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On $p^{\omega+n}$-projective abelian p-groups

by

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Recently, a great deal of attention has been given to totally projective and p^{α}-projective primary groups. The first authors together with M. Rafiq have studied $p^{\omega+1}$-projective p-groups in [2]. In this article we establish a few characterizations and properties of $p^{\omega+n}$-projective groups. We also show that every p-group contains subgroups of the form $\mathrm{Tor}(A, P)$ where P is the Prüfer group and A is a subgroup of the given group. All groups considered are abelian primary groups.

Let α be an ordinal number. A subgroup H of G is said to be p^{α}-pure in G if the short exact sequence:

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0\ ,$$

corresponds to an element of $p^{\alpha}\,\mathrm{Ext}(G/H, H)$. A p-group G is said to be p^{α}-projective if $p^{\alpha}\,\mathrm{Ext}(G, X)=0$ for all group X. In this article we restrict ourselves to the case $\alpha=\omega+n$ where ω is the first infinite ordinal and n is a natural number.

The symbol $\bigoplus_c$ denotes direct sums of cyclic groups. Most of the notation is the same as in [4].

I. Purity and $p^{\omega+n}$-projectivity.

In this section, we consider various properties involving pure subgroups. The following result makes it easier to handle $p^{\omega+n}$-purity.

LEMMA 1.1 ([5]). *A subgroup C of E is $p^{\omega+n}$-pure in E if and only if there exists a subgroup M in E such that*:

$$M \cap C=0,\ (M+C)/C=(E/C)[p^n],\ \text{and}\ \ (M+C)/M$$

is pure in E/M.

We apply this immediately to obtain a simple method of producing $p^{\omega+n}$-pure subgroups.

LEMMA 1.2. *Let H be a pure subgroup of G. Then $H/H[p^n]$ is $p^{\omega+n}$-pure in $G/H[p^n]$.*

Proof. In Lemma 1.1, let $E=G/H[p^n]$, $C=H/H[p^n]$ and $M=G[p^n]/H[p^n]$. It is a routine matter to see that the conditions there are satisfied.

LEMMA 1.3 ([5]). *Let C be a $p^{\alpha+1}$-pure subgroup of E. If E is p^{α}-projective then C is a summand of E, and thus C and E/C are p^{α}-projective.*

We establish next a generalization of a result in [1].

THEOREM 1.4. *Let K be a pure subgroup of G, such that $G/K[p^m]$ is $p^{\omega+n}$-projective for some n, $0\leq n<m$ then G/K is $p^{\omega+n}$-projective. Further, if K is $p^{\omega+n}$-pure in G then it is a summand of G and G is $p^{\omega+n}$-projective.*

Proof. By Lemma 1.2, $K/K[p^m]$ is $p^{\omega+m}$-pure in $G/K[p^m]$ and since $n<m$ it is also $p^{\omega+n+1}$-pure. But $G/K[p^m]$ is $p^{\omega+n}$-projective and an immediate application of Lemma 1.3 and the isomorphism theorems yeld that G/K is $p^{\omega+n}$-projective. If K is also $p^{\omega+n}$-pure then K is a summand of G. Now $K/K[p^m]\simeq p^mK$ is a subgroup of a $p^{\omega+n}$-projective group and therefore it is $p^{\omega+n}$-projective and K also is $p^{\omega+n}$-projective (see remarks after Lemma 2.2). Since $G\simeq K\oplus G/K$ we see that G is $p^{\omega+n}$-projective.

THEOREM 1.5. *Let K be a pure subgroup of G such that G/K is $p^{\omega+n}$-projective. Then G/H is isomorphic to G/K for every pure subgroup H of G satisfying $H[p^{n+1}]=K[p^{n+1}]$.*

Proof. Let $B=K[p^n]$. By Lemma 1.2 the sequence $0\rightarrow K/B\rightarrow G/B\rightarrow G/K\rightarrow 0$ is $p^{\omega+n}$-pure exact and since G/K is $p^{\omega+n}$-projective it must be split exact and K/B is a summand of G/B. Now $H[p^{n+1}]=K[p^{n+1}]$ implies that $H[p^n]=B$ and $(H/B)[p]=H[p^{n+1}]/B=(K/B)[p]$. A well known result shows that H/B is also a summand of G/B. In fact if $G/B=K/B\oplus R/B$ then $G/B=H/B\oplus R/B$ and $G/H\simeq G/K$.

II. Some chracterizations of $p^{\omega+n}$-projective p-groups

We recall first two important results:

LEMMA 2.1 ([5]). *A group E is $p^{\omega+n}$-projective if and only if there exists a subgroup $B\subset E[p^n]$ such that $E/B=\oplus_c$.*

LEMMA 2.2 ([2]). *Let G be a group. There exists an exact sequence $0\rightarrow K\rightarrow G\rightarrow M\rightarrow 0$, where $M=\oplus_c$ and $p^nK=0$ if and only if there exists an exact sequence $0\rightarrow S\rightarrow F\rightarrow G\rightarrow 0$ where $p^nS=0$ and $F=\oplus_c$.*

These two results have numerous consequences. In particular we see that every subgroup of a $p^{\omega+n}$-projective group is likewise $p^{\omega+n}$-projective and a group G is $p^{\omega+n}$-projective if p^mG is $p^{\omega+n}$-projective for some natural number m.

The following results parallel those obtained in [2] for $p^{\omega+1}$-projective p-groups.

THEOREM 2.3. *Let* $0\to K\to F\to G\to 0$ *be a pure exact sequence where* $F=\bigoplus_c$. *Then,* G *is* $p^{\omega+n}$*-projective if and only if it is isomorphic to a summand of* $F/K[P^n]$.

Proof. If G is $p^{\omega+n}$-projective, let $B=K[p^n]$ then, by Lemma 1.2 the sequence $0\to K/B\to F/B\to G\to 0$ is $p^{\omega+n}$-pure and therefore K/B is a summand of F/B. Conversely if G is isomorphic to a summand of F/B, it is $p^{\omega+n}$-projective since by Lemma 2.2 F/B is $p^{\omega+n}$-projective.

THEOREM 2.4. *Let* G *be a group and* $n\geq 1$. G *is* $p^{\omega+n+1}$*-projective if and only if there exists a* $p^{\omega+n}$*-projective group* H, *such that* $G\simeq H/S$ *where* $S\subset H[p]$.

Proof. By Lemma 2.2, if G is $p^{\omega+n+1}$-projective then $G\simeq F/B$ where $F=\bigoplus_c$ and $p^{n+1}B=0$. Let $H=F/pB$ and $S=B/pB$ then $p^n(pB)=0$ so H is $p^{\omega+n}$-projective and $pS=0$. Conversely if $G\simeq H/S$ where $S\subset H[p]$ and H is $p^{\omega+n}$-projective then by Lemma 2.2 $H\simeq F/K$ where $F=\bigoplus_c$ and $p^nK=0$ let $S\simeq M/K$ where $M\subset F$ then $p^{n+1}M=0$ and $F/M\simeq G$ and therefore G is $p^{\omega+n+1}$-projective.

COROLLARY 2.5. *If* G *is* $p^{\omega+n}$*-projective and* $S\subset G[p^m]$ *then* G/S *is* $p^{\omega+n+m}$*-projective.*

THEOREM 2.6. *Let* G *be a group and* $n\geq 1$. *Then* G *is* $p^{\omega+n+1}$*-projective if and only if there is a subgroup* S *of* $G[p]$ *such that* G/S *is* $p^{\omega+n}$*-projective.*

Proof. If G is $p^{\omega+n+1}$-projective then there exists $A\subset G[p^{n+1}]$ such that $G/A=\bigoplus_c$. Let $S=A[p]$ then G/S is $p^{\omega+n}$-projective by Lemma 2.1. Conversely if $S\subset G[p]$ and G/S is $p^{\omega+n}$-projective then there exists $K/S\subset G/S$ such that $(G/S)/(K/S)=\bigoplus_c$ and $p^n(K/S)=0$. Therefore $G/K=\bigoplus_c$, $p^{n+1}K=0$ and the result follows from Lemma 2.1.

III. An imbedding theorem.

It is well known that not every group is contained in a group of the form $\mathrm{Tor}(A, B)$ where A and B are reduced. However we show that every unbounded group G contains a subgroup of the form $\mathrm{Tor}(A, P)$ where P is the reduced Prüfer group and A is a subgroup of G. Furthermore if $G\neq\bigoplus_c$ then A can be chosen so that $\mathrm{Tor}(A, P)\neq\bigoplus_c$. But first we recall a result in [2].

LEMMA 3.1 ([2]). *Let* G *be a group such that* $G\neq\bigoplus_c$ *and* $G^1=0$. *Then* G *contains a subgroup* B *such that* $B=H\oplus K$ *where* $H\neq\bigoplus_c$, $K=\bigoplus_c$ *and fin* $r(K)=|H|$.

We need also

LEMMA 3.2 ([6]). *If* $0 \to A \to B \to C \to 0$ *is a* p^α*-pure exact sequence with* $\alpha \geq w$ *and* E *is any group then*

$$0 \to \mathrm{Tor}(A, E) \to \mathrm{Tor}(B, E) \to \mathrm{Tor}(C, E) \to 0$$

is a p^α*-pure exact sequence.*

LEMMA 3.3. *Let* $B = \bigoplus_{n=1}^{\infty} Z(p^n)$ *and* P *the Prüfer group, then* B *contains a copy of* $\mathrm{Tor}(B, P)$ *and there exists a summand* A *of* P *such that* P *contains a copy of* $\mathrm{Tor}(A, P) \neq \bigoplus_c$.

Proof. $\mathrm{Tor}(B, P)$ is isomorphic to a countable direct sum K of copies of B and B contains a copy of K. By [3] (pb. 19 a, p. 143), we can write $P = A \oplus B'$ where $B' = \bigoplus_c$ is unbounded. Now, there exists a pure exact sequence $0 \to B \to P \to Z(p^\infty) \to 0$ where B is high in P. Let $S = B[p]$, then the sequence $0 \to B/S \to P/S \to Z(p^\infty) \to 0$ is by Lemma 1.2, $p^{\omega+1}$-pure exact. But $B/S \simeq B$ and $P/S \simeq P$ thus there exists a $p^{\omega+1}$-pure exact sequence

$$0 \to B \to P \to Z(p^\infty) \to 0 .$$

By Lemma 3.2, $0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, P) \to \mathrm{Tor}(A, Z(P^\infty)) \to 0$ is $P^{\omega+1}$-pure exact, but $\mathrm{Tor}(A, Z(p^\infty)) \simeq A$ which is $p^{\omega+1}$-projective since P is $p^{\omega+1}$-projective, therefore $\mathrm{Tor}(A, P) \simeq A \oplus \mathrm{Tor}(A, B)$. However $\mathrm{Tor}(A, B)$ is a countable $\bigoplus_c$ so that B' contains a copy of it and P contains a copy of $\mathrm{Tor}(A, P)$ which is clearly not $\bigoplus_c$.

THEOREM 3.4. *Let* G *be an unbounded group. Then* G *contains a subgroup of the form* $\mathrm{Tor}(A, P)$ *where* P *is the Prüfer reduced group and* A *is a subgroup of* G. *Moreover if* $G \neq \bigoplus_c$, *then* A *can be chosen so that* $\mathrm{Tor}(A, P) \neq \bigoplus_c$.

Proof. We break the proof into three cases:
1) $G = \bigoplus_c$ 2) $G^1 \neq 0$ and 3) $G \neq \bigoplus_c$ and $G^1 = 0$.

Case 1 and 2 are easily settled by Lemma 3.3. Indeed the groups respectively in these cases contain a copy of B and a copy of P.

Case 3. If $G \neq \bigoplus_c$ and $G^1 = 0$ then by Lemma 3.1 there exists subgroups M and N of G such that $M \neq \bigoplus_c$ and $N = \bigoplus_c$, fin $r(N) = |M|$ and $M \cap N = 0$. Furthermore M can be chosen to be $p^{\omega+1}$-projective by Theorem 2.9 p. 205 in [2]. Thus using the $p^{\omega+1}$-pure exact sequence $0 \to B \to P \to Zp^\infty \to 0$ obtained in Lemma 3.3 we derive that $\mathrm{Tor}(M, P) \simeq \mathrm{Tor}(M, B) \oplus M$. Now $\mathrm{Tor}(B, M)$ is $\bigoplus_c$, a copy of which can be found in N and thus a copy of $\mathrm{Tor}(M, P)$ exists in $M \oplus N \subset G$, clearly $\mathrm{Tor}(M, P) \neq \bigoplus_c$.

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