof of the proposition for Case 1 of (2.1;1). $\gamma_m <_j \gamma$ since apr $(0, j, \gamma_m) = \mu <_i \gamma$. Suppose $h \leq l \leq j$ and σ is an l-section of γ_m. Then $j < i$ assumes that σ is either an l-section of δ_m, in which case $\sigma <_l \delta$ and $\sigma <_l \delta_{m+1}$ by the induction hypothesis, σ is an l-section of α or σ is μ. In any case $\sigma <_l \gamma$ and $\sigma <_l \gamma_{m+1}$ are obvious.

Case 2. $\gamma = \text{apr}((0, k+1), j, \gamma)$. Then it can be easily shown that $\mu = (i, 0, \alpha) = \text{apr}((0, k+1), i, \mu)$ and $\text{apr}((0, k), j, \mu) = \text{apr}((0, k), j, \gamma)$.

LEMMA 2. *$\text{apr}((0, k+1), j, \gamma_m) = \mu$, $v_2(j, \gamma_m) = i_0$ and $\gamma_m = \text{apr}((1, 1), \gamma_m)$.*

Proof. $\delta_m <_j \delta <_j \mu$. So if we let ρ be a component of $\alpha_m = \alpha\binom{\delta}{\delta_m}$, $\rho <_j \mu$. Therefore, either $v_0(j, \rho) < v_0(j, \mu)$, or $=$holds and there is a p, $1 \leq p \leq k$, such that

$$\text{apr}((0, p), j, p) = \text{apr}((0, p), j, \mu)\,,$$

and $\text{apr}((0, p+1), j, \rho) <_{i+1} \text{apr}((0, p+1), j, \mu)$. $\alpha_m \# (i_0, 0, \mu) <_i \alpha$ is established as in Lemma 1, so $\gamma_m <_{i+1} \mu$. Therefore $\mu = \text{apr}((0, k+1), j, \gamma_m)$. From this follow other equations.

The proof of the proposition for Case 2 is carried out in a manner similar to Case 1. Use the fact that $\text{apr}((0, k), j, \gamma_m) = \text{apr}((0, k), j, \gamma)$ and $\text{apr}((0, k+1), j, \gamma_m) = \mu <_{i+1} \gamma = \text{apr}((0, k+1), j, \gamma)$.

Case 3. $\gamma = \text{apr}((n, k+1), j, \gamma)$ for some $n > 0$ and $v_{n+1}(j, \gamma) = i$. We can show that $\mu = \text{apr}((n, k+1), j, \mu)$ and $\text{apr}((n, k), j, \mu) = \text{apr}((n, k), j, \gamma)$.

LEMMA 3. *$\text{apr}((n, k+1), j, \gamma_m) = \mu$, $v_{n+2}(j, \gamma_m) = i_0$ and $\gamma_m = \text{apr}((n+1, 1), j, \gamma_m)$.*

Proof. As in Lemma 2, for any ρ a component of α_m, $\rho <_j \mu$. $\gamma_m <_{i+1} \mu$ is also established in a similar manner. So the desired equations follow.

The proof of the proposition for Case 3 follow from Lemma 3.

(2.1;2) $\gamma = (i, 0, \alpha+1)$ where $j = i$ and $i = i_0 + 1$. Let (i, δ) be the next sp and let $\{\delta_m\}_m$ be its reduction sequence. $\alpha_m = \alpha\binom{\delta}{\delta_m}$. $\mu <_l \gamma$ for every l. $\alpha_m \# (i_0, 0, \mu) <_i \alpha$ since δ is an $(i_0, 0)$-dominant of α. So $\rho_m = (i, 0, \alpha_m \# (i_0, 0, \mu)) <_i \gamma$. Let $h \leq l < i$ and let σ be an l-section of ρ_m. Then either σ is an l-section of α, σ is an l-section of δ_m or $\sigma = \mu$. In any case $\sigma <_l \gamma$. So $\rho_m <_l \gamma$ for every such γ. Other relations are easily established.

(3.1) $\gamma = \text{apr}(0, j, \gamma) = (i, a, \alpha)$, α is marked and $a \neq 0$. The immediate successor of (j, γ) is (i, α), so the proposition holds for α. In particular $\alpha_m <_i \alpha$ and $\alpha_m <_i \alpha_{m+1}$. Therefore

$$(i, a, \alpha_m) <_l (i, a, \alpha) \quad \text{and} \quad (i, a, \alpha_m) <_l (i, a, \alpha_{m+1})$$

if $l \geq i$. Suppose $h=h(\gamma)\leq l<i$ and σ is an l-section of (i, a, α_m). Then $h\geq h(\alpha)$ and σ is an l-section of α_m. So by the induction hypothesis $\sigma<_l\alpha$ and $\sigma<_l\alpha_{m+1}$. So

$$(i, a, \alpha_m)<_l(i, a, \alpha_{m+1}) \quad \text{and} \quad (i, a, \alpha_m)<_l(i, a, \alpha)$$

for all $l\geq h$. h-active values of (i, a, α_m) (except (i, a)) are $h(\alpha)$-active, hence the induction hypothesis applies. For the second term of γ_m in both cases of (3.1), the desired conditions can be proved easily.

(4.1;1) In order to deal with (4.1;1), we first prove the following lemma.

LEMMA 4. *$\mathrm{apr}((0, k), j, \gamma_m)=\alpha$, $v_2(j,\gamma_m)=i_0$ and $\gamma_m=\mathrm{apr}((1, 1), j, \gamma_m)$.*

Proof. By the induction hypothesis, $\alpha_m<_j\alpha$ since $h\leq j\leq i$, where $h=h(\alpha)$. Also, $\alpha=\mathrm{apr}((0, k), j, \alpha)$ (cf. Proposition 1.10). So $\alpha_m<_j\alpha$ implies that either $v_0(j, \alpha_m)<v_0(j, \alpha)$ and the indicator of the latter is i, or there is a p, $1\leq p\leq k$, such that

$$\mathrm{apr}((0, p-1), j, \alpha_m)=\mathrm{apr}((0, p-1), j, \alpha)$$

and

$$\mathrm{apr}((0, p), j, \alpha_m)<_i \mathrm{apr}((0, p), j, \alpha)\ .$$

On the other hand $\mathrm{apr}((0, k), j, (i_0, 0, \alpha))=\alpha$. Therefore $\mathrm{apr}((0, k), j, \gamma_m)=\alpha$ and α_m j-omits α. The last two equations follow from this.

Now the proof of the proposition for (4.1;1). $\gamma_m<_j\gamma$ since they share the same $(0, k)^{\text{th}}$ approximation (cf. Lemma 4 above) and γ_m has no $(0, k+1)^{\text{th}}$ approximation, while γ does have the $(0, k+1)^{\text{th}}$ approximation.

Suppose $h\leq l\leq j$ and σ is an l-section of γ_m. Then, since $j<i$, σ is either an l-section of α_m, in which case $\sigma<_l\alpha$ and $\sigma<_l\alpha_{m+1}$ by the induction hypothesis, or $\sigma=\alpha$, hence $l=i_0=j$, in which case $\sigma<_l\gamma$ and $\sigma<_l\gamma_{m+1}$ are obvious, or σ is an l-section of α, in which case $\sigma<_l\alpha<_l\gamma$ and $\sigma<_l\alpha<_l\gamma_{m+1}$. This also implies $\gamma_m<_l\gamma$ for every l such that $h\leq l\leq j$.

(4.1;2) $\alpha<_l\gamma$ for every l such that $l\leq i$. $a=\mathrm{apr}((0, k), i, \gamma)$, so the outermost indicator of α is i, and hence $\alpha_m \# (i_0, 0, \alpha)<_i\alpha$. Therefore $\rho_m=(i, a, \alpha_m \# (i_0, 0, \alpha))<_i\gamma$. Other relations can be easily proved.

Note that for this case the upper bound for l is necessary.

(3.2) and [2°] of (3.3), (4.3) and (5.2). $\gamma=(i, a, \alpha)=(i, a, \alpha' \# \gamma')$. The immediate successor is (i, γ'). By the induction hypothesis, $\gamma_m'<_i\gamma'$ and $\gamma_m'<_i\gamma_{m+1}'$. So

$$(i, a, \alpha' \# \gamma_m')<_l(i, a, \alpha' \# \gamma')$$

and

$$(i, a, \alpha' \# \gamma_m')<_l(i, a, \alpha' \# \gamma_{m+1}')$$

if $l \geq i$. Suppose $h \leq l < i$ and σ is and l-section of $(i, a, \alpha' \# \gamma_m')$. If σ is an l-section of γ_m', then $\sigma <_l \gamma_{m+1}'$ and $\sigma <_l \gamma'$ by the induction hypothesis, so $\sigma <_l \gamma$ and $\sigma <_l \gamma_{m+1}$. If σ is an l-section of α', then $\sigma <_l \gamma$ and $\sigma <_l \gamma_{m+1}$ are obvious. Thus $\gamma_m <_l \gamma$ and $\gamma_m <_l \gamma_{m+1}$.

For (5.1.1;1), we first prove the following lemma.

LEMMA 5. $\mathrm{apr}((n, k), j, \gamma_m) = \alpha$; *if* $k=0$, *then* $v_{n+1}(j, \gamma_m) = i_0$ *and* $\gamma_m = \mathrm{apr}((n, 1), j, \gamma_m)$; *if* $k>0$, *then* $v_{n+2}(j, \gamma_m) = i_0$ *and* $\gamma_m = \mathrm{apr}((n+1, 1), j, \gamma_m)$.

Proof. By the induction hypothesis, $\alpha_m <_j \alpha$ since $h \leq j \leq i$ where $h = h(\alpha)$. Also $\alpha = \mathrm{apr}((n, k), j, \alpha)$ since $\alpha = \mathrm{apr}((n, k), j, \gamma)$ (cf. Proposition 1.11). So $\alpha_m <_j \alpha$ implies that there is a $(p, l) \leq (n, k)$ such that $\mathrm{apr}((q, s), j, \alpha_m) = \mathrm{apr}((q, s), j, \alpha)$ for any $(q, s) < (p, l)$ and $\mathrm{apr}((p, l), j, \alpha_m) <_{i(p+1)+1} \mathrm{apr}((p, l), j, \alpha)$. There is a unique occurrence of α in γ_m, and indicator occurring in γ_m immediately outside of α is $i_0 < i$. If $k=0$, then $\alpha = \mathrm{apr}(n, j, \gamma_m)$ $(j<i)$, hence $i_0 = v_{n+1}(j, \gamma_m)$ and $\gamma_m = \mathrm{apr}((n, 1), j, \gamma_m)$, while if $k>0$, then $\alpha = \mathrm{apr}(n+1, j, \gamma_m)$, hence $i_0 = v_{n+2}(j, \gamma_m)$ and $\gamma_m = \mathrm{apr}((n+1, 1), j, \gamma_m)$.

Now the proof of the proposition for (5.1.1;1). $\gamma_m <_j \gamma$ since they share the same $(n, k)^{\text{th}}$ j-approximation (cf. Lemma 5 above) and γ_m has no $(n, k+1)^{\text{th}}$ j-approximation, while γ does have one.
Suppose $h \leq l \leq j$ and σ is an l-section of γ_m. Since $j \leq i_0$, σ is either an l-section of α_m, in which case $\sigma <_l \alpha$ and $\sigma <_l \alpha_{m+1}$ by the induction hypothesis, or $\sigma = \alpha$, hence $l = i_0 = j$ or σ is an l-section of α in which cases $\sigma <_l \gamma$ and $\sigma <_l \gamma_{m+1}$ are obvious. This also implies $\gamma_m <_l \gamma$ for any l such that $h \leq l \leq j$.

(5.1.1;2) $\alpha = \mathrm{apr}((n, k), i, \gamma)$ and $i = v_{n+1}(j, \gamma)$, so the outermost indicator of α is i. The proof of this case can be carried out in a manner similar to the proof for (4.1;2). Note that here too $l \leq i$ is necessary.

(2′) and (5.3) (j, γ) is static. We prove the following statement: let (p, γ) be either (j, γ) or a tsp for (j, γ). Then the proposition holds for (p, γ) with h determined by (j, γ).

Suppose the proposition has been shown for the immediate successor of (p, γ), say (q, γ). That is, 1~5 hold for every l satisfying $h \leq l \leq q$. But $p<q$, so we can restate this for every l satisfying $h \leq l \leq p$.

This completes the proof of Proposition 2.1.

§ 2.3. Invariance of some approximations; non-critical case.

We can now show that our intuitive idea of reduction is realized in the formal definition: if, in (j, γ), γ is the $(n, k+1)^{\text{th}}$ j-approximation of itself, then the $(n, k)^{\text{th}}$ j-approximation is preserved under reduction. The critical case is still being excluded.

PROPOSITION 2.2. *Recall that when an o.d.* γ *is not connected, we define* apr $((n, k), j, \gamma)$ *to be the* $(n, k)^{\text{th}}$ *j-approximation of the j-greatest component of* γ. *Now let* (j, γ) *be a sp or a tsp (of* $(j_0, \tilde{\alpha})$*) and suppose that* $\gamma=\text{apr}((n, k+1), j, \gamma)$ *for some* (n, k). *Then* $\text{apr}((n, k), j, \gamma_m)=\text{apr}((n, k), j, \gamma)$ *where* $\{\gamma_m\}_m$ *is the reduction sequence for* γ.

Before getting into the proof, we prove some general lemmas.

LEMMA 2.1. *Let j be an indicator;* $\gamma=(i, a, \alpha)$ *and* δ *be an o.d. of one of the following forms with* $(l, b)<(i, a)$ *and* $j\leq l$: (l, b, α), $(l, b, \alpha+1)$, $(l, b, \cdots, (l, b, (i, a, \beta))\cdots)$ *when* $a=\beta+1$, *and* $(l, b, (l, b, \cdots, (l, b, \alpha)\cdots))$. *If* $\gamma=\text{apr}((n, k+1), j, \gamma)$, *then* $\text{apr}((n, k), j, \delta)=\text{apr}((n, k), j, \gamma)$.

Proof. First note that

(1) $$\text{apr}((n, k), j, \gamma) \text{ is contained in } \alpha .$$

This includes the case where $\text{apr}((n, k), j, \gamma)=\alpha$. Put $p=v_{n+1}(j, \gamma)$. Then $j\leq p\leq i$. The condition $(l, b)<(i, a)$ and the forms of δ imply that $v_0(j, \delta)=v_0(j, \gamma)$, hence (1) implies

(2) $$\text{apr}(0, j, \delta)=\text{apr}(0, j, \gamma) .$$

If $(n, k)=(0, 0)$, then this will do. If $(n, k)\neq(0, 0)$, then $\text{apr}((0, 1), j, \gamma)$ is contained in α, so

(3) $$\text{apr}((0, 1), j, \delta)=\text{apr}((0, 1), j, \gamma).$$

Suppose $n=0$. Continue the same reasoning as for (3). We obtain

(4) $$\text{apr}((0, m), j, \gamma)=\text{apr}((0, m), j, \delta) \quad \text{for every } m=1, 2, \cdots, k$$

by induction on m. So as a special case of (4), we have $\text{apr}((n, k), j, \delta)=\text{apr}((n, k), j, \gamma)$ when $n=0$.

Suppose $n>0$. Then with a reasoning similar to that for (4), we can establish

(5) $$\text{apr}((q, m), j, \gamma)=\text{apr}((q, m), j, \delta) \quad \text{for every } (q, m)\leq(n, 0)$$

by induction on (q, m). If $k=0$, then this will do.

Suppose $k>0$. Let $\eta=\text{apr}((n, 1), j, \gamma)$. Then

(6) η is contained in α, hence η is a j-subsection of δ .

(Recall that $j\leq l$.) By the definition of η,

(7) $$\gamma<_{p+1}\eta .$$

Let σ be any j-subsection of δ which contains η (cf. (6)). Then it follows that

(8) $$\sigma<_{p+1}\eta .$$

(5) with $(q, m)=(n, 0)$, (6), (7) and (8) establish

(9) $$\eta=\text{apr}((n, 1), j, \delta) .$$

If $k=1$, this will do. Otherwise we can show

(10) $$\text{apr}((q, m), j, \gamma)=\text{apr}((q, m), j, \delta) \quad \text{for all } (q, m)$$

such that $(n, 1)\leq(q, m)\leq(n, k)$,

With a reasoning similar to that for (9).

As a special case of (10) we obtain $\text{apr}((n, k), j, \gamma)=\text{apr}((n, k), j, \delta)$.

LEMMA 2.2. *Let (j, γ) be a sp or a tsp which is not the last reduction pair and let η be a proper j-subsection of γ which is (at some stage) marked. (Here η denotes a particular occurrence of η in γ) (This condition implies that η is connected and is not 0.) Let $\{\gamma_m\}_m$ be the reduction sequence for (j, γ). Then there is a j-active occurrence of η in γ_m.*

Note that marking of η is not necessarily taken place at the stage (j, γ); it may be marked at an earlier or a later stage.

Proof. We prove the folloing statement.

(*) Let (p, δ) be a sp which is a successor of (j, γ) or a tsp of a successor of (j, γ) (including the case where $(p, \delta)=(j, \gamma)$) satisfying that δ contains the η (a j-active, marked sub-o.d. of γ) and that [1°] does not apply to any sp between (j, γ) and (p, δ) except perhaps to (p, δ) itself. Let $\{\delta_m\}_m$ be the reduction sequence for (p, δ). Then δ_m contains a j-active occurrence of η.

The lemma is just a special case of (*): let $\gamma=\delta$, hence $\gamma_m=\delta_m$.

The conditions in (*) imply that such (p, δ)'s form a sequence of consecutive sp's starting with (j, γ), that η is j-active in δ and that once one hits a (p, δ) which [1°] applies to he no longer considers the next sp. By virtue of 2) of Proposition 1.7, this also implies that (3.2) never applies between (j, γ) and (p, δ).

The proof of (*) is carried out by induction on the number of sp's and tsp's between (j, γ) and the last pair determined by (*). Let $\delta=(i, a, \delta')$. There is a j-active occurrence of η in δ, hence

1) $$j\leq i.$$

We first consider the induction step.

1. Neither (2′) nor (5.3) applies to (p, δ). ((p, δ) may or may not be a tsp.) Let $\delta'=\lambda\#\kappa$ where (i, κ) is the next sp. Then (i, κ) satisfies the same condition as (p, δ).

If none of (2.1), (4.1) or (5.1.1) is the case, then by Proposition 1.13 δ_m contains a component of the form $\xi_m=(i, a, \lambda\#\kappa_m)$, where $\{\kappa_m\}_m$ is the reduction sequence for κ. By the induction hypothesis there is a j-active occurrence of η in κ_m, hence in ξ_m (cf. 1) above), hence in δ_m.

Suppose (2.1;1) is the case: $\delta=(i, 0, \alpha+1)$ and $\delta_m=(i, 0, \alpha_m \# (i_0, 0, \mu))$ where $\mu=(i, 0, \alpha)$ and $\alpha_m=\alpha\binom{\delta'}{\delta_m{}'}$. $\delta_m{}'$ contains a j-active occurrence of η. So by 1) the same occurrence of η is j-active in δ_m.

Consider (2.1;2). $\delta_m=\mu \# \cdots \# \mu \# (i, 0, \alpha_m \# (i_0, 0, \mu))$. $j\leq i$ by 1) and both μ and α_m contain j-active occurrences of η. This will do.

Suppose (4.1;1) is the case. $\delta_m=(i, a, \delta_m{}' \# (i_0, 0, \delta'))$ where $\{\delta_m{}'\}_m$ is the reduction sequence for δ'. $\delta_m{}'$ contains a j-active occurrence of η, so by 1) the same occurrence of η is j-active in δ_m. For (4.1;2), both δ' and $\delta_m{}'$ contain j-active occurrences of η. (5.1.1) is treated in the same manner.

2. (2′) or (5.3) applies to (p, δ). The conclusion of (*) is an immediate consequence of the induction hypothesis.

Now let us consider the basis for (*). There are three cases.

3. δ is the last reduction place, $\delta'=\eta$ or $\delta'=\eta+1$.

4. [1°] applies to (p, δ).

5. $\delta'=\lambda \# \kappa$ where κ is the next sod and η occurs in λ.

3. Here either δ is the last reduction place or one of (2.1), (3.1), (4.1) and (5.1.1) applies.

(2.1;1) $\delta_m=(i, 0, a_m \# (i_0, 0, \mu))$. Here $\alpha=\eta$, hence $\mu=(i, 0, \eta)$. If we can show that

$$2)\qquad j\leq i_0,$$

then η is j-active in δ_m. For the proof of 2), first let $\delta=\gamma$. Then $j\leq p<i$ (due to the condition of (2.1;1)), so $j\leq i_0$. If δ is a proper sub-o.d. of γ, then by 2) of Proposition 1.4 there is an indicator in γ connected to δ and satisfying $q\leq p$, hence $q<i$. In order that η (a sub-o.d. of δ) be j-active in γ, $j\leq q$. So $j<i_0$.

(2.1;2) $j\leq i$ and μ contains η i-active.

(3.1) δ_m contains a component of the form (i, a_m, η) or $(i, b, (i, b, \cdots (i, b, \eta) \cdots))$ (cf. Definition 1.4). So η is j-active in δ_m (cf. 1) above).

(4.1;1) and (5.1.1;1) can be treated in a manner similar to (2.1;1).

(4.1;2) $\delta_m=\delta' \# \cdots \# \delta' \# (i, 0, \delta_m{}' \# (i_0, 0, \delta'))$. δ' is η, hence η is j-active in δ_m.

(5.1.1;2) is treated in the same manner.

(2.2) $\delta=(i, a, \delta'+1)$. For c.1)~c.2.3) the proposition is obvious since $j\leq i$. For c.2.4) we can show that $j\leq p$ $(<i)$ in a manner similar to 2) above. For c.2.5) $j\leq i_0$. In any case there is a j-active $\eta=\delta'$ in δ_m.

(4.2) In d.3), $\delta_m=(i_m, 0, \delta')$. η is contained in δ'. $p<i$ by the assumption. As was proved for (2.1;1) above, $j\leq t$. Therefore $j<i_m$, hence η is j-active in δ_m. For d.4), η is j-active in $\alpha(=\delta')$.

(5.1.2) e.3) and e.4) can be treated in a manner similar to d.3) and d.4) respectively. Other cases of the last reduction place are easily treated.

4. Let us restate the situation. The sp under consideration is (p, δ) where $\delta=(t, b, \lambda \# \nu)$, $\nu=(i, 0, \alpha)$, $t>i$ and (3.2) applies to (t, ν). Only (3.3), (4.3) or (5.2) can apply to (p, δ) (cf. Definition 1.1). So by Proposition 1.6 ν is not marked. This implies, in particular, that ν is itself not η. $\delta_m=\kappa_m \# (t, b, \lambda \# \nu_m)$ where ν is i-active in κ_m. If η occurs in λ, then η is j-active in $(t, b, \lambda \# \nu_m)$. Suppose η occurs in ν (hence in α). Then $j \leq i < t$ (cf. 1)). So in any of the cases of [1°] (cf. Definition 1.4) η is j-active in κ_m.

5. In this case one of the following applies to (p, δ): (2.1), [2°] of (3.3), (4.3) and (5.2). In and of these cases, λ is j-active in δ_m, hence so is η (cf. Definitions 1.3 and 1.4; also 3 above for (2.1)).

LEMMA 2.3. *Let (j, γ) be a sp or a tsp and let η be a connected o.d. such that there are at least two j-active occurrences of η in γ. (We shall denote such occurrences of η by $\bar{\eta}$.) Let $\{\gamma_m\}_m$ be the reduction sequence for γ. Then there is a j-active occurrence of η in γ_m.*

Proof. As in Lemma 2.2, we prove the following statement:

(*) Let (p, δ) be a sp which is a successor of (j, γ) (including the case where $(p, \delta)=(j, \gamma)$) and which contains at least two occurrences of $\bar{\eta}$ which are j-active in γ (hence in δ also). Let $\{\delta_m\}_m$ be the reduction sequence for δ. Then δ_m contains a j-active η.

For (2.2), (4.2) and (5.1.2) see the proof of (*) in Lemma 2.2. If δ is not the last reduction place and none of (2.1), (4.1) and (5.1.1) applies to δ, then δ_m contains a component of the form $\xi_m=(i, a, \lambda \# \kappa_m)$ where $\delta=(i, a, \lambda \# \kappa)$. If λ contains an $\bar{\eta}$, then it is j-active in ξ_m, hence in δ_m. If not, there are at least two occurrences of $\bar{\eta}$ in κ, hence by the induction hypothesis there is a j-active occurrence of η in κ_m. This occurrence of η is j-active in ξ_m, hence in δ_m. As for (2.1), (4.1) and (5.1.1), we can show $j \leq i_0$ in the same way as for 2) in Lemma 2.2. From this the required property follows.

LEMMA 2.4. *Suppose $\gamma=$*apr$((n, k+1), j, \gamma)$, *δ is connected and $\delta <_j \gamma$ and there is a j-active occurrence of* apr$((n, k), j, \gamma)$ *in δ. Then* apr$((n, k), j, \delta)=$apr$((n, k), j, \gamma)$.

Proof. Since $\delta <_j \gamma$, $v_0(j, \delta) \leq v_0(j, \gamma)$. But there is a j-active occurrence of $v_0(j, \gamma)$ in δ (in apr$((n, k), j, \gamma)$), so $=$ holds. $\delta <_j \gamma$ then implies that apr$(0, j, \delta) \leq_i$ apr$(0, j, \gamma)$ where i is the indicator of $v_0(j, \gamma)$. But apr$(0, j, \gamma)$ occurs j-active in δ (in apr$((n, k), j, \gamma)$); so $=$ holds. Suppose we have shown that $v_{l+1}(j, \delta)=v_{l+1}(j, \gamma)$ $(=p)$ and apr$((l, m), j, \delta)=$apr$((l, m), j, \gamma)$, where $(l, m)<(n, k)$. Suppose the

$(l, m+1)^{\text{th}}$ j-approximation exists. Then $\delta <_j \gamma$ implies apr $((l, m+1), j, \delta) \leq_{p+1}$ apr apr $((l, m+1), j, \gamma)$. But the latter occurs j-active in δ (in apr $((n, k), j, \gamma)$). So $=$ holds. If $(l, m+1)^{\text{th}}$ approximation does not exist, then $v_{l+2}(j, \delta) \leq v_{l+2}(j, \gamma)$ and the latter occurs j-active in δ, hence $=$. Thus we reach apr $((n, k), j, \delta) =$ apr $((n, k), j, \gamma)$.

Now the proof of Proposition 2.2. It is carried out case by case according to Definitions 1.3 and 1.4. First consider the last reduction pairs.

(1) and c.1) (of (2.2)) are irrelevant.

c.2) $\gamma = (i, a, \alpha+1)$.

c.2.1) $(i, a_m) < (i, a)$. So by Lemma 2.1 (with $(l, b, (i, a, \beta))$).

c.2.2) $(i, b) < (i, a)$. So by Lemma 2.1 (with $(l, b, \cdots, (l, b, (i, a, \beta)) \cdots)$).

c.2.3) By Lemma 2.4 since $(i, a, \alpha) <_t \nu = (i, a, \alpha+1)$ and the outermost indicator of ρ_m is smaller then $i = t$ for each case.

c.2,4) and c.2,5) by Lemma 2.1.

d.1)-d.3) and e.1)-e.3) : all by Lemma 2.1; for instance in d.3) use the fact that $t < i_m$ is assumed.

d.4) and e.4) : If $\nu =$ apr $((n, k+1), t, \nu)$ then apr $((n, k), t, \nu) = \alpha =$ apr $((n, k), t, \nu_m)$ (cf. Propositions 1.10 and 1.11 for the first equation; the latter equation is obvious from the definition of ν_m).

Induction step.

If apr $((n, k), j, \gamma)$ is marked (at some stage), then it occurs in ν_m as j-active (Lemma 2.2). This and $\gamma_m <_j \gamma$ satisfy the condition of Lemma 2.4; so the proposition. If apr $((n, k), j, \gamma)$ is not marked, then there are at least two j-active occurrences of it,for, a unique, j-active occurrence of apr $((n, k), j, \gamma)$ in γ would be marked at the stage (j, γ) due to the condition that $\gamma =$ apr $((n, k+1), j, \gamma)$. So by Lemma 2.3 there is a j-active occurrence of it in γ_m. This, $\gamma_m <_j \gamma$ and Lemma 2.4 prove the proposition.

PROPOSITION 2.3. *Let (j, γ) be a sp such that $\gamma =$ apr $((0, k+1), j, \gamma)$ and γ is not the last reduction place. Let $\{\gamma_m\}_m$ be the reduction sequence for (j, γ). Then one of the following holds*: (2.1;1) *or* (4.1;1) *applies to (j, γ) and $\gamma_m =$ apr $((1, 1), j, \gamma_m)$*; (2.1;2) *applies, $\gamma_m = \mu \# \cdots \# \mu \# \rho_m$ and $\mu =$ apr $((0, k+1), j, \gamma_m)$*; (4.1;2) *applies, $\gamma_m = \alpha \# \cdots \# \alpha \# \rho_m$ and $\alpha =$ apr $((0, k), j, \gamma_m)$*; [1°] *of* (4.3) *applies and $\gamma_m = \mu_m \# \kappa_m$ where $\mu_m =$ apr $((0, r+1), j, \mu_m)$ for some $r \geq k$ and $\kappa_m <_l \mu_m$ for every l, $j \leq l \leq i$*; [2°] of (4.3) applies and *$\gamma_m =$ apr $((0, r+1), j, \gamma_m)$ for some $r \geq k$.*

Proof. For (2.1;1) and (4.1;1), $\gamma_m =$ apr $((1, 1), j, \gamma_m)$ has been proved in § 2.2 (cf. Lemmas 2 and 4 there).

Consider (2.1;2). $\mu =$ apr $((0, k+1), j, \mu)$ is easily proved. We wish to show that $\rho_m <_i \mu$. $\rho_m = (i, 0, \alpha_m \# (i_0, 0, \mu))$ where $\alpha_m = \alpha\binom{\delta}{\delta_m}$. But

$\alpha_m \# (i_0, 0, \mu) <_i \alpha$, so $\rho_m <_i (i, 0, \alpha) = \mu$ is obvious.

Consider (4.1;2). $\alpha = \text{apr}((0, k), j, \alpha)$ by the assumption. α is of the form (i, b, α') where $b > 0$. Therefore $\rho_m <_{i+1} \alpha$. Also $\alpha_m \# (i_0, 0, \alpha) <_i \alpha$ was established before. So $\rho_m <_i \alpha$.

Now we are coming down to the case where (4.3) applies to (j, γ). For this case we first prove some lemmas.

LEMMA 2.5. *Let δ be a connected o.d. such that* $\text{apr}((0, k), j, \delta) = \rho$ *exists and $\delta \neq \rho$. If there is an occurrence of ρ, say $\bar{\rho}$, such that every indicator connected to $\bar{\rho}$ is the indicator of $v_0(j, \delta)$, then* $\delta = \text{apr}((0, r+1), j, \delta)$ *for some $r \geq k$.*

Let δ be a connected o.d., let $\bar{\rho}$ be a connected sub-o.d. of δ and let i be an indicater. If every indicator in δ connected to $\bar{\rho}$ is i, then we say that δ *satisfies (the condition)* ***C*** *(with $\bar{\rho}$ and i)*, or $\bar{\rho}$ *satisfies* ***C*** *for δ (with i).*

In particular, if (j, γ) is as in Proposition 2.3 where (4.3) applies to (j, γ), $\bar{\rho} = \text{apr}((0, k), j, \gamma)$, δ is a sub-o.d. of γ and i is the indicator of $v_0(j, \gamma)$, then the situation as above is described as "δ satisfies ***C***" or "$\bar{\rho}$ satisfies δ" without mentioning i.

LEMMA 2.6. 1) *Let* $\rho = \text{apr}((0, q), j, \delta)$ *for some δ, q and j, and let the indicator of $v_0(j, \delta)$ be i. Suppose δ satisfies* ***C*** *with ρ and i. If $j \leq i$, the* $\rho = \text{apr}((0, q), l, \delta)$ *for every l satisfying $j \leq l \leq i$. In such a situation, namely if* $\rho = \text{apr}((0, q), l, \delta)$ *for every l satisfying $j \leq l \leq i$, then we write* $\rho = \text{apr}((0, q), (j, i), \delta)$. $\text{apr}((0, q), (j, i), \delta_1) = \text{apr}((0, q), (j, i), \delta_2)$ *will mean* $\rho = \text{apr}((0, q), l, \delta_1) = \text{apr}((0, q), l, \delta_2)$ *for all l, $j \leq l \leq i$, where ρ is common to all such l.*

2) *Let (j, γ) be as in Proposition* 2.3 *where* (4.3) *applies and let* $\rho = \text{apr}((0, k), j, \gamma)$. *Let δ be a j-subsection of γ which satisfies* ***C*** *and let i be the indicator of $v_0(j, \gamma)$. Then* $\text{apr}((0, k), (j, i), \delta) = \rho$ *and* $\delta = \text{apr}((0, q), (j, i), \delta)$ *for some $q \geq k$.*

3) *If $j \leq i$, $\delta = (i, a, \lambda_0 \# \lambda)$ where λ_0 is an i-greatest component of $\lambda_0 \# \lambda$ and if* $\delta = \text{apr}((0, q+1), j, \delta)$, *then* $\text{apr}((0, q), j, \lambda_0) = \text{apr}((0, q), j, \delta)$ *and* $\lambda_0 = \text{apr}((0, r), j, \lambda_0)$ *for some $r \geq q$.*

LEMMA 2.7. *Let (j, γ) be as in Proposition* 2.3 *where* (4.3) *applies to (j, γ) and let* $\rho = \text{apr}((0, k), j, \gamma)$. *Let $\bar{\rho}$ denote any occurrence of ρ satisfying* ***C*** *for γ and let i be the indicator of $v_0(j, \gamma)$. Then for every (p, δ) a sp succeeding (j, γ) such that δ properly contains $\bar{\rho}$, $p = i$ unless $(p, \delta) = (j, \gamma)$, $j \leq i$,* $\text{apr}((0, k), j, \delta) = \rho = \text{apr}((0, k), (j, i), \delta)$ *and* $\delta = \text{apr}((0, q+1), (j, i), \delta)$ *for some $q \geq k$.*

Proof. If $(p, \delta) = (j, \gamma)$, then the equations are obvious from 1) of Lemma 2.6. For a (p, δ) satisfying the condition, if we can show that $p = i$ and δ is j-active in γ, then the first equations follow immediately from 2) of Lemma 2.6. Also $\delta=\text{apr}((0,q),(j,i),\delta)$ for some $q\geq k$. But $\delta\neq\rho$ by the assumption, so Lemma 2.5 and 1) of Lemma 2.6 imply $q\geq k+1$.

Suppose for (p,δ) it has been proved that $p=i$ and δ is j-active in γ. $\delta=\text{apr}((0,q+1),j,\delta)$, $q\geq k$, and $\text{apr}((0,k),j,\delta)=\rho$ as shown above. Therefore $\delta=(i,b,\delta')$ for some b and δ'. (4) applies to (p,δ), hence the next sp, if it exists, is (i,ε) where ε is an i-least component of δ'. Therefore ε is j-active in γ.

LEMMA 2.8. *Let (j,γ) be as in Proposition 2.3, let i be the indicator of $v_0(j,\gamma)$ and let $\rho=\text{apr}((0,k),j,\gamma)$. Suppose (4.3) applies to (j,γ). Let (p,δ) be a proper successor of (j,γ) such that δ satisfies **C**. Then by Lemma 2.7 $p=i$, $\delta=\text{apr}((0,q+1),(j,i),\delta)$ for some $q\geq k$ and $\text{apr}((0,k),(j,i),\delta)=\rho$. Let $\{\delta_m\}_m$ be the reduction sequence for (p,δ). Then one of the following holds.*

[1°] of (4.3) applies to (p,δ) (hence $\delta=(t,b,\lambda\#\nu)$ where $t=i$ and ν is an i-least component of $\lambda\#\nu$), $\delta_m=\kappa_m\#\mu_m$ where $\mu_m=(t,b,\lambda\#\nu_m)=\text{apr}((0,r+1),(j,i),\mu_m)$ for some $r\geq q$, $\text{apr}((0,q),(j,i),\mu_m)=\text{apr}((0,q),(j,i),\delta_m)=\text{apr}((0,q),(j,i),\delta)$ and $\kappa_m<_l\mu_m$ for every l, $j\leq l\leq i$.

When $\alpha<_l\beta$ for every l, $j\leq l\leq i$, we write $\alpha<_{j,i}\beta$.

[2°] of (4.3), one of c.1), c.2.1) and c.2.2) of (2.2) or one of d.1) and d.2) of (4.2) applies to (p,δ), $\delta_m=\text{apr}((0,r+1),(j,i),\delta_m)$ for some $r\geq q$ and $\text{apr}((0,q),(j,i),\delta_m)=\text{apr}((0,q),(j,i),\delta)$.

c.2.3) of (2.2) applies to (p,δ) and $\delta_m=\tilde{\delta}\#\cdots\#\tilde{\delta}\#\rho_m$, where $\tilde{\delta}=\text{apr}((0,q+1),(j,i),\tilde{\delta})$ and $\text{apr}((0,q),(j,i),\tilde{\delta})=\text{apr}((0,q),(j,i),\delta)$.

d.4) of (4.2) applies to (p,δ) and $\delta_m=\tilde{\delta}\#\cdots\#\tilde{\delta}\#\rho_m$, where $\tilde{\delta}=\text{apr}((0,q),j,\delta)=\text{apr}((0,q),(j,i),\delta_m)$ and ρ_m is either empty or $\rho_m=(i_m,0,\tilde{\delta})$ where $i_m\uparrow i$ or $\rho_m=(i_0,0,\cdots,(i_0,0,\tilde{\delta})\cdots)$ where $i=i_0+1$.

(2.1;2) applies to (p,δ), and $\delta_m=\tilde{\delta}\#\cdots\#\tilde{\delta}\#\rho_m$ where, if $\delta=(i,0,\alpha+1)$, then $\tilde{\delta}=\text{apr}((0,q+1),(j,i),\tilde{\delta})=(i,0,\alpha)$, $\rho_m=(i,0,\alpha_m\#(i_0,0,\tilde{\delta}))$ and $\text{apr}((0,q),j,i),\tilde{\delta})=\text{apr}((0,q),(j,i),\delta)$.

(4.1;2) applies to (p,δ) and $\delta_m=\tilde{\delta}\#\cdots\#\tilde{\delta}\#\rho_m$ where, if $\delta=(i,0,\alpha)$, then $\tilde{\delta}=\alpha=\text{apr}((0,q),j,\delta)$ and $\rho_m=(i,0,\alpha_m\#(i_0,0,\alpha))$.

Proof. Recall first that $p=i$ and $\rho=\text{apr}((0,k),(j,i),\delta)$. Also $\text{apr}((0,q),(j,i),\delta_m)=\text{apr}((0,q),(j,i),\delta)$ is an immediate consequence of Proposition 2.2 and 1) of Lemma 2.6.

Case 1. (p,δ) is the last reduction pair. $\delta=\text{apr}((0,q+1),i,\delta)$, hence (2.2) or (4.2) applies.

(2.2) Let $\delta=(i,b,\delta'+1)$. $p=i$ forces that c.2) with $t=i$ applies.

c.2.1) and c.2.2) $\delta_m=(i,b_m,(i,b,\delta'))$ where $b_m\uparrow b$, $\delta_0=(i,c,(i,b,\delta'))$ and $\delta_{m+1}=(i,c,\delta_m)$ where $b=c+1$, or $\delta_m=(i,b,\delta')\#\cdots\#(i,b,\delta')$. All the indicators connected to δ' are i and b_m, $c<b$. There is an occurrence of $\bar{\rho}$ such that every indicator in δ' connected to $\bar{\rho}$ is i. So it is easy

to show from apr $((0, q), (j, i), \delta_m) = $ apr $((0, q), (j, i), \delta)$ that apr $((0, q+1), (j, i), \delta_m) = (i, b, \delta')$, and $\delta_m = $ apr $((0, q+2), (j, i), \delta_m)$ for the first two cases and $(i, b, \delta') = \tilde{\delta} = $ apr $((0, q+1), (j, i), \tilde{\delta})$ for the last.

c.2.3) $\delta_m = \mu \# \cdots \# \mu \# \rho_m$, where $\mu = (i, 0, \delta')$ and ρ_m either is empty or assumes one of the forms (i_0, a_m, μ) and $(i_0, e, \cdots, (i_0, e, \mu \# \cdots \# \mu) \cdots)$. So $\tilde{\delta} = \mu$ and the desired relations can be established without difficulty.

c.2.4) and c.2.5) cannot apply here.

(4.2) Let $\delta = (i, b, \delta')$. If d.1), then $\delta_m = (i, b_m, \delta')$. If d.2), then $\delta_m = (i, c, \cdots, (i, c, \delta') \cdots)$. If d.4), then $b = 0$ and $\delta_m = \delta' \# \cdots \# \delta' \# \rho_m$. Here $\delta' = $ apr $((0, q), (j, i), \delta)$, b_m, $c < b$. So for the first two cases $\delta_m = $ apr $((0, q+1), (j, i), \delta_m)$ is obvious. For the third case the required condition obviously holds.

Case 2. (p, δ) is not the last reduction pair, but is the last successor of (j, γ) satisfying the condition. Namely, if $\delta = (i, b, \lambda \# \nu)$ and (i, ν) is the next sp, then either $\nu = \bar{\rho}$ or in ν for any occurrence of $\bar{\rho}$ there is an indicator which is less than i and is connected to it. Since $p = i$, (2.1;1) and (4.1;1) cannot apply to (p, δ), and that leaves (2.1;2), (4.1;2) and (4.3).

For (2.1;2), the desired relations are obvious.

For (4.1;2), $\alpha = \rho$ must be the case since α is connected. So the relations are obvious with $q = k$.

[1°] $\delta = (t, b, \lambda \# \nu)$ where $t = i$ and ν is a t-least component of $\lambda \# \nu$. λ contains apr $((0, q), j, \delta)$. $\mu_m = (t, b, \lambda \# \nu_m)$ where $t = i$. So μ_m satisfies C with apr $((0, q), j, \delta)$ and i. $\delta = $ apr $((0, q+1), j, \delta)$. $\kappa_m <_{j,i} \mu_m$ is obvious. So by Proposition 2.2 apr $((0, q), j, \mu_m) = $ apr $((0, q), j, \delta)$ (since $\delta_m = \kappa_m \# \mu_m$ and $\kappa_m <_{j,i} \mu_m$). Therefore by Lemma 2.5 $\mu_m = $ apr $((0, r+1), j, \mu_m)$ for some $r \geq q$. From this follows other desired equations.

[2°] $\delta_m = (i, b, \lambda \# \nu_m)$ and $\delta_m = $ apr $((0, q+1), (j, i), \delta_m)$ etc. are proved as for the μ_m in [1°].

Case 3. The induction step. $\delta = (i, a, \delta' \# \varepsilon)$ and (i, ε), the next sp, satisfies the same condition as (p, δ). In particular $\varepsilon = $ apr $((0, s+1), (j, i), \varepsilon)$ for some $s \geq k$. By the induction hypothesis one of the following holds.

Case 3.1 [1°] of (4.3) applies to (i, ε), $\varepsilon_m = \tilde{\kappa}_m \# \tilde{\mu}_m$, $\tilde{\mu}_m = $ apr $((0, u+1), (j, i), \tilde{\mu}_m)$ for some $u \geq s$, apr $((0, s), (j, i), \tilde{\mu}_m) = $ apr $((0, s), (j, i), \varepsilon)$ and $\tilde{\kappa}_m <_{j,i} \tilde{\mu}_m$.

Case 3.2. [2°] of (4.3), one of c.1), c.2.1) and c.2.2) of (2.2) or one of d.1) and d.2) of (4.2) applies to (i, ε). $\varepsilon_m = $ apr $((0, u+1), (j, i), \varepsilon_m)$ for some $u \geq s$ and apr $((0, s), (j, i), \varepsilon_m) = $ apr $((0, s), (j, i), \varepsilon)$.

Case 3.3. c.2.3) of (2.2) applies and $\varepsilon_m = \tilde{\varepsilon} \# \cdots \# \tilde{\varepsilon}$ where $\tilde{\varepsilon} = $ apr $((0, s+1), (j, i), \tilde{\varepsilon})$ and apr $((0, s), (j, i), \tilde{\varepsilon}) = $ apr $((0, s), (j, i), \varepsilon)$.

Note that if i is a successor element, then ε cannot be the $(0, s+1)^{\text{th}}$ j-approximation since the outermost indicators of components of ε' are $<i$.

Case 3.4. d.4) of (4.2) applies and $\varepsilon_m=\tilde{\varepsilon}\#\tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}\#\tilde{\varepsilon}_m$ where $\tilde{\varepsilon}=$ apr $((0, s), j, \varepsilon)=$ apr $((0, s), (j, i), \varepsilon_m)$ and $\tilde{\varepsilon}_m$ is either empty or is connected and $\tilde{\varepsilon}_m=(i_m, 0, \tilde{\varepsilon})$ where $i_m \uparrow i$ or $\tilde{\varepsilon}_m=(i_0, 0, \cdots, (i_0, 0, \tilde{\varepsilon})\cdots)$ where $i=i_0+1$.

Case 3.5. (2.1;2) applies to (i, ε), $\varepsilon_m=\tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}\#\nu_m$ where, if $\varepsilon=(i, 0, \varepsilon'+1)$, then $\tilde{\varepsilon}=(i, 0, \varepsilon')=$ apr $((0, s+1), (j, i), \tilde{\varepsilon})$, $\nu_m=(i, 0, \varepsilon'_m\#(i_0, 0, \tilde{\varepsilon}))$ and apr $((0, s), (j, i), \tilde{\varepsilon})=$ apr $((0, s), (j, i), \varepsilon)$.

Case 3.6. (4.1;2) applies to (i, ε) and $\varepsilon_m=\tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}\#\nu_m$ where, if $\varepsilon=(i, 0, \varepsilon')$, then $\tilde{\varepsilon}=\varepsilon'=$ apr $((0, s), j, \varepsilon)$ and $\nu_m=(i, 0, \varepsilon_m'\#(i_0, 0, \varepsilon'))$.

Since (p, δ) is not the last reduction pair, one of (4.3), (2.1;2) and (4.1;2) can apply to it. Furthermore the assumption that (i, ε) satisfies the same condition as (p, δ) lmplies that the outermost indicator of ε is i. Therefore Proposition 1.7 inhibits [1°] of (4.3) for (p, δ), hence we have only to consider [2°] for (4.3).

[2°] of (4.3) $\delta=(i, a, \delta'\#\varepsilon)$. Suppose δ' is not empty. Let δ_0 be its i-greatest component. apr $((0, q), j, \delta_0)=$ apr $((0, q), j, \delta)=$ apr $((0, q, j, \delta_m)$ and $\delta_0=$ apr $((0, t), j, \delta_0)$ for some $t\geq q\geq k$ (cf. 3) of Lemmas 2.6 and Proposition 2.2). So $\delta_m(=(i, a, \delta'\#\varepsilon_m))$ satisfies $\boldsymbol{C}$ with apr $((0, q), j, \delta)$ and i. This and Lemma 2.5 yield that $\delta_m=$ apr $((0, r+1), j, \delta_m)$ for some $r\geq q$. Other equations are now obvious.

Suppose δ' is empty.

Case 3.1. [1°] of (4.3) applies to ε. By the induction hypothesis $\varepsilon_m=\tilde{\kappa}_m\#\tilde{\mu}_m$, where $\tilde{\mu}_m=$ apr $((0, u+1), (j, i), \tilde{\mu}_m)$ for some $u\geq s$, apr $((0, s), (j, i), \tilde{\mu}_m)=$ apr $((0, s), (j, i), \varepsilon)$ and $\tilde{\kappa}_m<_{j,i}\tilde{\mu}_m$. Since ε is not marked, $s+1>q$, or $s\geq q$, hence $u\geq q$. Therefore apr $((0,q), j, \varepsilon_m)=$ apr $((0,q), j, \delta)$, and this implies that δ_m satisfies $\boldsymbol{C}$ with apr $((0, q), j, \delta)$ and i. So by Lemma 2.5 $\delta_m=$ apr $((0, r+1), j, \delta_m)$ for some $r\geq q$.

Case 3.2. is dealt with in a manner similar to Case 3.1; only replace $\tilde{\mu}_m$ by ε_m and do not consider $\tilde{\kappa}_m$.

Case 3.3. $\varepsilon_m=\tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}$ where $\tilde{\varepsilon}=$ apr $((0, s+1), (j, i), \tilde{\varepsilon})$ and apr $((0, s), (j, i), \tilde{\varepsilon})=$ apr $((0, s), (j, i), \varepsilon)$. Here $\delta=(i, a, (i, 0, \varepsilon'+1))$, $\varepsilon=(i, 0, \varepsilon'+1)$ and $\tilde{\varepsilon}=(i, 0, \varepsilon')$. So $q=s$. $\delta_m=$ apr $((0, q+1), j, \delta_m)$.

Case 3.4. $\delta_m=(i, a, \tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}\#\tilde{\varepsilon}_m)$ where $\tilde{\varepsilon}=$ apr $((0, q), j, \varepsilon)=$ apr $((0, q), j, \delta)$ and $\tilde{\varepsilon}_m$ (if not empty) is either $(i_m, 0, \tilde{\varepsilon})$ or $(i_0, 0, \cdots, (i_0, 0, \tilde{\varepsilon})\cdots)$. So $\delta_m=$ apr $((0, q+1), j, \delta_m)$.

Case 3.5. As for Case 3.3, $q=s$ can be established. $\delta_m=(i, a, \varepsilon_m)$, where $\varepsilon_m = \tilde{\varepsilon} \# \cdots \# \tilde{\varepsilon} \# \nu_m$, $\tilde{\varepsilon} = (i, 0, \varepsilon') = \text{apr}((0, s+1), j, \tilde{\varepsilon})$ and $\nu_m = (i, 0, \varepsilon_m' \# (i_0, 0, \tilde{\varepsilon}))$. $\varepsilon_m' <_{j,i} \tilde{\varepsilon}$ and $s = q$ imply that $\text{apr}((0, q), j, \varepsilon_m') \leq \text{apr}((0, q), j, \varepsilon_m')$. Let η be a j-subsection of $\tilde{\varepsilon}$ which properly contains $\xi = \text{apr}((0, q), j, \tilde{\varepsilon})$ $(= \text{apr}((0, q), j, \delta) = \text{apr}((0, q), j, \delta_m))$ i-active. Then $\eta \leq_{i+1} \tilde{\varepsilon} <_{i+1} \delta_m$ since $\tilde{\varepsilon} = \text{apr}((0, q+1), j, \tilde{\varepsilon})$ and $\varepsilon' <_i \varepsilon_m$. If $\text{apr}((0, q), j, \varepsilon_m') <_j \xi$, then ε_m' omits ξ. Suppose $\text{apr}((0, q), j, \varepsilon_m') = \xi$ and let η be a j-subsection of ε_m' which properly contains ξ i-active. Then $\eta \leq_{i+1} \text{apr}((0, q+1), j, \varepsilon_m')$. So $\varepsilon_m' <_j \tilde{\varepsilon}$ implies that $\eta <_{i+1} \tilde{\varepsilon} <_{i+1} \delta_m$. If η is ν_m, then $\eta <_{i+1} \delta_m$ since $\varepsilon_m' \# (i_0, 0, \tilde{\varepsilon}) <_i \varepsilon_m$. This proves that $\delta_m = \text{apr}((0, q+1), j, \delta_m)$.

Case 3.6. Here too $s = q$. $\delta_m = (i, a, \tilde{\varepsilon} \# \cdots \# \tilde{\varepsilon} \# \nu_m)$ where $\tilde{\varepsilon} = \text{apr}((0, q), j, \varepsilon)$ and $\nu_m = (i, 0, \tilde{\varepsilon}_m \# (i_0, 0, \tilde{\varepsilon}))$. $\tilde{\varepsilon}_m <_{j,i} \tilde{\varepsilon}$. So $\delta_m = \text{apr}((0, q+1), j, \delta_m)$ can be established in a manner similar to Case 3.5.

For (2.1;2) and (4.1;2) the desired relations are obvious.

Proof of Proposition 2.3 *when one of* (2.1;2), (4.1;2) *and* (4.3) *applies.*

Let (p, δ) in Lemma 2.8 be the immediate succssor of (j, γ). Then one of the cases listed there holds. Repeating the proof for Case 3 there for (j, γ) and $q = k$, we obtain the desired results.

PROPOSITION 2.4. *Let (j, γ) be a sp such that $\gamma = \text{apr}((n, k+1), j, \gamma)$ where $n > 0$ and γ is not the last reduction place. Let $\{\gamma_m\}_m$ be the reduction sequence for (j, γ). Then one of the following holds for (j, γ).*

(2.1;1) *apqlies and* $\gamma_m = \text{apr}((n+1, 1), j, \gamma_m)$.

(2.1;2) *applies and* $\gamma_m = \mu \# \cdots \# \mu \# \rho_m$ *where* $\mu = \text{apr}((n, k+1), j, \mu)$.

(5.1.1;1) *applied and* $\gamma_m = \text{apr}((n, 1), j, \gamma_m)$ *if* $k = 0$ *and* $\gamma_m = \text{apr}((n+1, 1), j, \gamma_m)$ *if* $k < 0$.

(5.1.1;2) *applies and* $\gamma_m = \tilde{\gamma} \# \cdots \# \tilde{\gamma} \# \rho_m$ *where* $\tilde{\gamma} = \text{apr}((n, k), j, \gamma)$.

$[1_\circ]$ *of* (5.2) *applis to* (j, γ) *and* $\gamma_m = \mu_m \# \kappa_m$ *where* $\mu_m = \text{apr}((n, r+1), j, \mu_m)$ *for some* $r \geq k$ *and* $\kappa_m <_{j, i_{n+1}} \mu_m$. *Here* $i_{n+1} = v_{n+1}(j, \gamma)$.

$[2°]$ *of* (5.2) *applies and* $\gamma_m = \text{apr}((n, r+1), j, \gamma_m)$ *for some* $r \geq k$.

(5.3) *applies to* (j, γ). *Then one of the following holds.*

γ_m *is connected and* $\gamma_m = \text{apr}((n, r+1), j, \gamma_m)$ *for some* $r \geq k$.

$\gamma_m = \mu_m \# \kappa_m$ *where* μ_m *and* κ_m *are connected,* $\kappa_m <_{j, i_{n+1}} \mu_m$ *and* $\mu_m = \text{apr}((n, r+1), j, \mu_m)$ *for some* $r \geq k$.

$\gamma_m = \tilde{\gamma} \# \cdots \# \tilde{\gamma} \# \tilde{\gamma}_m$ *where* $\tilde{\gamma} = \text{apr}((n, k), j, \gamma)$ *and* $\tilde{\gamma}_m$ *may be empty.*

(2′) *applies to* (j, γ). *Then one of the following holds.*

$$\gamma_m = \text{apr}((n, r+1), j, \gamma_m) \quad \textit{for some} \quad r \geq k\,.$$

$$\gamma_m = \tilde{\gamma} \# \cdots \# \tilde{\gamma} \# \rho_m \quad \textit{where} \quad \tilde{\gamma} = \text{apr}((n, k+1), j, \tilde{\gamma})\,.$$

Proof. If (2.1;1) is the case, then the equation has been proved as Lemma 3 in § 2.2. For (2.1;2) and (5.1.1;2), the equations are obvious.

If (5.1.1;1) is the case, then the equations have been proved as Lemma 5 in § 2.2. Therefore suppose that either (5.2), (2′) or (5.3) applies. Note first that $\rho=\mathrm{apr}\,((n, k), j, \gamma)=\mathrm{apr}\,((n, k), j, \gamma_m)$ (Proposition 2.2) and, if we let i_{n+1} be $v_{n+1}(j, \gamma)$, then the indicator connected immediately to ρ is i_{n+1}.

We shall first prove some lemmas.

LEMMA 2.9. *Let δ be a connected o.d. such that* $\mathrm{apr}\,((n, k), j, \delta)=\rho$ *exists where $n>0$ and $\delta\neq\rho$. If there is an occurrence of ρ say $\bar{\rho}$, such that every indicator connected to $\bar{\rho}$ is $\geq v_{n+1}(j, \delta)$, then* $\delta=\mathrm{apr}\,((n, r+1), j, \delta)$ *for some $r\geq k$.*

Let δ be a connected o.d., let $\bar{\rho}$ be a connected sub-o.d. of δ and let i be an indicator. If every indicator in δ connected to $\bar{\rho}$ is $\geq i$, then we say that δ *satisfies* (*the condition*) ***D*** (*with $\bar{\rho}$ and i*), or $\bar{\rho}$ *satisfies **D** for δ* (*with i*).

We shall henceforth assume $n>0$.

LEMMA 2.10. 1) *Let* $\rho=\mathrm{apr}\,((n, q), j, \delta)$ *for some δ, q and j, and let $i_{n+1}=v_{n+1}(j, \delta)$ (which is assumed to exist). Then $j\leq i_{n+1}$ and* $\rho=\mathrm{apr}\,((n, q), l, \delta)$ *for every l satisfying $j\leq l\leq i_{n+1}$.*

$\rho=\mathrm{apr}\,((n, q), (j, i_{n+1}), \delta)$ *is used to describe such a situation.*

2) *Let (j, γ) be as in Proposition* 2.4 *where* (5.2), (2′) *or* (5.3) *applies, let $i_{n+1}=v_{n+1}(j, \gamma)$ and let* $\rho=\mathrm{apr}\,((n, k), j, \gamma)$. *Let δ be a j-subsection of γ which satisfies **D** with ρ and i_{n+1}. Then* $\mathrm{apr}\,((n, k), (j, i_{n+1}), \delta)=\rho$ *and* $\delta=\mathrm{apr}\,((n, q), (j, i_{n+1}), \delta)$ *for some $q\geq k$.*

3) *If $j\leq i=i_{n+1}$, $\delta=(i, a, \lambda_0\,\#\,\lambda)$ where λ_0 is an i-greatest component of $\lambda_0\,\#\,\lambda$ and if* $\delta=\mathrm{apr}\,((n, q+1), j, \delta)$, *then* $\mathrm{apr}\,((0, q), j, \lambda_0)=\mathrm{apr}\,((n, q), j, \delta)$ *and* $\lambda_0=\mathrm{apr}\,((n, r), j, \lambda_0)$ *for some $r\geq q$.*

LEMMA 2.11. *Let (j, γ) be as in Proposition* 2.4 *where* (5.2), (2′) *or* (5.3) *applies to (j, γ) and let* $\rho=\mathrm{apr}\,((n, k), j, \gamma)$. *Let $\bar{\rho}$ denote any occurrence of ρ satisfiying **D** for γ with $i_{n+1}=v_{n+1}(j, \gamma)$. Then for every (p, δ) a sp succeeding (j, γ) such that δ properly contains $\bar{\rho}$, $p\geq i_{n+1}$ unless $(p, \delta)=(j, \gamma)$,* $\mathrm{apr}\,((n, k), j, \delta)=\rho=\mathrm{apr}\,((n, k), (j, i_{n+1}), \delta)$ *and* $\delta=\mathrm{apr}\,((n, q+1), (j, i_{n+1}), \delta)$ *for some $q\geq k$.*

Proof. If $(p, \delta)=(j, \gamma)$, then the equations are obvious from 1) of Lemma 2.10. For a (p, δ) a proper successor of (j, γ) satisfying the condition, if we can show that $p\geq i_{n+1}$ and δ is j-active in γ, then the first equations follow immediately from 2) of Lemma 2.10. Also $\delta=\mathrm{apr}\,((n, q), (j, i_{n+1}), \delta)$ for some $q\geq k$. But $\delta\neq\rho$ by the assumption, so by Lemma 2.9 and 1) of Lemma 2.10 $q\geq k+1$.

Suppose for (p, δ) it has been proved that $p\geq i_{n+1}$ and δ is j-active in γ. $\delta=\text{apr}((n, q+1), j, \delta), q\geq k$ and $\text{apr}((n, k), j, \delta)=\rho$ as shown above. Therefore $\delta=(i, b, \delta')$ for some $i\geq i_{n+1}$, b and δ'. The next sp (if existent) is either (i, ε) where ε is component of δ' or is (t, δ) where $t\geq p$. In either case the indicator is $\geq i_{n+1}\geq j$ and the sod is j-active in γ.

LEMMA 2.12. *Let (j, γ) be as in Proposition 2.4, let $i_{n+1}=v_{n+1}(j, \gamma)$ and let $\rho=\text{apr}((n, k), j, \gamma)$. Suppose (2′), (5.2) or (5.3) applies to (j, γ). Let (p, δ) be a proper, non-static successor of (j, γ) such that δ satisfies* **D**. *Then by Lemma 2.11 $p\geq i_{n+1}$, $\delta=\text{apr}((n, q+1), (j, i_{n+1}), \delta)$ for some $q\geq k$ and $\text{apr}((n, k), (j, i_{n+1}), \delta)=\rho$. Suppose (3.2) does not apply to any sp between (j, γ) and (p, δ), (p, δ) inclusive. Let $\{\delta_m\}_m$ be the reduction sequence for (p, δ). Then one of the following holds.*

[1°] *of* (3.3), (4.3) *or* (5.2) *applies to (p, δ) (hence $\delta=(t, b, \lambda \# \nu)$ where $t\geq i_{n+1}$ and ν is a t-least component of $\lambda \# \nu$), $\delta_m=\kappa_m \# \mu_m$ where $\mu_m=(t, b, \lambda \# \nu_m)=\text{apr}((n, r+1), (j, i_{n+1}), \mu_m)$ for some $r\geq q$, $\text{apr}((n, q), (j, i_{n+1}), \mu_m)=\text{apr}((n, q), (j, i_{n+1}), \delta_m)=\text{apr}((n, q), (j, i_{n+1}), \delta)$ and $\kappa_m<_{j, i_{n+1}}\mu_m$.*

[2°] *of* (3.3), (4.3) *or* (5.2), (2.2) *except* c.1) c.2.3) *and* c.2.5), (4.2) *except* d.4), (5.1.2) *except* e.4), (2.1;1), (4.1;1) *or* (5.1.1;1) *applies, $\delta_m=\text{apr}((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r\geq q$ and $\text{apr}((n, q), (j, i_{n+1}), \delta_m)=\text{apr}((n, q), (j, i_{n+1}), \delta)$.*

c.1) *or* 1° *of* c.2.3) *applies and $\delta_m=\tilde{\delta} \# \cdots \# \tilde{\delta}$ where $\tilde{\delta}_m=\text{apr}((n, q+1), (j, i_{n+1}), \tilde{\delta})$ and $\text{apr}((n, q), (j, i_{n+1}), \tilde{\delta})=\text{apr}((n, q), (j, i_{n+1}), \delta)$.*

2° *or* 3° *of* c.2.3) *applies, $i=p<i_{n+1}$ and $\delta_m=\tilde{\delta} \# \cdots \# \tilde{\delta} \# \rho_m$ where $\tilde{\delta}=\text{apr}((n, q+1), (j, i_{n+1}), \tilde{\delta})$, $\rho_m=\text{apr}((n, r+1), (j, i_{n+1}), \rho_m)$ for some $r\geq q$, and $\text{apr}((n, q), (j, i_{n+1}), \tilde{\delta})=\text{apr}((n, q), (j, i_{n+1}, \rho_m)=\text{apr}((n, q), (j, i_{n+1}), \delta)$.*

c.2.5) *applies, $i>i_{n+1}$ and $\delta_m=\text{apr}((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r\geq q$ and $\text{apr}((n, q), (j, i_{n+1}), \delta_m)=\text{apr}((n, q), (j, i_{n+1}), \delta)$.*

d.4) *of* (4.2) *or* e.4) *of* (5.1.2) *applies and $\delta_m=\tilde{\delta} \# \cdots \# \tilde{\delta} \# \rho_m$ where $\tilde{\delta}=\text{apr}((n, r), j, \delta)=\text{apr}((n, r), (j, i_{n+1}), \tilde{\delta})$ for some $r\geq q$ and either ρ_m is empty or $\delta=(i, 0, \tilde{\delta})$, $p=i$ and $\rho_m=(j_m, 0, \tilde{\delta})$ where $j_m \uparrow i$ or $\rho_m=(j_0, 0, \cdots, (j_0, 0, \tilde{\delta}) \cdots)$ where $i=j_0+1$.*

(3.1) *applies and $\delta_m=\kappa_m \# \mu_m$, where if $\delta=(i, a, \alpha)$, then $\mu_m=(i, a, \alpha_m)$ and $\kappa_m=(i, a_m, \alpha)$ with $a_m \uparrow a$ or $\kappa_m=(i, b, \cdots, (i, b, \alpha) \cdots)$ with $a=b+1$, and $\kappa_m=\text{apr}((n, r+1), (j, i_{n+1}), \kappa_m)$ and $\text{apr}((n, r), (j, i_{n+1}), \kappa_m)=\text{apr}((n, r), (j, i_{n+1}), \delta)$ for some $r\geq q$.*

(2.1;2) *applies and $\delta_m=\tilde{\delta} \# \cdots \# \tilde{\delta} \# \rho_m$ where $\tilde{\delta}=\text{apr}((n, q+1), (j, i_{n+1}), \tilde{\delta})$ and $\text{apr}((n, q), (j, i_{n+1}), \delta_m)=\text{apr}((n, q), (j, i_{n+1}), \delta)$.*

(4.1;2) *or* (5.1.1;2) *applies and $\delta_m=\tilde{\delta} \# \cdots \# \tilde{\delta} \# \rho_m$ where $\tilde{\delta}=\text{apr}((n, r), j, \delta)=\text{apr}((n, r), (j, i_{n+1}), \delta_m)$ for some $r\geq q$.*

Proof. Recall first that $p\geq i_{n+1}$ and $\rho=\text{apr}((n, k), (j, i_{n+1}), \delta)$. Also apr $((n, q), (j, i_{n+1}), \delta_m)$=apr $((n, q), (j, i_{n+1}), \delta)$ is an immediate consequence of Proposition 2.2 and 1) of Lemma 2.10. Also, once (p, δ) hits [1°], then we stop.

Case 1. (p, δ) is the last reduction pair.

(2.2) Let $\delta=(i, b, \delta'+1)$. Recall that $p \geq i_{n+1}$ and $i \geq i_{n+1}$.

c.1) and 1° of c.2.3) $p \geq i$. $\delta_m=(i, b, \delta')\#\cdots\#(i, b, \delta')$. Let $\tilde{\delta}=(i, b, \delta')$ and ρ_m be empty. Since $i \geq i_{n+1}$ and apr $((n, q), (j, i_{n+1}), \delta_m)$= apr $((n, q), (j, i_{n+1}), \delta)$, it is obvious that $\tilde{\delta}$=apr $((n, q+1), (j, i_{n+1}), \delta)$= apr $((n, q+1), (j, i_{n+1}), \delta_m)$.

c.2.1) and c.2.2) $\delta_m=(i, b_m, (i, b, \delta'))$ or $\delta_0=(i, c, (i, b, \delta'))$ and $\delta_{m+1}=(i, c, \delta_m)$ as the case may be. All the indicators connected to δ' are i. There is an occurrence of $\bar{\rho}$ such that every indicator in δ' connected to $\bar{\rho}$ is $\geq i_{n+1}$. So it is evident from apr $((n, q), (j, i_{n+1}), \delta_m)$=apr $((n, q), (j, i_{n+1}), \delta)$ that δ_m=apr $((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r \geq q$.

2° or 3° of c.2.3) applies. Then $i=p>i_{n+1}$ must be the case since $i=i_0+1$ and there is no $(i_0, 0)$-dominant of δ. For $\tilde{\delta}$, the relations are proved as for 1° of c.2.3). For ρ_m, notice that $i>i_0 \geq i_{n+1}$ $(=v_{n+1}(j, \gamma))$. Therefore we can follow the proof for c.2.1) and c.2.2) above.

c.2.4) $\delta_m=(t_m, 0, (i, 0, \delta'))$ where $t_m>p$. Since $p \geq i_{n+1}$, $t_m \geq i_{n+1}$. Now follow the proof for c.2.1) and c.2.2).

c.2.5) $i=i_0+1>i_{n+1}$ must be the case (cf. 2° or 3° of c.2.3) as above), so $i_0 \geq i_{n+1}$. Now follow the proof for c.2.1) and c.2.2).

(4.2) Let $\delta=(i, b, \delta')$. If d.1), then $\delta_m=(i, b_m, \delta')$. If d.2), then $\delta_m=(i, c, \cdots, (i, c, \delta')\cdots)$. If d.3), then $a=0$, $p<i$ and $\delta_m=(j_m, 0, \delta')$. If d.4), then $b=0$ and $\delta_m=\delta'\#\cdots\#\delta'\#\rho_m$. Here δ'=apr $((n, r), (j, i_{n+1}), \delta')$ for some $r \geq q$ and ρ_m takes a form as described. For the first two cases, δ_m=apr $((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r \geq q$ is obvious. For d.3), $j_m>p \geq i_{n+1}$, so δ_m=apr $((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r \geq q$. For d.4), the desired result holds as is mentioned above.

(5.1.2) $\delta_m=(i, b_m, \delta')$ (e.1)), $\delta_0=(i, b, \delta')$ and $\delta_{m+1}=(i, b, \delta_m)$ (e.2)), $b=0$, $p<i$ and $\delta_m=(j_m, 0, \delta')$ (e.3)) or $\delta_m=\delta'\#\cdots\#\delta'\#\rho_m$ where $\rho_m=(j_m, 0, \delta')$ or $\rho_m=(j_0, 0, \cdots, (j_0, 0, \delta')\cdots)$ or ρ_m is empty (e.4)). For e.1) and e.2), δ_m=apr $((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r \geq q$ is obvious. For e.3), $j_m>p \geq i_{n+1}$, hence the same holds. For e.4) δ'=apr $((n, r), (j, i_{n+1}), \delta')$ for some $r \geq q$.

Case 2. (p, δ) is not the lest reduction pair, but is the last successor of (j, γ) satisfying the condition. Namely if $\delta=(i, b, \lambda\#\nu)$ and (i, ν) is the next sp, then either $\nu=\bar{\rho}$ or there is no desirable occurrence of ρ in ν.

(2.1;1) Let $\delta=(i, 0, \lambda\#\nu+1)$. $\nu=\bar{\rho}$ is not possible, for then $i=i_{n+1}$, and $p<i$ due to the condition of (2.1;1) on the one hand, while $p \geq i$

as was mentioned above on the other hand. Therefore ν contains no desirable ρ (hence ν j-omits ρ). Therefore a $\bar{\rho}$ occurs in λ, hence in $\delta_m=(i, 0, \lambda \# \nu_m \# (i_0, 0, \mu))$. Since apr $((n, q), j, \delta_m)=$apr $((n, q), j, \delta)$ and there is an occurrence of $\bar{\rho}$ in δ_m satisfying $\boldsymbol{D}$ with i_{n+1}, $\delta_m=$ apr $((n, r+1), j, \delta_m)$ for some $r\geq q$.

For (2.1;2), (4.1;2) and (5.1.1;2) the proposition is obvious.

(3.1) In this case λ is empty, so $\nu=\bar{\rho}$, and hence $i=i_{n+1}$. $\kappa_m=(i, b_m, \rho)$ or $(i, c, \cdots, (i, c, \rho)\cdots)$. It is obvious that $\kappa_m=$apr $((n, k+1), (j, i_{n+1}), \kappa_m)$ and $\rho=$apr $((n, k), (j, i_{n+1}), \kappa_m)$.

(3.2) does not apply to (p, δ), since in order to reach (3.2) there must be a predecessor which [1°] applies to.

(4.1;1) is not possible for this case, for if (4.1) applied, then ν would be ρ since λ is empty, so $i=i_{n+1}$ and $p=i_{n+1}$. But $p<i$ is the condition for (4.1;1).

(5.1.1;1) is not possible, for then $\delta=(i_{n+1}, 0, \rho)$ and $p<i_{n+1}$, which is not the case.

[2°] (of (3.3), (4.3) or (5.2)). $\delta=(i, b, \lambda \# \nu)$. We claim that there is an occurrence of $\bar{\rho}$ satisfying $\boldsymbol{D}$ with i_{n+1}. For, if ν omits ρ, then the claim is obvious. Suppose $\nu=\rho$ and $\nu<_i \nu_0$, where ν_0 is a component of λ. Since $\nu=\rho$, $i=i_{n+1}$, and hence $\nu<_i \nu_0$ implies that apr $((n, k), (j, i_{n+1}), \nu_0)=\rho$ and there is an occurrence of ρ in ν_0 satisfying $\boldsymbol{D}$ with i_{n+1}. Therefore in fact λ_0 contains apr $((n, q), (j, i_{n+1}), \delta)$ i_{n+1}-active. $\delta_m=(i, b, \lambda \# \nu_m)$. Since δ_m contains apr $((n, q), j, \delta)$ and $\delta_m<_j \delta$, apr $((n, q), j, \delta_m)=$apr $((n, q), j, \delta)$. Since apr $((n, q), j, \delta)$ is contained in λ i_{n+1}-active, $\delta_m=$apr $((n, r+1), j, \delta_m)$ for some $r\geq q$.

[1°] (of (3.3), (4.3) or (5.2)). $\delta=(t, b, \lambda \# \nu)$, $\delta_m=\kappa_m \# \mu_m$, $\mu_m=(t, b, \lambda \# \nu_m)$. $\nu=(i, 0, \alpha)$ and $t>i$. If $\nu=\rho$, then $t=i_{n+1}\leq i$. So this is not the case, hence λ contains $\bar{\rho}$, hence apr $((n, q), j, \delta)$ also. $\mu_m<_j \delta$. So apr $((n, q), j, \mu_m)=$apr $((n, q), j, \delta)$ and $\mu_m=$apr $((n, r+1), j, \mu_m)$ for some $r\geq q$. $\kappa_m<_{j, i_{n+1}} \mu_m$ is obvious from the situation.

Case 3. The induction step. $\delta=(i, a, \delta' \# \varepsilon)$ and (i, ε) is the next sp. If (i, ε) is non-static, then the induction hypothesis holds for (i, ε). If (i, ε) is static, then consider the next sp, (v, ε), for which the induction hypothesis holds. Here $v>i$. In particular $\varepsilon=$apr $((n, s+1), (j, i_{n+1}), \varepsilon)$ for some $s\geq k$. By the induction hypothesis one of the following holds.

Case 3.1. [1°] of (3.3), (4.3), or (5.2) applies to (i, ε) (or (v, ε)), $\varepsilon_m=\tilde{\kappa}_m \# \tilde{\mu}_m$, $\tilde{\mu}_m=$apr $((n, u+1), (j, i_{n+1}), \tilde{\mu}_m)$ for some $u\geq s$. apr $((n, s), (j, i_{n+1}), \tilde{\mu}_m)=$apr $((n, s), (j, i_{n+1}), \varepsilon)$ and $\tilde{\kappa}_m<_{j, i_{n+1}} \tilde{\mu}_m$.

Case 3.2. [2°] of (3.3), (4.3) or (5.2), c.2.1) and c.2.2) of (2.2), (4.2) except d.4), (5.1.2) except e.4), (2.1;1), (4.1;1) or (5.1.1;1) applies,

$\varepsilon_m = \text{apr}((n, u+1), (j, i_{n+1}), \varepsilon_m)$ for some $u \geq s$ and $\text{apr}((n, s), (j, i_{n+1}), \varepsilon_m) = \text{apr}((n, s), (j, i_{n+1}), \varepsilon)$.

Case 3.3. c.1) or c.2.3) of (2.2) applies and $\varepsilon_m = \bar{\varepsilon} \# \cdots \# \bar{\varepsilon}$ where $\varepsilon = (t, b, \varepsilon' + 1)$, $\tilde{\varepsilon} = (t, b, \varepsilon')$ and $\bar{\varepsilon} = \text{apr}((n, s+1), (j, i_{n+1}), \tilde{\varepsilon})$.

Case 3.4. d.4) of (4.2) or e.4) of (5.1.2) applies and $\varepsilon_m = \bar{\varepsilon} \# \cdots \# \bar{\varepsilon} \# \tilde{\varepsilon}_m$ where $\bar{\varepsilon} = \text{apr}((n, s), j, \varepsilon)$ and $\tilde{\varepsilon}_m$ is either empty or connected and the following holds: $\varepsilon = (i, 0, \tilde{\varepsilon})$ and $\varepsilon_m = (t_m, 0, \tilde{\varepsilon})$ when $t_m \uparrow t$ or $\varepsilon_m = (t_0, 0, \cdots, (t_0, 0, \tilde{\varepsilon}) \cdots)$ when $t = t_0 + 1$.

Case 3.5. (3.1) applies and $\varepsilon_m = \tilde{\kappa}_m \# \tilde{\mu}_m$, where $\tilde{\kappa}_m = \text{apr}((n, u+1), (j, i_{n+1}), \tilde{\kappa}_m)$ and $\text{apr}((n, u), (j, i_{n+1}), \tilde{\kappa}_m) = \text{apr}((n, u), (j, i_{n+1}), \varepsilon)$ for some $u \geq s$.

Case 3.6. (2.1;2) applies and $\varepsilon_m = \bar{\varepsilon} \# \cdots \# \bar{\varepsilon} \# \nu_m$ where $\bar{\varepsilon} = \text{apr}((n, s+1), (j, i_{n+1}), \tilde{\varepsilon})$ and $\text{apr}((n, s), (j, i_{n+1}), \varepsilon_m) = \text{apr}((n, s), (j, i_{n+1}), \varepsilon)$.

Case 3.7. (4.1;2) or (5.1.1;2) applies and $\varepsilon_m = \bar{\varepsilon} \# \cdots \# \bar{\varepsilon} \# \nu_m$ where $\bar{\varepsilon} = \text{apr}((n, u), j, \varepsilon) = \text{apr}((n, u), (j, i_{n+1}), \varepsilon_m)$ for some $u \geq s$.

Since (p, δ) is not the last reduction pair, only one of the following can apply to it: [2°] of (3.3), (4.3) or (5.2), (2.1), (4.1), (5.1.1) or (3.1).

(α) [2°] of (3.3), (4.3) or (5.2). Suppose first δ' is not empty. Then $\text{apr}((n, q), j, \delta') = \text{apr}((n, q), j, \delta) = \text{apr}((n, q), j, \delta_m)$ (by 3) of Lemma 2.10 and Proposition 2.2). Therefore δ_m satisfies $\boldsymbol{D}$ with $\text{apr}((n, q), j, \delta)$ and i_{n+1}. From this and Lemma 2.9 follows that $\delta_m = \text{apr}((n, r+1), j, \delta_m)$ for some $r \geq q$.

Suppose next δ' is empty.

Case 3.1 applies to (i, ε) or (v, ε). Since ε is not marked, $s+1 < q$, or $s \geq q$. Therefore $\text{apr}((n, s), j, \tilde{\mu}_m) = \text{apr}((n, s), j, \varepsilon)$ implies $\text{apr}((n, q), j, \tilde{\mu}_m = \text{apr}((n, q), j, \varepsilon) = \text{apr}((n, q), j, \delta)$. This implies that δ_m satisfies $\boldsymbol{D}$ with $\text{apr}((n, q), j, \delta)$ and i_{n+1}. So by Lemma 2.9 $\delta_m = \text{apr}((n, r+1), j, \delta_m)$ for some $r \geq q$.

Case 3.2 applies to (i, ε) or (v, ε). The argument immediate above goes through with ε_m in the place of $\tilde{\mu}_m$.

Case 3.3. $\delta_m = (i, a, \bar{\varepsilon} \# \cdots \# \bar{\varepsilon})$ where $\bar{\varepsilon} = \text{apr}((n, s+1), (j, i_{n+1}), \tilde{\varepsilon})$, $\text{apr}((n, q), j, \bar{\varepsilon}) = \text{apr}((n, q), j, \delta)$ and $s \geq q$. So the result follows.

Case 3.4. $\delta_m = (i, a, \bar{\varepsilon} \# \cdots \# \bar{\varepsilon} \# \tilde{\varepsilon}_m)$ where $\bar{\varepsilon} = \text{apr}((n, s), j, \varepsilon)$ and $s \geq q$. So $\text{apr}((n, q), j, \delta) = \text{apr}((n, q), j, \bar{\varepsilon})$ is contained in δ_m. Since $\delta_m <_j \delta$, this implies $\text{apr}((n, q), j, \delta_m) = \text{apr}((n, q), j, \delta)$. When $\tilde{\varepsilon}_m$ is not empty, $i = i_{n+1}$, hence $\tilde{\varepsilon}_m <_i \bar{\varepsilon}$ and $\delta_m = \text{apr}((n, r+1), (j, i_{n+1}), \delta_m)$ follows for some $r \geq q$.

Case 3.5. $s \geq q$ and $u \geq s$, hence $u \geq q$. Therefore $\text{apr}((n, q), (j, i_{n+1}), \tilde{\kappa}_m) = \text{apr}((n, q), (j, i_{n+1}), \delta)$. So δ_m satisfies $\boldsymbol{D}$ with $\text{apr}((n, q), j, \delta)$ and i_{n+1}. So $\delta_m = \text{apr}((n, r+1), j, \delta_m)$ for some $r \geq q$.

Case 3.6. $\delta_m=(i, a, \tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}\#\nu_m)$ where $\tilde{\varepsilon}=\text{apr}((n, s+1), (j, i_{n+1}), \tilde{\varepsilon})$, $\text{apr}((n, q), j, \tilde{\varepsilon})=\text{apr}((n, q), j, \delta)$ and $s\geq q$. If $\varepsilon=(t, 0, \varepsilon'+1)$, then $\tilde{\varepsilon}=(t, 0, \varepsilon')$ and $\nu_m=(t, 0, \varepsilon_m'\#(t_0, 0, \tilde{\varepsilon}))$. $\varepsilon_m'<_{j,t}\tilde{\varepsilon}$ and $t\geq i_{n+1}\geq j$. So $\delta_m=\text{apr}((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r\geq s\geq q$.

Case 3.7. (4.1;2) or (5.1.1;2) applies. $\delta_m=(i, a, \tilde{\varepsilon}\#\cdots\#\tilde{\varepsilon}\#\nu_m)$. This can be dealt with in a manner similar to Case 3.6.

(β) (4.1;1) or (5.1.1;1). $\delta_m=(i, 0, \varepsilon_m\#(i_0, 0, \varepsilon))$ where $i=i_0+1$. $i>p\geq i_{n+1}$. So $i_0\geq i_{n+1}$. $\varepsilon_m<_{j,i}\varepsilon$ and $\varepsilon=\text{apr}((n, s+1), (j, i_{n+1}), \varepsilon)$ where $s\geq k$. It follows that $s+1\geq q$, and hence $\delta_m=\text{apr}((n, u+1), j, \delta_m)$ for some $u\geq q$ is obvious.

(γ) (3.1) $\delta_m=\kappa_m\#\mu_m$ where $\delta=(i, a, \varepsilon)$, $\mu_m=(i, a, \varepsilon_m)$ and $\kappa_m=(i, a_m, \varepsilon)$ or $\kappa_m=(i, b, \cdots, (i, b, \varepsilon)\cdots)$. Since $\delta=\text{apr}((n, q+1), j, \delta)$ for some $q\geq k$, ε contains $\text{apr}((n, q), j, \delta)$ i_{n+1}-active. Therefore $\kappa_m=\text{apr}((n, r+1), j, \kappa_m)$ for some $r\geq q$ is obvious.

(δ) (2.1;1) $\delta_m=(i, 0, \varepsilon_m'\#(i_0, 0, \mu))$, where $\delta=(i, 0, \varepsilon'+1)$, $\mu=(i, 0, \varepsilon')$ and ε_m' is obtained from ε' by replacing ε by ε_m. $i>p\geq i_{n+1}$. So $i_0\geq i_{n+1}$ and hence δ contains $\text{apr}((n, q), (j, i_{n+1}), \delta)$ i_{n+1}-active. $\delta_m<_j\delta$. So $\text{apr}((n, q), (j, i_{n+1}), \delta_m)=\text{apr}((n, q), (j, i_{n+1}), \delta)$ and $\delta_m=\text{apr}((n, r+1), (j, i_{n+1}), \delta_m)$ for some $r\geq q$.

(ε) (2.1;2), (4.1;2) or (5.1.1;2). The desired relations are obvious.

Proof of Proposition 2.4 *where* (5.2) *or* (5.3) *applies to* (j, γ).

Let (p, δ) in Lemma 2.12 be the immediate successor of (j, γ). Then one of the cases listed there holds. Repeating the proof for Case 3 there for (j, γ) and $q=k$, we obtain the result. If (5.3) applies to (j, γ), then start with its immediate successor, and let (p, δ) be the immediate successor of the latter.

§3. The reduction sequence of an o.d. converges to it; non-critical cases.

Let $\tilde{\alpha}$ be a connected o.d., let j_0 be an indicator and let $\{\tilde{\alpha}_m\}_m$ be the reduction sequence for $(j_0, \tilde{\alpha})$. The objective of this section is to establish that $\{\alpha_m\}_m$ is truly a fundamental sequence for $\tilde{\alpha}$, namely $\{\tilde{\alpha}_m\}_m$ converges to $\tilde{\alpha}$ with respect to j_0.

THEOREM 2. *The reduction sequence for* $(j_0, \tilde{\alpha})$ *converges to* $\tilde{\alpha}$ *from below with respect to* $<_{j_0}$. *Namely given any* $\tilde{\beta}$ *snch that* $\tilde{\beta}<_{j_0}\tilde{\alpha}$, *there is an* m *such that* $\tilde{\beta}<_{j_0}\tilde{\alpha}_m$.

Theorem 1 provides with the "from below" part. The m in Theorem 2 can be determined primitive recursively from $\tilde{\alpha}$, $\tilde{\beta}$ and j_0.

The proof of Theorem 2 consists of four parts: §§3.1–3.4. The idea can be explained as follows. Given an o.d. $\tilde{\beta}$ satisfying $\tilde{\beta}<_{j_0}\tilde{\alpha}$, we first define a sequence of connected sub-o.d's of $\tilde{\beta}$ (which will be called *comparison factors*, or *cmf*) subject to the sp's and the tsp's of $\tilde{\alpha}$. Then we can show that for each sp or tsp of $\tilde{\alpha}$, say (j, γ), and its corresponding cmf of $\tilde{\beta}$, say δ, there is an m such that $\delta <_j \gamma_m$. Theorem 2 is a special case of this fact.

The lemmas and definitions below should save some cumbersom descriptions of various situations.

LEMMA 3.1. *Let α and β be two o.d.'s for which* apr $((n, k), j, \beta) <_j$ apr $((n, k), j, \alpha)$ *holds for some* (n, k). *Then* $\beta <_j \alpha$.

Proof. Let $\delta =$ apr $((n, k), j, \beta)$ and $\gamma =$ apr $((n, k), j, \alpha)$. $\delta <_j \gamma$ holds if and only if one of the following holds.

1°. $v_0(j, \delta) < v_0(j, \gamma)$ or $v_0(j, \delta) = v_0(j, \gamma)$ and apr $(0, j, \delta) <_i$ apr $(0, j, \gamma)$, where i is the indicator of $v_0(j, \gamma)$.

2°. There is a number $l \geq 1$ such that $(0, l) \leq (n, k)$ and apr $((0, l-1), j, \delta) =$ apr $((0, l-1), j, \gamma)$ and either $v_{(0,l)}(j, \delta) < v_{(0,l)}(j, \gamma)$ or $v_{(0,l)}(j, \delta) = v_{(0,l)}(j, \gamma)$ and apr $((0, l), j, \delta) <_{i+1}$ apr $((0, l), j, \gamma)$, where i is the indicator of $v_0(j, \delta)$ $(= v_0(j, \gamma))$.

3°. There is a pair of numbers (m, l), where $m > 0$ and $(m, l+1) \leq (n, k)$ such that apr $((m, l), j, \delta) =$ apr $((m, l), j, \gamma)$ and either $l = 0$ and $v_{m+1}(j, \delta) < v_{m+1}(j, \gamma)$, or $l > 0$, $v_{m+1}(j, \delta) = v_{m+1}(j, \gamma)$ $(= p)$ and apr $((m, l+1), j, \delta) <_{p+1}$ apr $((m, l+1), j, \gamma)$.

On the other hand, apr $((m, l), j, \alpha) =$ apr $((m, l), j, \gamma)$ and apr $((m, l), j, \beta) =$ apr $((m, l), j, \delta)$ if $(m, l) \leq (n, k)$. So 1°-3° supply a sufficient condition for $\beta <_j \alpha$.

DEFINITION 3.1. Suppose $\alpha =$ apr $((n, k+1), j, \alpha)$ and one of the following holds for α and β.

1) apr $((m, l), j, \beta) <_j$ apr $((m, l), j, \alpha)$ for some $(m, l) \leq (n, k)$.

2) apr $((m, l), j, \beta) =$ apr $((m, l), j, \alpha)$, apr $((m, l+1), j, \alpha)$ exists and apr $((m, l+1), j, \beta)$ does not exist for some $(m, l+1) \leq (n, k)$.

3) apr $((n, k), j, \beta) =$ apr $((n, k), j, \alpha)$ and apr $((n, k+1), j, \beta)$ exists.

3.1) $n = 0$ and $v_{(0,k+1)}(j, \beta) < v_{(0,k+1)}(j, \alpha)$.

3.2) $n > 0$, $k = 0$ and $v_{n+1}(j, \beta) < v_{n+1}(j, \alpha)$.

Then α is said to (j-) *dominate* β *at an early stage.*

PROPOSITION 3.1. *If α j-dominates β at an early stage, then $\beta <_j \alpha$. Suppose $\beta <_j \alpha$ and $\alpha =$* apr $((n, k+1), j, \alpha)$. *Then either α j-dominates β at an early stage or* apr $((n, k), j, \beta) =$ apr $((n, k), j, \alpha)$, $v_{n+1}(j, \beta) = v_{n+1}(j, \alpha)$ $(= p)$ and apr $((n, k+1), j, \beta) <_{p+1}$ apr $((n, k+1), j, \alpha)$ *when* $n > 0$.

In $n = 0$, we should replace the second equation with $v_{(0,k+1)}(j, \beta) = v_{(0,k+1)}(j, \alpha)$ and let p be the indicator of $v_0(j, \alpha)$ $(= v_0(j, \beta))$.

Proof. From Definition 3.1 and Lemma 3.1.

DEFINITION 3.2. Consider two o.d.s's α and β and an indicator i. Suppose $\alpha=\alpha_1 \# \cdots \# \alpha_m$ and $\beta=\beta_1 \# \cdots \# \beta_l$, where $\alpha_1, \cdots, \alpha_m$ are components of α and $\alpha_1 {}_i\geq \alpha_2 {}_i\geq \cdots {}_i\geq \alpha_m$, and $\beta_1, \cdots, \beta_l$ are components of β and $\beta_1 {}_i\geq \beta_2 {}_i\geq \cdots {}_i\geq \beta_l$. The following two conditions, (ω) and (θ), are necessary and sufficient in order that $\beta <_i \alpha$ hold. We will refer to those two conditions throughout this section.

(ω) $m>l$ and $\alpha_1=\beta_1, \cdots, \alpha_l=\beta_l$.

(θ) There is k such that $u\leq l, m$ and $\alpha_1=\beta_1, \cdots, \alpha_{u-1}=\beta_{u-1}$ and $\beta_u <_i \alpha_u$.

§ 3.1. The definition of comparison factors.

DEFINITION 3.3. Let $\tilde{\beta}$ and $\tilde{\alpha}$ be connected o.d.'s satisfying $\tilde{\beta} <_{j_0} \tilde{\alpha}$. We will define a sequence of connected sub-o.d.'s of $\tilde{\beta}$ subject to the sp's and tsp's of $\tilde{\alpha}$; those sub-o.d.'s are called *comparison factors*, or *cmf's of* $\tilde{\beta}$ (*relative to* $(j_0, \tilde{\alpha})$). It is possible that two successive cmf's are identical. Notice that we are assuming that $\tilde{\beta}$ is connected.

Let $\tilde{\beta}$ be the first cmf of $\tilde{\beta}$. Suppose a cmf, say δ has been defined corresponding to a sp or a tsp of $\tilde{\alpha}$, say (j, γ). The next step is defined according to cases. Here we need not distinguish between [1°] and [2°]; the cases are exhausted in (1)-(5).

A. (j, γ) is the last reduction pair of $\tilde{\alpha}$. Stop. δ is the last cmf or $\tilde{\beta}$.

B. (j, γ) has a successor.

(2.1;1) $\gamma=(i, 0, \alpha+1)$, $i=i_0+1$ and $j<i$. Put $\mu=(i, 0, \alpha)$. Suppose that the components of α are $\alpha_1, \alpha_2, \cdots, \alpha_m$, which are arranged in the non-increasing order with respect to i.

(2.1;1.a) $\gamma=\text{apr}(0, j, \gamma)$.

Case (1) $v_0(j, \delta)<(i, 0)$, $v_0(j, \delta)=(i, 0)$ and $\text{apr}(0, j, \delta) <_i \mu$ or $\delta=\mu$. Stop. δ is the last cmf.

Case (2) $\text{apr}(0, j, \delta)=\mu$ and $\text{apr}((1, 1), j, \delta)$ exists. Let it be (p, b, β) and suppose $p<i$. Stop. δ is the last cmf.

Case (3) $\text{apr}(0, j, \delta)=\mu$ and $\text{apr}((1, 1), j, \delta)=(p, b, \beta)$ where $(p, b)=(i, 0)$. Suppose that the components of β are $\beta_1, \beta_2, \cdots, \beta_l$, which are arranged in the non-increasing order with respect to p. Apply Definition 3.2 to α and β with respect to i. (ν) applies. Let the next sod of γ be α_w.

(ν.1) $u<w$. Stop. δ is the last cmf.

(ν.2) $u=w$. Let β_u be the next cmf.

(2.1;1.b) $\gamma=\text{apr}((0, k+1), j, \gamma)$.

Case (1) μ j-dominates δ at an early stage or $\delta=\mu$. Stop. δ is the last cmf.

Case (2) apr $((0, k+1), j, \delta)=\mu$ and apr $((1, 1), j, \delta)=(p, b, \beta)$. $v_2(j, \delta)<i_0$ or $v_2(j, \delta)=i_0$ and $p=i_0$. Stop. δ is the last cmf.

Case (3) apr $((0, k+1), j, \delta)=\mu$. apr $((1, 1), j, \delta)=(p, b, \beta)$, $v_2(j, \delta)=i_0$ $(p, b)=(i, 0)$.

(ν) holds for α, β and i. Let α_w be the next sod of γ.

(ν.1) $u<w$. Stop. δ is the last cmf.

(ν.2) $u=w$. Let β_u be the next cmf.

Case (4) apr $((0, k+1), j, \delta)=\gamma$, $v_2(j, \delta)=i_0$, $p=i$ and $b>0$.

Case (4.1) The next sod, say ρ, is the first component of α, viz. $\rho=\alpha_1$. Let apr $((1, 1), j, \delta)$ $(=\delta_{(1,1)})$ be the next cmf.

Case (4.2) The next sod is α_w where $w>1$. Stop. δ is the last cmf.

(2.1;1.c) $\gamma=$apr $((n, k+1), j, \gamma)$ where $n>0$.

Case (1) μ j-dominates δ at an early stage or $\delta=\mu$. Stop.

Case (2) $\mu=$apr $((n, k+1), j, \delta)$, apr $((n+1, 1), j, \delta)=(p, b, \beta)$ exists, and either $v_{n+2}(j, \delta)<i_0$ or $v_{n+2}(j, \delta)=i_0$ and $p=i_0$. Stop.

Case (3) apr $((n, k+1), j, \delta)=\mu$, apr $((n+1, 1), j, \delta)=(p, b, \beta)$ exists, $v_{n+2}(j, \delta)=i_0$ and $(p, b)=(i, 0)$.

(ν.) holds.

(ν.1) $u<w$. Stop.

(ν.2) $u=w$. Let β_u be the next cmf.

Case (4) As in Case (3) except that $(p, b)>(i, 0)$.

Case (4.1) The next sod is the first component of α, namely α_1. Let apr $((n+1, 1), j, \delta)$ $(=\delta_{(n+1,1)})$ be the next cmf.

Case (4.2) The next sod is α_w where $w>1$. Stop.

(2.1;2) $\gamma=(i, 0, \alpha+1)$, $i=i_0+1$, there is an $(i_0, 0)$-dominant in α and $j=i$.

We consider the following "transitory" condition (***T***): the immediate predecessor of (j, γ) is transitory, the corresponding cmf is δ_0, and if we let $j_1, j_2, \cdots, j_{u_0}$ be the sequence of indicators defined for (2′) in Definition 1.1, then $j=i=j_{u_0}$, $\delta<_{j_v}\gamma$ if $v+1=u_0$, $\gamma=$apr $((n, k+1), j_v, \gamma)=\mu=(i, 0, \alpha)$ and $\delta^*=$apr $((n, k+2), j_v, \delta)$ exists. Put $\delta^*=(p, b, \beta)$.

Case (1) (***T***) holds and $(p, b)=(i, 0)$.

(ν) applies to α and β with respect to i and $u=w$ (cf. (2.1;1) of this definition for u and w). Let β_u be the next cmf.

Case (2) (***T***) holds, and $(p, b)>(i, 0)$ and $w=1$. Let δ^* be the next cmf.

Case (3) All other cases. Stop. δ is the last cmf.

(ν) holds.

(ν.1) $u<w$. Stop.

(ν.2) $u=w$. Let β_u be the next cmf.

Case (4) As in Case (3) except that $(p, b)>(i, 0)$.

Case (4.1) The next sod is the first component of α, namely α_1. Let apr $((n+1), 1), j, \delta) (=\delta_{(n+1,1)})$ be the next cmf.

Case (4.2) The next sod is α_w where $w>1$. Stop.

(2.1; 2) $\gamma=(i, 0, \alpha+1)$, $i=i_0+1$, there is an $(i_0, 0)$-dominant in α and $j=i$. We consider the following "transitory" condition (***T***): the immediate predecessor of (j, γ) is transitory, the corresponding cmf is δ_0, and if we let $j_1, j_2, \cdots, j_{u_0}$ be the sequence of indicators defined for (2′) in Definition 1.1, then $j=i=j_{u_0}$, $\gamma=\text{apr}((n, k+1), j_v, \gamma)$ for some $n>0$ and k where $u_0=v+1$, $\delta=\text{apr}((n, k+1), j_v, \delta_0)=(i, 0, \alpha)=\mu$, and $\delta^*=\text{apr}((n, k+2), j_v, \delta_0)$ exists. Put $\delta^*=(p, b, \beta)$.

Case (1) (***T***) holds and $(p, b)=(i, 0)$.

(ν) applies to α and β with respect to i and $u=w$ (cf. (2.1; 1) of this definition for u and w). Let β_u be the next cmf.

Case (2) (***T***) holds and $(p, b)>(i, 0)$ and $w=1$. Let δ^* be the next cmf.

Case (3) All other cases. Stop. δ is the last cmf.

Note that for this case the next cmf "steps back". This does not cause a trouble, since the corresponding sod decreases its complexity.

(3.1) or (3.2) $\gamma=\text{apr}(0, j, \gamma)=(i, a, \alpha)$; α is connected and marked. Let $\delta_0=\text{apr}(0, j, \delta)=(p, b, \beta)$. Suppose that the components of β are $\beta_1, \beta_2, \cdots, \beta_l$, which are arranged in the non-increasing order with respect to p.

Case (1). $(p, b)<(i, a)$. Stop. δ is the last cmf.

Case (2). $(p, b)=(i, a)$. Let β_1 be the next cmf.

(3.3) $\gamma=\text{apr}(0, j, \gamma)=(i, a, \alpha)$; not all the components of α are marked suppose that the components of α are $\alpha_1, \alpha_2, \cdots, \alpha_m$, which are ordered in the non-increasing order with respect to i. Let $\delta_0=\text{apr}(0, j, \delta)=(p, b, \beta)$. Recall that the next sp of $\tilde{\alpha}$ is (i, γ') where γ' is an i-least component of α, or $\gamma'=\alpha_m$.

Case (1). $(p, b)<(i, a)$. Stop.

Case (2). $(p, b)=(i, a)$. Apply Definition 3.2 to α and β.

(ω) Stop. δ is the last cmf.

(θ.1) α_u is not γ' (i.e. $u<m$). Stop. δ is the last cmf.

(θ.2) α_u is γ' or $u=m$. Let β_u be the next cmf.

(4.1) $\gamma = \text{apr}((0, k+1), j, \gamma) = (i, 0, \alpha)$, $\alpha = \text{apr}((0, k), j, \gamma)$ (cf. Proposition 1.10), $i = i_0 + 1$ and $j < i$. For this case, either 1) or 2) in Definition 3.1 holds.

Case (1) 1) or 2) with $(m, l) < (0, k)$ (i.e. $m = 0$ and $l < k$) or 2) with $(m, l) = (0, k)$ and $\delta = \text{apr}((0, k), j, \delta)$. Stop. δ is the last cmf.

Suppose $\text{apr}((0, k), j, \delta) = \text{apr}((0, k), j, \gamma)$ and $\delta \neq \text{apr}((0, k), j, \delta)$ $\text{apr}((0, k+1), j, \delta)$ cannot exist.

Case (2) $v_2(j, \delta) < i_0$. Stop. δ is the last cmf.

Case (3) $v_2(j, \delta) = i_0$. Let $\text{apr}((1, 1), j, \delta)$ be the next cmf. Notice that in this case the next cmf may be δ itself.

(4.1; 2) $j = i$. Consider the following condition (T'): (T') is described as (T) for (2.1; 2) except that $\text{apr}((n, k+1), j_v, \delta_0) = \alpha$ (in the place of $= \mu$) is required.

Case (1) (T') holds. $\delta^* = \text{apr}((n, k+2), j_v, \delta_0)$ is the next cmf.

Case (2) Not Case (1). Stop.

(4.3) $\gamma = \text{apr}((0, k+1), j, \gamma) = (i, a, \alpha)$.

Case (1) γ j-dominates δ at an early stage. Stop. δ is the last cmf.

Case (2) $\text{apr}((0, k), j, \delta) = \text{apr}((0, k), j, \gamma)$ and $v_{(0,k+1)}(j, \gamma) = v_{(0,k+1)}(j, \delta)$ $(= a)$. Let $\text{apr}((0, k+1), j, \delta) = (i, a, \beta)$. Apply Definition 3.2 to α and β.

(ω) Stop. δ is the last cmf.

(θ.1) $m < u$, so α_u is not γ'. Stop. δ is the last cmf.

(θ.2) $u = m$, so α_u is γ'. Let β_u be the next cmf.

(5.1.1) $\gamma = \text{apr}((n, k+1), j, \gamma) = (i, 0, \alpha)$, $\alpha = \text{apr}((n, k), j, \gamma)$ (cf. Proposition 1.11), $i = i_0 + 1$ and $j \leq i$.

(5.1.1; 1) $j < i$. For this case, either 1) or 2) in Definition 3.1 holds.

Case (1) 1) or 2) with $(m, l) < (n, k)$ or 2) with $(m, l) = (n, k)$ and $\delta = \text{apr}((n, k), j, \delta)$. Stop. δ is the last cmf.

Suppose $\text{apr}((n, k), j, \delta) = \text{apr}((n, k), j, \gamma)$ and $\delta \neq \text{apr}((n, k), j, \delta)$. $\text{apr}((n, k+1), j, \delta)$ does not exist.

Case (2) $k = 0$ and $v_{n+1}(j, \delta) < i_0$, or $k > 0$ and $v_{n+2}(j, \delta) < i_0$. Stop. δ is the last cmf.

Case (3) $k = 0$ and $v_{n+1}(j, \delta) = i_0$ or $k > 0$ and $v_{n+2}(j, \delta) = i_0$. Let $\delta_{(n,1)} = \text{apr}((n, 1), j, \delta)$ be the next cmf when $k = 0$ and let $\delta_{(n+1,1)} = \text{apr}((n+1, 1), j, \delta)$ be the next cmf when $k > 0$.

(5.1.1; 2) $j = i$. Stop.

(5.2) $\gamma = \text{apr}((n, k+1), j, \gamma) = (i, a, \alpha)$ and $i = v_{n+1}(j, \gamma)$ where $n > 0$.

Case (1) γ j-dominates δ at an early stage. Stop. δ is the last cmf.

Case (2) $\mathrm{apr}((n, k), j, \delta)=\mathrm{apr}((n, k), j, \gamma)$ and $v_{n+1}(j, \delta)=v_{n+1}(j, \gamma)=i$. Let $\mathrm{apr}((n, k+1), j, \delta)=(i, b, \beta)$.

Case (2.1) $b<a$. Stop. β is the last cmf.

Case (2.2) $b=a$. Apply Definition 3.2 to α and β.
(ω) Stop. δ is the last cmf.
(θ.1) $u<m$, so α_u is not γ'.
(θ.2) $u=m$, so α_u is γ'. Let β_u be the cmf.

(5.3) and (2′) $\gamma=\mathrm{apr}((n, k+1), j, \gamma)=(i, a, \alpha)$, $n>0$ ($=(i, 0, \alpha+1)$ for (2′)), and $i>v_{n+1}(j, \gamma)$.

Let $j_1, j_2, \cdots, j_{u_0}$ be the sequence of indicators defined in (5.3) and (2′) of Definition 1.1. We define cmf's of $\tilde{\beta}$ corresponding to the tsp's, $(j_1, \gamma), (j_2, \gamma), \cdots, (j_{u_0}, \gamma)$.

Case (1) γ j-dominates δ at an early stage. Stop. δ is the last cmf.

Case (2) $\mathrm{apr}((n, k), j, \delta)=\mathrm{apr}((n, k), j, \gamma)$, $v_{n+1}(j, \delta)=v_{n+1}(j, \gamma)$ and $\mathrm{apr}((n, k+1), j, \delta)$ exists. Let $\mathrm{apr}((n, k+1), j, \delta)$ be the next cmf, corresponding to the tsp (j_1, γ). In this case the next cmf may be δ itself.

Suppose we have defined $\eta_1, \cdots \eta_v$, cmf's corresponding to $(j_1, \gamma), \cdots, (j_v, \gamma)$, and $v<u_0$. So $\gamma=\mathrm{apr}((r, s+1), j_v, \gamma)$ for some r and s, and (5.3) or (2′) applies to (j_v, γ).

Case (v. 1) γ j_v-dominates η_v at an early stage. Stop. η_v is the last cmf.

Case (v. 2) $\mathrm{apr}((r, s), j_v, \eta_v)=\mathrm{apr}((r, s), j_v, \gamma)$, $v_{r+1}(j_v, \eta_v)=v_{r+1}(j_v, \gamma)$ and $\mathrm{apr}((r, s+1), j_v, \eta_v)$ exists. Let $\eta_{v+1}=\mathrm{apr}((r, s+1), j_v, \eta_v)$ be the next cmf, corresponding to (j_{v+1}, γ). Here also the next cmf may be η_v itself.

This completes the definition of cmf's.

PROPOSITION 3.2. *Suppose $\tilde{\beta}$ is connected, $\tilde{\beta}<_{j_0}\tilde{\alpha}$ and δ is a cmf of $\tilde{\beta}$ corresponding to (j, γ), where (j, γ) is a sp or a tsp of $\tilde{\alpha}$. Then δ is connected and:*

I. *In each case of Defiition* 3.3, *all the possibilities are covered, hence the definition is complete; and*

II. $\delta<_j\gamma$.

Proof. Suppose for (j, γ) δ has been defined, and the proposition holds for (j, γ) and δ. It is obvious that at each stage δ is connected. Therefore we shall consider I and II.

Basis: for $\tilde{\beta}$ and $(j_0, \tilde{\alpha})$, II is assumed.

(2.1;1.a) I. As was proved in §2, $\mu = \mathrm{apr}\,(0, j, \mu)$. $\delta <_j \gamma = \mathrm{apr}\,(0, j, \gamma)$ by the induction hypothesis (II). Therefore either $v_0(j, \delta) < (i, 0)$ or $v_0(j, \delta) = (i, 0)$ and $\delta_0 = \mathrm{apr}\,(0, j, \delta) = (i, 0, \delta') <_i \gamma = (i, 0, \alpha+1)$. So $\delta' <_i \alpha+1$ or $\delta' \leq_i \alpha$. Therefore $\delta_0 \leq_i (i, 0, \alpha) = \mu$. If $\delta_0 = \mu$ and $\delta \neq \delta_0$, then $\mathrm{apr}\,((0, 1), j, \delta)$ cannot exist and $\mathrm{apr}\,((1, 1), j, \delta)$ exists. $v_2(j, \delta) \leq i_0$ and, if $\mathrm{apr}\,((1, 1), j, \delta) = (p, b, \beta)$, then $(p, b) \leq (i, 0)$. Suppose $(p, b) = (i, 0)$. Then $(i, 0, \beta) <_i \mu = \delta_0$ must hold, or $\beta <_i \alpha$. If $\alpha = \beta \# \beta'$ for some β' then α would contain itself. So (ω) in Definition 3.2 does not hold. So only (ν) is possible. In order to claim that $(\nu.1)$ and $(\nu.2)$ are the only possibilities, we prove a lemma.

LEMMA. *If $\mu = (i, 0, \alpha) = \mathrm{apr}\,(0, j, \mu)$, $\mathrm{apr}\,(0, j, \delta) = \mu$, $\mathrm{apr}\,((1, 1), j, \delta) = (i, 0, \beta)$, $v_2(j, \delta) = i_0$ and (ν) applies to α, β and i, then $w \geq u$, where $\alpha_w = \gamma'$, an i-least $(i_0, 0)$-dominant of α and $\alpha_1 = \beta_1, \cdots, \alpha_{u-1} = \beta_{u-1}$ and $\beta_u <_i \alpha_u$.*

Proof. Suppose $u > w$. There is a component of β, say β_x, in which μ is i_0-active. $x \geq u > w$. Since $u > w$, the outermost value of α_w is $\leq (i_0, 0)$. On the other hand the outermost indicator $\beta_x \geq i_0$. This and $\beta_x \leq_i \beta_u <_i \alpha_u$ force that $\beta_x = (i_0, 0, \beta'')$, $\beta_u = (i_0, 0, \beta')$ and $\alpha_u = (i_0, 0, \alpha')$ and $\beta'' <_{i_0} \beta' <_{i_0} \alpha'$. But α' is i_0-active in β'', hence $\alpha' \leq_{i_0} \beta''$. So $w \geq u$.

From the lemma. The claim is obvious.

II. When the next cmf exists, $u = w$. So $\gamma' = \alpha_u$ and $\beta_u <_i \alpha_u$.

(2.1; 1.b) $\mu = \mathrm{apr}\,((0, k+1), j, \mu)$ (cf. §2). Suppose $v_0(j, \delta) = v_0(j, \gamma)$. $\mathrm{apr}\,((0, k), j, \delta) = \mathrm{apr}\,((0, k), j, \gamma)$ and $\mathrm{apr}\,((0, k+1), j, \delta) = \delta_{(0,k+1)}$ exists. Then $\delta_{(0,k+1)} \leq_i (i, 0, \alpha) = \mu$ is proved as for (2.1; 1.a). Suppose $\delta_{(0,k+1)} = (i, 0, \alpha) \neq \delta$. Then $\mathrm{apr}\,((0, k+2), j, \delta)$ does not exists but $\mathrm{apr}\,((1, 1), j, \delta) = \delta_{(1,1)} = (p, b, \beta)$ does exist. $v_2(j, \delta) \leq i_0$ and $p = i$ or $p = i_0$. Suppose $v_2(j, \delta) = i_0$ and $p = i$. For this case we need the following lemma.

LEMMA. *Let η be a connected o.d. If $j < i$, $v_0(j, \eta) = i$, $\alpha_1 = \mathrm{apr}\,(1, j, \eta) = \mathrm{apr}\,((0, k), j, \eta)$ for some k, and $\eta \neq \alpha_1$ (hence η properly contains an occurrence of α_1), then $\eta <_i \alpha_1$.*

Proof. The definitions $\alpha_1 = \mathrm{apr}\,(1, j, \eta)$ and $v_0(j, \eta) = i$ evoke the equations $\mathrm{apr}\,((0, p), i, \alpha_1) = \mathrm{apr}\,((0, p), j, \alpha_1) = \mathrm{apr}\,((0, p), j, \eta)$ for every p such that $0 \leq p \leq k$. $v_0(j, \eta) = i$ and $\eta \neq \alpha_1$ imply that α_1 cannot be i-active in η. Consider the i-approximations of η: $\eta_0 = \mathrm{apr}\,(0, i, \eta)$ and $\eta_{(0,p)} (= \mathrm{apr}\,((0, p), i, \eta))$ for some $p = 0, 1, \cdots$. Note that the i-approximations of η are j-active in η. The very definition of α_0 (as the 0^{th} j-approximation of η) implies $\eta_0 \leq_{i+1} \alpha_0$. If $\eta_0 <_{i+1} \alpha_0$, then $\eta <_i \alpha_1$, regarding α_0 as the 0^{th} i-approximation of α_1. If $\eta_0 = \alpha_0$, then $v_0(i, \eta) = i$ and there is an i-active occurrence of α_0 in η, which is j-active as well. So due to the definition of $\alpha_{(0,1)}$ (as the $(0, 1)^{\text{th}}$ j-approximation of η), $\eta_{(0,1)} \leq_{i+1} \alpha_{(0,1)}$. If $<_{i+1}$, then with the same reasoning as for 0^{th} approximations (and the fact that $\eta_0 = \alpha_0$), $\eta <_i \alpha_1$ (regarding $\alpha_{(0,1)}$ as the $(0, 1)^{\text{th}}$ i-approximation of α_1). Otherwise, there is an i-active occurrence of $\alpha_{(0,1)}$ in η. Continue this reasoning. Unless there is a number p such that $p \leq k$, $\eta_{(0,p-1)} = \alpha_{(0,p-1)}$ and $\eta_{(0,p)} <_{i+1} \alpha_{(0,p)}$, we eventually reach the equation $\eta_{(0,k)} = \alpha_{(0,k)} = \alpha_1$. This means that α_1 is i-active in η, yielding a contradiction. Therefore there is a p satisfying the condition as above, which implies $\eta <_i \alpha_1$.

Let α_1 be μ. η be $\delta_{(1,1)}$ and k be $k+1$ in the lemma immediately above. Then by the lemma $\delta_{(1,1)} <_i \mu$.

Case (3) I. $(p, b) = (i, 0)$. Then $(i, 0, \beta) <_i (i, 0, \alpha)$, or $\beta <_i \alpha$. As for (2.1; 1.a), it can be shown that only (ν) is possible. Then a lemma similar to that for (2.1; 1.a) holds: under the circumstances $w \geq u$. From this follows that either (ν.1) or (ν.2) holds.

II. $\beta_u <_i \alpha_u = \gamma'$.

Case (4.1) By the lemma proved above, $\delta_{(1,1)} = (i, b, \beta) <_i \mu = (i, 0, \alpha)$. Since $b > 0$, this implies $\delta_{(1,1)} <_i \alpha$, or $\delta_{(1,1)} \leq_i \rho = \gamma'$. But $\delta_{(1,1)}$ contains ρ properly, so $<_i$ must hold.

Case (4.2) No further cmf.

It is obvious that all the possibilities have been exhausted.

(2.1; 1.c) $\delta <_j \gamma$ and $\mu = \text{apr}((n, k+1, j, \mu)$. It can be easily shown that if γ j-dominates δ at an early stage, then so does μ. Suppose $\text{apr}((n, k), j, \delta) = \text{apr}((n, k), j, \gamma)$, $v_{n+1}(j, \delta) = i_0$ and $\text{apr}((n, k+1), j, \delta)$ exists. Then $\text{apr}((n, k+1), j, \delta) = \delta_{(n,k+1)} \leq_{i,i+1} \mu$, for $\delta_{(n,k+1)} <_{i_{n+1}+1} \gamma$ and $i = i_{n+1}$ imply that $\delta_{(n,k+1)} = (i, 0, \delta')$, hence $\delta' \leq_i \alpha$. Suppose $\text{apr}((n, k+1), j, \delta) = \mu$ and $\delta \neq \mu$. $\text{apr}((n+1, 1), j, \delta) = \delta_{(n+1,1)}$ exists. We need the following lemma.

LEMMA. *Let η be a connected o.d. and j and i be indicators such that $j < i$ and $i = i_0 + 1$.*

1) *If η properly contains an occurrence of $\alpha_n = \text{apr}(n, j, \eta)$ and $v_{n+1}(j, \eta) = i_0$, then $\eta <_i \alpha_n$.*

2) *If η properly contains an occurrence of $\alpha_{n+1} = \text{apr}((n+1, j, \eta) = \text{apr}((n, k), j, \eta)$ for some $k > 0$, and $v_{n+2}(j, \eta) = i_0$, then $\eta <_i \alpha_{n+1}$.*

Proof. The distinction between 1) and 2) is only a matter of subscripts, so we shall prove 2) as an example.

The condition $v_{n+2}(j, \eta) = i_0$ evokes the relation $v_{n+1}(j, \eta) \geq i$, hence the equation $(\alpha_{(n,p)} =) \text{apr}((n, p), i, \alpha_{n+1}) = \text{apr}((n, p), j, \eta)$ for all p such that $0 \leq p \leq k$. The condition that η properly contains an occurrence of α_{n+1} and $v_{n+2}(j, \eta) = i_0 < i$ imply that α_{n+1} cannot be i-active in η. Consider the i-approximations of η; $\eta_{(p,l)} = \text{apr}((p, l), i, \eta)$. Note that the

i-approximations of η are j-active in η. Let $\iota_p = v_p(j, \eta)$, $p=0, 1, \cdots, n+2$. If $0 \leq p \leq n+1$, then $\iota_p = v_p(i, \alpha_{n+1})$. From the definition of α_0 $(=\text{apr}(0, j, \eta))$, $\eta_0 = \text{apr}(0, i, \eta) \leq_{\iota_0+1} \alpha$. If $<$, then $\eta <_i \alpha_{n+1}$, regarding α_0 as $\text{apr}(0, i, \alpha_{n+1})$. If $=$, then there is an i-active occurrence of α_0 in η. It is also j-active. So from the definition of $\alpha_{(0,1)} (=\text{apr}((0, 1), j, \eta))$, $\eta_{(0,1)} \leq_{\iota_0+1} \alpha_{(0,1)}$, If $<$, then $\eta <_i \alpha_{n+1}$; othewise there is an i-active occurrence of $\alpha_{(0,1)}$ in η. Continue this reasoning.

Suppose, with this reasoning, we reach $\eta_{(n,k)} = \alpha_{(n,k)} = \alpha_{n+1}$. This would mean that there is an i-active occurrence os α_{n+1}, yielding a contradiction. Therefore there must be a $(p, l) \leq (n, k)$ at which the equality break down. Let (p, l) be the first such. Then either $\text{apr}((p, l), i, \eta)$ does not exist, or $\text{apr}((p, l), i, \eta)$ exists with $l=1$ $v_{p+1}(i, \eta) < \iota_{p+1}$, or $v_{p+1}(i, \eta) = \iota_{p+1}$ and $\eta_{(p,l)} <_{\iota(p+1)+1} \alpha_{(p,l)}$. In any case $\eta <_i \alpha_{n+1}$.

In the lemma immediately above, let η be $\delta_{(n+1,1)}$, α_{n+1} be μ and k be $k+1$. Then $\delta_{(n+1,1)} <_i \mu$.

Case (3) As has been seen, either (ν.1) or (ν.2).

Case (4.1) $\delta_{(n+1,1)} <_i \gamma'$ is proved as for (2.1;1.b).

(2.1;2) Cases (1) and (2). The condition (***T***) demands that $v_{n+1}(j_v, \gamma) = v_{n+1}(j, \delta_0) = i_0$. Since $\mu = \text{apr}((n, k+1), j_v, \mu)$ whose $n>0$, $\delta^* <_i \mu$. So for Case (1) $\beta <_i \alpha$, and hence $\beta_u = \beta_w <_i \alpha_w$ (the next sod) by definition. For Case (2), $\delta^* <_i \mu$. But $\mu <_{i+1} \delta^*$. So $\delta^* \leq_i \alpha$. But α is a part of δ^*, hence $\delta^* <_i \alpha$. δ^* is connected, so this means $\delta^* <_i \alpha_1$.

(3.1) and (3.2) I. $\delta <_j \gamma$ by the induction hypothesis (II), hence $(p, b) \leq (i, a)$, so either Case (1) or Case (2) must hold.

II. If Case (2), $\delta <_j \gamma$ implies $\delta_0 <_i \gamma$, hence $\beta <_i \alpha$. Since α is marked, this implies $\beta_1 <_i \alpha$. Recall that the next sp is (i, α) and the next cmf is β_1. Therefore II is proven for the next stage.

(3.3) If Case (2), $\beta <_i \alpha$; hence for (θ.2) $\beta_u <_i \alpha_u = \gamma'$ by definition.

(4.1;1) $\delta <_j \gamma$ by the induction hypothesis.

I. If $\text{apr}((0, k), j, \delta) = \text{apr}((0, k), j, \gamma) = \alpha$, then $\text{apr}((0, k+1), j, \delta)$ cannot exist. So either $\delta = \text{apr}((0, k), j, \delta)$ or $\text{apr}(2, j, \delta)$ exists and $v_2(j, \delta) \leq i_0$.

II. Case (3). The next sp is (i, α) and the cmf is $\text{apr}((1, 1), j, \delta)$ $(=\delta_{(1,1)})$.

Taking $\delta_{(1,1)}$ as the η in the Lemma proven for (2.1;1.b), we obtain $\delta_{(1,1)} <_i \text{apr}((0, k), j, \delta) = \text{apr}((0, k), j, \gamma) = \alpha$.

(4.1;2) Case (1) $\delta^* <_i \alpha$ from the definition. α is the next sod.

(5.1.1;2) is dealt with exactly in the same manner.

(4.3) $\delta <_j \gamma$ (the induction hypothesis) and proposition 3.1 yield that either Case (1) or the equations of Case (2) hold and $\text{apr}((0, k+1), j, \delta) <_{i+1} \text{apr}((0, k+1), j, \gamma)$, viz. $(i, a, \beta) <_{i+1} (i, a, \alpha)$. This implies $\beta <_i \alpha$, hence the sufficiency of (ω) and (θ), and $\beta_u <_i \alpha_u = \gamma'$ when (θ.2) holds.

(5.1.1.; 1) $\delta <_j \gamma$ by the induction hypothesis.

I. If apr $((n, k), j, \delta)=$apr $((n, k), j, \gamma)=\alpha$ (cf. Proposition 1.11), then apr $((n, k+1), j, \delta)$ cannot exist. So either $\delta=$apr $((n, k), j, \delta)$ or apr $(n+1, j, \delta)$ exists and $v_{n+1}(j, \delta)\leq i_0$ (when $k=0$) or apr $(n+2, j, \delta)$ exists and $v_{n+2}(j, \delta)\leq i_0$ (when $k>0$).

II. Case (3). The next sp is (i, α) and the cmf is either apr $((n, 1), j, \delta)$ $(=\delta_{(n,1)})$ (when $k=0$) or apr $((n+1, 1), j, \delta)$ $(=\delta_{(n+1,1)})$ (when $k>0$).

Let us consider the latter case. Taking $\delta_{(n+1,1)}$ as the η in the Lemma (Proven for (2.1;1.c), we obtain $\delta_{(n+1,1)} <_i$ apr $((n, k), j, \delta)=\alpha$.

(5.2) I. Suppose Case (1) does not hold. Then apr $((n, k), j, \delta)=$ apr $((n, k), j, \gamma)$ and $v_{n+1}(j, \delta)=v_{n+1}(j, \gamma)$ $(=i)$. Let apr $((n, k+1), j, \delta)=$ (p, b, β). Since $\delta <_j \gamma$, Proposition 3.1 implies $(p, b, \beta) <_{i+1} (i, a, \alpha)$, hence $p\leq i$. On the other hand, $p\geq i$ by the definition of apr $((n, k+1), j, \delta)$ (since $v_{n+1}(j, \delta)=i$). So $p=i$ is the only possibillity. This implies that either $b<a$ or $b=a$ and $\beta <_i \alpha$.

II. If $(\theta.2)$ is the case, $\beta_u <_i \alpha_u=\gamma'$.

(5.3) and (2′) I. Note that if Case (1) does not apply, then apr $((n, k+1), j, \delta)$ exists.

II. If Case (2), $\eta_v <_{j_v} \gamma$ implies $\eta_{v+1} <_{j_{v+1}} \gamma$, wheae $j_{v+1}=v_{r+1}(j_v, \eta_v)+1$. This completes the proof of Proposition 3.2.

§ 3.2 Proof of Theorem 2

Theorem 2 follows from the proposition stated below. This is a basic property possessed by the cmf's.

PROPOSITION 3.3. *Let* $\tilde{\beta}$ *be a connected o.d. satisfying* $\tilde{\beta} <_{j_0} \tilde{\alpha}$. *Let* (j, γ) *be a sp or a tsp of* $(j_0, \tilde{\alpha})$ *and let* δ *be the cmf of* $\tilde{\beta}$ *corresponding to* (j, γ). *Let* $\{\gamma_m\}_m$ *be the reduction sequence for* γ. *Then one of the cases* (1*)~(3*) *below holds.*

(1*) *There exists an* m *and a component of* γ_m, *say* ρ_m, *satisfying* $\delta <_j \rho_m$.

(2*) $\gamma_m=\alpha \# \cdots \# \alpha \# \rho_m$, *where either* ρ_m *is empty* (b.3), d.4.2) *and* e.4.2)) *or* ρ_m *is connected and assumes one of the forms* $\rho_m=(k_m, 0, \alpha)$ *where* $k_m \uparrow i$ (*d*.4.1) *and* e.4.1)) *and* $\rho_m=(i, 0, \alpha_m) \# (i_0, 0, \alpha))$ *where* $\gamma=(i, 0, \alpha)$ *and* $i=i_0+1$ ((4.1;2) *and* (5.1.1;2)), *and* $\delta \leq_j \alpha$.

(3*) $\gamma_m=u \# \cdots \# \mu \# \rho_m$, *where* $\gamma=(i, a, \alpha+1)$ *and* $\mu=(i, a, \alpha)$ *and either* ρ_m *is empty* (c.1) *and* 1° *of* c.2.3)) *or assumes the form* $(i, 0, \alpha_m \# (i_0, 0, \mu))$ ((2.1;2)).

(2′) and (5.3) can fit to one of the forms given above.

The proof of Proposition 3.3 will be presented in §§ 3.3 and 3.4. Here we shall complete the proof of Theorem 2 assuming Proposition 3.3.

THEOREM 2. *The reduction sequence for* $(j_0, \tilde{\alpha})$ *converges to* $\tilde{\alpha}$ *from below with respect to* $<_{j_0}$.

Proof. 1. Let $\tilde{\beta}$ be a connected o.d. satisfying $\tilde{\beta}<_{j_0}\tilde{\alpha}$. Then as a special case of Proposition 3.3, with $\delta=\tilde{\beta}$ and $(j, \gamma)=(j_0, \tilde{\alpha})$, $\tilde{\beta}<_{j_0}\rho_m\leq_{j_0}\tilde{\alpha}_m$ (in case of (1*)) or $\tilde{\beta}\leq_{j_0}\alpha<_{j_0}\tilde{\alpha}_1$ (in case of (2*)) or $\tilde{\beta}\leq_{j_0}(i, a, \alpha)<_{j_0}\tilde{\alpha}_1$.

2. Let $\tilde{\beta}=\beta_1 \# \cdots \# \beta_q$ and $\tilde{\beta}<_{j_0}\tilde{\alpha}$. We may assume β_1 is a j_0-greatest component of $\tilde{\beta}$. Then $\beta_1<_{j_0}\alpha$. So 1. applies. If (1*) is the case, then $\beta_1<_{j_0}\rho_m$ where ρ_m is connected. So $\beta_p<_{j_0}\rho_m$ for every $p=1, 2, \cdots, q$, and hence $\tilde{\beta}<_{j_0}\rho_m\leq_{j_0}\tilde{\alpha}_m$. If (2*) is the case, then $\beta_1\leq_{j_0}\alpha$. So $\beta_p\leq_{j_0}\alpha$ for every p, and hence $\tilde{\beta}<_{j_0}\alpha \# \alpha \# \cdots \# \alpha=\tilde{\alpha}_q$. (3*) can be dealt with likewise.

This completes the proof of Theorem 2, and hence the existence of fundamental sequences.

§ 3.3. Proposition 3.3—Basis

Let us restate here what we are going to prove:

Let δ be the last cmf of (a connected) $\tilde{\beta}$ corresponding to (j, γ). Then the statements in Paoposition 3.3 hold for δ (cf. §3.2).

Proof. We look for the cases where the construction of cmf's stops (cf. Definition 3.3). Note that δ is connected.

A. (j, γ) is the last reduction pair of $\tilde{\alpha}$, hence $\gamma=\nu$ and $j=t$ in the notation of Definition 1.1.

(1) a.1) For this case we show that for any connected $\delta<_j\gamma$ $(=(i, a, 0)=\nu)$, $\delta<_j\gamma_m$ for some m. $\rho_m=\gamma_m(=\nu_m)$. If $\delta<_j\gamma$, $v_0(j, \delta)<(i, a)$ must hold. If the indicator of $v_0(j, \delta)$ is $<i$, then $\delta<_j\nu_0$. If $=i$, then there is an m such that the second element of $v_0(j, \delta)<a_m$. So $\delta<_j\nu_m=\rho_m$. So (1*) is the case.

a.2.) As in a.1), suppose $\delta<_j\gamma(=\nu)$ where δ is connected. $v_0(j, \delta)<(i, a)$, viz. $v_0(j, \delta)\leq(i, b)$. If $<$, then $\delta<_j\nu_0$. If $v_0(j, \delta)=(i, b)$, then we define the nesting number of j-active (i, b)'s in δ as follows. Let $\eta=(p, c, \eta')$ be a j-active sub-o.d. of δ such that η' does not contain any j-active occurrence of (i, b). The nesting number of η, denoted by $n(\eta)$, is 0 if $(p, c)<(i, b)$ and is 1 if $(p, c)=(i, b)$. (Note that $(p, c)\leq(i, b)$.) Let $\eta=(p, c, \eta')$ be a j-active sub-o.d. of δ where η' contains a j-active (i, b). If $(p, c)<(i, b)$, then $n(\eta)=\max(n(\eta_1), \cdots, n(\eta_k))$, where $\eta_1, \cdots, \eta_k$ are the components of η'; if $(p, c)=(i, b)$, then $n(\eta)=\max(n(\eta_1), \cdots, n(\eta_k))+1$.

Let $n_0=n(\delta)$. $n_0\geq 1$. We shall show that $\delta<_j\nu_{n_0+1}$ by induction on the complexity of δ.

Suppose first $n_0=1$. We will prove that $\delta<_l\nu_1$ for all l such that $j\leq l\leq i$. If δ is of the form (i, b, δ'), then there is no l-active (i, b) in δ'. So $v_0(j, \delta')<(i, b)$, which implies $\delta'<_l\nu_0$; in particular $\delta'<_i\nu_0$, from which follows $\delta<_l\nu_1$ for any $l\geq i$. Let $j\leq l<i$ and let η be an l-section of δ. $v_0(l, \eta)<(i, b)$. So $\eta<_l\nu_0<_l\nu_1$, hence $\delta<_l\nu_1$ for all l,

$j \le l < i$. If $\delta = (p, c, \delta')$ where $(p, c) < (i, b)$, then $j \le p \le i$, for δ contains a j-active (i, b), and $v_0(j, \delta) = (i, b)$. $\delta <_j \nu_1$ is obvious if $l > p$. Let $l = p$. $\delta' <_p \nu_1$ by the induction hypothesis. So $\delta <_p \nu_2$ Let $j \le l < p$ and η be an l-section of δ. By the induction hypothesis $\eta <_l \nu_1$. So $\delta <_l \nu_1$ for all such l.

Suppose next $n_0 > 1$. Again we shall show that (from the induction hypothesis) $\delta <_l \nu_{n_0}$ for all l such that $j \le l \le i$. If $\delta = (i, b, \delta')$ then the nesting number of each component of $\delta' \le n_0 - 1$. So $\delta' <_l \nu_{n_0-1}$ by the indution hypothesis. In particular $\delta' <_i \nu_{n_0-1}$ (using the fact that $\{\nu_m\}_m$ is an increasing sequence with respect to any l, $j \le l \le i$). So $\delta <_i \nu_{n_0}$. Let $j \le l < i$ and let η be an l-section of δ. Then $n(\eta) < n_0$. So by the induction hypothesis $\eta <_l \nu_{n_0}$; hence $\delta <_l \nu_{n_0}$. If δ is of the form (p, c, δ') where $(p, c) < (i, b)$, then $\delta <_l \nu_{n_0}$ for any $l > p$. Let $j \le l \le p$ and let η be an l-section of δ. Then by the induction hypothesis $\eta <_l \nu_{n_0}$, hence $\delta <_l \nu_{n_0}$. In any case $\rho_m = \nu_m$, hence (1*).

a.3.1) $j = t > i$ and $h > i$.

1°. $i_m \uparrow I$. $\nu_m = (i, b, (i_m, 0, 0)) = \mathrm{apr}(0, j, \nu_m)$ and $v_0(j, \nu_m) = (i, b)$. If $\delta <_j \nu$, then $v_0(j, \delta) < (i, a)$, i.e., $v_0(j, \delta) \le (i, b)$. If $<$, then $\delta <_j \nu_0$. If $=$, consider $\mathrm{apr}(0, j, \delta) = (i, b, \delta')$. Let p be the greatest indicator occurring in δ'. There is an m such that $p < i_m$. Then evidently $\delta' <_i (i_m, 0, 0)$, so $\mathrm{apr}(0, j, \delta) <_i \mathrm{apr}(0, j, \nu_m)$, which implies $\delta <_j \nu_m (= \rho_m)$.

2°. I has the maximum element ι and $a_m \uparrow A$. $\nu_m = \mathrm{apr}(0, j, \nu_m) = (i, b, (\iota, a_m, 0))$. Suppose $\delta <_j \nu$. Then $v_0(j, \delta) \le (i, b)$. If $<$, then $\delta <_j \nu_0$. If $=$, then $\mathrm{apr}(0, j, \delta) = (i, b, \delta')$. There is an m such that all the values occurring in δ' are $< (\iota, a_m)$. Then $\delta' <_i (\iota, a_m, 0)$, so $\mathrm{apr}(0, j, \delta) <_i \mathrm{apr}(0, j, \nu_m)$ $(= \nu_m)$. Therefore $\delta <_j \nu_m (= \rho_m)$.

3°. I has the maximum element ι and A has the maximum element e. $\kappa_m = (\iota, e, \cdots, (\iota, e, 0), \cdots)$ and $\nu_m = (i, b, \kappa_m)$. $\mathrm{apr}(0, j, \nu_m) = \nu_m$. Suppose $\delta <_j \nu$. If $v_0(j, \delta) < (i, b)$, then $\delta <_j \nu_1$. If $=$, let $\mathrm{apr}(0, j, \delta)$ (i, b, δ'). We can define the nesting number of i-active occurrences of (ι, e) in δ' in the same manner as the nesting number in a.2). Let it be n_0. Note that all the occurrences of (ι, e) are l-active in κ_m for every l. We can show as for a.2) that $\delta' <_l \kappa_{n_0}$ for every l such that $l \ge i$. Then $\delta <_j \nu_{n_0}$.

b) This case can be dealt with in a manner similar to a).

b.2.2;1) 1°. $\nu_m = (i_0, e, (i_m, 0, 0)) = \mathrm{apr}(0, j, \nu_m)$ and $v_0(j, \nu_m) = (i_0, e)$. If $\delta <_j \nu$, then $v_0(j, \delta) < (i, 0)$ or $\le (i_0, e)$. If $<$, then $\delta <_j \nu_0$. If $=$, consider $\mathrm{apr}(0, j, \delta) = (i_0, e, \delta')$. Let p be the greatest indicator occurring in δ' and suppose $p < i_m$. The evidently $\delta' <_{i_0} (i_m, 0, 0)$, so $\mathrm{apr}(0, j, \delta) <_{i_0} \nu_m$.

2°. $\kappa_m = (\iota, e, \cdots, (\iota, e, 0) \cdots)$ and $\nu_m = (i_0, e, \kappa_m)$. $\mathrm{apr}(0, j, \nu_m) = \nu_m$. Suppose $\delta <_j \nu$. If $v_0(j, \delta) < (i_0, e)$, then $\delta <_j \nu_0$. If $=$, let $\mathrm{apr}(0, j, \delta) = (i_0, e, \delta')$. Define the nesting number of i_0-active occurrences of (ι, e) in δ' (cf. a.2)). Let it be n_0. All the occurrences of (ι, e) are l-active in κ_m for every l. $\delta' <_l \kappa_{n_0+1}$ for every $l \geq i_0$ (cf. a.2)). Then $\delta <_j \nu_{n_0+1}$.

b.2.2;3) $j \leq i_0$ $\delta <_j \gamma$. If $v_0(j, \delta) < (i_0, e)$, then $\delta <_j \nu_0$ Suppose $=$. Define the nesting number of j-actiue (i_0, e)'s. Then follow the proof of a.2).

(2) c.1) Suppose $\delta <_j \nu$ for a connected δ. If $v_0(j, \delta) < (i, a)$ or $v_0(j, \delta) = (i, a)$ and apr $(0, j, \delta) <_i (i, a, \alpha)$, then $\delta <_j \nu_0 (= (i, a, \alpha))$. Suppose $v_0(j, \delta) = (i, a)$ and apr $(0, j, \delta) = (i, a, \delta')$. Then $\delta <_j (i, a, \alpha+1)$ implies $\delta' <_i \alpha+1$, viz. $\delta' \leq_i \alpha$. So apr $(0, j, \delta) \leq_i (i, a, \alpha)$. $j > i$ and $v_0(j, \delta) = (i, a)$ imply that $\delta =$ apr $(0, j, \delta)$. So if apr $(0, j, \delta) \neq (i, a, \alpha)$, then $\delta = (i, a, \alpha)$. Therefore (3*) holds.

c.2.1) We will work out this case in two steps: the case where $j = i$ and the case where $j \leq i$ in general.

$j = i$. We need some lemmas for this case.

LEMMA (1). *Suppose δ is connected and $(i, a, \alpha) \leq_i \delta <_i (i, a, \alpha+1)$. Then $\delta <_{i+1} (i, a, \alpha+1)$.*

Proof. Under the assumption, $\alpha <_i \delta$. So $\alpha+1 <_i \delta$ (since δ is connected). Thus $(i, a, \alpha+1) <_{i+1} \delta$ would imply $(i, a, \alpha+1) <_i \delta$, contradicting the assumption.

LEMMA (2) *Suppose δ is connected and $(i, a, \alpha) <_i \delta <_i (i, a, \alpha+1)$. Then δ is of the form (i, b, δ') where $b < a$ and $(i, a, \alpha) \leq_i \delta' <_i (i, a, \alpha+1)$.*

Proof. Let δ be of the form (k, b, δ'). $(i, a, \alpha) <_i \delta$ implies $k \geq i$. Suppose $k > i$. Then $(i, a, \alpha+1) <_l \delta$ for all $l > i$. Therefore $\delta <_i (i, a, \alpha+1)$ implies $\delta \leq_i \alpha+1$, so $\delta \leq_i (i, a, \alpha)$, contradicting the assumption. So $k = i$.

Suppose $b = a$. If $\delta <_{i+1} (i, a, \alpha)$, then $\delta' <_i \alpha$. This implies $\delta <_i (i, a, \alpha)$, contradicting the assumption. So $(i, a, \alpha) <_{i+1} \delta <_{i+1} (i, a, \alpha+1)$ (by Lemma (1)), from which follows $\alpha <_i \delta' <_i \alpha+1$. This is impossible. Thus $b \neq a$. If $b > a$, then $(i, a, \alpha+1) <_{i+1} \delta$, contradicting Lemma (1). So $b < a$. $\delta' <_i (i, a, \alpha+1)$ is obvious. $(i, b, \delta') <_{i+1} (i, a, \alpha)$, since $b < a$. So in order that $(i, a, \alpha) <_i \delta$, $(i, a, \alpha) \leq_i \delta'$ must hold.

Now back to c.2.1). Suppose $\delta <_i (i, a, \alpha+1)$. Note first that $\{\nu_m\}_m$ is increasing for all $<_l$.

Case 1) $\delta \leq_i (i, a, \alpha)$. Then $\delta <_i (i, a_0, (i, a, \alpha)) = \nu_0$ is obvious. (1*).

Case 2) $(i, a, \alpha) <_i \delta <_i (i, a, \alpha+1)$. Let Γ be the set of connected o.d.'s of δ defined as follows. δ belongs to Γ. If γ is in Γ and η is an i-section of γ, then each component of η is in Γ. Only those are in Γ. If γ belongs to Γ and there is no i-section of γ, then we say that γ is a simplest member or Γ.

We shall prove for each γ in Γ that there is an m such that $\gamma <_i (i, a_m, (i, a, \alpha)) (= \nu_m)$, by induction on the complexity of γ. As a special case, $\delta <_i \nu_m$ for some m.

From the definition and the assumption, $\gamma <_i (i, a, \alpha+1)$ for any γ in Γ. Let γ be a simplest member of Γ. Suppose γ is of the form (p, b, γ'). $\gamma \leq_i (i, a, \alpha)$, for otherwise $p=i$ by Lemma (2), hence each component of γ' belongs to Γ, contradicting the choice of γ. So $\gamma <_i (i, a_0, (i, a, \alpha)) = \nu_0$ Suppose $\gamma = (p, b, \gamma')$ where γ' contains a member of Γ. If $(i, a, \alpha) <_i \gamma$, then Lemma (2) implies $p=i$ and $b<a$, hence γ' is an i-section of γ. By the induction hypothesis, for each η a component of γ', there is an m (depending on η) such that $\eta <_i \nu_m$. Let m_0 be the greatest such m. Then $\gamma' <_i \nu_{m_0}$. Since $b<a$, there is an m such that $a_m > b$ and $a_m > a_{m_0}$. Pick such an m. $\gamma <_l \nu_m$ if $l>i$, and $\gamma' <_i \nu_{m_0} <_i \nu_m$. So $\gamma <_i \nu_m$. This completes the case where $j=i$, and we have seen that (1*) holds.

Now we drop the condition $j=i$. Let δ be an arbitrary connected o.d.

We prove that

(*) Given and $j \leq i$ for which $\delta <_j \nu = (i, a, \alpha+1)$ holds, there is an m such that $\delta <_j \nu_m$.

The proof is by induction on the complexity of δ within which by induction on the number of indicators satisfying the following condition (C): an indicator l satisfies (C) if $j<l\leq i$ and there is an l-sectin of δ or ν. The number of such indicators depends on j (when δ is fixed) and if l satisfies (C) for j, then the number corresponding to l is less than that which corresponds to j. Therefore the induction hypothesis applies to l (presuming that $\delta <_l \nu$). When $\delta=0$, $\delta <_j \nu_0$. Suppose δ is not 0. The case where $j=i$ is done. So suppose $j<i$. Let l be the least indicator such that $j<l\leq i$ and there is an l-section of δ or ν. By the induction hypothesis (*) holds for l. Suppose $\delta <_j (i, a, \alpha+1)$.

Case 1) $\delta <_l \nu$. Then by (*) for l, $\delta <_l \nu_m$ for some m. Let η be a j-section of δ. Then $\eta <_j \nu$, so by the induction hypothesis $\eta <_j \nu_n$ for some n. Let $m_0 = \max(m, n)$. $\{\nu_m\}_m$ is increasing with respect to l and j, so $\delta <_l \nu_{m_0}$ and $\eta <_j \nu_{m_0}$. Therefore by the choice of l, $\delta <_j \nu_{m_0}$.

Case 2) $\nu <_l \delta$. Since $\delta <_j \nu$, there is a j-section of $(i, a, \alpha+1)$, hence of α, say η, such that $\delta \leq_j \eta$. η is a j-section of ν_0, so $\delta <_j \nu_0$.

c.2.2) $a=b+1$.

Suppose $\delta <_j \nu = (i, b+1, \alpha+1)$. This case can be dealt with in a manner similar to the proof for c.2.1). Note that $\{\nu_m\}_m$ is $<_l$-increasing for every l.

$j=i$. If $\delta \leq_i (i, b+1, \alpha)$, then $\delta <_i \nu_0$. If $(i, b+1, \alpha) <_i \delta <_i (i, b+1, \alpha+1)$, then by Lemma (2) (with $a=b+1$) δ is of the form (i, c, δ') where $c \leq b$ and the components of $\delta' <_i (i, b+1, \alpha+1)$. By the induction hypothesis, $\delta' <_i \nu_m$ for some m. (See the proof for c.2.1) for detail.) So $\delta <_i \nu_{m+1}$.

$j<i$. The proof for c.2.1) goes through. In any case (1*) holds.
c.2.3) $a=0$ and $t=i$.

LEMMA (3) *If* δ *is connected and* $\delta<_i \nu$ $(=(i, 0, \alpha+1))$, *then* $\delta \leq_i (i, 0, \alpha)$.

Proof. Suppose $\delta=(k, b, \delta')<_i \nu$. If $k>i$ or $k=i$ and $b>0$, then $\nu<_{i+1}\delta$, hence by Lemma (1) (proven for c.2.1)) $\delta<_i (i, 0, \alpha)$. If $k=i$ and $b=0$, then $\delta<_i \nu$ implies $\delta'<_i \alpha+1$ or $\delta' \leq_i \alpha$. Therefore $\delta' \leq_i (i, 0, \alpha)$. If $k<i$, then $\delta<_i (i, 0, \alpha)$ is obvious.

Lemma (3) proves that (3*) holds for this case.
c.2.4) $a=0$, $j=t<i$ and $i_m \uparrow i$. Suppose $\delta<_t \nu=(i, 0, \alpha+1)$.

Case 1. $\nu=\text{apr}(0, j, \nu)$. Then $\mu=(i, 0, \alpha)=\text{apr}(0, j, \mu)$ (cf. Case 1 of 2.1 in the proof of Proposition 2.1). $\delta_0=\text{apr}(0, j, \delta)<_i \nu$. So by Lemma (3) above. $\delta_0 \leq_i (i, 0, \alpha)=\mu$. If $\delta_0<_i \mu$ or $\delta_0=\mu=\delta$, then $\delta<_j \nu_0$. Suppose $\delta_0=\mu\neq\delta$. Then apr $((0, 1), j, \delta)$ cannot exist and $v_2(j, \delta)<i$. So $v_2(j, \delta)<i_m<i$ for some m. It is evident that $\nu_m=\text{apr}((1, 1), j, \nu_m)$, $\text{apr}(0, j, \nu_m)=\mu$ and $v_2(j, \nu_m)=i_m$. So $\delta<_j \nu_m$.

Case 2. $\nu=\text{apr}((0, k+1), j, \nu)$ for some k. Then $\mu=(i, 0, \alpha)=\text{apr}((0, k+1), j, \mu)$ and $\text{apr}((0, k), j, \mu)=\text{apr}((0, k), j, \nu)$. Also $\text{apr}((0, k+1), j, \nu_m)=\mu$, $v_2(j, \nu_m)=i_m$ ($j<i_m$ is assumed.) and $\nu_m=\text{apr}((1, 1), j, \nu_m)$. Suppose $\delta<_j \nu$. Then either ν j-dominates δ at an early stage or $\text{apr}((0, k), j, \delta)=\text{apr}(0, k), j, \nu)$ and $\text{apr}((0, k+1), j, \delta)<_{i+1} \nu$ (cf. Proposition 3.1). If the former is the case, then $\delta<_j \mu$, hence $\delta<_j \nu_0$ is obvious. Suppose $\text{apr}((0, k), j, \delta)=\text{apr}((0, k), j, \nu)$ and $\text{apr}((0, k+1), j, \delta)$ exists. $\text{apr}((0, k+1), j, \delta)<_{i+1} \nu$ and $v_1(j, \nu)=v_1(j, \delta)=i$ imply that $\text{apr}((0, k+1), j, \delta)<_i \nu$. Therefore by Lemma (3) $\text{apr}((0, k+1), j, \delta)\leq_i \mu$, hence $\leq_{i+1}$. If $<_{i+1}$ holds, or $=$ and $\delta=\text{apr}((0, k+1), j, \delta)$, then $\delta<_j \nu_0$ is obvious. Suppose $\text{apr}((0, k+1), j, \delta)=\mu\neq\delta$. Then $v_2(j, \delta)<i_m$ for some m, hence $\delta<_j \nu_m$.

Case 3. $\nu=\text{apr}((n, k+1), j, \nu)$ for some $n>0$ and some k. Then $\mu=\text{apr}((n, k+1), j, \mu)$ and $\text{apr}((n, k), j, \mu)=\text{apr}((n, k), j, \nu)$. Also $\text{apr}((n, k+1), j, \nu_m)=\mu$. $v_{n+2}(j, \nu_m)=i_m$ and $\nu_m=\text{apr}((n+1, 1), j, \nu_m)$. ($j<i_m$ is assumed.) Suppose $\delta<_j \nu$. Suppose $\text{apr}((n, k), j, \delta)=\text{apr}((n, k), j, \nu)$ and $\text{apr}((n, k+1), j, \delta)<_{i+1} \nu$. Recall that $i=v_{n+1}(j, \nu)=v_{n+1}(j, \delta)$. Then $\text{apr}((n, k+1), j, \delta)<_i \nu$. So by Lemma (3) $\text{apr}((n, k+1), j, \delta)\leq_i \mu$, hence $\leq_{i+1}$. As for Case 2, now follows $\delta<_j \nu_m$ for some m.

c.2.5) $a=0$, $j<i$ and $i=i_0+1$. Since the requirement for (2.2) is that there be no $(i_0, 0)$-dominant of γ, the outermost indicator of every component of α in $<i$. Also (2′) is not the case. Therefore $\nu=\text{apr}(0, j, \nu)$ and the outermost $(i, 0)$ is the only j-active $(i, 0)$ (a greatest j-active value). Therefore $\mu=\text{apr}(0, j, \mu)$ and, if $\delta<_j \nu$, then $\delta_0=$

apr $(0, j, \delta) <_i \nu$. As for Case 1 of c.2.4) $\delta_0 \leq_i \mu$. Suppose $\delta_0 = \mu \neq \delta$. If $v_2(j, \delta) < i_0$, then $\delta <_j \nu_0$. Suppose $v_2(j, \delta) = i_0$. Put apr $((1, 1), j, \delta) = (p, b, \beta)$.

Suppose $(p, b) = (i, 0)$. Then $(i, 0, \beta) <_i \mu = (i, 0, \alpha)$, hence $\beta <_i \alpha$. It is not the case that $\alpha = \beta \# \beta'$ for some β', since β contains α. Therefore (ν) applies to β, α and i. Let $\beta = \beta_1 \# \cdots \# \beta_u \# \cdots \# \beta_{l_1}$ and $\alpha = \alpha_1 \# \cdots \# \alpha_u \# \cdots \# \alpha_{l_2}$ where the components are ordered in the non-increasing manner with regards to i. Here $\alpha_1 \# \cdots \# \alpha_{u-1} = \beta_1 \# \cdots \# \beta_{u-1}$ and $\beta_u <_i \alpha_u$. There is an l, $u \leq l \leq l_1$, such that μ is i_0-active in β_l. The outermost values of the components of β are $\leq (i_0, 0)$, since those of α are, so β_u is of the form $(i_0, 0, \beta')$, α_u is of the form $(i_0, 0, \alpha')$ and β_l is of the form $(i_0, 0, \beta'')$. $\beta_l <_i \beta_u <_i \alpha_u$ then implies that $\beta'' <_{i_0} \beta' <_{i_0} \alpha'$. But α' is i_0-active in β''. So $\alpha' \leq_{i_0} \beta''$. Therefore $\beta'' <_{i_0} \alpha'$ is impossible. Namely $p < i$, or $p = i_0$, must be the case.

1°. $(p, b) < (i_0, a_m)$ for some m, hence $(p, b, \beta) <_i \nu_m =$ apr $((1, 1), j, \nu_m)$, which implies $\delta <_j \nu_m$.

2°. Let η be an i_0-subsection of apr $((1, 1), j, \delta)$ which contains μ properly. Then η is of the form (i_0, c, η'). (Follow the proof for $p = i_0$ given as above.) We shall show that $\eta <_{i_0, i} \nu_m$ for some m for any such η. If each component of η' either omits μ or is μ, then $\eta' <_{i_0} \mu \# \cdots \# \mu$ for some number of μ's, hence $\eta <_{i_0, i} \nu_m$. Suppose each component of η' which contains μ properly is $<_{i_0, i} \nu_m$ for some m. We may assume that m is common to all those. Since μ is i_0-active in ν_m, a component of η' which omits $\mu <_{i_0} \nu_m$. Therefore $\eta' <_{i_0} \nu_m$, hence $\eta <_{i_0, i} \nu_{m+1}$.

It is easily seen that apr $(0, j, \nu_m) = \mu$, $v_2(j, \nu_m) = i_0$ and $\nu_m =$ apr $((1, 1), j, \nu_m)$. So apr $((1, 1), j, \delta) <_i \nu_m$, which implies $\delta <_j \nu_m$.

In any case (1*) holds for c.2.4) and c.2.5).

(4.2) $\nu =$ apr $((0, k+1), t, \nu) = (i, a, \alpha)$ and $\alpha =$ apr $((0, k), t, \nu)$ (cf. Proposition 1.10). Therefore if $\delta <_t \nu$ for a connected δ, then ν t-dominates δ at an early stage (cf. Definition 3.1). For this case we state a little lemma.

Lemma. apr $((0, k), t, \nu_m) =$ apr $((0, k), t, \nu)$ $(= \alpha)$ and $\nu_m =$ apr $((0, k+1), t, \nu_m)$ (d.1) and d.2)), $\nu_m =$ apr $((1, 1), t, \nu_m)$ (d.3)), $\nu_m = \alpha \# \cdots \# \alpha$ or $\nu_m = \alpha \# \cdots \# \alpha \# \rho_m$ where the indicator of $v_0(t, \rho_m) < i$ (d. 4)).

So if apr $((0, k), t, \delta) =$ apr $((0, k), t, \nu)$ $(= \alpha)$ and apr $((0, k+1), t, \delta)$ $(= (i, c, \beta))$ exists, then $c = v_{(0, k+1)}(t, \delta) < a$.

d.1) If 1) or 2) of Definition 3.1 is the case, then $\delta <_t \nu_0$ is obvious. Suppose 3.1) is the case.

There is an m such that $c < a_m$. Since $\nu_m =$ apr $((0, k+1), t, \nu_m)$, $\delta <_t \nu_m (= \rho_m)$. (1*).

d.2) Suppose 3.1) is the case. $c < a$ means $c \leq b$. $\nu_m =$ apr $((0, k+1), t, \nu_m)$. So, if $c < b$, then $\delta <_t \nu_0$. Suppose $c = b$. If we can claim that

(*) for every t-subsection of apr $((0, k+1), t, \delta)$, say η, which contains apr $((0, k), t, \delta)$ $(=\alpha)$ properly, there is an m (depending on η) such that $\eta<_{i,i+1}\nu_m$.

Then as a special case apr $((0, k+1), t, \delta)<_i\nu_m$ for some m, hence $\delta<_t\nu_m$. (1*).

Proof of (*). Let $\eta=(i, d, \eta')$ where $d\leq b$. If each component of η' either omits α or is α, then $\eta'<_i\alpha\#\cdots\#\alpha$. So $\eta<_i\nu_m$. If a component of η' contains an occurrence of α properly, then there is an m such that $\eta'<_i\nu_m$. Then it is obvious that $\eta<_{i,i+1}\nu_{m+1}$.

d.3) In this case, if apr $((0,k),t,\delta)=$ apr $((0,k),t,\nu)$, then apr $((0,k+1), t, \delta)$ cannot exist, for $\nu=(i, 0, \alpha)$ and $\alpha=$ apr $((0, k), t, \nu)$. So 1) or 2) of Definition 3.1 applies. If apr $((m, l), t, \delta)<_t$ apr $((m, l), t, \nu)$ for some $(m, l)\leq(0, k)$, then $\delta<_t\nu_0$. Suppose apr $((0, k), t, \delta)=$ apr $((0, k), t, \nu)=\alpha$. Let $p=v_2(t, \delta)$. There exists an m such that $p<i_m$. $\nu_m=$ apr $((1, 1), t, \nu_m)$ and $i_m=v_2(t, \nu_m)$. So $\delta<_t\nu_m$. (1*).

d.4) Since $t=i$, there cannot be apr $((1, 1), t, \delta)$. So if apr $((0, k), t, \delta)=\alpha$, then the only possibility is that $\delta=\alpha$. So (2*) holds.

(5.1.2) $\nu=$ apr $((n, k+1), t, \nu)=(i, a, \alpha)$, $a\neq0$, $\alpha=0$ and $t=i$, or $a=0$, $t<i$ and $i_m\uparrow i$. $i=v_{n+1}(t, \nu)$ and $\alpha=$ apr $((n, k), i, \nu)$. Suppose $\delta<_t\nu$.

e.1) If ν t-dominates δ at an early stage, then $\delta<_t\nu_0$, since apr $((n, k), t, \nu_0)=\alpha$ and $v_{n+1}(t, \nu_0)=i$. Suppose apr $((n, k+1), t, \delta)$ exists, apr $((n, k), t, \delta)=\alpha$ and $v_{n+1}(t, \nu_0)=i$. Let apr $((n, k+1), t, \delta)=(p, c, \delta')$. Then $p\geq i$, $(p, c, \delta')<_{i+1}(i, a, \alpha)$, so $p=i$ and $c\leq a$. If $c=a$, then $\delta'<_i\alpha$ must hold, but δ' contains α as an i-subsection. So $c<a$. There is an m such that $c<a_m$. $\delta<_t\nu_m$. (1*).

e.2) If ν t-dominates δ, then $\delta<_t\nu_0$. Suppose apr $((n, k+1), t, \delta)$ exists, apr $((n, k), t, \delta)=\alpha$ and $v_{n+1}(t, \delta)=i$. Let apr $((n, k+1), t, \delta)=(i, c, \delta')$. Then $c<a$, viz. $c\leq b$. If $c<b$, then $\delta<_t\nu_0$. If $c=b$, then we can prove a statement similar to the (*) in d.2): Let η be any t-subsection of apr $((n, k+1), t, \delta)$ which properly contains α. Then there is an m such that $\eta<_{i,i+1}\nu_m$. (we ought to remark that if $\eta=(q, d, \eta')$ contains an i-active α, then $q=i$.) In particular $(i, c, \delta')<_{i+1}\nu_m$ for some m. This implies $\delta<_t\nu_m$. (1*).

e.3) If 1) or 2) of Definition 3.1 with $(m, l)<(n, k)$ or 2) with $(m, l)=(n, k)$ and $\delta=$ apr $((n, k), t, \delta)$, then $\delta<_t\nu_0$. Suppose apr $((n, k), t, \delta)=\alpha$ and $\delta\neq$ apr $((n, k), t, \delta)$. Note that apr $((n, k+1), t, \delta)$ does not exist.

Case (1) $k=0$. Let $v_{n+1}(t, \delta)=p$. Then $p<i$, hence $p<i_m=v_{n+1}(t, \nu_m)$ for some m. $\delta<_t\nu_m$ (1*)

Case (2). $k>0$. Consider spr $((n+1, 1), t, \delta)$ (which exists under the circumstances) and let $p=v_{n+2}(t, \delta)$. $p<i$. There is an m such that $p<i_m$. $v_{n+2}(t, \nu_m)=i_m$, hence $\delta<_t\nu_m$. (1*).

e.4) $a=0$ and $t=i$. Suppose apr $((n, k), t, \delta)=\alpha$ and apr $((n, k+1), t, \delta)$ $(=(p, b, \delta'))$ exists. Since $\nu=(i, 0, \alpha)$, $p<i$; but then α would not be

t-active in δ, contradicting the situation. So ν dominates δ at an early stage, and either $\delta <_t \alpha$. (when δ omits α) or $\delta = \alpha$. (2*) holds.

B. (j, γ) is not the last reduction pair (but δ is the last cmf).

(2.1;1) (2.1;1.a) $\gamma = (i, 0, \alpha+1) = \mathrm{apr}(0, j, \gamma)$. $\mu = (i, 0, \alpha) = \mathrm{apr}(0, j, \mu)$. Let (i, ρ) be the next sp. ρ is an i-least $(i_0, 0)$-dominant of γ. $\{\rho_m\}_m$ has been defined so as to satisfy Proposition 2.1. See also Lemma 1 for (2.1) in the proof of Proposition 2.1.

Suppose $\delta <_i \gamma$. As in Case 1 of c.2.4), $\delta_0 = \mathrm{apr}(0, j, \delta) \leq_i \mu$. If $\delta_0 <_i \mu$ or $\delta_0 = \mu = \delta$, then $\delta <_j \gamma_0$ is obvious. When $\delta_0 = \mu$, $\mathrm{apr}((0, 1), j, \delta)$ cannot exist and $v_2(j, \delta) < i$. If $v_2(j, \delta) < i_0$, then $\delta <_j \gamma_0$. Now consider the case where $\delta_0 = \mu \neq \delta$ and $v_2(j, \delta) = i_0$. Let $\mathrm{apr}((1, 1), j, \delta) = (p, b, \beta)$. If $(p, b) < (i, 0)$, then $\delta <_j \gamma_0$. So let us assume that $(p, b, \beta) = (i, 0, \beta)$. $(i, 0, \beta) <_i \mu = (i, 0, \alpha)$ must hold, so $\beta <_i \alpha$. As for c.2.5), if we let $\beta = \beta_1 \# \cdots \# \beta_{l_1}$ and $\alpha = \alpha_1 \# \cdots \# \alpha_{l_2}$, then $\alpha_1 \# \cdots \alpha_{u-1} = \beta_1 \# \cdots \# \beta_{u-1}$ and $\beta_u <_i \alpha_u$ for some u, and there is an $l \geq u$ such that μ is i_0-active in β.

The lemma for (2.1;1.a) in the proof of Proposition 3.2 claims that

$$(*) \qquad \text{if} \quad \rho = \alpha_w, \quad \text{then} \quad w \geq u .$$

By (*), if $w > u$, theu $\alpha_1, \cdots \alpha_u$ remain invariant in $\alpha\binom{\rho}{\rho_m} \# (i_0, 0, \mu)$, so $(\beta <_i \alpha\binom{\rho}{\rho_0} \# (i_0, 0, \mu)$, and hence $(i, 0, \beta) <_i \gamma_0$, or $\delta <_j \gamma_0$.

(2.1;1.b) $\gamma = \mathrm{apr}((0, k+1), j, \gamma)$. $\mathrm{apr}((0, k), j, \mu) = \mathrm{apr}((0, k), j, \gamma)$ and $\mu = \mathrm{apr}((0, k+1), j, \mu)$. Suppose $\delta <_j \gamma$. As for the preceding cases, only the case where $\mathrm{apr}((0, k+1), j, \delta) = \mu \neq \delta$ and $v_2(j, \mu) = i_0$ matters (cf. Case 2 of c.2.4)). Let $\mathrm{apr}((1, 1), j, \delta) = (p, b, \beta)$. If $p = i_0$, then $\delta <_j \gamma_0$. So suppose $p = i$.

Case (3) $(p, b) = (i, 0)$ and $w > u$. Then $\beta <_i \alpha_0 \# (i_0, 0, \mu)$ and $\delta <_j \gamma_0$.

Case (4.2) $p = i, b > 0$ and $w > 1$. Let α^* be a component of α in which $\mathrm{apr}((0, k), j, \mu)$ is i-active. (Such a component exists.) If in α_1 (an i-greatest component of α) $\mathrm{apr}((0, k), j, \mu)$ is not i-active, then α_1 omits $\mathrm{apr}((0, k), j, \mu)$, hence $\alpha_1 <_i \mathrm{apr}((0, k), \mu) \leq_i \alpha^*$, contradicting the fact that α_1 is an i-greatest component. So $\mathrm{apr}((0, k), j, \mu)$ is i-active in α_1. We claim that

$$(**) \qquad \mathrm{apr}(1, 1), j, \delta) <_i \mathrm{apr}((0, k) j, \mu) .$$

From (**) $\mathrm{apr}((1, 1), j, \delta) <_i \alpha_1$, hence $\mathrm{apr}((1, 1), j, \delta) <_i \alpha\binom{\rho}{\rho_0} \# (i_0, 0, \mu)$, and $\mathrm{apr}((1, 1), j, \delta) <_i \gamma_0$ is obvious. Therefore $\delta <_j \gamma_0$.

For (**), note that $\mu = (i, 0, \alpha) = \mathrm{apr}((0, k+1), j, \mu)$. Therefore $\mathrm{apr}((0, k), j, \mu)$ is not i-active in $\mathrm{apr}((1, 1), j, \delta)$. Using this fact and following the proof of the Lemma for (2.1;1.b) in Proposition 3.2, we can conclude (**).

(2.1;1.c) $\gamma = \mathrm{apr}((n, k+1), j, \gamma)$ where $n > 0$. $\mu = \mathrm{apr}((n, k+1), j, \mu)$. Suppose $\delta <_j \gamma$. Let us consider the case where $\mathrm{apr}((n, k+1), j, \delta) =$

$\mu \neq \delta$ and $v_{n+2}(j, \delta)=i_0$. Let apr $((n+1, 1), j, \delta)=(p, b, \beta)$. If $p=i_0$, then $\delta <_j \gamma_0$. So suppose $p \geq i$.

Case (3) $(p, b)=(i, 0)$ and $w>u$. $\delta <_j \gamma_0$.

Case (4.2) $(p, b)>(i, 0)$ and $w>1$. In α_1, an i-greatest component of α, apr $((n, k), j, \mu)$ is i-active. We claim that

(***) $$\text{apr}((n+1, 1), j, \delta) <_i \text{apr}((n, k), j, \mu) .$$

From (***) apr $((n+1, 1), j, \delta) <_i \alpha_1$ hence apr $((n+1, 1), j, \delta) <_i \gamma_0$, or $\delta <_j \gamma_0$.

For (***), note that $\mu=(i, 0, \alpha)=$ apr $((n, k+1), j, \mu)$ and $i=v_{n+1}(j, \delta)$. Therefore apr $((n, k), j, \mu)$ is not i-active in apr $((n+1, 1), j, \delta)$. Using this fact and following the proof of the Lemma for (2.1;1.c) in Proposition 3.2, we can conclude (**).

(1*) for those cases.

(2.1;2) The proof for c.2.3) goes through and (3*) holds for this case.

For this case the induction step is irrelevant. The same comment for (4.1;2) and (5.1.1;2).

(3.1) Case (1) $v_0(j, \delta) < v_0(j, \gamma)=(i, a)$. (Recall that $j>i$.) γ_m contains a component of the form (i, a, α_m) (cf. Proposition 1.13), and $v_0(j, (i, a, \alpha_m))=(i, a)$. So $\delta <_j (i, a, \alpha_0)$.

(3.2) Case (1) $v_0(j, \delta) < v_0(j, \gamma)=(i, 0)$. $v_0(j, \gamma_m)=(i, 0)$. So $\delta <_i \gamma_0$.

(3.3) [1°] $\gamma_m=\kappa_m \# \mu_m$ for some κ_m, where $\mu_m=(t, b, \lambda \# \nu_m)$.

Case (1) $v_0(j, \delta) < (t, b)=v_0(j, \mu_m)$. So $\delta <_j \mu_0$. (If $j \leq t$, the maximum, j-active value of $\mu_m \leq$ that of $\gamma=(t, b)$ (cf. Proposition 2.1).) Therefore put $\rho_m=\mu_m$ and $m=0$.

Case (2) (ω) $\gamma=(t, b, \lambda \# \nu)$. $\lambda=\delta' \# \lambda'$ for some δ' and λ', where δ' is not empty while λ' may be empty, and apr $(0, j, \delta)=(t, b, \delta')$. $\gamma_0= \kappa_0 \# \mu_0$ where $\mu_0=(t, b, \lambda \# \nu_0)$. Therefore $(t, b, \delta') <_t \mu_0$

Since $v_0(j, \mu_0) \leq v_0(j, \gamma)=(t, b)$ (cf. Proposition 2.1), $\mu_0 \leq_t$ apr $(0, j, \mu_0)$. So apr $(0, j, \delta) <_t$ apr $(0, j, \mu_0)$, hence $\delta <_j \mu_0$ Put $\rho_0=\mu_0$.

(θ.1) $\delta <_j \mu_0=\rho_0$.

(3.3) [2°] Case (1) $v_0(j, \delta) < (i, a)=v_0(j, \gamma_0)$. So $\delta <_j \gamma_0$.

Case (2) (ω) apr $(0, j, \delta)=(i, a, \delta')$, where $\alpha=\delta' \# \alpha' \# \gamma'$; $\gamma_0=(i, a, \delta' \# \alpha' \# \gamma'_0)$. So apr $(0, j, \delta) <_i \gamma_0 \leq_i$ apr $(0, j, \gamma_0)$ (since the j-active values of $\gamma_0 \leq (i, a)$; cf. Proposition 2.1). So $\delta <_j \gamma_0$.

(θ.1) $\delta <_j \gamma_0$; proved as for (ω).

(4.1;1) $\alpha=$ apr $(0, k), j, \gamma_m)$ (Proposition 2.2).

Case (1) $\delta <_j \gamma_0$.

Case (2) apr $((0, k), j, \delta)=$ apr $((0, k), j, \gamma_0)=\alpha$. $v_2(j, \delta) < i_0=v_2(j, \gamma_0)$. So $\delta <_j \gamma_0$.

(4.1;2) The proof for d.4) goes through. (2*) holds.

(4.3) [1°] $\gamma_m = \kappa_m \# \mu_m$ where $\mu_m = (t, b, \lambda \# \nu_m)$.

Case (1) apr $((0, k), j, \mu_m) =$ apr $((0, k), j, \gamma)$ (Proposition 2.2) and $v_{(0,k+1)}(j, \mu_m) = b$. So $\delta <_j \mu_0$, hence put $\rho_0 = \mu_0$.

Case (2) (ω) See [1°] of (3.3). apr $((0, k+1), j, \delta) <_t \mu_0$ and $\mu_0 \leq_t$ apr $((0, k+1), j, \mu_0)$.

(θ.1) $\delta <_j \mu_0$.

(4.3) [2°] See [2°] of (3.3).

(5.1.1;1) Case (1) $\alpha =$ apr $((n, k), j, \gamma_m)$. So $\delta <_j \gamma_0$.

Case (2) apr $((n, k), j, \delta) =$ apr $((n, k), j, \gamma_0) = \alpha$ and $v_{n+1}(j, \delta) < v_{n+1}(j, \gamma_0)$ $(= i_0)$ or $v_{n+2}(j, \delta) < v_{n+2}(j, \gamma_0)$ $(= i_0)$ (as the case may be). So $\delta <_j \gamma_0$.

(5.1.1;2) The proof for e.4) goes through. (2*) holds.

(5.2) [1°] Case (1) apr $((n, k), j, \mu_m) =$ apr $((n, k), j, \gamma)$ and $v_{n+1}(j, \mu_m) = t$ (cf. Propositions 2.2 and 2.4). So $\delta <_j \mu_0$.

Case (2.1) $\delta <_j \mu_0$.

Case (2.2) (ω) $\delta <_j \mu_0$.

(θ.1) $\delta <_j \mu_0$.

(5.2) [2°] See [2°] of (3.3).

(5.3) and (2′) Case (1) or Case (v.1) Let ρ_m be a component of γ_m for which apr $((n, k), j, \rho_m) =$ apr $((n, k), j, \gamma)$. Then $\delta <_j \mu_0$.

§ 3.4. Proposition 3.3—Induction step.

Let δ be the cmf corresponding to (j, γ). Then we show Proposition 3.3 for δ and (j, γ), assuming the same for the next cmf.

(2.1;1.a) Case (3) (ν.2) $u = w$. $\beta_u <_i \alpha_u$ $(= \eta)$. By the induction hypothesis, $\beta_u <_i \rho_m$ for some m where ρ_m is a connected component of η_m, hence $\beta <_i \alpha\binom{\eta}{\eta_m} \# (i_0, 0, \mu)$ (when (1*) applies to (i, η)), or $\beta_u \leq_i \rho$ and $\eta_m = \rho \# \cdots \# \rho \# \rho_m$, hence $\beta_u \# \beta_{u+1} \# \cdots \# \beta_{l_1} <_i \eta_m$ for some m and $\beta <_i \alpha\binom{\eta}{\eta_m} \# (i_0, 0, \mu)$ (when (2*) or (3*) applies). In any case apr $((1, 1), j, \delta) <_i \gamma_m$ (= apr $((1, 1), j, \gamma_m)$). So $\delta <_j \gamma_m$.

(2.1;1.b) Case (3) (ν.2) See above.

Case (4.1) As for Case (4.2), apr $((1, 1), j, \delta) <_i$ apr $((0, k), j, \mu) <_i \alpha_1$ $(= \eta)$. In a manner similar to (2.1;1.a) above, we can claim by the induction hypothesis that apr $((1, 1), j, \delta) <_i \eta_m$ for some m, hence apr $((1, 1), j, \delta) <_i \gamma_m$, or $\delta <_j \gamma_m$.

(2.1;1.c) Case (3) (ν.2) See (2.1;1.a)

Case (4.1) See (2.1;1.b).

In any case (1*) holds.

(3.1) and (3.2) Case (2). As for (2.1), $\beta_1 <_i \alpha$ implies $\beta <_i \alpha_m$ for some m. Therefore $(i, a, \beta)<_i (i, a, \alpha_m)$. Also $\mathrm{apr}((0, j, (i, a, \alpha_m))=(i, a, \alpha_m)$ (since $j>i$). So $\delta<_j (i, a, \alpha_m)<_j \gamma_m$.

(3.3) [1°] Case (2) (ν.2) $\gamma=(t, b, \lambda \# \nu)$ and the next sp is (t, ν). By the induction hypothesis applied to β_u and (t, ν), $\beta<_t \lambda\#\nu_m$ for some m. This is proved as for (2.1;1). So $\mathrm{apr}(0, j, \delta)<_t \mu_m=(t, b, \lambda \# \nu_m)$. On the other hand $\mathrm{apr}(0, j, \mu_m)\ {}_t{\geq} \mu_m$ (cf. Case (2) (ω) of (3.3) [1°]), hence $\mathrm{apr}(0, j, \delta)<_t \mathrm{apr}(0, j, \mu_m)$ and $\delta<_j \mu_m$, where $\gamma_m=\kappa_m \# \mu_m$.

[2°] Case (2) (θ.2) As in the preceding cases $\beta<_i \alpha_m$ for some m. $(i, a, \alpha_m)<_j (i, a, \alpha)$ (Proposition 2.1), so $v_0(j, \gamma_m)=(i, a)$ and $(i, a, \alpha_m)\leq_i \mathrm{apr}(0, j, \gamma_m)$. So $\mathrm{apr}(0, j, \delta)=(i, a, \beta)<_i (i, a, \alpha_m)\leq_i \mathrm{apr}(0, j, \gamma_m)$, which implies $\delta<_j \gamma_m$.

(4.1;1) Case (3) $\mathrm{apr}((1, 1), j, \delta)$ $(=\delta_{(1,1)})<_i \alpha_m$ for some m, be the induction hypothesis. $\delta_{(1,1)}<_i \alpha_m<_i \gamma_m$ is obvious. So $\delta_{(1,1)}<_i \gamma_m$, from which follows $\delta<_j \gamma_m$, since $\gamma_m=\mathrm{apr}((1, 1), j, \gamma_m)$ and $v_2(j, \gamma_m)=i_0$ where $i=i_0+1$.

(4.3) [1°] Case (2) (θ.2) The induction hypothesis holds for β_u and ν, so $\beta<_t \lambda \# \nu_m$ some m. So $(t, b, \beta)<_{t+1} (t, b, \lambda \# \nu_m)$. $\mathrm{apr}((0, k), j, \gamma_m)=\mathrm{apr}((0, k), j, \gamma)$ and $\mu_m=(t, b, \lambda \# \nu_m)\leq_{t+1} \mathrm{apr}((0, k+1), j, \mu_m)$ (cf. Proposition 2.3). So $\delta<_j \mu_m$.

[2°] Case (2) (θ.2). See [1°].

(5.1.1;1) Case (3) Let us consider the case where $k>0$, induction hypothesis $\delta_{(n+1,1)}<_i \alpha_m$ for some m. So $\delta_{(n+1,1)}<_i \gamma_m$ (for $\alpha_m<_i \gamma_m$). From this follows $\delta<_j \gamma_m$, since $\gamma_m=\mathrm{apr}((n+1, 1), j, \gamma_m)$ (Proposition 2.4) and $v_{n+2}(j, \gamma_m)=i_0$, where $i=i_0+1$. The case where $k=0$ can be treated similarly.

(5.2) [1°] (θ.2) From the induction hypothesis $\beta<_i \alpha_m$ for some m, so $(i, a, \beta)<_{i+1} (i, a, \alpha_m)=\mu_m$. $\mu_m=\mathrm{apr}((n, r+1), j, \mu_m)$ for some $r\geq k$ (cf. Proposition 2.4), and hence $\mathrm{apr}((n, k+1), j, \delta)<_{i+1} \mu_m\leq_{i+1} \mathrm{apr}((n, k+1), j, \mu_m)$. So $\delta<_j \mu_m$.

[2°] This can be dealt with in a manner similar to [1°]; use Proposition 2.4.

(5.3) or (2′) Case (2) By the inductive hypothesis either (1*) $\mathrm{apr}((n, k+1), j, \delta)<_j \rho_m$ for some ρ_m a component of γ_m, or (2*) $\mathrm{apr}((n, k+1), j, \delta)\leq_{j_1} \rho$ and $\gamma_m=\rho \# \cdots \# \rho \# \rho_m$, or $\mathrm{apr}((n, k+1), j, \delta)\leq_{j_1} (i, 0, \alpha)$ where $\gamma=(i, 0, \alpha+1)$ and $j_1=i$ (c.2.3.)).

Suppose (1*) is the case. Here $j_1=i_{(n+1)}+1$, $\mathrm{apr}((n, k), j, \delta)=\mathrm{apr}((n, k), j', \gamma)$ and $v_{n+1}(j, \delta)=v_{n+1}(j, \gamma)=i_{n+1}$. ρ_m is either γ_m (where γ_m is connected) or μ_m. This can be confirmed by going over the foregoing part of this proof. In either case $\rho_m=\mathrm{apr}((n, r+1), j, \rho_m)$ for some $r\geq k$, since the γ_m for j and the one for j_1 are the same (cf. Definition 1.4 and Proposition 2.4). Therefore $\mathrm{apr}((n, k+1), j, \delta)<_{j_1} \rho_m\leq_{j_1} \mathrm{apr}((n, k+1), j, \rho_m)$, which implies $\delta<_j \rho_m$.

Suppose (2*) is the case. Then $\gamma_m=\alpha \# \cdots \# \alpha \# \rho_m$ where $\gamma=$

(i, a, α). From the conditions on (j, γ), b.3), d.4.2) and e.4.2) can be eliminated immediately, for the indicators of the tsp's for (j, γ) and if i is a limit indicator, then $j_{u_0}<i$, hence d.4.1) and e.4.1) are also eliminated. Therefore only (4.1;2) and (5.1.1;2) are left for (j_{u_0}, γ). Namely $j_{u_0}=i=i_0+1$, $a=0$, α is connected and $\rho_m=(i, 0, \alpha_m \# (i_0, 0, \alpha))$.

Suppose first that (4.1;2) applies to (j_{u_0}, γ) and $u_0=1$. Then $v_{n+1}(j, \gamma)=i_0$, and $\text{apr}((n, k+1), j, \delta)<_j \gamma$. Put $\tilde{\delta}=\text{apr}((n, k+1), j, \delta)$. By the induction hypothesis $\tilde{\delta}\leq_i \alpha$. Since $i>v_{n+1}(j, \gamma)$, α properly contains $\text{apr}((n, k), j, \gamma)=\rho$. So $\alpha=\text{apr}((n, q+1), j, \alpha)$ for some $q\geq k$. We claim that $\rho_m=\text{apr}((n, q+2), j, \rho_m)$ and $\text{apr}((n, q+1), j, \rho_m)=\alpha$. Then, $\tilde{\delta}<_i \alpha$ implies $\tilde{\delta}<_i \text{apr}((n, k+1), j, \rho_m)$, so $\delta<_j \rho_0$ when $\tilde{\delta}<_i \alpha$.

In order to establish the claim, it suffices to show that $\rho_m<_i \alpha$, for $\alpha_m<_{j,i} \alpha$ and $(i_0, 0, \alpha)<_i \rho_m$. Since (4.1;2) applies to (i, γ), $\text{apr}((0, s+1), i, \gamma)$ and $\alpha=\text{apr}((0, s), i, \gamma)$. So $\alpha=(i, b, \alpha')$ for some $b>0$. Therefore $\rho_m<_{i+1} \alpha$ and $\alpha_m \# (i_0, 0, \alpha)<_i \alpha$. So $\rho_m<_i \alpha$.

Next suppose $\tilde{\delta}=\alpha$. Then $q=k$. If there is no $\text{apr}((n, k+2), j, \delta)$, then this immediately implies $\tilde{\delta}<_j \rho_0$. So, suppose $\text{apr}(n, k+2), j, \delta)=\delta^*$ exists. $\delta^*<_i \alpha$ $(=\tilde{\delta})$. By the induction hypothesis $\delta^*<_i \alpha_m$ for some m, hence $\delta^*<_i \rho_m$. This implies that $\delta<_j \rho_m$ for some m. In any case (1*) holds.

Next suppose $u_0>1$ and consider (j_v, γ) where $v+1<u$. Suppose $\delta=\text{apr}((n, k+1), j_v, \gamma)$ and $\delta<_{j_v} \gamma$. This and the condition for this case imply that $\text{apr}((n, k+1), j_v, \delta)<_{j_{v+1}} \gamma$. Therefore by the induction hypothesis $\tilde{\delta}=\text{apr}((n, k+1), j_v, \delta)<_{j_{v+1}} \rho_m$ for some m. If we can establish that $\rho_m=\text{apr}((n, q+1), j_v, \rho_m)$ and $\text{apr}((n, k), j_v, \rho_m)=\text{apr}((n, k), j_v, \tilde{\delta})$ for some $q\geq k$, then $\tilde{\delta}<_{j_{v+1}} \text{apr}((n, k+1), j_v, \rho_m)$. So $\delta<_{j_v} \rho_m$.

For the desired relations, note that as above α properly contains $\text{apr}((n, k), j_v, \gamma)=\rho$. It is j_v-active in ρ_m and $\rho_m<_{j_{v+1}} \gamma$ can be easily established. On the other hand $\gamma<_{j_{v+1}} \rho$. So $\rho_m<_{j_{v+1}} \rho$. $(i_0, 0, \alpha)<_{j_{v+1}} \gamma<_{j_{v+1}} \rho$. $\alpha_m<_{j,i} \alpha$, hence in particular $\alpha_m<_{j_{v+1}} \alpha<_{j_{v+1}} \rho$. Those inequalities guarantee that $\rho_m=\text{apr}((n, q+1), j_v, \rho_m)$ for some $q\geq k$ and $\text{apr}((n, k), j_v, \rho_m)=\text{apr}((n, k), j_v, \tilde{\delta})=\rho$.

In any case (1*) holds.

Next, let us consider the case where (5.1.1;2) applies. Suppose first that $u=1$. $\tilde{\delta}\leq_i \alpha$ and $\alpha=\text{apr}((n, q+1), j, \alpha)$ for some $q\geq k$. If we can claim that $\rho_m=\text{apr}((n, q+2), j, \rho_m)$ and $\text{apr}((n, q+1), j, \rho_m)=\alpha$, then the proof for (4.1;2) goes through.

Since (5.1.1;2) applies to (i, γ), $\gamma=\text{apr}((r, s+1), i, \gamma)$ and $\alpha=\text{apr}((r, s), i, \gamma)$. $i=v_{r+1}(i, \gamma)$. So $\alpha=(p, b, \alpha')$ for some $p\geq i$ and $\gamma<_{i+1} \alpha$. This means that either $(p, b)>(i, 0)$, or $(p, b)=(i, 0)$ and $\alpha<_i \alpha'$. But the latter is impossible. So $(d, b)>(i, 0)$, hence $\rho_m<_{i+1} \alpha$. From this follows $\rho_m<_i \alpha$, and hence the claim.

Finally, suppose (3*) is the case. Then $\gamma_m=\mu \# \cdots \# \mu \# \rho_m$ where

$\gamma=(i, a, \alpha+1)$ and $\mu=(i, a, \alpha)$, c.1) can be eliminated. Let (j_{u_0}, γ) be the next sp. Then either 2° or 3° of c.2.3) or (2.1;2) applies to (j_{u_0}, γ). In any of those cases, $j_{u_0}=i=i_0+1$.

Suppose first that (2.1;2) applies to (j_{u_0}, γ) and $u_0=1$. Let $\mathrm{apr}((n, k+1), j, \delta)=\tilde{\delta}$. Then $\tilde{\delta}<_{j_1}\gamma$. By the induction hypothesis $\tilde{\delta}\leq_i \mu$. $\rho_m=(i, 0, \alpha_m \# (i_0, 0, \mu))$. We shall first show that $\rho_m= \mathrm{apr}((n, k+2), j, \rho_m)$ and $\mathrm{apr}((u, k+1), j, \rho_m)=\mu$. Here $i_0=v_{n+1}(j, \gamma)$, $\mu=\mathrm{apr}((n, k+1), j, \mu)$, $\alpha_m<_{j,i}\alpha$ and $\alpha_m \# (i_0, 0, \mu)<_i \alpha$. Therefore it suffices to show that $\rho_m<_i \mu$. But this follows immediately from $\alpha_m \# (i_0, 0, \mu)<_i \alpha$.

Now suppose $\tilde{\delta}<_i \mu$. Then $\delta<_j \rho_0$ is obvious. So let us assume that $\tilde{\delta}=\mu$. If $\mathrm{apr}((n, k+2), j, \delta)$ does not exist, then $\delta<_j \rho_0$ is obvious. So suppose that $\delta^*=\mathrm{apr}((n, k+2), j, \delta)$ exists. Let $\delta^*=(p, b, \beta)$. $\delta^*<_i \mu$ and $p\geq i_0$. We shall show that $\delta^*<_i \rho_m$ for some m. Then follows $\delta<_j \rho_m$.

Case 1. $p=i_0$. $\delta^*<_i \rho_0$ is obvious.

For the subsequent cases, notice that if $(p, b)=(i, 0)$, then $\beta<_i \alpha$.

Case 2. $(p, b)=(i, 0)$ and $w>u$ (cf. Definition 3.3 for w and u). Then $\beta<_i \alpha$ implies $\beta<_i \alpha_0 \# (i_0, 0, \mu)$, hence $\delta^*<_i \rho_0$.

Case 3. $(p, b)=(i, 0)$ and $w=u$. By the induction hypothesis $\beta<_i \alpha_m$ for some m, hence $\delta^*<_i \rho_m$.

Case 4. $(p, b)>(i, 0)$. For this case $\mu<_{i+1}(p, b, \beta)$. So in order that $(p, b, \beta)<_i \mu$, $(p, b, \beta)\leq_i \alpha$. But α is a part of (p, b, β), so $(p, b, \beta)<_i \alpha$. Therefore $(p, b, \beta)<_i \alpha_m$ for some m, hence $\delta^*<_i \rho_m$.

Next suppose $u_0>1$ and consider (j_v, γ) where $v+1<u_0$. Suppose $\gamma=\mathrm{apr}((n, k+1), j_v, \gamma)$ and $\delta<_{j_v}\gamma$. As for (2*) above, it suffices to show that $\rho_m=\mathrm{apr}((n, q+1), j_v, \rho_m)$ for some $q\geq k$. $\mu=\mathrm{apr}((n, k+1), j_v, \mu)$ and $v_{n+1}(j_v, \mu)<i_0$. $\alpha_m<_{j_v,i}\alpha$, hence $\mathrm{apr}((n, k), j_v, \rho_m)=\mathrm{apr}((n, k), j_v, \mu)$ and this is j_v-active. So the result.

If c.2.3) applies to (j_{u_0}, γ), then $j_{u_0}=i$, $\gamma_m=\mu \# \cdots \# \mu \# \rho_m$ and either $\rho_m=(i_0, a_m, \mu)$ or $\rho_m=(i_0, e, \cdots, (i_0, e, \mu \# \cdots \# \mu) \cdots)$.

If $u_0=1$, then $j_1=i$, $v_{n+1}(j, \gamma)=i_0$, $\tilde{\delta}=\mathrm{apr}((n, k+1), j, \delta)<_{j_1}\gamma$, and hence $\tilde{\delta}\leq_i \mu$. It is obvious that $\rho_m=\mathrm{apr}((n, k+2), j, \rho_m)$ and $\mathrm{apr}((n, k+1), j, \rho_m)=\mu$. If $\tilde{\delta}<_i \mu$ or $\tilde{\delta}=\mu$ and there is no $\mathrm{apr}((n, k+2), j, \delta)$, then $\delta<_j \rho_0$. Suppose $\tilde{\delta}=\mu$ and $\delta^*=\mathrm{apr}((n, k+2), j, \delta)$ exists. $\delta^*<_i \mu$. Put $\delta=(p, b, \beta)$. Suppose $(p, b)>(i, 0)$. Then in order that $\delta^*<_i \mu$, $\delta^*\leq_i \alpha$. But there is no $(i_0, 0)$-dominant of α, so this is impossible. Therefore $(p, b)\leq(i, 0)$. If $(p, b)=(i, 0)$, then $\beta<_i \alpha$ must hold, but a contradiction can be deduced from this as for c.2.5) in this proof. So $p=i_0$ (since $v_{n+1}(j, \gamma)=v_{n+1}(j, \delta)=i_0$). For 2°, $b<a_m$ for some m, hence $\delta<_j \rho_m$. For 3°, let η be an arbitrary i_0-subsection of δ^* which contains μ properly. Then η is of the form (i_0, c, η'). As for

Case 2° of c.2.5), we can show that $\eta <_{i_0, i} \rho_m$ for some m. So in particular $\delta^* <_i \rho_m$, and hence $\delta <_j \rho_n$.

When $u_0 > 1$, for (j_v, γ) where $v+1 < u_0$, follow the proof for (2.1;2) above. $q = k+1$ here.

§ 3.5. When α is not connected.

Just a word about the fundamental sequence of $(j_0, \tilde{\alpha})$ where $\tilde{\alpha}$ is not connected. Suppose $\tilde{\alpha}$ consists of the components $\alpha_1, \alpha_2, \cdots, \alpha_n$, where those components are ordered in the non-increasing order with respect to j_0. Let $\{\beta_m\}_m$ be the fundamental sequence for (j_0, α_n). Then $\{\alpha_1 \# \alpha_2 \# \cdots \# \alpha_{n-1} \# \beta_m\}_m$ can be taken as the fundamental sequence for $(j_0, \tilde{\alpha})$.

§ 4. Critical cases

We shall first consider a.3.2). The critical case a.3.2), is subject to the following condition (cf. § 1): let (t, ν) be the last reduction pair of $\tilde{\alpha}$ with respect to j_0. Then ν is of the form $(i, b+1, 0)$ and $t > i$, and if we let h be the least indicator occurring in the sp's of $\tilde{\alpha}$, then $h \leq i$.

In order to define the reduction sequence for a critical case, we first locate a particular sp of $\tilde{\alpha}$, which we call (g, μ).

DEFINITION 4.1. Let g be the last indicator occurring in the sp's (including tsp's) of $\tilde{\alpha}$ such that $g \leq i$. Let the corresponding o.d. be μ.

This notation will be observed throughout this section.

LEMMA 4.1. *A sod of μ with respect to i is also a sod of $\tilde{\alpha}$ (with respect to j_0): it appears as a sod after μ. In particular if $g = i$, then the sp's of (i, μ) are exactly those of $(j_0, \tilde{\alpha})$ which are successors of (g, μ).*

Proof. Let us denote a sp of (i, μ) by (j, γ). The proof is by induction on the number of sp's of (i, μ) between (i, μ) and (j, γ). The first one is (i, μ) and μ is a sod of $\tilde{\alpha}$. Suppose (j', γ) is a sp of $(j_0, \tilde{\alpha})$. If γ is the last reduction place for the former or (j, γ) is static and the next sp is the last one, then we are done. If not, then the next sp or the second successor when (j, γ) is static which is strictly different from γ is (p, γ'), where $\gamma = (p, c, \delta)$ and γ' is either a p-least $(p_0, 0)$-dominant of γ when (2.1;1) applies to (j, γ) and $p = p_0 + 1$ (cf. 3) of 4) of Proposition 1.1).

This can be shown just by going over Definition 1.1. Recall that the last reduction place of $(j_0, \tilde{\alpha})$ is ν, hence (j', γ) must have a successor whose sod is strictly different from γ, and from the definition it must be (p, γ').

When $g = i$, the first ones to compare are the same: $(g, \mu) = (i, \mu)$.

For a (j, γ) to be the last reduction pair of (i, μ), either $\gamma=(p, c, 0)$, $=(p, c, \delta'+1)$ or $=(p, c, \delta)$ and δ is connected and marked. For the first two, γ must be the last reduction place of $(j_0, \tilde{\alpha})$ also (hence only the first one is possible). For the third case, δ is connected and marked for $(j_0, \tilde{\alpha})$ also, hence (j, γ) satisfies the same condition for $(j_0, \tilde{\alpha})$ (for being the last reduction pair), which means that $\gamma=\nu$, but this is impossible.

The proof of the lemma has an implication that the sp's of $(j_0, \tilde{\alpha})$ which are successors of (g, μ) are exactly the same as those of (i, μ) (as long as the latter are defined).

§4.1 The case where $g=i$.

First we deal with a special case of a.3.2); namely the case where $g=i$. So $g=i$ will be assumed throughout §4.1.

DEFINITION 4.2. Let μ^* be the figure obtained from μ by replacing the scanned ν by a new symbol, say X. Let μ^*_m be obtained from μ^* by applying Definition 1.4 to all the sp's (of $(j_0, \tilde{\alpha})$) between (g, μ) (or (i, μ)) and (t, ν), leaving X unchanged. In case the reduction results in more than one component, X will occur only in the inductive part. For instance, if $\gamma=(i, a, \alpha)$ and its reductian should assume the form $(i, a_m, \alpha)\#(i, a, \alpha_m)$, then γ^*_m assumes the form $(i, a_m, \alpha)\#(i, a, \alpha^*_m)$. As a consequence, there will be exactly one occurrence of X in μ^*_m. Note that Definition 1.4 concerns with the induction steps, so (t, ν) can be bxcluded from the consideration. By Lemma 4.1 we may regard Definition 1.4 as applied to the sp's of $(j_0, \tilde{\alpha})$ also.

m matters in μ^*_m when one of (3.1), [1°] (2.1;2), (4.1;2) and (5.1.1;2) applies.

Now define ν_m and μ_m simultaneously. $\nu_0=(i, b, 0)$; $\mu_0=\mu^*_0\binom{X}{\nu_0}$ where the right hand side is a figure obtained from μ^*_0 by replacing X by ν_0: $\nu_{m+1}=(i, b, \mu_m)$; $\mu_{m+1}=\mu^*_{m+1}\left(\begin{matrix}X\\ \nu_{m+1}\end{matrix}\right)$.

It is obvious that μ_m and ν_m are o.d.'s.

Let γ be a sod of $(j_0, \tilde{\alpha})$ which succeeds μ. Define $\gamma_m=\gamma^*\binom{X}{\nu_m}$. It is obvious that γ_m is a sub-o.d. of μ_m and that γ_m can be defined inductively as in Definition 1.4, starting with ν_m.

We shall show that $\{\gamma_m\}_m$ is the reduction sequence for (j, γ).

It can be easily shown that all the propositions (including Proposition 1.13) in §1 hold when a critical case is involved.

We claim that $\{\nu_m\}_m$ is the reduction sequence for (t, ν). Note that all the propositions in §1 hold when a critical case is involved. (This includes Proposition 1.13.)

LEMMA 4.2. *Suppose σ is an i-section of μ_m. Then σ is either*

μ_{m-1}, *an i-subsection of* μ *or* $\sigma=(p, 0, \sigma')$ *where* $(p, 0, \sigma'+1)$ *is an i-subsection of* μ (*for* (2.1)).

Proof. Consider a sod of (g, μ), say γ $(=(p, c, \delta))$. We shall show, by induction on the number of sod's succeeding γ, that if σ is an i-section of γ_m, then σ is either μ_{m-1}, an i-subsection of μ or $\sigma=(p, o, \sigma')$ where $\gamma=(p, 0, \sigma'+1)$ (which is an i-subsection of μ). As a special case, take $\gamma=\mu$, proving the lemma.

γ_m can assume one of the following forms: (p, c, δ_m) (This will be called λ_m), $\lambda_m \# (p, c_m, \delta)$, $\lambda_m \# (p, d, \cdots, (p, d, \delta)\cdots)$, where $c=d+1$, $(p, 0, \delta_m \# (p_0, 0, \delta))$ where $p=p_0+1$, $\lambda_m \# (p_m, c, \delta)$ where $p_m \uparrow p$, $\lambda_m \# (p_0, a_m, \delta)$ where $a_m \uparrow A$, $\lambda_m \# (p_0, e, \cdots, (p_0, e, \delta)\cdots)$ where e is the maximum element of A, $\delta \# \cdots \# \delta \# (p, 0, \delta_m \# (p_0, 0, \delta))$, $(p, 0, \delta_m \# (p_0, 0, \nu))$ and $\nu \# \cdots \# \nu \# (p, 0, \delta_m \# (p_0, 0, \nu))$ where $\gamma=(p, 0, \delta+1)$, and $\nu=(p, 0, \delta)$. Recall that by the choice of μ, $p>i$, hence $p_0 \geq i$ and we may assume that $p_m>i$. The sixth and the seventh cases are from [1°]; $\gamma=(p, c, \lambda \# \tilde{\nu})$, $\tilde{\nu}=(q, 0, \alpha)$, $p>q$ and $(p, \tilde{\nu})$ is the next sp. By definition of μ, $q>i$, hence $p_0>i$.

Suppose σ is an i-section of γ_m. If γ is ν, then γ_m is $\nu_m=(i, b, \mu_{m-1})$. So $\sigma=\mu_{m-1}$, and this satisfies the condition. Suspose γ properly contains ν. Then either σ is an i-section of δ_m, an i-section of δ, the δ in $(p, 0, \delta_m \# (p_0, 0, \delta))$ or the ν in $(p, 0, \delta_m \# (p_0, 0, \nu))$ when $p_0=i$. All the possibilities are exhausted with those form, for $p>i$ for all cases and $p_0>i$ for the sixth and the seventh cases. If σ is an i-section of δ_m, then the induction hypothesis applies. If σ is an i-section of δ, then it is an i-section of μ.

LEMMA 4.3. *Let* $l<i$ *and let* σ *be an l-section of* μ_m. *Then* σ *is an l-section of* μ.

Proof. As in Lemma 4.2, consider every (j, γ) a sp between (g, μ) and (t, ν) and its reduction $\{\gamma_m\}_m$. The lemma is proved by induction on m, within which by induction on the complexity of γ. Suppose $m=0$. Then $\gamma_0=\gamma^*\binom{X}{\nu_0}$, and ν_0, has no l-section, so it is trivially shown that an l-section of γ_0 is an l-section of γ which is an l-section of μ. Suppose next $m>0$. Following the notation in the proof of Lemma 4.2, $l<i<p$ implies that for any m μ_{m-1} cannot be σ. If σ is an l-section of μ_{m-1}, then by the induction hypothesis σ is an l-section of μ. Neither δ_m, δ nor ν occurring explicitly in γ_m can be σ, for $p_0>l$ and we may assume that $p_m>l$. Therefore for every m σ an l-section of γ_m is an l-section of γ, an l-section of μ_{m-1} or an l-section of δ_m. For the first case it is obvious that σ is an l-section of μ. An l-section of μ_{m-1} is an l-section of μ. If σ is an l-section of δ_m, then by the induction hypothesis (on the complexity) σ is an l-section of μ.

PROPOSITION 4.1. *Let (j, γ) be any sp between (g, μ) and (t, ν), where (t, ν) is included but (g, μ) is excluded. Then, for every such that $i<l\leq j$, $\gamma_m <_l \gamma$ and, if σ is an l-section of γ_m, then $\sigma <_l \gamma$.*

Proof. Recall that $\nu_m=(i, b, \mu_{m-1})$, where $\mu_{m-1}=0$ if $m=0$. $\nu_m <_l \nu = (i, b+1, 0)$ is obvious for every $l>i$. There is no l-section of ν_m for such l.

Suppose (j, γ) is a sp between (t, ν) and (g, μ), (g, μ) being excluded. Let $\gamma=(p, c, \delta)$. $p>i$. There is a sp (p, γ'), where γ' is a component of δ (cf. 1) of Proposition 1.4) $\gamma'_m <_p \gamma'$ by the induction hypothesis. If the transition to (p, γ') is not by [1°], (2.1), (3.1), (4.1), (5.1.1), (2') or (5.3), then $\gamma_m=(p, c, \delta_m)$. So $\gamma_m <_l (p, c, \delta)=\gamma$ is obvious for every $l\geq p$. Let $i<l<p$ and let σ be an l-section of γ_m. Then δ is an l-section of δ_m, hence $\sigma >_l \delta$ (by the induction hypothesis). But δ is l-active in γ, so $\sigma <_l \gamma$, and this implies that $\gamma_m <_l \gamma$ for any such l.

Suppose the transition is by [1°]. Then γ_m assumes one of the following forms: $(p, c_m, \delta) \# (p, c, \delta_m)$ where $c_m \uparrow c$, $(p, d, \cdots, (p, d, \delta)\cdots) \# (p, c, \delta_m)$ where $c=d+1$, $(p_m, c, \delta) \# (p, c, \delta_m)$ where $c=0$ and $p_m \uparrow p$, $(q, a_m, \delta) \# (p, c, \delta_m)$ where $p=q+1, c=0$ and $a_m \uparrow A$, and $(q, e, \cdots, (q, e, \delta)\cdots) \# (p, c, \delta_m)$ where $c=0$, $p=q+1$ and e is the maximum element of A. In any case, $\gamma_m <_l \gamma$ is obvious if $l>p$, for $\delta_m <_p \delta$ by the induction hypothesis. For $l=p$, an l-section of γ_m is either δ_m or $(p, d, \cdots, (p, d, \delta)\cdots)$. $\delta_m <_p \delta <_p \gamma$ is obvious. $(p, d, \cdots, (p, d, \delta)\cdots) <_p \gamma$ can be proved by induction. So in any case $\sigma <_p \gamma$, hence $\gamma_m <_p \gamma$. Suppose $i<l<p$. If the outermost indicator in a component of γ_m is p, then an l-section of it is either an l-section of δ, in which case $\sigma <_l \delta <_l \gamma$, or an l-section of δ_m, in which case $\sigma <_l \gamma$ by the induction hypothesis. If the outermost indicator is q, where $p=q+1$, and $l=q$, then σ is either δ or $(q, e, \cdots, (q, e, \delta)\cdots)$. The latter being $<_q \gamma$ can be shown by induction. If $i<l<q$, then the argument for the first case goes through. If the outermost indicator is p_m, and if $p_m>l$, then an l-section is an l-section of δ. In any case $\sigma <_l \gamma$, hence $\gamma_m <_l \gamma$.

Suppose the transition is by (2.1;1). Then $\gamma_m=(p, 0, \delta'_m \# (p_0, 0, \kappa))$ where $p=p_0+1$, $\gamma=(p, 0, \delta'+1)$, $\kappa=(p, 0, \delta')$ and $\delta'_m=\delta'\begin{pmatrix}\delta\\ \delta_m\end{pmatrix}$. $\delta'_m \# (p_0, 0, \kappa) <_p \delta'$ as has been proven earlier, hence $\gamma_m <_l \gamma$ if $p\geq l$. If $l=p$, σ is $\delta'_m \# (p_0, 0, \kappa)$, so $\sigma <_p \delta' <_p \gamma$. Suppose $i<l<p$. If $l=p_0$ and σ is κ, then $\kappa <_l \gamma$ is obvious. For other cases the proposition follows from the induction hypothesis.

If the transition is by (2.1;2), then $\gamma_m=\kappa \# \cdots \# \kappa \# \rho_m$, where $\rho_m=(p, 0, \delta'_m \# (p_0, 0, \kappa))$. The proposition is trivial, having proved the proposition for (2.1;1).

Suppose the transition is by (3.1). Then $\gamma_m=(p, c, \delta_m) \# (p, c_m, \delta)$ or $\gamma_m=(p, c, \delta_m) \# (p, d, (p, d, \cdots, (p, d, \delta)\cdots))$. In either of those cases,

$\gamma_m <_l \gamma$ is obvious from the induction hypothesis if $l > p$. For $l = p$, an l-section of γ_m is either δ_m or δ (the first case), or is either δ_m or $(p, d, \cdots, (p, d, \delta) \cdots)$ (the second case). For the first case the proposition is obvious (by the induction hypothesis). For the second case, we can show that $\gamma_m <_p \gamma$ by induction on m. If $i < l < p$, a similar argument as above goes through.

Suppose the transition is by (4.1;1). Then $\gamma_m = (p, 0, \delta_m \# (q, 0, \delta))$ where $p = q+1$. $\delta_m <_p \delta$ and $(q, 0, \delta) <_p \delta$ (since in this case the outermost indicator of δ is p). So $\gamma_m <_l \gamma$ if $l \geq p$, and when σ is $\delta_m \# (q, 0, \delta)$, $\sigma <_p \delta <_p \gamma$. Let $i < l < p$. If $l = q$ and b is δ, $\sigma <_l \gamma$ is obvious. For other cases, the proposition can be proved from the induction hypothesis.

The transition is by (4.1;2). Then $\gamma_m = \delta \# \cdots \# \delta \# (p, 0, \delta_m \# (q, 0, \delta))$. Note that here $j = p$. For the last term in γ_m, the proposition is proved as for (4.1;1). For δ, $\delta <_p \gamma$, and other conditions can be shown trivially. Notice that here the upper bound for l (namely j) is necessary.

If the transition is by (5.1.1;1), then $\gamma_m = (p, 0, \delta_m \# (q, 0, \delta))$ where $p = q+1$, and the outermost indicator of δ is $\geq p$. The rest of the proof for (4.1;1) goes through.

If the transition is by (5.1.1;2), then $\gamma_m = \delta \# \cdots \# \delta \# (p, 0, \delta_m \# (q, 0, \delta))$. Follow the proof for (4.1;2).

For (5.3) and and (2′), let (r, γ) be the next sp. Then $r > j$. Therefore the propostion follows from the induction hypothesis.

PROPOSITION 4.2. *For every l, $\mu_m <_l \mu$ and, if σ is an l-section of μ_m, then $\sigma <_l \mu$.*

Also an h-active value of $\mu_m \leq$ the corresponding value of μ.

Proof. The last statement is obvious. We shall show the first part by induction on m. Let $\mu = (p, c, \delta)$. Recall that $p > i$ and there is a sp (p, γ'), where γ' is a component of δ.

$m = 0$. μ_0 assumes one of the following forms: (p, c, δ_0) (which we denote by λ_0), $\lambda_0 \# (p, c_0, \delta)$, $\lambda_0 \# (p, d, \delta)$ where $c = d+1$, $(p, 0, \delta_0 \# (q, 0, \delta))$ where $p = q+1$, $\lambda_0 \# (p_0, c, \delta)$ where $p_m \uparrow p$, $\lambda_0 \# (q, a_0, \delta)$ where $a_m \uparrow A$, $\lambda_0 \# (q, e, \delta)$ where e is the maximum element of A, and $(p, 0, \delta'_0 \# (q, e, \rho))$ where $p = q+1$, $\delta = \delta' + 1$ and $\rho = (p, 0, \delta')$. Note that the condition $p > i$ prevents (2.1;2), (4.1;2) and (5.1.1;2). We may assume $p_0 > i$. $q \geq i$ by the definition of μ. The sixth and the seventh cases come from [1°]; $\mu = (p, c, \lambda \# \nu)$, $\nu = (r, 0, \alpha)$, $p > r$ and (p, ν) is the next sp. Therefore $r > i$, hence $q > i$. The last cast is by (2.1;1). $\lambda_0 <_l \mu$ for any $l \geq p$ since $\delta_0 <_p \delta$ (Proposition 4.1). In each case except the first, the fourth and the last, the second term being $<_l \mu$ for $l \geq p$ is obvious. The fourth and the last cases will be dealt with at the end. A p-section of any of those o.d.'s is either $<_p \delta$ or $= \delta$, hence $<_p \mu$. Let $i < l < p$ and let σ be an l-section of μ_0. Then either σ is an l-section of δ_0, and l-section of δ, $l=q$ and $\sigma=\delta$ or $l=p_0$ and $\sigma=\delta$. If the first, then $\sigma<_l\delta$ from Proposition 4.1. In any of those cases, $\sigma<_l\mu$ is obvious. Let $l\leq i$ and let σ be an l-section of μ_0. Then σ is an l-subsection of μ (cf. Lemma 4.2 with $m=0$ and Lemma 4.3). So $\sigma<_l\mu$, hence $\mu_0<_l\mu$.

Consider the fourth case. This case is possible when $\mu=(p, c, \delta)$ satisfies that δ is connected and the outermost indicator of δ is $\geq p$. So by Proposition 4.1 $\delta_0\#(q, 0, \delta)<_p\delta$, which implies $\mu_0<_l\mu$ for every $l\geq p$. $\delta_0\#(q, 0, \delta)<_p\mu$. Suppose $i<l<p$ and σ is an l-section of μ_0. Then either σ is an l-section of δ_0 or δ, or σ is δ (and $l=q$). If the first is the case, then by Proposition 4.1 $\sigma<_l\delta_0<_l\delta<_l\mu$. For the latter two, $\sigma\leq_l\delta<_l\mu$. Suppose $l\leq i$. Then Lemmas 4.1 and 4.2 apply.

Consider the last case. Namely $\mu=(p, 0, \delta+1)$ and $\mu_0=(p, 0, \delta_0'\#(q, 0, \rho))$ where $\rho=(p, 0, \delta')$. $\delta_0'\#(q, 0, \rho)<_p\delta'$ by Proposition 4.1 (since δ' has a $(q, 0)$-dominant). So $\mu_0<_l\mu$ if $l\geq p$. $\delta_0'\#(q, 0, \rho)<_p\mu$. Suppose $i<l<p$ and σ is an l-section of μ_0. Then either σ is an l-section of δ_0' or of ρ, or σ is ρ. Then by Proposition 4.1 $\sigma<_l\mu$. If $l\leq i$, then Lemmas 4.2 and 4.3 apply.

$m+1$. $\mu_{m+1}<_l\mu$ is proved as above for $l>i$, applying Proposition 4.1 with $m+1$. Suppose σ is an i-section of μ_{m+1}. Then σ is either μ_m or an i-subsection of μ (Lemm 4.1). For the former case, $\sigma=\mu_m<_i\mu$ by the induction hypothesis. For the latter case, $\sigma<_i\mu$ is known. So $\mu_{m+1}<_i\mu$. Let $l<i$ and σ be an l-section of μ_{m+1}. By Lemma 4.2, σ is an l-subsection of μ, hence $\sigma<_l\mu$. Therefore $\mu_{m+1}<_l\mu$ for every l.

PROPOSITION 4.3. *Let (j, γ) be a sp which is a predecessor of (g, μ). For every $l\leq j$, $\gamma_m<_l\gamma$ and, if σ is an l-section of γ_m, then $\sigma<_l\gamma$.*

Proof. For (g, μ), this has been proved in Proposition 4.2. If γ properly contains μ, then then the proof of Proposition 2.1; induction step goes through (cf. §2.2). Note that the reason why there is a lower bound of l in Proposition 2.1 is the existence of the case a.3.1), which is irrelevant here. Therefore Proposition 4.3 holds for all l.

As a special case of Proposition 4.3, we obtain $\tilde{\alpha}_m<_{j_0}\tilde{\alpha}$.

PROPOSITION 4.4. *Let (j, γ) be a sp or a tsp and suppose that* $\gamma=\mathrm{apr}((n, k+1), j, \gamma)$ *for some* (n, k) (where the last reduction pair for $\tilde{\alpha}$ is a critical case). *Then* $\mathrm{apr}((n, k), j, \gamma_m)=\mathrm{apr}((n, k), j, \gamma)$. (See Proposition 2.2)

Proof. Lemmas 2.1–2.4 for Proposition 2.2 in §2.3 hold. (In fact Lemma 2.1 is irrelevant here.) In the proofs of Lemmas 2.2 and 2.3 the ν here cannot satisfy the condition on δ, hence the peculiarity of ν does not affect the proof. Since $\nu=(i, b+1, 0)$ itself cannot satisfy the condition on γ, only the induction step of the proof of Proposition 2.2 matters. We have proved $\gamma_m <_j \gamma$ in Propositions 4.1–4.3, so Lemma 2.4 can be applied.

PROPOSITION 4.5. *Propositions* 2.3 *and* 2.4 *in* §2.3 *hold when the last reduction pair is a critical case.*

Proof. The proofs of Propositions 2.3 and 2.4 go through. The (p, δ)'s there are not the (t, ν) of this section.

In order to show that $\{\tilde{\alpha}_m\}_m$ converges to $\tilde{\alpha}$ (with respect to j_0), we need a proposition similar to Proposition 3.3.

Suppose $\tilde{\beta}$ is connected and $\tilde{\beta} <_{j_0} \tilde{\alpha}$. We can define cmf's of $\tilde{\beta}$ as in Definition 3.3 (§3.1). Our task is now to prove.

PROPOSITION 4.6. *Let* (j, γ) *be a sp of* $(j_0, \tilde{\alpha})$ *such that* γ *contains* μ *and let* δ *be the corresponding cmf of* $\tilde{\beta}$. *Then Proposition* 3.3. *holds for* (j, γ) *and* δ. (See §3.2.)

If we can show that $\{\mu_m\}_m$ coverges to μ with respect to g $(=i)$, then, following the proof in §3.3, we can claim the proposition for such (j, γ). Therefore, it suffices to prove the following.

PROPOSITION 4.7. *Suppose* β *is connected and* $\beta <_i \mu$ *(where* $g=i$*). Let* (j, γ) *be a sp of* $\tilde{\alpha}$ *such that* γ *is contained in* μ *and* δ *be the corresponding cmf (of* β *relative to* (i, μ)*). (*$\gamma=\mu$ *is inclusive.) Then Proposition* 3.3 *holds for* (j, γ) *and* β. (See §3.2.)

From this, in particular, $\beta <_i \mu_m$ for some m when β is a cmf of $(j_0, \tilde{\alpha})$ corresponding to (g, μ).

LEMMA 4.4. *Suppose* $\beta <_i \mu$ *and the cmf's of* β *(relative to* (i, μ)*) are defined as long as the sp's of* μ *are defined (hence the last cmf corresponds to* (t, ν)*). If the last cmf is of the form* (i, b, η), *then* $\eta <_i \mu$.

Proof. We shall show that under the circumstances every cmf (of β) is i-active in β. This implies in particular that (i, b, η) is i-active in β. So $\eta <_i (i, b, \eta) <_i \beta <_i \mu$.

The assertion can be proved by going over the cases in Definition 3.3. Let (j, γ) be a sp or a tsp of (i, μ) and δ be the corresponding cmf. Note that $j \geq i$; this is so by 2) of Proposition 1.4 and the fact that all the indicators connected to ν are $>i$. Suppose δ is i-active in β. We first consider the cases except (2.1;2), (4.1;2) and (5.1.1;2). Except for those cases, when the next cmf is defined, we first take a j-approximation of δ, say δ^*. δ^* is j-active in δ, hence i-active in β. Suppose $\gamma=(q, d, \gamma')$. Then $q>i$ (q is connected to ν) and $\delta^*=(q, d, \delta')$. The next cmf is a component of δ' except for the following cases: Case (4.1) of (2.1;1.b), Case (4.1) of (2.1;1.c), Case (3) of (4.1;1), Case (3) of

(5.1.1;1), (2′) and (5.3). δ' is i-active in δ^*, so in β also. For the exceptional cases except for (2′) and (5.3), the next cmf is a j-approximation of δ, which is j-active, hence is i-active in δ.

Consider Case (v.2) of (2′) or (5.3). $\delta=\eta_v$ and $j_v=j$. $\eta_{v+1}=\mathrm{apr}((r, s+1), j_v, \eta_v)$ is the next cmf. η_{v+1} is j_v-active in η_v and η_v is i-active in β. Since $j_v \geq i$, this means that η_{v+1} is i-actiue in β.

Now consider the cases which have been excluded so far: Case (2) of (2.1;2), Case (1) of (4.1;2) and Case (1) of (5.1.1;2). Due to the conditions of those cases, $(j, \gamma) \neq (i, \mu)$. The next cmf is a j_v-approximation of δ, but $j_v \geq i$. So it is i-active in δ, hence in β also.

Lemma 4.4 supplies us with the means how to employ the induction on the complexity of β for the proof of Proposition 4.7. Namely, we prove the proposition by induction on the complexity of β, within which by induction on the number of cmf's between β and its last cmf.

Assume the proposition for all the connected o.d.'s which are simpler than β.

Basic in Definition 3.3. If B is the case, then the proof in § 3.3 goes thraugh. Suppose the last cmf, say $\delta=(p, c, \eta)$, corresponds to (t, ν). Note that $t>i$ and ν_m and μ_m are connected. If $(p, c)<(i, b)$, then $\delta <_t \nu_0$. If $=$, then $\eta <_i \mu$ by Lemma 4.4. By the induction hypothesis the proposition holds for η, so in particular $\eta <_i \mu_m$ for some m. This implies $\delta <_t \nu_{m+1}$.

Induction steps. Following the proof in § 3.4, we can show Proposition 3.3 for any (j, γ) and the corresponding δ satisfying the condition. In particular $\beta <_i \mu_{m+1}$.

This completes the proof of Proposition 4.7, hence we are done with the critical case when $g=i$.

§ 4.2. The case where $g<i$.

Next we deal with the case where $g<i$.

LEMMA 4.5. *Consider the sp's of* (i, μ). *By a remark after Lemma* 4.1, *the sp's of* (i, μ) (except (i, μ) itself) *are exactly those of* $(j_0, \tilde{\alpha})$ *succeeding* (g, μ) *as long as the former are defined. Therefore the last sp of* (i, μ) *is a successor of* (g, μ). *If it is* (t, ν), *then it is a critical case with* $g=i$ *according to* a.3.2), *and hence the argument in* § 4.1 *employed to* (i, μ) (in the place of $(j_0, \tilde{\alpha})$) *yields* $\{\nu_m\}_m$ *and* $\{\mu_m\}_m$ *as defined in Definition* 4.2 *and* $\{\mu_m\}_m$ *converges to* μ *with respect to* i. *If the last sp is not* (t, ν), *then it is a non-critical case and* $\{\mu_m\}_m$ *can be defined as in earlier sections. Define* $\nu_0=(i, b, 0)$; $\nu_{m+1}=(i, b, \mu_m)$. *The* ν_m*'s here are not the ones defined before.*

DEFINITIION 4.3. Let μ^* be the figure defined in Definition 4.2. Let ε_m^* be obtained from μ^* by applying Definition 1.4 to all the sp's of (g, μ), leaving X unchanged. Let $\varepsilon_m = \varepsilon_m^* \binom{X}{\nu_m}$ where $\nu_m = (i, b, \mu_{m-1})$. (For μ_m and ν_m, see Lemma 4.5 above.)

We wish to claim that $\{\varepsilon_m\}_m$ serves as the reduction sequence for (g, μ).

If (j, γ) is a sp of (g, μ), then $\{\gamma_m\}_m$ will denote the corresponding reduction sequence induced from $\{\varepsilon_m\}_m$.

We can equivalently define γ_m as follows. If γ is ν, then let $\gamma_m = \nu_m$. If γ is not ν, then apply Definition 1.4, depending on the induction hypothesis.

LEMMA 4.6. *$\mu_m <_l \mu$ for all l such that $l \leq i$.*

Proof. If (i, μ) is a critical case, hence the last sp is (t, ν) (cf. Lemma 4.5), then by Proposition 4.2 the lemma holds for all l. If it is not a critical case, then it is not the case a.3.1) or b.2.2;2), since the last sp is a sp of $(j_0, \tilde{\alpha})$ succeeding (g, μ) (cf. Lemma 4.5). So from the proof of Proposition 2.1, $\mu_m <_l \mu$ for all l such that $l \leq i$. (In the proof of Proposition 2.1, the only cases where the lower bound has to be placed a.3.1) and b.2.2;2).)

The following lemma is a parallel to Lemmas 4.2 and 4.3.

LEMMA 4.7. 1) *Suppose σ is an i-section of ε_m. Then σ is either μ_{m-1} or an i-subsection of μ.*

2) *Let $l < i$ and suppose σ is an l-section of ε_m. Then $\sigma <_l \mu$.*

Proof. 1) Consider a sod of (g, μ), say γ $(=(p, c, \delta))$. We shall show, by induction on the number of sod's succeeding γ, that if σ is an i-section of γ_m, then σ is either μ_{m-1} or an i-subsection of μ. As a special case, take $\gamma = \mu$, proving the lemma.

γ_m can assume one of the forms listed in the proof of Lemma 4.2. The conditions on the indicators there are valid here also. Suppose σ is an i-section of γ_m. If γ is ν, then $\gamma_m = \nu_m = (i, b, \mu_{m-1})$. So $\sigma = \mu_{m-1}$, which meets the condition. Suppose γ porperly contains ν. Then the same situation as in the proof of Lemma 4.2 holds.

2) The proof is much the same as the proof of Lemma 4.3 with ε_m in the place of μ_m, concluding for each case that $\sigma <_l \mu$. The only case which does not arise in Lemma 4.3 is the case where σ is an l-section of μ_m (the μ_m as considered here in § 4.2). Suppose (i, μ) is a critical case. Then, by Lemma 4.1 we may apply Lemma 4.3, hence σ is an l-sectisn of μ. If (i, μ) is not a critical case, then § 2 applies and by Proposition 2.1 $\sigma <_l \mu$. (The reason why there is a bound for l in Proposition 2.1 is the existence of a.3.1) and b.2.2;2), which are irrelevant here (cf. Lemma 4.1).)

We can establish the proposition below, following the proof of Proposition 4.1.

PROPOSITION 4.8. *Let (j, γ) be any sp (of $(j_0, \tilde{\alpha})$) between (g, μ) and (t, ν) where (t, ν) is included but (g, μ) is excluded. Then for any l such that $i<l\leq j$, $\gamma_m<_l\gamma$ and, if σ is an l-section of γ_m, then $\sigma<_l\gamma$. Also $\{\gamma_m\}_m$ is an increasing sequence with respect to l.*

PROPOSITION 4.9. *Any h-active value of $\varepsilon_m\leq$ the corresponding value of μ. For every l, $\varepsilon_m<_l\mu$ and, if σ is an l-section of ε_m, then $\sigma<_l\mu$. We can also show that $\{\varepsilon_m\}_m$ is an increasing sequence with respect to g.*

Proof. We prove only the second part. Let $\mu=(p, c, \delta)$. $p>i$ and there is sp (p, γ') where γ' is a component of δ.

ε_m assumes one of the following forms. (p, c, δ_m) which we shall denote by λ_m, $\lambda_m\#(p, c_m, \delta)$, $\lambda_m\#(p, d, \cdots, (p, d, \delta)\cdots)$ where $c=d+1$, $(p, 0, \delta_m\#(p_0, 0, \delta))$ where $p=p_0+1$, $\lambda_m\#(p_m, c, \delta)$ where $p_m\uparrow p$, $\lambda_m\#(p_0, a_m, \delta)$ where $a_m\uparrow A$, $\lambda_m\#(p_0, e, \cdots, (p_0, e, \delta)\cdots)$ where e is the maximum element of A, and $(p, 0, \delta'_m\#(p_0, 0, \nu))$ where $p=p_0+1$, $\mu=(p, 0, \delta'+1)$ and $\nu=(p, 0, \delta')$. $p>i>q$, so $p_0\geq i$ and we may assume that $p_m>i$. The condition $g<p$ excludes the cases (2.1;2), (4.1;2) and (5.1.1;2). Since $\delta_m<_p\delta$ by Proposition 4.8, $\varepsilon_m<_l\mu$ is obvious when $l\geq p$ except for the fourth and the last cases, which will be considered later. Suppose $i<l<p$ and σ is either an l-section of δ_m (In this case $\sigma<_l\delta$ by Proposition 4.8.), an l-section of δ (In this case $\sigma<_l\delta$ is obvious.), δ itself or $(p_0, e, \cdots, (p_0, e, \delta)\cdots)$. In any case $\sigma<_l\mu$. Therefore $\varepsilon_m<_l\mu$ for all such l. Suppose $l\leq i$ and let σ be an l-section of ε_m. Then $\sigma<_l\mu$ by Lemmas 4.6 and 4.7, and $\varepsilon_m<_l\mu$ for all such l.

Consider the fourth case; this case is possible when $\mu=(p, c, \delta)$, where δ is connected and the outermost indicator of δ is $\geq p$. So by Proposition 4.8 $\delta_m\#(p_0, 0, \delta)<_p\delta$. This implies $\varepsilon_m<_l\mu$ for every $l\geq p$ and $\delta_m\#(p_0, 0, \delta)<_p\mu$. Suppose $i<l<p$ and σ is an l-section of ε_m. Then σ is either an l-section of δ_m or of δ, or δ itself. In any case $\sigma<_l\mu$ (cf. Proposition 4.8). If $l\leq i$, then apply Lemmas 4.6 and 4.7.

Consider the last case, namely (2.1;1) applies to (g, μ). By Proposition 4.8 and the condition of (2.1;1), $\delta'_m\#(p_0, 0, \nu)<_p\delta'$. This implies $\varepsilon_m<_l\mu$ for every $l\geq p$ and $\delta'_m\#(p_0, 0, \nu)<_p\mu$. Suppose $i<l<p$ and σ is an l-section of ε_m. Then σ is either an l-section of δ'_m or ν. In either case $\sigma<_l\mu$ by Proposition 4.8. If $l\leq i$, then apply Lemmas 4.6 and 4.7.

PROPOSITION 4.10. *Let (j, γ) be a sp such that γ contains μ. For every l, $\gamma_m<_l\gamma$ and, if σ is an l-section of γ_m, then $\sigma<_l\gamma$.*

Proof. For (g, μ), this has been proved in Proposition 4.9. If γ properly contains μ, then the proof of Proposition 2.1; induction step goes through.

As a special case, $\tilde{\alpha}_m <_{j_0} \tilde{\alpha}$.

Next we with to show that $\tilde{\alpha}_m$ converges to $\tilde{\alpha}$.

PROPOSITION 4.11. *Let* (j, γ) *be a sp or a tsp* (of $(j_0, \tilde{\alpha})$) *and suppose that* $\gamma = \mathrm{apr}\,((n, k+1), j, \gamma)$ *for some* (n, k) (where the last reduction *pair for* $\tilde{\alpha}$ *is a critical case and* $g<i$). *Then* $\mathrm{apr}\,((n, k), j, \gamma_m) = \mathrm{apr}\,((n, k), j, \gamma)$.

Proof. The proof of Proposition 4.4 goes through for this case also. We have proved $\gamma_m <_j \gamma$ in Propositions 4.8~4.10.

PROPOSITION 4.12. *Propositions* 2.3 *and* 2.4 *in* §2.3 *hold when the last reduction pair is a critical case and* $g<i$.

Proof. See the proof of Proposition 4.5.

Suppose now $\tilde{\beta} <_{j_0} \tilde{\alpha}$ for a connected $\tilde{\beta}$. We can define cmf's of β as in §3. Evidently, Proposition 3.2 holds.

If we can show that $\{\varepsilon_m\}_m$ converges to μ with respect to g, then, following the proof in §3.3 we can conclude the desired convergence. Therefore it suffices to prove the following.

PROPOSITION 4.13. *Let* β *be a connected o.d. such that* $\beta <_g \mu$. *Let* (j, γ) *be a sp of* $\tilde{\alpha}$ *such that* γ *is contained in* μ *and let* δ *be the corresponding cmf of* β. *Then Proposition* 3.3 *holds for* (j, γ) *and* δ *relative to* (g, μ). (See §3.2.) *Note that the sp's of* $(j_0, \tilde{\alpha})$ *succeeding* (g, μ) *are exactly those of* (g, μ) (cf. Proposition 1.9).

As a special case, Proposition 3.3 holds for β and (g, μ).

The following lemma is a crucial point for the proof of the proposition.

LEMMA 4.8. *Suppose* $\beta <_g \mu$ *and the cmf's of* β *are defined as long as the sp's of* $(j_0, \tilde{\alpha})$ *are defined, hence the last cmf of* β *corresponds to* (t, ν). *If the last cmf* (of β) *is of the form* (j, b, η), *then* $\eta <_i \mu$.

Proof. We first locate a particular sub-o.d. of β, which will be denoted by β^*, as follows.

1) $\mu = \mathrm{apr}\,((n, k), g, \mu)$ and $\mathrm{apr}\,((n, k), g, \beta)$ exists. $\beta^* = \mathrm{apr}\,((n, k), g, \beta)$.

2) $\mu = \mathrm{apr}\,((0, k+1), g, \mu)$ and (4.1;1) applies. $\beta^* = \mathrm{apr}\,((1, 1), g, \beta)$.

3) $\mu = \mathrm{apr}\,((n, k+1), g, \mu)$ and (5.1.1;1) applies. $\beta^* = \mathrm{apr}\,((n, 1), g, \beta)$ when $k=0$ and $\beta^* = \mathrm{apr}\,((n+1, 1), g, \beta)$ when $k>0$.

4) $\mu = (p, 0, \delta'+1)$ and (2.1;1) applies. $\beta^* = (p, 0, \delta')$.

Note that one of 1)~4) holds ($g<p$ excludes (2.1;2), (4.1;2) and (5.1.1;2)) and that the term in the right hand side of each equation exists under our assumption.

In any of these cases β^* is g-active in β. So $\beta <_g \mu$ implies

$\beta^* <_g \mu$.

Using this fact, we first show that

1° $\beta^* <_i \mu$

and then that

2° η is i-active in β^*.

From 1° and 2° follows $\eta <_i \mu$.

1°. For 4), $\beta^* <_i \mu$ is obvious. We shall consider 1)~3). Suppose, contrary to the claim, that $\mu <_i \beta^*$. ($\mu = \beta^*$ is impossible.) In order that $\beta^* <_g \mu$, there must be an indicator l_0 such that $g \leq l_0 < i$ and that there is an l_0-section of μ, say κ, which satisfies $\beta^* \leq_{l_0} \kappa$.

We claim, under the assumption $\mu <_i \beta^*$, that 1.1° there is a (j, γ) a sp (of $(j_0, \tilde{\alpha})$) succeeding (g, μ) such that γ is of the form $(p, c, \lambda \# \gamma')$, where (p, γ') is a sp and κ is an l_0-section of λ (hence λ is not empty).

Suppose there is not such (j, γ). Then at each stage of defining sp's κ is an l_0-section of the sod, for all the indicators connected to the last reduction place are $> i > l_0$. In the end the sod must be ν (the last reduction place), which has no l_0-section. So there must be a (j, γ) claimed as above.

From 1.1°, we shall infer that 1.2° under the circumstances κ is an l_0-section of β^*.

When 1.2° is established, we can conclude that $\kappa <_{l_0} \beta^* \leq_{l_0} \kappa$, a contradiction. Therefore there can be no such l_0, which enforces $\beta^* <_i \mu$.

The proof of 1.2° is carried out exploiting the (j, γ) whose existence has been established in 1.1°.

Let γ be of the form $(p, c, \lambda \# \gamma')$, where κ is an l_0-section of λ, and suppose $\gamma = \mathrm{apr}((r, s), j, \gamma)$. $p > i$. Let δ be the cmf of β corresponding to (j, γ). Since (j, γ) is not the last sp, the next cmf of β can be defined. (4.1) and (5.1.1) cannot apply to (j, γ) since $\lambda \# \gamma'$ is not connected. Also we may assume that neither (2′) nor (5.3) is the case with (j, γ). (2.1) will be considered later. Let us consider $\rho = \mathrm{apr}((r, s), j, \delta)$ $(= (p, c, \lambda \# \delta'))$. Since κ is an l_0-section of λ in ρ also and $p > i$, κ is an l_0-section of ρ. Consider (2.1;1). $\gamma = (p, 0, \lambda \# \gamma' \# 0)$ and δ contains $\rho = (p, 0, \lambda \# \gamma')$ j-active. κ is an l_0-section of ρ.

If (2.1;2) is the case, then δ contains ρ as above j_v-active for some j_v. (2.1;2) is possible only if $(j, \gamma) \neq (g, \mu)$. If (j, γ) is not an immediate successor of (g, μ) then $j_v > i$, and hence $j_v > l_0$. Therefore κ is an l_0-sections of δ. If (j, γ) is an immediate successor of (g, μ), (as a tsp), then $\mu = \gamma = (j, 0, \lambda \# \gamma' \# 0)$ and $j_v = g$. Due to the condition of (2.1;2) having the next cmf, $\mu = \mathrm{apr}((n, k+1), g, \mu)$ for some $n > 0$ and $\mathrm{apr}((n, k+1), g, \beta) = \delta = (p, 0, \lambda \# \gamma')$. But by definition $\beta^* = \delta$ here (cf. 1) in the definition of β^*), so κ is an l_0-section of β^*. (The conclusion is proved directly for this case.)

For the cases other than the latter of (2.1;2), ρ is j-active in δ and $j>i$ unless (j, γ) is (g, μ). So this means that κ is an l_0-section of δ.

If (j, γ) is (g, μ), we consider β^* instead of β, so $\rho=\beta^*$, hence κ is an l_0-section of β^*, which completes the proof.

Suppose $(j, \gamma)\neq(g, \mu)$. Consider (q, η) a sp or a tsp between (g, μ) and (j, γ). Let δ be the corresponding cmf. We shall show by induction, on the number of sp's and tsp's between (q, η) and (j, γ) that κ is an l_0-section of δ.

If $(q, \mu)=(j, \gamma)$, then the proposition has been shown above. Suppose $(q, \eta)\neq(j, \gamma)$, except for (2.1;2), (4.1;2) and (5.1.1;2), ρ, the next cmf, is either a q-approximation of δ or a component of λ in (p, c, λ) where (p, c, λ) is a q-approximation of δ. By the induction hypothesis, κ is an l_0-section of ρ. If $(q, \eta)\neq(g, \mu)$, then $p, q>i>l_0$. So in either case it is obvious that κ is an l_0-section of δ. If $(q, \eta)=(g, \mu)$, then $p>i>l_0$ still holds and we consider $\rho=\beta^*$. So κ is an l_0-section of β^*.

Suppose one of the excluded cases applies. Then we first take a j_v-approximation for some j_v. If $j_v\geq i$, the proof immediately above goes through. If $j_v<i$, then $j_v=g$, or $(q, \eta)=(g, \mu)$. So we consider β^*, and κ is an l_0-section of β^*.

This completes the proof of 1.2°, hence of Lemma 4.8.

Now we shall finish up with the critical case by proving Proposition 4.13.

Proof of Proposition 4.13. Suppose $(j, \gamma)=(t, \nu)$. Then the $j=t$-active values of $\delta\leq(i, b)$, since $\delta<_t\nu$ $(=(i, b+1, 0))$ (Proposition 3.2). If $<$, then $\delta<_t\nu_0=(i, b, 0)$, for $t>i$. If $=$, then δ is of the form (i, b, η) and by Lemma 4.8 $\eta<_i\mu$. Therefore by Lemma 4.5, there is an m such that $\eta<_i\mu_m$. From this follows $\delta<_t\nu_{m+1}$. Taking this as the basis and following the proof of Proposition 3.3, we obtain the desired result.

§ 4.3. The second critical case, b. 2.2;2).

This case is subject to the following condition: let (t, ν) be the last reduction pair of $\tilde{\alpha}$ with respect to j_0. Then ν is of the form $(i, 0, 0)$, $i=i_0+1$, $t>i_0$ and if we let h be the least indicator occurring in the sp's of $\tilde{\alpha}$, then $h\leq i_0$.

This case can be treated parallel to the first critical case, a.3.2). We have only to replace i by i_0 and (i, b) by (i_0, e). We shall only point out a few things.

Let g be the last indicator occurring in the sp's of $\tilde{\alpha}$ such that $g\leq i_0$. Let the corresponding o.d. be μ. The sp (g, μ) will act as the corresponding one in Definition 4.1.

For Lemma 4.1, sp's of (i_0, μ) be considered.

In § 4.1, $g=i_0$ be assumed. ν_m and μ_m are defined as in Definition

4.2 with the following modification: $\nu_0=(i_0, e, 0)$ and $\nu_{m+1}=(i_0, e, \mu_m)$.

In Lemma 4, suppose $\beta<_{i_0}\mu$ and that the last cmf is of the form (i_0, e, η).

Replacing i by i_0 and (i, b, η) by (i_0, e, η), the rest of the material in § 4.1 goes through.

In § 4.2, $g<i_0$ be assumed.

In Lemma 4.5, if the last sp of (i_0, μ) is (t, ν), then it is a critical case with $g=i_0$ according to b.2.2;2). So $\{\nu_m\}_m$ and $\{\mu_m\}_m$ are defined as for the case (i_0, μ). Now follow the argument, replacing i by i_0 and (i, b) by (i_0, e).

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