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# Finite transformation semigroups and automata

by

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## Abstract

The finite transformation semigroups are investigated. The main object which is treated here is the wreath product of transformation semigroups. The properties of the wreath product and the transformation semigroup are applied to homomorphisms and automorphisms of automata.

## Introduction

In this paper, if no confusion will arise, a semigroup and a group always mean a transformation semigroup and a permutation group, respectively.

$M^\Omega$ is the semigroup of all transformations of a set $\Omega$ into itself. $S^\Omega$ is the symmetric group on $\Omega$. If $|\Omega|=n$, $M^\Omega$ and $S^\Omega$ are denoted by $M^n$ and $S^n$, respectively.

Let $J$ be a subsemigroup of $M^\Omega$. $J$ is called transitive on $\Omega$, if for any pair $a, b \in \Omega$ there exists an element $x \in J$ such that $(a)x=b$.

For each $x \in M^\Omega$ we may define the mapping matrix $P(x)$, which is $|\Omega|\times|\Omega|$ type matrix whose entry in position $(i, j)$ is 1 if $(i)x=j$ and zero otherwise, where $i, j \in \Omega$. In case $x$ is a permutation, $P(x)$ is called a permutation matrix.

For the subsemigroup $T$ of $M^\Omega$ we define two subsemigroups of $M^\Omega$:

$$C(T)=\{x \in M^\Omega \mid xt=tx \text{ for all } t \in T\}\ ,$$
$$\mathfrak{C}(T)=\{x \in S^\Omega \mid xt=tx \text{ for all } t \in T\}\ .$$

*Definition* 1. Let $T$ be a semigroup on $\Omega$. $\{B_i \mid i=1, \cdots, r\}$ is called a complete block system of $T$, if the following (1) and (2) hold,

(1) $\Omega=B_1+\cdots+B_r$ and $B_i \cap B_j=\phi$ for all pair $i, j (i \neq j)$,

(2) Every element of $T$ either maps all letters of a set $B_i$ into $B_i$ or into another set $B_j$.

Every semigroup $T$ on $\Omega$ has the trivial block systems $\Omega$ and $\{\{a\} \mid a \in \Omega\}$.

PROPOSITION 1. *Let $G$ be a group on $\Omega$ and $J$ be a subsemigroup of $C(G)$. Then, the set of $G$-orbits is a complete block system of $J$.*

*Proof.* Let $\{\Omega_i\}$ be the collection of $G$-orbits. If $(u)x \in \Omega_j$ for $x \in J$ and $u \in \Omega_i$, then since there is some $g \in G$ with $(u)g = v$ for arbitrary $v \in \Omega_i$, we have $(v)x = (u)gx = (u)xg \in (\Omega_j)g = \Omega_j$. Thus $(\Omega_i)x \subseteqq \Omega_j$ holds.

PROPOSITION 2. *If $J$ is a transitive semigroup on $\Omega$, then $\mathfrak{C}(J)$ is semiregular on $\Omega$.*

For the proof, see Lemma 1 of [1].

*Definition* 2. An (finite) automaton $A$ is a triple, $A = (\Omega, I, N)$, where $\Omega$ is a nonempty finite set (of states), $I$ is a semigroup (of inputs) and $N$ is a (next state) function from $\Omega \times I$ into $\Omega$ such that $N(a, xy) = N(N(a, x), y)$ for all $x, y \in I$ and all $a \in \Omega$.

*Definition* 3. An automaton $A = (\Omega, I, N)$ is called connected, if for any $a \in \Omega$ there exist some $x \in I$ and some $b \in \Omega - \{a\}$ such that $N(a, x) = b$ or $N(b, x) = a$. Especially, if for any pair $a, b \in \Omega$ there exists some $x \in I$ such that $N(a, x) = b$, then $A$ is called strongly connected.

*Definition* 4. Let $A = (\Omega, I, N)$ and $B = (\Delta, J, L)$ be two automata. A function $(\eta, \xi)$: $A \longrightarrow B$ is a homomorphism of $A$, if the next conditions hold,

(1) $\xi: I \longrightarrow J$, is one-to-one and onto,

(2) $\eta: \Omega \longrightarrow \Delta$, is a mapping,

(3) $N(a, x)\eta = L((a)\eta, (x)\xi)$ for all $a \in \Omega$ and all $x \in I$. If $\eta$ is, in addition, one-to-one and onto, $(\eta, \xi)$: $A \longrightarrow B$ is called an isomorphism, and $A$ and $B$ are called to be isomorphic to each other, denoted by $A \approx B$. An isomorphism $(\eta, 1)$: $A \longrightarrow A$ is called an automorphism of $A$.

The set of all automorphisms of an automaton $A$ forms a group, denoted by $G(A)$. In this case, the elements $(\eta, 1) \in G(A)$ are denoted simply by $\eta$. Let $A = (\Omega, I, N)$ be an automaton and $(\eta, \xi)$ be a homomorphism of $A$. If $(\eta, \xi)$ is neither an isomorphism nor $|(\Omega)\eta| \neq 1$, then $(\eta, \xi)$ is called a proper homomorphism of $A$.

*Definition* 5. An automaton $A = (\Omega, I, N)$ is called a permutation automaton, if $x^*$: $a \longrightarrow N(a, x)$, $a \in \Omega$, is a permutation on $\Omega$ for all $x \in I$.

Let $A = (\Omega, I, N)$ be an automaton and $x, y \in I$. Define the equivalence relation $\sim$ on $I$ by $x \sim y$ if and only if $N(a, x) = N(a, y)$ for all $a \in \Omega$. We denote by $\bar{x}$ the set of all $y \in I$ such that $x \sim y$ and by $I(A)$ the set of all such classes. $I(A)$ forms a semigroup under the natural operation, i.e., $\bar{x} \cdot \bar{y} = \overline{xy}$. For each $\bar{x} \in I(A)$ we assign the mapping $x^*$: $a \longrightarrow N(a, x)$ where $a \in \Omega$. This is one-to-one mapping and we have $N(a, x) = (a)x^*$. When no confusion will arise, we denote $N(a, x)$ or $(a)x^*$ by $(a)x$ simply and we do not distinguish $I(A)$ from $I(A)^* = \{x^* \mid x \in I(A)\}$.

## Wreath product and automata

For an arbitrary abstract semigroup $J$, we have the right regular representation $\eta$ of $J$ such that

$$(a)\eta_1 \colon x \longrightarrow xa, \text{ where } a, x \in J .$$

This representation is generalized in the following way. Put $\Delta=\{1, 2, \cdots, n\}$ and $J\times\Delta=\{(x, i) \mid x\in J, i\in\Delta\}$, and to each $a\in J$ we assign the transformation on $J\times\Delta$

$$(a)\eta_n \colon (x, i) \longrightarrow (xa, i) .$$

Then we have the right semiregular representation $\eta_n$ of $J$. For an arbitrary positive integer $n$, $(J)\eta_1 \approx (J)\eta_n$. Further, if $J$ is a group, $J\approx(J)\eta_n$ holds.

PROPOSITION 3. *If $G$ is a regular group on $\Omega$, then $\mathfrak{C}(G)$ is regular on $\Omega$ and $C(G)=\mathfrak{C}(G)$. Furthermore $\mathfrak{C}(G)$ is isomorphic to $G$.*

*Remark* 1. If $G$ is abelian, $\mathfrak{C}(G)=G$, [4]. If $G$ is not abelian, for $g_i\in G$ and $h_i\in\mathfrak{C}(G)$ the mapping $g_i\longrightarrow h_i$ is an inverse isomorphism as permutation group, where $(1)g_i=(1)h_i=i$ for some fixed $1\in\Omega$.

PROPOSITION 4. *Let $G$ be a semiregular group of order $k$ on $\Omega$ and $G_1$ be the constituent of $G$ on a $G$-orbit. Then $x\in C(G)$ if and only if $P(x)$ satisfies the following condition:*

(1) *$P(x)=(X_{ij})$, where $X_{ij}\in P(\mathfrak{C}(G_1))$ or $X_{ij}=0$, $k\times k$ type zero matrix, for $1\leqq i, j\leqq |\Omega|/k$.*

(2) *For each $i$, $i=1, 2, \cdots, |\Omega|/k$, there exists a unique number $t$ such that $X_{it}\in P(\mathfrak{C}(G_1))$.*

COROLLARY 1. *Let $G$ be a semiregular group on $\Omega$ and $\Omega_1, \Omega_2, \cdots, \Omega_n$ be the all $G$-orbits. Then we have*

(1) *For an arbitrary subsemigroup $J$ of $C(G)$, $\bar{\Omega}=\{\Omega_i \mid i=1, 2, \cdots, n\}$ is a complete block system of $J$.*

(2) *For any fixed element $x\in C(G)$ and any pair $s, t\in\Omega_i (s\neq t)$, $(s)x\neq(t)x$ holds.*

Let $\Omega=\{a, b, \cdots, c\}$ and $\Delta=\{1, 2, \cdots, n\}$ be two finite set and let $H$ and $K$ be subsemigroup of $M^\Omega$ and $M^\Delta$, respectively. We define the semigroup $\tilde{H}=H\times H\times\cdots\times H$ ($n$-product) on $\Omega\times\Delta$ by

$$(d, i)[h_1, h_2, \cdots, h_n]=((d)h_i, i) ,$$

for all $(d, i)\in\Omega\times\Delta$ and all $[h_1, h_2, \cdots, h_n]\in\tilde{H}$. The element $\varphi\in K$ acts naturally on $\Omega\times\Delta$ by

$$(d, i)\varphi=(d, (i)\varphi) .$$

The semigroup generated by $\tilde{H}$ and $K$ is called the wreath product of

$H$ by $K$, and denoted by $H \wr K$. The product $[h_1, h_2, \cdots, h_n]\cdot\varphi$ is written by $[h_1, h_2, \cdots, h_n\colon \varphi]$. Since $\varphi\cdot[h_1, h_2, \cdots, h_n]=[h_{(1)\varphi}, h_{(2)\varphi}, \cdots, h_{(n)\varphi}\colon \varphi]$, we have

$$[h_1, h_2, \cdots, h_n\colon \varphi]\cdot[g_1, g_2, \cdots, g_n; \eta]=[h_1 g_{(1)\varphi}, h_2 g_{(2)\varphi}, \cdots, h_n g_{(n)\varphi}\colon \varphi\eta] .$$

From Proposition 4 we have the following Proposition.

PROPOSITION 5. *Let $G$ be a semiregular on $\Omega$ and $G_1$ be the constituent of $G$ on a $G$-orbit. If $|\Omega|/|G|=n$, then*

$$C(G)=\mathfrak{S}(G_1)\wr M^n .$$

Let $H_1$ and $H_2$ be two transformation semigroups on $\Omega_1$ and $\Omega_2$, respectively. If $\Omega_1 \cap \Omega_2=\phi$, the semigroup $H_1+H_2$ on $\Omega_1 \cup \Omega_2$ is defined by

$$(a)(h_1+h_2)=\begin{cases}(a)h_1 & \text{if } a\in\Omega_1 ,\\ (a)h_2 & \text{if } a\in\Omega_2 ,\end{cases}$$

where $h_i \in H_i$, $i=1, 2$. This definition generalizes in the obvious way to the case of more than two factors.

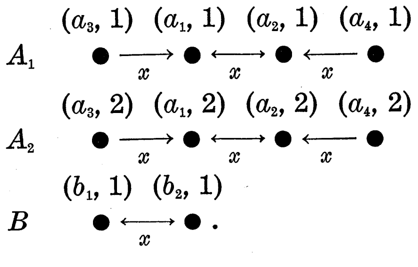
*Definition* 6. Let $A_i=(\Omega_i, I, N_i)$, $i=1, 2, \cdots, n$, be automata such that $\Omega_i \cap \Omega_j=\phi$ $(i\neq j)$. The sum of outomata, $A=A_1+A_2+\cdots+A_n$, is the automaton $A=(\cup_{i=1}^{n} \Omega_i, I, N)$ where $N(a, x)=N_i(a, x)$ if $a\in\Omega_i$. If $A_1 \approx A_i$ for any $i$, the sum of automata $A=A_1+A_2+\cdots+A_n$ is denoted by $nA_1$.

We notice that the theorem in chapter 14 of [5] is also true for finite automaton.

PROPOSITION 6. *Let $A$ be an automaton such that $A=n_1A_1+n_2A_2+\cdots+n_rA_r$, where $A_i$ is connected for any $i$. Then*

$$G(A)=G(A_1)\wr S^{n_1}+G(A_2)\wr S^{n_2}+\cdots+G(A_r)\wr S^{n_r} .$$

*Example* 1. We consider the following state diagrams:


$(a_3, 1)$ $(a_1, 1)$ $(a_2, 1)$ $(a_4, 1)$
$A_1$
$x$
$x$
$x$
$(a_3, 2)$ $(a_1, 2)$ $(a_2, 2)$ $(a_4, 2)$
$A_2$
$x$
$x$
$x$
$(b_1, 1)$ $(b_2, 1)$
$B$
$x$

$A=2A_1+B$ is a sum of automata. $\tau=(1, 2)$ is an isomorphism from $A_i$ to $A_j$ $(1\leqq i\neq j\leqq 2)$ by

$$(a, i)\tau=(a, (i)\tau)=(a, j) .$$

Put $h=(a_1, a_2)(a_3, a_4)$, then $G(A_1)=G(A_2)=\langle h\rangle$ by

$$(a, i)h=((a)h, i),\ i=1, 2\ .$$

In the same manner, $G(B)=\langle k=(b_1, b_2)\rangle$. Thus $G(A)=\langle h\rangle\wr S^2+\langle k\rangle\wr S^1$.

COROLLARY 2. *Let $A=(\Omega, I, N)$ be an automaton such that $G(A)$ is a nontrivial semiregular group. Then $A$ is connected excepting the case $A=2A_1$ for some connected automaton $A_1$ such that $G(A_1)=1$.*

*Proof.* We may write $A=n_1A_1+\cdots+n_rA_r$ where $r\geqq 1$, $n_i\geqq 1$ and $A_i$ is connected for all $i$.

Suppose $r\geqq 2$. If $n_i\geqq 2$ for some $i$, then $G(A)$ is not semiregular. Thus $n_i=1$ for all $i$ and $A=A_1+\cdots+A_r$. Since $G(A)\neq 1$ and $G(A)=G(A_1)+\cdots+G(A_r)$, $G(A)$ is not semiregular. Thus we have $r=1$ and $A=n_1A_1$. If $n_1\geqq 3$, then $G(A)$ is not semiregular. If $n_1=2$, then $G(A)=G(A_1)\wr S^2$. Since $G(A)$ is semiregular, we have $G(A_1)=1$.

*Definition* 7. Let $H$ and $K$ be subsemigroups of $M^\Omega$ and $M^\Delta$, respectively. Further let $J$ be a subsemigroup of $H\wr K$. An automaton $(J\colon H\wr K)=(\Omega\times\Delta, J, N)$ is defined by $N((a, i), x)=(a, i)x$ for all $(a, i)\in\Omega\times\Delta$ and $x\in J$.

Let $G_1$ be a regular group on $\Omega_1$ and $|\Omega_1|=n$, and put $G=\{[g, g, \cdots, g\colon 1]\mid g\in G_1\}$. If $J$ is a subsemigroup of $\mathfrak{S}(G_1)\wr M^n$, then $\mathfrak{S}(J)\supseteqq G$. Thus $G$ is contained in the automorphism group of $(J\colon \mathfrak{S}(G_1)\wr M^n)$.

PROPOSITION 7. *If $A=(\Omega, I, N)$ is an automaton with $n$ states and $G=G(A)$ is semiregular on $\Omega$ of order $k$. Then there exists a subsemigroup $J$ of $\mathfrak{S}(G_1)\wr M^{n/k}$ such that $(J\colon \mathfrak{S}(G_1)\wr M^{n/k})\approx A$,* where $G_1$ is a constituent of $G$ on a $G$-orbit.

*Proof.* Let $\Omega_1, \Omega_2, \cdots, \Omega_r, (r=n/k)$, be the $G$-orbit on $\Omega$. The constituent $G_i$ of $G$ on $\Omega_i$, $i=1, 2, \cdots, r$, is isomorphic to $G$. Since $G_i$ is isomorphic to $G_j$ as permutation group for any $i$ and $j$, we may assume that if we put $\Omega_i=\{a_{1,i}, a_{2,i}, \cdots, a_{k,i}\}$, $i=1, 2, \cdots, r$, then

$$g=\begin{pmatrix}\cdot\cdot & a_{t,1} & \cdot\cdot\\ & a_{j,1} & \end{pmatrix}\begin{pmatrix}\cdot\cdot & a_{t,2} & \cdot\cdot\\ & a_{j,2} & \end{pmatrix}\cdots\begin{pmatrix}\cdot\cdot & a_{t,r} & \cdot\cdot\\ & a_{j,r} & \end{pmatrix}$$

holds for all $g\in G$.

Define the mapping $\eta\colon a_{t,i}\longrightarrow(a_t, i)$, then $\eta$ is a one-to-one mapping from $\Omega$ onto $\Omega_1'\times\Delta$, where $\Omega_1'=\{a_1, a_2, \cdots, a_k\}$ and $\Delta=\{1, 2, \cdots, r\}$. Since $I(A)\subseteqq C(G)$, $\bar{\Omega}=\{\Omega_i\mid i=1, 2, \cdots, r\}$ is a complete block system of $I(A)$. For each $x\in I$ we define the mapping $\varphi_x$ on $\Delta$ by $(i)\varphi_x=j$, if $(\Omega_i)x\subseteqq\Omega_j$. Let $P(x)=(X_{s,t})$, where $X_{s,t}$ is $k\times k$ type matrix for each $s$ and $t$ $(1\leqq s, t\leqq r)$. Define the mapping $\rho\colon I(A)\longrightarrow\mathfrak{S}(G_1)\wr M^\Delta$ by

$$(\bar{x})\rho=[g_1, g_2, \cdots, g_n\colon \varphi_x]\ ,$$

where $P(g_i)=X_{i,m}$ for $i=1, 2, \cdots, n$. Then $N(a_{t,i}, x)=(a_{t,i})x=a_{u,v}$, where $a_u=(a_t)g_i$ and $v=(i)\varphi_x$. Thus

$$\begin{aligned}(N(a_{t,i}, x))\eta&=(a_u, v)\\ &=(a_t, i)[g_1, g_2, \cdots, g_n\colon \varphi_x]\\ &=(a_{t,i})\eta[g_1, g_2, \cdots, g_n\colon \varphi_x]\,.\end{aligned}$$

By putting $J=\langle I(A)\rho\rangle$ we have an isomorphism from $A$ onto $(J\colon \mathfrak{S}(G_1)\wr M^{n/k})$. The proposition is proved.

Let $G_1$ be regular on $\Omega_1$ and $\Delta=\{1, 2, \cdots, n\}$. For a subsemigroup $J\subseteqq\mathfrak{S}(G_1)\wr M^\Delta$ we define the following sets $\hat{J}$ and $J^{(j)}$.

$$\hat{J}=\{\varphi\in M^\Delta\,|\,[*, \cdots, *\colon\varphi]\in J\}\,.$$

Let $M_i=\{\eta\in M^\Delta\,|\,(i)\eta=i\}$. We put

$$J^{(i)}=\{h_i\in\mathfrak{S}(G_1)\,|\,[*, \cdots, \underset{\substack{\uparrow\\ i\text{-th}}}{h_i}, \cdots, *\colon\eta]\in J,\ \eta\in M_i\}\,.$$

PROPOSITION 8. *Let $H$ be a regular group on $\Omega$ and $J$ be a subsemigroup of $H\wr M^\Delta$, where $\Delta$ is finite. Then $J$ is transitive on $\Omega\times\Delta$ if and only if* (1) *and* (2) *hold:*

(1) *$\hat{J}$ is transitve on $\Delta$,*

(2) *$J^{(i)}=H$ for all $i=1, 2, \cdots, |\Delta|$.*

*Proof.* Suppose that $J$ is transitive on $\Omega\times\Delta$. Then for any pair $(a, i), (b, i)\in\Omega\times\Delta$ there exists $x\in J$ with $(a, i)x=(b, i)$. Thus $J^{(i)}\neq\phi$ and we have (1) and (2).

Conversely if (1) and (2) hold, then for arbitrary pair $(a, i), (b, j)\in\Omega\times\Delta$ there exists some $x=[g_1, g_2, \cdots, g_n\colon\eta]\in J$ such that $(i)\eta=j$ from (1). Since $H$ is transitive, there is $h_j\in H$ such that $((a)g_i)h_j=b$. By (2) there is some $y=[h_1, \cdots, h_j, \cdots, h_n\colon\tau]\in J$ with $(j)\tau=j$. Thus we have $xy\in J$ and $(a, i)xy=(b, j)$.

## Homomorphism and automorphism of automata

The imprimitive permutation groups are reduced to transitive groups of smaller degree by Proposition 7.2 of [4]. This is generalized to the representation of transformation semigroup which has the complete block system.

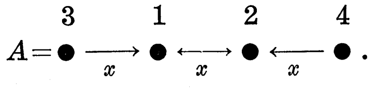
Let $J$ be a semigroup on $\Omega$ and let $\bar{\Omega}=\{\Omega_i\,|\,i=1, 2, \cdots, r\}$ be a complete block system of $J$. Then we may define the action of $x\in J$ on $\bar{\Omega}$ by $(\Omega_i)x=\Omega_j$, if $(\Omega_i)x\subseteqq\Omega_j$.

PROPOSITION 9. *The automaton $A=(\Omega, I, N)$ has a proper homomorphic image if and only if $I(A)$ has a complete nontrivial block system.*

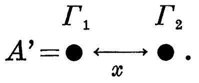
*Proof.* Let $\bar{\Omega}=\{\Omega_i\,|\,i=1, 2, \cdots, r\}$ be a complete nontrivial block system of $I(A)$. We define the function $\bar{N}\colon \bar{\Omega}\times I\rightarrow\bar{\Omega}$ by $\bar{N}(\Omega_i, x)=\Omega_j$, if $(\Omega_i)x\subseteqq\Omega_j$. Since $\bar{\Omega}$ is a complete block system of $I(A)$, $\bar{A}=(\bar{\Omega}, I, \bar{N})$ is an automaton. The mapping $\eta\colon \Omega\rightarrow\bar{\Omega}$ is defined by $(a)\eta=\Omega_i$, if $a\in\Omega_i$. Then $\eta$ is a proper homomorphism from $A$ onto $\bar{A}$, since $r\neq 1, |\Omega|$.

Conversely, suppose that $A$ has a proper homomorphic image $A'=(\Omega', I, N')$. For $u_i\in\Omega'$ we put $\Gamma(u_i)=\{a\in\Omega\,|\,(a)\eta=u_i\}$. Then $\Omega=\Gamma(u_1)+\cdots+\Gamma(u_r)$ and $\Gamma(u_i)\cap\Gamma(u_j)=\phi$, if $i\neq j$. Since $N(a, x)\eta=N'((a)\eta, x)=N'(u_i, x)$ for all $a\in\Gamma(u_i)$ and $x\in I$, we have $\Gamma(u_i)x\subseteqq\Gamma((u_i)x)=\Gamma(u_j)$ for some $j$. Thus $\{\Gamma(u_i)\,|\,i=1, \cdots, r\}$ is a complete nontrivial block system of $I(A)$.

*Example* 2. Consider the next state diagram:



Since $\{1, 4\}x=\{2\}$ and $\{2, 3\}x=\{1\}$, $\{\Gamma_1=\{1, 4\}, \Gamma_2=\{2, 3\}\}$ is a complete block system of $I(A)=\langle x\rangle$. The next $A'$ is a proper homomorphic image of $A$.



Let $J$ be an semigroup on $\Omega$ and $H$ be a subgroup of $\mathfrak{G}(J)$, furthermore $\bar{\Omega}=\{\Omega_1, \Omega_2, \cdots, \Omega_r\}$ be the set of $H$-orbits. Suppose that $(u)x\in\Omega_j$ for some $u\in\Omega_i$ and $x\in J$. Since for any $v\in\Omega_i$ there is $h\in H$ with $(u)h=v$, we have $(\Omega_i)x\subseteqq\Omega_j$. Thus $\bar{\Omega}$ is a complete block system of $J$. Therefore we have a representation of $J$ on $\bar{\Omega}$.

*Definition* 8. Let $A=(\Omega, I, N)$ be an automaton and $G(A)$ be its automorphism group. Then for a subgroup $H$ of $G(A)$, the factor automaton $A/H$, is $A/H=(\Omega_H, I, N_H)$ where $\Omega_H$ is the set of $H$-orbits and $N_H(\bar{s}, x)=\overline{N(s, x)}$ for all $\bar{s}\in\Omega_H$ and all $x\in I$.

PROPOSITION 10. *Let $A=(\Omega, I, N)$ be a strongly connected automaton and $H$ be a normal subgroup of $G(A)$. Then, $G(A)/H$ is isomorphic to a subgroup of $G(A/H)$. Furthermore, if $|\Omega|=|G(A)|$, then $G(A)/H\approx G(A/H)$ holds.*

For the proof, see [2].

The next Proposition is due to Ito [3]. The proof given here is different from that given by [3].

PROPOSITION 11. *Let $A=(\Omega, I, N)$ be a strongly connected automaton such that $|\Omega|=p|G(A)|$, where $p$ is a prime unmber, and $H$ be* *a normal subgroup of* $G(A)$. *Then if* $A$ *is not a permutation automaton,* $G(A)/H \approx G(A/H)$.

*Proof.* Since $A$ is strongly connected, $A/H$ is strongly connected. Thus $G(A/H)$ is semiregular on $\Omega_H$, where $\Omega_H$ is the set of $H$-orbits. The number of $H$-orbits is $p|G(A)|/|H|$. Since $|G(A)/H| \leqq |G(A/H)|$ by Proposition 10, $|G(A/H)|=|G(A)/H|$ or $p|G(A)/H|$. If $|G(A/H)|=|G(A)/H|$, then $G(A/H) \approx G(A)/H$.

Suppose that $|G(A/H)|=p|G(A)/H|$. Then $G(A/H)$ is regular on $\Omega_H$. $I(A)$ is a semiregular group on $\Omega_H$ by Proposition 3. Since $I(A)$ is transitive on $\Omega_H$, $I(A)$ is a regular group on $\Omega_H$. Let $\Gamma$ be a $H$-orbit. Since $\Gamma$ is contained some $G(A)$-orbit, by Corollary 1 we have $(s)x \neq (t)x$ for any pair $s, t \in \Gamma$ $(s \neq t)$ and for $x \in I(A)$. Thus $I(A)$ is a permutation group on $\Omega$.

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