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Special linear group and generating functions

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1. Introduction

In this paper we aim at obtaining certain generating functions by using Lie theory method. The process in short involves introducing linear differential operators which form a 3-dimensional Lie algebra isomorphic tot he Lie algebra $sl(2)$ [2, p. 8]. Based on these operators local multiplier representation $[T(g)f](x,y)g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$, is determined. Thereafter, by choosing $f(x, y)$ in certain ways this multiplier representation leads to generating functions for Laguerre functions.

2. Differential operators and multiplier representation

(2.1) $$u(x)=L_{m+n}^{(\gamma)}(x)$$

is a solution of [3; p. 204]

(2.2) $$x\frac{d^2u}{dx^2}+(1+\gamma-x)\frac{du}{dx}+(m+n)u=0 \ .$$

By substituting $y(\partial/\partial y)$ for n we construct the following partial differential equation.

(2.3) $$\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+y\right]f(x, y)=0$$

$f(x, y)=y^n u(x)$ is a solution of (2.3). Now, we introduce the first order partial differential operators

(2.4) $$\begin{aligned} J^3&=y\frac{\partial}{\partial y}+m+\frac{\gamma+1}{2} \ , \\ J^+&=xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+(m+\gamma+1-x)y \ , \\ J^-&=xy^{-1}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-my^{-1} \ , \end{aligned}$$

obeying the commutation relations

(2.5) $$[J^3, J^\pm]=\pm J^\pm, \quad [J^+, J^-]=2J^3 \ .$$

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These J-operators from the basis of a Lie algebra isomorphic to the Lie algebra $sl(2)$ [2].

The Casimir operator

$$(2.6)\qquad C=J^+J^-+J^3J^3-J^3 = x^2\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+mx+\frac{\gamma^2-1}{4}$$

commutes with J^3, J^+ and J^-. (2.3) may be rewritten as

$$(2.7)\qquad Cf(x, y)=\left(\frac{\gamma^2-1}{4}\right)f(x, y)\ .$$

To determine the multiplier representation induced by J operators, we need to compute the expression [2; p. 3] $e^{a'J^-}e^{b'J^+}e^{c'J^3}$

$$(2.8)\qquad e^{a'J^-}e^{b'J^+}e^{c'J^3}f(x, y)=\exp\left\{\left(\frac{m+\gamma+1}{2}\right)c'\right\} \times\exp\left\{\frac{-b'xy}{(1+a'b'-b'y)}\right\}(1+a'b'-b'y)^{-(\gamma+1+m)}\left(1-\frac{a'}{y}\right)^m \times f\left(\frac{xy}{(y-a')(1+a'b'-b'y)}, \frac{y-a'}{(1+a'b'-b'y)}e^{c'}\right)\ .$$

The complex parameters a', b' and c' are related to $g\in SL(2)$,

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad ab-bc=1\quad \text{by [2; p. 8]}$$

$$e^{c'/2}=a,\quad a'=-c/a,\quad b'=-ab\ .$$

Therefore, for g in a sufficiently small neighbourhood of the identity element $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\in SL(2)$

$$(2.9)\qquad [T(g)f](x, y)=\exp\left(\frac{bxy}{d+by}\right)(d+by)^{-(\gamma+1+m)}\left(a+\frac{c}{y}\right)^m \times f\left(\frac{xy}{(c+ay)(d+by)}, \frac{c+ay}{d+by}\right),$$

$|by/d|<1,\ -\pi<\arg a,\ \arg d<\pi,\ ad-bc=1.$

3. Generating functions

A. We choose $f(x, y)$ to be a common eigenfunction of the operators C and $J^3J^3+(\gamma'-\gamma-2m-1)J^3-J^+$.

Let $f(x, y)$ satisfy the simultaneous equations

$$(3.1)\qquad Cf(x, y)=\left(\frac{\gamma^2-1}{4}\right)f(x, y)\ ,$$

$$[J^3J^3+(\gamma'-\gamma-2m-1)J^3-J^+]f(x, y)=\left(\frac{m+\gamma+1}{2}\right)\left(\gamma'-m-\frac{\gamma+1}{1}\right)f(x, y)$$

which may be rewritten as

$$\left[x\frac{\partial^2}{\partial x^2}+(\gamma+1-x)\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-(m+\gamma+1)\right]e^x f(-x, y)=0\,, \tag{3.2}$$

$$\left[y\frac{\partial^2}{\partial y^2}+(\gamma'+1-y)\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}-(m+\gamma+1)\right]e^x f(-x, y)=0\,.$$

These equations have a solution [1; 234]

$$e^x f(-x, y)=\psi_2(m+\gamma+1; \gamma+1, \gamma'+1; x, y) \tag{3.3}$$

where

$$\psi_2(\alpha; \gamma, \gamma'; x, y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}x^m y^n}{m!\,n!\,(\gamma)_m(\gamma')_n}\,.$$

We rewrite (3.3) as

$$f(x, y)=e^x\psi_2(m+\gamma+1; \gamma+1, \gamma+1; -x, y)\,. \tag{3.4}$$

Therefore

$$[T(g)f](x, y)=\exp\left(\frac{axy}{c+ay}\right)(d+by)^{-(\gamma+1+m)} \tag{3.5}$$

$$\times\left(a+\frac{c}{y}\right)^m\psi_2\left[m+\gamma+1; \gamma+1, \gamma'+1; \frac{-xy}{(c+ay)(d+by)}, \frac{c+ay}{(d+by)}\right],$$

$$\left|\frac{by}{d}\right|<1,\quad -\pi<\arg a,\ \arg d<\pi,\quad ad-bc=1\,.$$

$[T(g)f](x, y)$ satisfies

$$C[T(g)f](x, y)=\left(\frac{\gamma^2-1}{4}\right)[T(g)f](x, y)\,. \tag{3.6}$$

(3.5) has an expansion of the form

$$[T(g)f](x, y)=\sum_{n=-\infty}^{\infty}k_n(g)L_{m+n}^{(\gamma)}(x)y^n\,. \tag{3.7}$$

Putting $x=0$, this gives

$$k_n(g)=a^m(-b)^n d^{-(\gamma+m+n+1)}\frac{\Gamma(1+m+n)}{(1+\gamma)_m} \tag{3.8}$$

$$\times\sum_{p=0}^{\infty}\frac{(-m)_p(1+\gamma+m+n)_p}{p!\,\Gamma(1+n+p)}\left(\frac{bc}{ad}\right)^p$$

$$\times{}_2F_2\left[\begin{matrix}-n-p, & 1+m\\ \gamma'+1, & 1+m-p\end{matrix}; \frac{a}{b}\right].$$

Thus the generating function (3.7) becomes

$$(3.9)\qquad \exp\left(\frac{axy}{c+ay}\right)\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^{m}$$
$$\times\psi_2\left[m+\gamma+1;\gamma+1,\gamma'+1;\frac{-xy}{(c+ay)(d+by)},\frac{c+ay}{(d+by)}\right]$$
$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m+n)(-b/d)^n y^n}{(1+\gamma)_m}L_{m+n}^{(\gamma)}(x)\sum_{p=0}^{\infty}\frac{(-m)_p(1+\gamma+m+n)_p}{p!\,\Gamma(1+n+p)}$$
$$\times{}_2F_2\left[\begin{matrix}-n-p, & 1+m\\ \gamma'+1, & 1+m+p\end{matrix};\frac{a}{b}\right],$$
$$\left|\frac{c}{ay}\right|<1,\ \left|\frac{by}{d}\right|<1,\ ad-bc=1,$$

where the terms corresponding to $n=-1,-2,\cdots$ are well defined because of the relation

$$(3.10)\qquad \underset{\mu\to k}{L_t}\sum_{p=0}^{\infty}\frac{(1+\gamma+m+\mu)_p(-m)_p}{p!\,\Gamma(1+\mu+p)}\left(\frac{bc}{ad}\right)^p{}_2F_2\left[\begin{matrix}-\mu-p, & -1-\gamma-m\\ \gamma'+1, & 1+m-p\end{matrix};\frac{a}{b}\right]$$
$$=\frac{(1+\gamma+m-k)_k(-m)_k}{k!}\left(\frac{bc}{ad}\right)^k\sum_{p=0}^{\infty}\frac{(1+\gamma+m)(-m+k)_p}{p!(1+k)_p}\left(\frac{bc}{ad}\right)^p$$
$$\times{}_2F_2\left[\begin{matrix}-p, & 1+m\\ \gamma'+1, & 1+m-p-k\end{matrix};\frac{a}{b}\right],\qquad k=1,2,3,\cdots$$

Special Cases:

(3.9) gives the following special cases:

$$(3.11)\qquad \frac{-xy}{e^{w-y}}\left(1-\frac{w}{y}\right)^m\psi_2\left(m+\gamma+1;\gamma+1,\gamma'+1;\frac{xy}{w-y},w-y\right)$$
$$=\sum_{n=-\infty}^{\infty}\frac{(1+n)_m}{(1+\gamma)_m(1+\gamma')_n}L_{m+n}^{(\gamma)}(x)\,{}_2F_2\left[\begin{matrix}1+\gamma+m+n, & 1+\lambda-\gamma+n\\ 1+n, & \gamma'+1+n\end{matrix};w\right]$$

and where the terms corresponding to $n=-1,-2,\cdots$ are well defined because of the relation of the type (3.10);

$$3.12\qquad e^x(1-y)^{-(\gamma+1+m)}\psi_2\left(m+\gamma+1;\gamma+1,\gamma'+1;\frac{-x}{1-y},\frac{-wy}{1-y}\right)$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(1+n+m)}{(\gamma+1)_m(\gamma'+1)_n}L_{m+n}^{(\gamma)}(x)L_n^{(\gamma')}(w)y^n$$

B. We choose $f(x,y)$ to be a common eigenfunction of the operator C and $J^-J^3-(m+(\gamma+1)/-2)J^-J^3$.

Let $f(x,y)$ satisfy the simultaneous equations

$$(3.13)\qquad \begin{aligned}&Cf(x,y)=\left(\frac{\gamma^2-1}{4}\right)f(x,y),\\ &\left[J^-J^3-\left(m+\frac{\gamma+1}{2}\right)J^--J^3\right]f(x,y)=0.\end{aligned}$$

which may be rewritten as

$$(3.14)\qquad \begin{aligned}&\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+m\right]f(x, y)=0\,,\\ &\left[-y\frac{\partial^2}{\partial y^2}+x\frac{\partial^2}{\partial x\partial y}-(1+m+y)\frac{\partial}{\partial y}-\left(m+\frac{\gamma+1}{2}\right)\right]f(x, y)=0\end{aligned}$$

These equations have a solution [1; 235]

$$(3.15)\qquad f(x, y)=H_4\left[-m,\ m+\frac{\gamma+1}{2};\ 1+\gamma;\ x,\ y\right]$$

where

$$H_4[\alpha, \gamma; \delta; x, y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\gamma)_n}{m!\,n!\,(\delta)_m}x^m y^n\,.$$

Therefore

$$(3.16)\qquad \begin{aligned}[T(g)f](x, y)&=\exp\left(\frac{bxy}{d+by}\right)(d-by)^{-(\gamma+1+m)}\left(a+\frac{c}{y}\right)^m\\ &\times H_4\left[-m,\ m+\frac{\gamma+1}{2};\ 1+\gamma;\ \frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}\right]\\ &\left|\frac{by}{d}\right|<1,\ -\pi<\arg a,\ \arg d<\pi,\ ad-bc=1\,.\end{aligned}$$

By the similar analysis as we did in sec. **A.** we obtain the generating function

$$(3.17)\qquad \begin{aligned}&e^{(bxy/d+by)}\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^m\\ &\times H_4\left[-m,\ m+\frac{\gamma+1}{2};\ 1+\gamma;\ \frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}\right]\\ &=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m+n)}{(1+\gamma)_m}\left(\frac{-by}{d}\right)^n L_{m+n}^{(\gamma)}(x)\\ &\times\sum_{p=0}^{\infty}\frac{(-m)_p(1+m+n+\gamma)_p}{p!\,\Gamma(1+n+p)}\left(\frac{bc}{ad}\right)^p\\ &\times {}_2F_2\left[\begin{matrix}-n-p,\ m+\dfrac{\gamma+1}{2}\\ m+1-p,\ 1+\gamma+m\end{matrix};\ -\frac{a}{b}\right],\\ &\left|\frac{c}{ay}\right|<1,\ \left|\frac{by}{d}\right|<1,\ ad-bc=1\,,\end{aligned}$$

where the terms corresponding to $n=-1, -2, \cdots$ are well defined because of the relation (3.10).

Special Cases:

(3.18)
$$e^{(-xy/1-y)}(1-y)^{-(1+\gamma+m)}H_4\left[-m,\ m+\frac{\gamma+1}{2};\ 1+\gamma;\ \frac{x}{1-y},\ \frac{-wy}{1-y}\right]$$
$$=\sum_{n=-\infty}^{\infty}\frac{(1+n)_m}{(1+\gamma)_m}L_{m+n}^{(\gamma)}(x)\,{}_2F_2\left[\begin{matrix}-n,\ m+\frac{\gamma+1}{2}\\ m+1,\ \ 1+m+\gamma\end{matrix};\ w\right]y^n$$

where the terms corresponding to $n=1, -2, \cdots$ are well defined because of the relation of type 3.10.;

(3.19)
$$\left(1-\frac{w}{y}\right)^m H_4\left[-m,\ m+\frac{\gamma+1}{2};\ 1+\gamma;\ \frac{xy}{w-y},\ w-y\right]e^w$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(1+m)}{(1+\gamma)_m}L_{m+n}^{(\gamma)}(x)L_{m+(\gamma-1/2)}^{(n)}(w)y^n$$

C. We choose $f(x, y)$ to be a common eigenfunction of the operator C and $(-J^-J^3+((m+\gamma+1)/2)J^-+J^3)$.

Let $f(x, y)$ satisfy the simultaneous equations

(3.20)
$$Cf(x, y)=\left(\frac{\gamma^2-1}{4}\right)f(x, y)\ ,$$
$$\left(-J^-J^3+\left(\frac{m+\gamma+1}{2}\right)J^-+J^2\right)f(x, y)=\left(\frac{m+\gamma-1}{2}\right)f(x, y)\ .$$

This may be rewritten as

(3.21)
$$\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+m\right]f(x, y)=0\ ,$$
$$\left[y\frac{\partial^2}{\partial y^2}-x\frac{\partial^2}{\partial x\partial y}+(1+m+y)\frac{\partial}{\partial y}+1\right]f(x, y)=0\ .$$

These equations have a solution [1; 235]

(3.22)
$$f(x, y)=H_5[-m;\ 1+\gamma;\ x,\ y]$$

where

$$H_5[\alpha;\ \delta;\ x,\ y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}}{m!\ n!\ (\delta)_m}x^m y^n\ .$$

The generating function that we obtain now is

(3.23)
$$\exp\left(\frac{bxy}{d+by}\right)\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^m$$
$$\times H_5\left[-m;\ 1+\gamma;\ \frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{(d+by)}\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+m+n)}{(1+\gamma)_{m+n}} \left(\frac{-by}{d}\right)^n L_{m+n}^{(\gamma)}(x) \sum_{p=0}^{\infty} \frac{(-m)_p}{p!} \left(\frac{bc}{ad}\right)^p$$
$$\times L_{n+p}^{(\gamma+m)}\left(\frac{-a}{b}\right) .$$

where the terms corresponding to $n=-1, -2, \cdots$ are well defined because of the relation of the type (3.10).

Special Cases:

(3.24) $$\exp\left(\frac{-xy}{1-y}\right)(1-y)^{-(\gamma+1+m)} H_5\left[-m; 1+\gamma; \frac{x}{1-y}, \frac{-wy}{1-y}\right]$$
$$= \sum_{p=0}^{\infty} \frac{\Gamma(1+m+n)}{(1+\gamma)_{m+n}} L_{m+n}^{(\gamma)}(x) L_n^{(\gamma+m)}(w) y^n$$

and

(3.25) $$\left(1-\frac{w}{y}\right)^m H_5\left[-m; 1+\gamma; \frac{-xy}{w-y}, w-y\right]$$
$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+m)}{\Gamma(1+n)(1+\gamma)_m} L_{m+n}^{(\gamma)}(x) {}_0F_1[-m; 1+n; -w] y^n .$$

D. Choose $f(x, y)$ to be a common eigenfunction of the operator C and $[J^3J^3 - J^-J^3 + (m+(\gamma+1)/2)J^- - (m+1)J^3]$.

Let $f(x, y)$ satisfy the simultaneous equations

(3.26) $$Cf(x, y) = \left(\frac{\gamma^2-1}{4}\right) f(x, y) ,$$
$$\left[J^3J^3 - J^-J^3 + \left(m+\frac{\gamma+1}{2}\right)J^- - (m+1)J^3\right] f(x, y) = 0$$

which may be rewritten as

(3.27) $$\left[x\frac{\partial^2}{\partial x^2} + (1+\gamma-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + m\right] f(x, y) = 0 ,$$
$$\left[y(y+1)\frac{\partial^2}{\partial y^2} - x\frac{\partial^2}{\partial x \partial y} + \{1+m+(m+\gamma+1)y\}\frac{\partial}{\partial y} + \left(\frac{\gamma-1}{2}\right)\left(m+\frac{\gamma+1}{2}\right)\right] f(x, y) = 0 .$$

These equations have a solution [1; 235]

(3.28) $$f(x, y) = H_{11}\left[-m, \frac{\gamma-1}{2}, m+\frac{\gamma+1}{2}; 1+\gamma; x, y\right]$$

where

$$H_{11}(\alpha, \beta, \gamma; \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_n(\gamma)_n}{m!\,n!\,(\delta)_m} x^m y^n \qquad |y|<1 .$$

Thus, we arrive at a generating function

$$\text{(3.29)}\quad \exp\left(\frac{bxy}{d+by}\right)\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^{m}$$

$$\times H_{11}\left[-m,\frac{\gamma-1}{2},m+\frac{\gamma+1}{2};\gamma+1;\frac{xy}{(c+ay)(d+by)},\frac{c+ay}{d+by}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m+n)}{(1+\gamma)_m}\left(\frac{-by}{d}\right)^{n}L_{m+n}^{(\gamma)}(x)$$

$$\times\sum_{p=0}^{\infty}\frac{(-m)_p(1+\gamma+m+n)_p}{p!\,\Gamma(1+n+p)}\left(\frac{bc}{ad}\right)^{p}$$

$$\times{}_3F_2\left[\begin{matrix}\frac{\gamma-1}{2},\ m+\frac{\gamma+1}{2},\ -n-p\\ m+1-p,\ 1+\gamma+m\end{matrix};\frac{a}{b}\right],$$

$$\left|\frac{c}{ay}\right|<1,\quad\left|\frac{by}{d}\right|<1,\qquad ad-bc=1\,.$$

where the terms corresponding to $n=-1,-2,\cdots$ are well defined because of the relation of the type 3.10.

Special Cases:

$$\text{(3.30)}\quad \exp\left(\frac{xy}{1-y}\right)(1-y)^{-(\gamma+m+1)}$$

$$\times H_{11}\left[-m,\frac{\gamma-1}{2},m+\frac{\gamma+1}{2};1+\gamma;\frac{x}{1-y},\frac{wy}{1-y}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{(1+n)_m}{(1+\delta)_m}L_{m+n}^{(\gamma)}(x)\,{}_3F_2\left[\begin{matrix}\frac{\gamma-1}{2},\ m+\frac{\gamma+1}{2},\ -n\\ m+1,\ 1+m+\gamma\end{matrix};w\right](y)^{n}\,.$$

and

$$\text{(3.31)}\quad \left(1-\frac{w}{y}\right)^{m}H_{11}\left[-m,\frac{\gamma-1}{2},m+\frac{\gamma+1}{2};1+\gamma;\frac{-xy}{w-y},w-y\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m)\left(\frac{\gamma-1}{2}\right)_n\left(m+\frac{\gamma+1}{2}\right)_n}{\Gamma(1+n)(1+\gamma)_m}L_{m+n}^{(\gamma)}(x)$$

$$\times{}_2F_1\left[\begin{matrix}m+n+\frac{\gamma-1}{2},\ +n\frac{\gamma+1}{2}\\ 1+n\end{matrix};-w\right]y^{n}$$

where the terms corresponding to $n=-1,-2,\cdots$ are well defined by a relation of the type (3.10).

References

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