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On Derived Rules of Intuitionistic Second Order Arithmetic

by

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The purpose of this paper is to prove the derived rules announced in [8].[1] Our tool proving the derived rules is a sort of locally formalized normal form theorem. This method gives unified proofs of various derived rules. We prove also an extended Church's rule by this method. Roughly speaking our normal form theorem maintains existence of an operation which transforms each finitary derivation to its infinitary recursive normal form.[2] The normal form theorem is proved easily by the fact that *normalization trees*, which are modified versions of "normaization" of [12], are well-founded. Note that in this paper "*well-foundness*" means that principle of induction on the tree is valid. So the essential part of the proof is a proof of the well-foundness of the trees. We carry out this by the method of [13], [14]. Since we employ the existential quantifier as a primitive symbol, a complication arises, however, there is no essential new technique in our proof. We indicate only how to modify the proof of [14]. The reader will find our proof is very simplified by restricting the language and may ask why we dare extend the language. Indeed there is no nessecity of the addition of the logical symbols for the proofs of derived rules. The reasons why we extends the language are (i) it simplifies the proofs of derived rules if once the normal form theorem is proved, (ii) as was shown in [1], the normal form theorem supplies a general method proving independence of arithmetical sentences in intuitionistic formal systems, however, it seems there is no published proof of the theorem (see 2.7.2).

In §1 we give syntax. In §2 we prove the locally formalized normal form theorem. In §3 we prove the derived rules. In an appendix we extend our results to some closely related systems including the intuitionistic higher order arithmetic with the axioms of extensionality.

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[1] In [2], [3] M. Beeson has given more general forms of [8]. His methods are quite different from ours. [2] was obtained before [8], however, our work is independent from him. [3] was obtained after a sight of [8].

[2] It is very easy to extend the normal form theorem to arbitrary infinitary proofs. The resulting normal form is recursive in the original infinitary proof. If the original proof is extremly restricted, then the resulting form is also (cf. [11]).

The author would like to express his thanks to Professor Beeson for his encouragement. Without his emphasizing the importance of the results in [8], the present paper may not have been published.

§1. Syntax

1.1. Second order arithmetic **S**

We now define a formal system **S** which is a varient of **HAS** of [17].

1.1.1. Language of **S**

As logical symbols we employ "$\curlywedge$", "$\forall^\sigma$", "$\exists^\sigma$" (σ is a type of first or second order), "&", "$\to$" and "λ". Individual constants and function constants are only 0 (zero) and S (succesor) respectively. Terms and formulae are constructed in the usual way. We adopt $s, t, u, t_1, \cdots$ as metanotations for terms. We use a notation $S^n t$ as follows:

$$\begin{cases} S^0 t = t \\ S^{n+1} t = S(S^n t)\ . \end{cases}$$

For predicate constants we employ a binary predicate constant $=$ (equal) and the constants F_k of 1.3.6 of [17].

1.1.2. Axioms and rules of **S**

1.1.2.1. Non-logical axioms and rules

Axioms and inferences for equality and S are as follows:

$$t=t \qquad \frac{A(s) \quad s=t}{A(t)} \text{ (equality rule)}$$

$$\frac{S^m t = S^n t}{\curlywedge}\ (m \neq n) \qquad \frac{St = Su}{t = u}\ .$$

Axioms and inferences for F_k are the same as 1.3.6 of [17], however, there is a luck in [17]. We must add the following rule:

$$\frac{F_k t_1 \cdots t_n s \quad F_k t_1 \cdots t_n t}{s=t}\ .$$

As induction rule we use the the following form:

$$\frac{A(0) \quad \begin{matrix}[A(a)] \\ \Sigma(a) \\ A(Sa)\end{matrix}}{\forall x^0 A(x)} \text{ (induction rule)}\ .$$

As usual the inferences except the equality rule and the induction rule are called *atomic rules*.

1.1.2.2. Logical rules

Logical rules are usual ones of natural deductions of intuitionistic second order logic, including rules for "λ".

1.2. Infinitary second order arithmetic **IS**

1.2.1. Language of **IS**

Language of **IS** are the same as **S** except that there is no free variables of type zero.

1.2.2. Axioms and rules of **IS**

1.2.2.1. Non-logical axioms and rules

Non-logical axioms and rules of **IS** are exactly the closed instances of the axioms and atomic rules of **S**. Note that the axiom of equality (i.e. $\forall^{(0)}X(\forall x^0 y^0(Xx \ \& \ x=y \ .\!\rightarrow\!. \ Xy)))$ is a theorem of **IS**.

1.2.2.2. Logical rules

Logical rules are the same as **S** except that instead of the rules $\forall^0$I and $\exists^0$E of **S** we use the followings:

$$\frac{\begin{matrix}\Sigma_0 & \Sigma_1 & \cdots \\ A(0) & A(S0) & \end{matrix}}{\forall x^0 A(x)}(\forall^0\mathrm{I}) \qquad \frac{\exists x A(x) \quad \begin{matrix} A(0) & A(S0) & \\ \Sigma_0 & \Sigma_1 & \cdots \\ C & C & \end{matrix}}{C}(\exists^0\mathrm{E})\,.$$

1.2.2.3. A *preproof* of **IS** is a well-founded, recursive tree whose node is labelled recursively by a non-empty finite sequence of formulae $\langle F_1, \cdots, F_n\rangle$ and a name of an inference rule of **IS**. (Cf. 3.2.9.) If a node is labelled by $\langle F_1, F_2, \cdots, F_n\rangle$, it is intended that F_1 is derived under the assumptions $F_2, \cdots, F_n$, i.e., the possible forms of assumptions which are open at the node are at most $F_2, \cdots, F_n$. A preproof is called a *proof* if at each node of the tree the labelled sequence is related to the sequences labelled to the predecessors of the node by the labelled rule. We may identify the proofs of **IS** with appropriate recursive functions as in [11] or elements of an inductively defined set of numbers as in [16].

1.3. Terms and formulae are called *numerical closed* if they have no free variables of type zero. Derivations are called numercal closed if they have no free variables of type zero except eigen variables. *From now on we assume terms, formulae and derivations are numercal closed as far as without provisory clauses.*

1.4. Con(Σ) is the formula F_1 and Asp(Σ) is the sequence $\langle F_2, \cdots, F_n\rangle$, where $\langle F_1, \cdots, F_n\rangle$ is labelled to $\langle\ \rangle$ of the Σ. We sometimes identify $\langle F_2, \cdots, F_n\rangle$ with $\{F_2, \cdots, F_n\}$. Rule(Σ) is the name of the last inference of Σ.

§2. Normal form theorem

2.1. Reduction

A formula is called *a cut* when it is the major premiss of an elimination rule and the conclusion of a non-elimination rule. *Main cut* and *main branch* are defined as in [12]. A formula is called *a* *semi cut* when it is the conclusion of a $\exists^{\sigma}$E-rule and the major premiss of an elimination rule. We say Σ is *reducible* when it has the main cut or a semi cut in the main branch.

For reducible Σ we define its *reduct form* Red(Σ) as follows:
(1) If Σ has the main cut, we reduce it by the proper reductions of 4.1.3 of [17] or the reductions according to the following contractions:

(i)
$$\frac{\dfrac{\begin{matrix}\Sigma_0 \\ A(0)\end{matrix}\quad \begin{matrix}A(a)\\ \Sigma_1(a)\\ A(sa)\end{matrix}}{\forall x^0 Ax}}{A(S^n0)} \ \text{contr.}\ \begin{matrix}\Sigma_0\\ \Sigma_1(0)\\ \vdots \\ \Sigma_1(S^i0)\\ \vdots\\ \Sigma_1(S^{n-1}0)\\ A(S^n0)\end{matrix}$$

(ii)
$$\frac{\dfrac{\begin{matrix}\Sigma_0 \\ A(S^m0)\end{matrix}\quad \begin{matrix}\Sigma_1\\ S^m0=S^n0\end{matrix}}{A(S^n0)}\quad \Sigma_2}{B} \ \text{contr.}\ \begin{cases} \dfrac{\Sigma_0 \quad \Sigma_2}{B} & \text{if} \quad m=n \\ \dfrac{\dfrac{\dfrac{\Sigma_1}{\curlywedge}}{A(S^n0)}\quad \Sigma_2}{B} & \text{if} \quad m\neq n \end{cases}$$

(iii)
$$\frac{\dfrac{\begin{matrix}\Sigma_0\\ \curlywedge\end{matrix}}{A}\quad \Sigma_2}{B} \ \text{contr.}\ \frac{\begin{matrix}\Sigma_0\\ \curlywedge\end{matrix}}{B} \qquad (\Sigma_2 \text{ may be void.})$$

Remark. In (i), (ii), (iii), the cuts are $\forall x^0 Ax$, $A(S^n0)$, A respectively.
(2) If Σ has no main cut but a semi cut in the main branch, Red(Σ) is the derivation gained by an application of the permutative reduction (see (C) of 4.1.3 of [17]) at the uppermost semi cut in the main branch.

2.2. Twig

Let Rule(Σ) be an elimination rule, Σ be not reducible and $\Sigma = \dfrac{\Sigma_1\Sigma_2}{A}$. *A twig* of Σ is a derivation gained by one of the following:
(1) If Π is a minor deduction (see 3.3 of [12]) of an inference in the main branch of Σ_1, then Π is a twig of Σ.
(2) If Rule(Σ) is not $\exists^0$E, then Σ_2 is a twig of Σ.
(3) If Rule(Σ) is $\exists^0$E and $\Sigma_2 = \begin{matrix}B(a)\\ \Sigma_2(a)\\ A\end{matrix}$, then $\begin{matrix}B(S^n0)\\ \Sigma_2(S^n0)\\ A\end{matrix}$ is a twig of Σ for each n.

2.3. Predecessor

For each derivation Σ which does not consist solely of one formula we define its *predecessors* as follows:
Case 1. Rule$(\Sigma) \in \{\to$I, &I, $\forall^{\sigma}$I$(\sigma\neq 0)$, $\exists^{\sigma}$I, λI, atomic rules, $\curlywedge$-rule$\}$. Π is

a predecessor of Σ iff it is a derivation of a premiss of last inference of Σ.

Case 2. Rule $(\Sigma) \in \{$induction rule, $\forall^0 I\}$.

Let

$$\Sigma = \frac{\begin{matrix} \Sigma_1 \\ A(0) \end{matrix} \quad \begin{matrix} A(a) \\ \Sigma_2(a) \\ A(Sa) \end{matrix}}{\forall x^0 A(x)} \quad \text{or} \quad \Sigma = \frac{\begin{matrix} \Sigma_3(a) \\ A(a) \end{matrix}}{\forall x^0 A(x)} .$$

Π is a predecessor of Σ iff it is one of the followings:

$$\begin{matrix} \Sigma_1 \\ \vdots \\ \Sigma_2(S^i 0) \\ \vdots \\ \Sigma_2(S^{n-1}0) \\ A(S^n 0) \end{matrix} \quad \text{or} \quad \Sigma_3(S^n 0) .$$

Case 3. Rule (Σ) is an elimination and Σ is reducible. Π is a predecessor of Σ iff Π is Red (Σ).

Case 4. Rule (Σ) is an elimination and Σ is not reducible. Π is a predecessor of Σ iff Π is a twig of Σ.

Case 5. Rule (Σ) is an equality rule.

Let $\Sigma = \dfrac{\begin{matrix} \Sigma_1 \\ A(S^m 0) \end{matrix} \quad \begin{matrix} \Sigma_2 \\ S^m 0 = S^n 0 \end{matrix}}{A(S^n 0)}$. Π is a predecessor of Σ iff $m=n$ and $\Pi=\Sigma_1$, or $m \neq n$ and $\Pi = \Sigma_2$.

2.4. Normalization tree

For each derivation of **S** we assign a tree which is labelled with proofs of **S** and is called *the normalization tree of the derivation.* Normalization tree is a modification of "normalization" of Martin-Löf [12]. *A normalization tree* is a function which satisfies the following conditions:

(1) $\{x: f(x) \neq 0\}$ is a tree, i.e. $f(x * m) \neq 0 \rightarrow f(x) \neq 0$.

(2) If $f(x) \neq 0$, then $f(x)$ is an index of a derivation.

(3) Let $f(x) \neq 0$ and $f(x) = \Sigma$.

Case 1. $f(x)$ does not consist solely of one formula.

Let $\Pi_1, \cdots, \Pi_i, \cdots$ $(i < a \leqq \omega)$ is an enumeration of the predecessors without repetitions, then

$$f(x * \langle i \rangle) = \begin{cases} \Pi_i & i < a \\ 0 & \text{otherwise.} \end{cases}$$

Case 2. $f(x)$ consist solely of one formula.

$$f(x * \langle i \rangle) = 0 \quad \text{for each } i$$

A normalization tree f is called *a normalization tree of* Σ provided

$f(\langle\ \rangle)=\Sigma$. All of normalization trees of a derivation are extensionally equal when the differences of enumerations of predecessors are ignored. We assume that for each derivation of S a recursive normalization tree of it is assigned in a definite way. We call it "*the normalization tree of the derivation*".

2.5. Normalizability

2.5.1. A derivation of S is said to be *normalizable* if f is the normalization tree of it and the tree $\{x: f(x)\neq 0\}$ is well founded. We can prove that derivations of S are normalizable. We carry out this by the same way as in [13], [14]. We indicate how the definitions of computability predicates and $\varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$ are modified. We state a lemma which corresponds to 4.1.6 of [17]. It is not so difficult (but tedious) to prove the normalizability of S from these as in [13], [14]. The details will be left to readers.

2.5.2. Computability predicates and $\varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$

In $\varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$ $u(\mathbf{x})$ *need not be numerical closed* but $u(\mathbf{t})$ must be numercal closed. We explain how the notions of [14] are modified, since [12] is a particular case of [14].

2.5.2.1. 5 of [14] is modified as follows:

(1) Terms, formulae and derivations are numerical closed. (Note that computability predicates of type zero are the numerals.)

(2) 5.2.2 of [14] is replaced by the following: A derivation Σ which ends with an elimination and is reducible satisfies α_F iff Red(Σ) satisfies α_F.

(3) 5.2.3 of [14] is replaced by the following: A derivation Σ which ends with an elimination except $\exists^r$E-rules and is not reducible satisfies α_F iff the twigs of Σ are normalizable.

(4) The following clauses are added.

5.2.5. A derivation Σ which ends with an $\exists^r$E-rule and is not reducible satisfies α_F iff the twigs of Σ are normalizable and the twigs obtained by the clauses (2), (3) of 2.2 satisfy α_F.

5.2.6. A derivation which ends with $\curlywedge$-rule or an atomic rule satisfies α_F iff it is normalizable.

5.2.7. A derivation which ends with the equality rule, say

$$\frac{\overset{\Sigma_1}{F(S^m 0)} \quad \overset{\Sigma_2}{S^m 0 = S^n 0}}{F},$$

satisfies α_F iff $m=n$ and Σ_1 satisfies α_F, or $m\neq n$ and Σ_2 is normalizable.

2.5.2.2. 6 of [14] is modified as follows:

(1) 6.1–6.3, 6.4.4, 6.5.4 for higher order quantifiers and 6.6 are leaved as they are. ($\curlywedge$ is ranked as a zero place predicate constant.)

(2) We now define a list of clauses (♯) which is used in the following

definitions.

(♯)

Case 1. Σ consists solely of one formula.

If $\Sigma = u(\mathbf{x})$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

Case 2. Rule $(\Sigma) = \curlywedge$-rule. If Σ is normalizable and Con $(\Sigma) = u(\mathbf{t})$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

Case 3. Rule (Σ) = equality rule. Let $\Sigma = \dfrac{\overset{\Sigma_1}{F(S^m0)} \quad \overset{\Sigma_2}{S^m0 = S^n0}}{F(S^n0)}$ and $F(S^n0) = u(\mathbf{t})$.

Subcase 1. If $m = n$ and $\Sigma_1 \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

Subcase 2. If $m \neq n$ and Σ_2 is normalizable, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

Case 4. Rule (Σ) is an elimination rule. Let Con $(\Sigma) = u(\mathbf{t})$.

Subcase 1. If Σ is reducible and Red $(\Sigma) \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

Subcase 2. If Σ is not reducible, Rule $(\Sigma) \neq \exists^\sigma$E-rule and all of the twigs of it are normalizable, then then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

Subcase 3. If Σ is not reducible, Rule $(\Sigma) = \exists^\sigma$E-rule, all of the twigs are normalizable and the twigs obtained by the cluses (2), (3) of 2.2 satisfy $\varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

(3) 6.4.1-6.4.3, 6.5.1-6.5.3 are replaced by (♯).

(4) 6.5.4 for the type zero is replaced by the following:

If Con $(\Sigma) = u(\mathbf{t})$, Rule $(\Sigma) \in \{\forall^0$I, induction rule$\}$ and for each natural number n the deduction which is the predecessor with the conclusion $F(S^n0)$ satisfies $\varphi_{F(x,\mathbf{x})}(S^n0, \boldsymbol{\alpha}_t)$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

(5) The following clauses are added as 6.7, 6.8, 6.9.

6.7. $u(\mathbf{x}) = \{\lambda \mathbf{y} F(\mathbf{x}, \mathbf{y})\}\mathbf{u}(\mathbf{x})$

6.7.1. (♯).

6.7.2. If Rule $(\Sigma) = \lambda$I-rule and the premiss of Σ satisfies the predicate $\varphi_{F(\mathbf{x},\mathbf{y})}(\boldsymbol{\alpha}_t, \varphi_{\mathbf{u}(\mathbf{x})}(\boldsymbol{\alpha}_t))$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

6.8. $u(\mathbf{x}) = F(\mathbf{x}) \& G(\mathbf{x})$

6.8.1. (♯).

6.8.2. If $\Sigma = \dfrac{\Sigma_1 \ \Sigma_2}{u(\mathbf{t})}$, Rule $(\Sigma) = \&$I, $\Sigma_1 \in \varphi_{F(\mathbf{x})}(\boldsymbol{\alpha}_t)$ and $\Sigma_2 \in \varphi_{G(\mathbf{x})}(\boldsymbol{\alpha}_t)$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

6.9. $u(\mathbf{x}) = \exists x^\sigma F(x, \mathbf{x})$

6.9.1. (♯).

6.9.2. If

$$\Sigma = \frac{\overset{\Sigma_1}{F(t^\sigma, \mathbf{t})}}{\exists x^\sigma F(x, \mathbf{t})},$$

Rule $(\Sigma) = \exists^\sigma$I-rule and $\exists \alpha_{t^\sigma}(\Sigma_1 \in \varphi_{F(a,\mathbf{x})}(\alpha_{t^\sigma}, \boldsymbol{\alpha}_t)$, then $\Sigma \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_t)$.

2.5.3. LEMMA 1. *Assume*

$$\Sigma(\mathbf{x}) = \frac{\begin{matrix}\Sigma_1(\mathbf{x}) & \cdots & \Sigma_n(\mathbf{x}) \\ u_1(\mathbf{x}) & \cdots & u_n(\mathbf{x})\end{matrix}}{u(\mathbf{x})},$$

Rule (Σ) *is an elimination rule,* Σ_1 *is the major premiss and all of the free variables of* Σ *occur in* $\mathbf{x}$. *Then* $\Sigma(\mathbf{t}) \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$ *provided that the following conditions are fulfilled.*

(a) $\forall i(\Sigma_i(\mathbf{t}) \in \varphi_{u_i(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}}))$.

(b) *If Rule* $(\Sigma) = \exists^\sigma$E-*rule and*

$$\Sigma(\mathbf{x}) = \frac{\begin{matrix} & F(a, \mathbf{x}) \\ \Sigma_1 & \Sigma_2(a, \mathbf{x}) \\ \exists x^\sigma F(x, \mathbf{x}) & u(\mathbf{x})\end{matrix}}{u(\mathbf{x})},$$

then $\forall \alpha_{t^\sigma} \forall \Pi \left(\textit{if } \begin{matrix} \\ \Pi \in \varphi_{F(a,\mathbf{x})}(\alpha_{t^\sigma}, \boldsymbol{\alpha}_{\mathbf{t}}), \end{matrix} \textit{ then } \begin{matrix}\Pi \\ \Sigma_2(t^\sigma, \mathbf{t}) \in \varphi_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})\end{matrix} \right)$.

Proof. By an induction over the generalized inductive definition used in the definition of $\Sigma_1(\mathbf{t}) \in \varphi_{u_1(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$. The details are left for readers.

2.5.4. THEOREM 1. *Every proof figure of* S *is normalizable.*

Proof. Similarly to 9.1 of [14]. Details are left for readers.

2.5.5. Let T^n be the set of the terms and formulae which have logical symbols at most n. (We adomit the formulae and terms are not numerical closed.) Let $\mathrm{T}^{(n)}$ be the set of formulae and terms which are obtained by repeated mutual substitutions from the elements of the set T^n (cf. 4.5.6 of [17]). $\mathrm{Pf}^n(\Sigma)$ $[\mathrm{IPf}^n(\Sigma)]$ means that Σ is a derivation of S [IS] and Σ has only terms and formulae of $\mathrm{T}^{(n)}$. Note that the Σ is numerical closed, however, formulae and terms of Σ may contain eigen variables. For each n we can define a formula $\varphi^{(n)}_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$ of S so that $\mathrm{S} \vdash \forall u(\mathbf{t})\{u(\mathbf{t}) \in \mathrm{T}^{(\bar{n})} \rightarrow \varphi^{(n)}_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$ satisfies the clauses 6.1–6.9 of 2.5.2.2$\}$. (The equality between computability predicates are expressed by the extensional equality.) Such a formula is define as $\mathrm{Sat}^{(n)}$ of 4.5.6 of [17], i.e., we firstly define as formula $\varphi^n_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$ which satisfies the 6.1–6.9 for the elements of T^n, and $\varphi^{(n)}_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$ is defined by repeated mutual substitutions from $\varphi^n_{u(\mathbf{x})}(\boldsymbol{\alpha}_{\mathbf{t}})$. Since in the proof of Theorem 1 we need only the clauses 6.1–6.9 and the definition of computability predicates, we can show that for each natural number n the formula $\forall \Sigma\{\mathrm{Pf}^{\bar{n}}(\Sigma) \rightarrow \Sigma$ is normarizable$\}$ is a theorem of S. Namely we have obtained the following lemma.

LEMMA 2. $\forall n[\mathrm{S} \vdash \forall \Sigma\{\mathrm{Pf}^{\bar{n}}(\Sigma) \rightarrow \Sigma$ *is normalizable*$\}]$.

2.6. Normal form

2.6.1. A proof figure is said to be *normal* if it has no cut and no semi cut. Σ_1 is said to be a normal form of Σ if Σ_1 is normal, $\mathrm{Asp}(\Sigma_1) \subseteq \mathrm{Asp}(\Sigma)$ and $\mathrm{Con}(\Sigma_1) = \mathrm{Con}(\Sigma)$.

LEMMA 3. $\mathbf{S} \vdash \{\mathrm{Pf}^n(\Sigma)$ & Σ *is normalizable* $\rightarrow$ *there is a derivation* Σ_1 *which is a normal form of* Σ *and satisfies* $\mathrm{IPf}^{\bar{n}}(\Sigma_1)\}$.

Proof. Let f be the normalization tree of Σ. By the method of §30 of [16], we can find a Gödel number of a normal form of Σ from a Gödel number of f. Note that if the (Rep) is added as an inference rule to **IS**, a primitive recursive function representing a normal form of Σ is defined naturally from f (cf. [11]).

2.6.2. THEOREM 2 (Locally Formalized Normal Form Theorem). $\forall n[\mathbf{S} \vdash \{\mathrm{Pf}^{\bar{n}}(\Sigma) \rightarrow$ *there is a derivation* Σ_1 *which is a normal form of* Σ *and satisfies* $\mathrm{IPf}^{\bar{n}}(\Sigma_1)\}]$.

Proof. It is evident from Lemma 2 and Lemma 3.

2.7. We finish this section with two remarks on the normalization trees.

2.7.1. *Remark* 1. Although we prove the derived rules by the infinitary normal forms in §3, we can prove those only by the wellfoundness of the normalization trees. However, such proofs are somewhat complicated and essentially same to the proofs of §3.

2.7.2. *Remark* 2. Let $\Sigma_1, \Sigma_2, \cdots$ be a primitive recursive enumeration of the proofs of $\mathbf{S}$ and f_i be the normalization tree of Σ_i. Since $\lambda ix.f_i(x)$ is primitive recursive, we can construct a primitive recursive tree $T = \{a : \exists ix(a = \hat{i} * x \;\&\; f_i(x) \neq 0)\}$. We define a primitive recursive g so that $g(n)$ is the n-th member of T in the order of natural numbers. We define $\prec$ so that $a \prec b \leftrightarrow g(a)$ is smaller than $g(b)$ in the Kleene-Brouwer ordering. Obviously $\prec$ is a primitive recursive well-ordering on the natural numbers. Let α be the order type of $\prec$. Then the following proposition is hold (we adopt the terminologies of [16]).

PROPOSITION 1. (i) *Each segment of* $\prec$ *is a provable well-ordering of* $\mathbf{S}$. (ii) *The order types of provable well-orderings of* $\mathbf{S}$ *are less than* α. (iii) *Every provably recursive function of* $\mathbf{S}$ *is* $\prec$*-recusive.* (iv) α *is the supremum of the ranks of the trees* $\{x : f_i(x) \neq 0\}$. (v) *For any* F *which is a formula of* **HA** *and a theorem of* $\mathbf{S}$ *we can find a segment* $\prec\cdot$ *of* $\prec$ *so that* F *is a theorem of* $\mathbf{HA} + TI(\prec\cdot)$.

§3. Derived rules

3.1. In this section we prove the derived rules. We use the explicit definability property (EP) of S repeatedly. It has been known EP of S cannot be proved in S. Hence some proofs of this section cannot be formalized in S. However, there are several methods avoiding the uses of EP in the metalevel. Actually we can prove our theorems in **HA.**[3] We will explain the formalizability breifly in Appendix 2. Note that we do not use the full effect of Theorem 2 in this section. In the proofs of this section we need only the following version of Theorem 2, i.e., if Σ is a proof of S and its formulae belong to T^n, then $S \vdash \{\Sigma$ has a normal sorm Σ_1 which satisfies $\mathrm{IPf}^{\bar{n}}(\Sigma_1)\}$.

For simplicity we restrict our derived rules to be closed. The restriction is not essential. It is easy to extend our proofs to the cases with parameters.

3.2. We list here the notations and terminologies which are used in this section. In a theorem we must define the real numbers in S. Since S has no principle of selection we define the real numbers as Cauchy sequences of rational numbers with modulus of convergence. This way is essentially same approach which is adopted in [4]. Since the approach of [4] seems to be accepted widely, there will be no objections to our definition.

3.2.1. *Strict positive parts* (*s.p.p.*) of a formula A (cf. [19]).

3.2.1.1. A is a *s.p.p.* of A.

3.2.1.2. If $C \rightarrow B$, $\forall x^\tau Bx$, $B \& C$, $C \& B$ is a *s.p.p.* of A, then B is a *s.p.p.* of A (τ is an arbitrary type).

3.2.1.3. $\{\lambda \mathrm{x} B\mathrm{x}\}\mathrm{t}$ is a *s.p.p.* of A, then $B\mathrm{t}$ is a *s.p.p.* of A.

3.2.2. $A \in \Phi$ if and only if A has no *s.p.p.* which is the form $\exists x Bx$(B is not a predicate constant) or Xt (X is a *bound* variable).

3.2.3. $\mathrm{NPf}(\Sigma) \equiv_{\mathrm{def}} \mathrm{IPf}(\Sigma)$ & Σ is normal.

$\mathrm{NPf}^n(\Sigma) \equiv_{\mathrm{def}} \mathrm{IPf}^n(\Sigma)$ & Σ is normal.

$\mathrm{NPv}^n(A) \equiv_{\mathrm{def}} \exists \Sigma(\mathrm{Asp}(\Sigma) = \emptyset \ \&\ \mathrm{Con}(\Sigma) = A \ \&\ \mathrm{NPf}^n(\Sigma))$.

$\mathrm{Pv}^n(A) \equiv_{\mathrm{def}} \exists \Sigma(\mathrm{Pf}^n(\Sigma) \ \&\ \mathrm{Con}(\Sigma) = A \ \&\ \mathrm{Asp}(\Sigma) = \emptyset)$.

3.2.4. $\mathfrak{A}(F, x^0, y^0) \equiv_{\mathrm{def}} F(x, y) \ \&\ \forall z^0(F(x, z) \rightarrow y = z)$.

$\mathfrak{F}(F) \equiv_{\mathrm{def}} \forall x^0 \exists y^0 \mathfrak{A}(F, x, y)$.

$\mathfrak{C}(F) =_{\mathrm{def}} \{\mathfrak{F}(F)\} \cup \{\mathfrak{A}(F, S^m 0, S^n 0)$: m, n are natural numbers$\}$.

3.2.5. We say F *is a function* provided $\mathfrak{F}(F)$ is true. We use $f, g, h, \cdots$ as variables for functions. The formulae which contain the function variables are considered as abbreviations in the sence of §74 of [11]. $\forall f A(f)$, $\exists f A(f)$, $\exists ! f A(f)$ mean $\forall F(\mathfrak{F}(F) \rightarrow A(F))$, $\exists F(\mathfrak{F}(F) \ \&\ A(F))$, $\exists F(\mathfrak{F}(F) \& \forall G((\mathfrak{F}(G) \& A(G)) \rightarrow \forall xy(G(x, y) \leftrightarrow F(x, y)))$ respectively. Although these explanations are not so exact, the readers will find that he can use the function variables and quantifiers with the axiom schema $\forall x^0 \exists ! y^0 A(x, y) \rightarrow \exists f \forall x^0 A(x, fx)$.

[3] In suitable formulations the derived rures will be formalizable in the quantifier free primitive recursive arithmetic. It seems that there is no essential difficulity. However, the full details of proofs of such theorems will be too long and tedious to be actually carried out.

3.2.6. We assume the variables are enumerated in a definite way. In the rest of the paper F, G are the first and second free variables of type (0, 0) in the enumeration. We often ignore the difference between free and bound variables, e.g. we use F as follows: $\forall F(\mathfrak{F}(F)\to A(F))$.

3.2.7. Definition of $\mathrm{Sat}^{(n)}$.

3.2.7.1. $\mathrm{Sat}^{(n)}(X; A)$ denotes $\mathrm{Sat}^{(n)}(X, \ulcorner A \urcorner)$ of 4.5.6 of [17].

3.2.7.2. $\mathrm{Sat}^{(n)}(f, g; A(F, G))$ denotes $\mathrm{Sat}^{(n)}(\lambda z\{\exists x(z=j(j(2, 1), \langle x, fx\rangle)\vee z=j(j(2, 2), \langle x, gx\rangle))\}; A(F, G))$. Namely $\mathrm{Sat}^{(n)}(f, g; A)$ means the interpretation of A, where F, G are interpreted by $fx=y$, $gx=y$ respectively. $\mathrm{Sat}^{(n)}(f; A(F))$ is defined in the same manner.

3.2.8. We identify a finite sequence of natural numbers with a natural number as 1.3.9 of [17]. The notations of 1.3.9 of [17], e.g. $\mathrm{lth}(x)$, $\langle x_0, \cdots, x_n\rangle$, $(x)_i$, $\cdots$, are used in the same manner as in [17]. Note that in the present section we use $\prec$ in the sence of 1.3.9 of [17]. It does not mean the ordering of the end of §2.

We introduce two notations for finite sequences as follows:

$$\begin{cases}\mathrm{seg}(x, 0)=\langle\ \rangle\\ \mathrm{seg}(x, n+1)=\mathrm{seg}(x, n)*(x)_n ,\end{cases}$$

$$\begin{cases}\hat{0}_0=\langle\ \rangle\\ \hat{0}_{n+1}=\hat{0}_n*\hat{0} .\end{cases}$$

3.2.9. *A tree* is a set of finite sequences of natural numbers with the following conditions:

(1) $\exists x\in T\forall y\in T(x\leqq y)$ (existence of the root),

(2) If x is the root of T, $z\in T$ and $x\leqq y\leqq z$, then $y\in T$ (the tree condition).

We assume that the root of the tree of a proof figure of **JS** is $\langle\ \rangle$.

3.2.10. $\Sigma, \Pi, \cdots$ are used to denote the proof figures of **IS**. Σ_x means the subproof figure of Σ determined by x, e.g. $\Sigma_{\langle\ \rangle}=\Sigma$, $\Sigma_{\langle i\rangle}=\Pi^i$, where

$$\Sigma=\frac{\Pi^1\Pi^2\cdots}{C} .$$

The notations $\Sigma_1, \Sigma_2, \cdots$ which are used in the above sections do not mean this notation. Note that the proof figure Σ_x has the root $\langle\ \rangle$ even if x is not $\langle\ \rangle$.

By $x\in\Sigma$ we mean x is an element of the tree of the proof figure Σ. Hence $\forall\Sigma(\langle\ \rangle\in\Sigma)$.

3.2.11. The notation $\bar{n}$ means S^n0.

3.2.12.

$$\varphi_1(\Sigma)=\begin{cases} n & \text{if Rule}(\Sigma)=\exists^0\text{I and } \Sigma=\dfrac{\overset{\Sigma_{\hat{0}}}{A\bar{n}}}{\exists x^0 Ax} \\ 0 & \text{otherwise.}\end{cases}$$

$$\varphi_2(\Sigma)=\begin{cases} n & \text{if Rule}(\Sigma)=\forall^0\text{E and } \Sigma=\dfrac{\overset{\Sigma_{\hat{0}}}{\forall x^0 Ax}}{A\bar{n}} \\ 0 & \text{otherwise.}\end{cases}$$

$$\varphi_3(\Sigma)=\begin{cases} \text{the index of } t^{(0,0)} & \text{if Rule}(\Sigma)=\exists^{(0,0)}\text{I and } \Sigma=\dfrac{\overset{\Sigma_{\hat{0}}}{At^{(0,0)}}}{\exists x^{(0,0)} Ax} \\ 0 & \text{otherwise.}\end{cases}$$

$$\varphi_4(\Sigma)=\begin{cases} n & \text{if Rule}(\Sigma)\in\{\text{atomic rules, \& I}\}, \text{Con}(\Sigma) \text{ is a quantifier free formula without free variables except } F \text{ and } \text{Sat}^{(1)}(f;\text{Con}(\Sigma)) \text{ is false.} \\ 0 & \text{otherwise,}\end{cases}$$

where $n=\mu i\{\text{Sat}^{(1)}(f;\text{Con}(\Sigma_{\hat{i}}))\}$.

Note that $\text{Sat}^{(1)}(f;\text{Con}(\Sigma))$, $\text{Sat}^{(1)}(f;\text{Con}(\Sigma_{\hat{i}}))$ are decidable in the above definition by the condition on Con (Σ), and the formulae Con (Σ), Con ($\Sigma_{\hat{i}}$) belong to $\text{Fm}^{(1)}$ (cf. 4.5.6 of [17]).

3.2.13. We assume rational numbers are embedded in natural numbers in a definite way in S.

$\boldsymbol{R}$ denotes the formula with free variable F such that $[\mathfrak{F}(F)$ & $\forall x\exists y\{F(x, y)$ & $(y)_0$ is an index of a rational number$\}$ & $\forall x\exists y\{F(x, y)$ & $\forall z_1 z_2>(y)_1(\exists u_1 u_2(F(z_1, u_1)$ & $F(z_2, u_2)$ & $d((u_1)_0, (u_2)_0, x)\}]$, where $d((u_1)_0, (u_2)_0, x)$ means that $|p-q|<1/x$ provided $(u_1)_0$, $(u_2)_0$ are the index of the rational numbers p, q respectively.

We call an element of $\boldsymbol{R}$ a real number. Namely a real number is a function f which may be seen as a pair of functions f_0, f_1 such that $\forall x(f_i(x)=(fx)_i)$ $(i=0, 1)$ and f_0 is a Cauchy sequence of rational numbers with the modulus of continuity f_1, i.e. $\forall x\forall y_1 y_2>f_1(x)(|f_0(y_1)-f_0(y_2)|<1/x)$. (Cf. *p*. 60 of [4]).

The identity between the real numbers are defined in the usual way. (See 2.2.1 of [9]). Note that in [9] the real numbers are the equivalence classes of real number-generators. So our real numbers are rather real number-generators than real numbers if we agree to [9]. We sometimes call a real number a real number-generater in order to refer the results of [9].

3.2.14. Let $\equiv$ be the formula which express the identity on the real numbers. A formula A is called a function from $\boldsymbol{R}$ to $\boldsymbol{R}$ in S, when $\text{S}\vdash\forall f\in\boldsymbol{R}\exists g\in\boldsymbol{R}(A(f,g)$ & $\forall h\in\boldsymbol{R}(A(f,h)\rightarrow g\equiv h)$ & $\forall fghk\in\boldsymbol{R}(A(f,g)$

$\& \ f \equiv h \ \& \ g \equiv k \ .\!\longrightarrow. \ A(h, k))$. If a, b are provably real numbers in S, i.e. $S \vdash a \in \boldsymbol{R}$, $S \vdash b \in \boldsymbol{R}$, then functions from $[a, b]$ to $\boldsymbol{R}$ in S are defined in the same way.

3.2.15. We omit the indication of type of variables, if the types of the variables are determined by the contexts.

In the contrary to this the omittion of types of *rules* of quantifiers, e.g. $\exists$E-rule, $\forall$E-rule, means the types of the rules are not specified. Hence if one says a rule is not $\exists$E-rule, he means the rule does not belong to the set $\{\exists^\sigma \mathrm{E}: \sigma$ is an arbitrary type$\}$.

3.2.16.
$$\begin{cases} \bar{f}0 =_{\mathrm{def}} \langle \ \rangle \\ \bar{f}(n+1) =_{\mathrm{def}} \langle f0, \cdots, fn \rangle \ . \end{cases}$$
$$(h \mid f)(x) \simeq y \equiv_{\mathrm{def}} h(x * \bar{f}(\min_z [h(x * \bar{f}z) > 0])) \dot{-} 1 = y \ .$$
$$f(g) \simeq y \equiv_{\mathrm{def}} f(\bar{g}(\min_z [f(\bar{g}z) > 0])) \dot{-} 1 = y \ .$$
$$!f(g) \equiv_{\mathrm{def}} \exists x (f(g) \simeq x) \ .$$
$$!(h \mid f) \equiv_{\mathrm{def}} \forall x \exists y ((h \mid f)(x) \simeq y) \ .$$
$$f \leq g \equiv_{\mathrm{def}} \forall x (fx \leq gx) \ .$$
$$[x](i) =_{\mathrm{def}} (x)_i \ .$$

3.2.17. We assume the major premiss of an elimination rule of IS place at the left end, e.g. $\dfrac{A \to B \quad A}{B}$.

3.3. THEOREM 3 (An Extended Church's Rule). *If A belongs to Φ and* $S \vdash \forall x(Ax \to \exists y B(x, y))$, *then* $S \vdash \exists e \forall x(Ax \to \exists y(T(e, x, y) \ \& \ B(x, Uy))$, *where T and U are the ones of Theorem IX of* [10].

Proof. If B is a predicate constant, then we can take e so that $\mu y B(x, y) \simeq U(\mu y T(e, x, y))$. Hence without loss of generality we may assume B is not a predicate constant. Since Φ is primitive recursive, we see $S \vdash A \in \Phi$ by the assumption of the theorem. By Theorem 2 and the assumption of the theorem we can find a natural number n so that $S \vdash \{\exists \Sigma(\mathrm{Npf}^{\bar{n}}(\Sigma) \ \& \ \mathrm{Con}(\Sigma) = \forall x(Ax \to \exists y B(x, y)) \ \& \ \mathrm{Asp}(\Sigma) = \emptyset\}$. It is provable in S that if $A \in \Phi$ and $\exists \Sigma(\mathrm{NPf}^n(\Sigma) \ \& \ \mathrm{Con}(\Sigma) = Ax \to \exists y B(x, y))$ $\& \ \mathrm{Asp}(\Sigma) = \emptyset)$, then $\exists e \forall x(Ax \to \exists y(T(e, x, y) \ \& \ B(x, y)))$. Thence we obtain the theorem. We indicate the proof only informally, however, it is not difficult to see the formalizability of it.

Let Σ be a normal proof of $\forall x(Ax \to \exists y B(x, y))$. We assume Σ satisfies $\mathrm{Pf}^n(\Sigma)$. Since Σ is normal Σ has the following form:

$$\Sigma = \dfrac{\overset{\Sigma_{\hat{0}}}{A0 \to \exists y B(0, y)} \quad \overset{\Sigma_{\hat{1}}}{A(S0) \to \exists y B(S0, y)} \quad \cdots}{\forall x(Ax \to \exists y B(x, y))}$$

$$\Sigma_{\hat{x}} = \cfrac{\begin{array}{c} A\bar{x} \\ \Sigma_{\hat{x}*\hat{0}} \\ \exists y B(\bar{x}, y) \end{array}}{A\bar{x} \rightarrow \exists y B(\bar{x}, y)} .$$

From now on Rule(x) and Con(x) denotes Rule(Σ_x) and Con(Σ_x) respectively. We define a subtree of T_x for each $\Sigma_{\hat{x}}$ so that

$$T_x = \{\hat{x} * \hat{0} * z \colon z \in \Sigma_{\hat{x}*\hat{0}} \ \& \ \forall y \prec z(\text{Rule}(\hat{x} * \hat{0} * y) = \exists^0\text{E}) \ \& \ \forall i < \text{lth}(z)((z)_i > 0)\} .$$

T_x is well-founded, since Σ is well-founded. Assume $m \in T_x$ and Rule$(m) = \exists^0$E-rule. Then the major premiss of the inference is the form $\exists y P y$, where P is a prime formula and $\exists y P y$ is closed. We prove this by a course of values induction on the length of m. Assume the statement is true for n $(n \prec m)$. Then abviously the major premiss $\Sigma_{m*\hat{0}}$ has no open assumption except $A\bar{x}$ and closed prime formulae. Since Σ_m is normal there is no $\exists$E-rule in the main branch of Σ_m. Since the top formula of the main branch is $A\bar{x}$, the major premiss of the last inference of Σ_m must be a *s.p.p.* of $A\bar{x}$. For $A\bar{x} \in \Phi$, the major premiss has the form $\exists y P y$, where P is a prime formula and $\exists y P y$ is closed. This completes the proof of the statement. Assume $m \in T_x$, Rule$(m) \neq \exists^0$E. Then we see that Rule(m) is $\exists^0$I-rule or $\curlywedge$-rule. Assume it is not true. Then Rule(m) is an elimination except the $\exists^0$E-rule. Hence the top formula of the main branch of Σ_m has $\exists y B(\bar{x}, y)$ or a formula whose outermost logical symbol is $\exists^\tau$ $(\tau \neq 0)$ as a *s.p.p.*. However, the top formula is $A\bar{x}$ and $A\bar{x} \in \Phi$. This is a contradiction. Hence Rule(m) is $\exists^0$I-rule or $\curlywedge$-rule.

We define a partial recursive function g so that

$$\begin{cases} g(0, x) \simeq \langle x, 0 \rangle \\ g(a+1, x) \simeq \begin{cases} g(a, x) * \langle \mu y \text{Tr}(\text{Con}(g(a, x) * \hat{0}), y) + 1 \rangle & \text{if } \forall i < a(\text{Rule}(g(i, x)) = \exists^0\text{E}) \\ 0 & \text{otherwise,} \end{cases} \end{cases}$$

where Tr(x, y) denotes the recursive predicate which is true if and only if x is an index of the form $\exists y P y$ and $P\bar{y}$ is a true closed prime formula. We define two partial recursive fundtions f and h so that $fx \simeq \mu y(g(y, x) \simeq 0)$ and $hx \simeq \varphi_1(fx)$. Assume $A\bar{x}$, then g is a totally defined function. We prove this by an induction on a. Firstly $g(0, x)$ is defined. Let $g(a, x)$ be defined and equals to zero. Then $\exists i < a(\text{Rule}(g(i, x)) \neq \exists^0\text{E})$. Then $g(a+1, n)$ is defined and zero. Assume $g(a, n)$ is defined and not zero. Then we see $\forall i < a(\text{Rule}(g(i, x)) = \exists^0\text{E})$ is true from the definition of g. By the above analyses of Σ the assumptions of Σ_z $(z \leqq g(a, x))$ are closed prime formulae or $A\bar{x}$. By induction on the length of z, we can prove all of the assumptions of Σ_z $(z \leqq g(a, x))$ are true. Since $\mathrm{IPf}^n(\Sigma_{g(a,x)})$, we see $\mathrm{Sat}^{(n)}(\varnothing; \mathrm{Con}(g(a, x)))$ by the partial reflection principle. Let $\mathrm{Con}(g(a, x))$ be $\exists y Py$. Since $\mathrm{Sat}^{(n)}(\varnothing; \exists y Py)$, there is a natural number y_0 which satisfies Py_0. Hence by the definition of g we conclude $g(a+1, x)$ is defined. Hence g is totally defined. Note that $g(a, x)$ is a node of Σ. Since Σ is well-founded, there is the least number m ($\neq 0$) that satisfies $g(m-1, x) \not< g(m, x)$. From the definition of g, it is evident that $g(m, x)=0$. Hence we now see if Ax, then fx and hx is defined. Moreover by the analyses of the structure of Σ, $\mathrm{Rule}(g(m, x))$ is the $\exists^0$I-rule or the $\curlywedge$-rule. However, the $\curlywedge$-rule is impossible, for all assumptions of $\Sigma_{g(m,x)}$ are true. Hence $\mathrm{Rule}(g(m, x))$ is the $\exists^0$I-rule. It is evident that $\mathrm{Con}(g(m, x) * \hat{0}) = B(\bar{x}, \overline{hx})$. Since all assumptions of $\Sigma_{g(m,x)*\hat{0}}$ are true, $B(\bar{x}, \overline{hx})$ is true. Let e be an index of h. Then we get the desired conclusion.

3.4. The following lemma is probable in S. In the proofs of the remaining theorems we use this lemma in S.

LEMMA 4. *If* $\mathrm{Npf}(\Sigma)$, $\mathrm{Asp}(\Sigma)=\varnothing$, $\mathrm{Con}(\Sigma)=\forall f \exists x^\tau A(f, x)$ *and* $\exists x A(F, x)$ *does not belong to the set* $\{\exists y^0 \mathfrak{A}(F, \bar{m}, y): m$ *is a natural number*$\}$, *then the followings are true.*

(1) Σ *has the following form:*

$$\cfrac{\cfrac{\begin{array}{c}\mathfrak{F}(F)\\ \Sigma_{\hat{0}_2}\\ \exists x^\tau A(F, x)\end{array}}{\mathfrak{F}(F) \to \exists x A(F, x)}}{\forall f \exists x A(f, x)},$$

and $\mathrm{Asp}(\Sigma_{\hat{0}_2}) \subseteq \{\mathfrak{F}(F)\}$.

(2) *Let* Π *indicate* $\Sigma_{\hat{0}_2}$. *If* $a \in \Pi$, *then* a *satisfies the following properties*:

Case 1. *If* $\forall b \leqq a(\mathrm{Rule}(\Pi_b)=\exists^0\mathrm{E})$, *then for all* Π_b *whose* b *satisfies* $b \leqq a$ *are the following forms*:

$$\frac{\dfrac{\mathfrak{F}(F)}{\exists y \mathfrak{A}(F, \bar{m}, y)} \qquad \begin{array}{c}\mathfrak{A}(F, \bar{m}, \bar{n})\\ \cdots\ \Pi_{a*\langle n+1\rangle}\ \cdots\\ \exists x A(F, x)\end{array}}{\exists x A(F, x)}\,(\exists^0\mathrm{E}),$$

and $\mathrm{Asp}(\Pi_{a*\langle n+1\rangle}) \subseteq \mathfrak{C}(F)$.

Case 2. *If* $\exists b \leqq a(\mathrm{Rule}(\Pi_b) \neq \exists^0\mathrm{E})$ *and* $\mu y(\mathrm{Rule}(\Pi_{\mathrm{seg}(a,y)}) \neq \exists^0\mathrm{E})=m$, *then* $\Pi_{\mathrm{seg}(a,m)}$ *is one of the following forms*:

(i) $$\frac{\begin{array}{c}\Pi_{\mathrm{seg}(a,m)*\hat{0}}\\ \curlywedge\end{array}}{\forall x A(F, x)}$$

(ii) $$\frac{\begin{array}{c}\Pi_{\mathrm{seg}(a,m)*\hat{0}}\\ A(F, t)\end{array}}{\exists x A(F, x)}$$

(iii) $$\frac{\mathfrak{F}(F)}{\exists y \mathfrak{A}(F, \bar{n}, y)},$$

In cases (i), (ii) $\Pi_{\mathrm{seg}(a,m)}$ *is a minor premiss of an application of* $\exists^0$E-*rule or* a *equals to* $\langle\ \rangle$, *and* Asp $(\Pi_{\mathrm{seg}(a,m)*\hat{0}})\subseteq\mathfrak{C}(F)$. *In the case* (iii) $a\neq\langle\ \rangle$ *and* $\Pi_{\mathrm{seg}(a,m)}$ *is the major premiss of an application of* $\exists^0$E-*rule.*

Proof. (1) is evident, since Σ is normal and Asp $(\Sigma)=\varnothing$. Assume the condition of Case 1 of (2). We prove Case 1 by a course-of-values induction on the length of b. Let the statement of Case 1 be true for the finite sequences $b'\prec b$. By the induction hypothesis we see the open assumptions of the major deduction of the application of $\exists^0$E-rule at b belong to $\mathfrak{C}(F)$. The top formula of the main branch of Π_b must be an open assumption, hence it belongs to $\mathfrak{C}(F)$. The major premiss of the last inference of Π_b must be a *s.p.p.* of the top formula, since the main branch has no application of $\exists$E-rule. Hence the possible top formula is only $\mathfrak{F}(F)$ and the possible main branch is only the sequence $\mathfrak{F}(F)$, $\exists y\mathfrak{A}(F, \bar{m}, y)$ (m is a natural number). Hence Π_b has the form of the figure of Case 1. Now we prove Case 2. Assume the condition of Case 2. By Case 1, Asp $(\Pi_{\mathrm{seg}(a,m)})$ is a subset of $\mathfrak{C}(F)$. Hence if Rule $(\Pi_{\mathrm{seg}(a,m)})$ is an elimination, then the top formula of the main branch of $\Pi_{\mathrm{seg}(a,m)}$ belongs to $\mathfrak{C}(F)$. By similar argument of the proofs of Case 1 we can see $\Pi_{\mathrm{seg}(a,m)}$ has the form of (iii). If Rule $(\Pi_{\mathrm{seg}(a,m)})$ is not an elimination, then $\Pi_{\mathrm{seg}(a,m)}$ is obviously one of the forms (i), (ii). In the cases (i), (ii) $\Pi_{\mathrm{seg}(a,m)}$ is not a major premiss of an elimination rule, since Σ is normal. Hence it must be a minor premiss of $\exists^0$E-rule or $a=\langle\ \rangle$. In the case (iii) a is not $\langle\ \rangle$, since $\exists y\mathfrak{A}(F, \bar{n}, y)\neq\exists xA(F, x)$. Hence it is a premiss of an application of $\exists^0$E-rule whose conclusion is $\exists xA(F, x)$. Hence it is the major premiss; otherwise $\exists xA(F, x)$ equals to $\exists y\mathfrak{A}(F, \bar{n}, y)$.

3.5. THEOREM 4 (Bar Induction Rule). *Let* H1, H2 *and* H3 *be the formulae*

$$\forall fn[A(f, n)\rightarrow A(f, n+1)]$$
$$\forall fn[A(f, n)\rightarrow Q(\bar{f}(n))]$$

and

$$\forall x[(\forall yQ(x*\hat{y}))\rightarrow Qx]\ ,$$

respectively. If $\mathrm{S}\vdash\forall f\exists nA(f, n)$, *then* $\mathrm{S}\vdash\forall Q^{(0)}[\{\mathrm{H1}\ \&\ \mathrm{H2}\ \&\ \mathrm{H3}\}\rightarrow\forall xQ(x)]$.

Proof. If $\exists nA(F, n)$ is a form $\exists y^0\mathfrak{A}(F, \bar{m}, y)$, then the theorem is evident. So we may assume the form $\forall f\exists nA(f, n)$ satisfies the condition of Lemma 4. Assume $\mathrm{Pv}^n(\forall f\exists nA(f, n))$. Then $\mathrm{S}\vdash\mathrm{NPv}^{\bar{n}}(\forall f\exists nA(f, n))$. It is sufficient to prove $\forall xQx$ from H1-H3 and $\mathrm{NPv}^n(\forall f\exists nA(f, n))$ in S. Note that we can prove the condition of Lemma 4 in S.

Assume $\mathrm{NPf}^n(\Sigma)$, $\mathrm{Asp}(\Sigma)=\varnothing$ and $\mathrm{Con}(\Sigma)=\forall f\exists nA(f, n)$. Rule (a), Con (a), Asp (a), $\varphi_1(a)$ and $\varphi_2(a)$ mean Rule (Σ_a), Con (Σ_a), Asp (Σ_a), $\varphi_1(\Sigma_a)$ and $\varphi_2(\Sigma_a)$ respectively. We define a subtree T of Σ so that $a\in T$ if and only if $a\in\Sigma$, $\mathrm{lth}(a)\geq 2$, $\forall i(2\leq i<\mathrm{lth}(a)\dot{-}1\rightarrow\mathrm{Rule}(\mathrm{seg}(a, i))=\exists^0\mathrm{E})$ and $\mathrm{Rule}(a)\in\{\exists^0\mathrm{E}, \curlywedge\text{-rule}, \exists^0\mathrm{I}\}$. Note that T is well-founded, since Σ is well-founded. For each element a of T we assign a finite set $\mathfrak{G}(a)$ of pairs of numbers as follows:

$$\mathfrak{G}(a)=_{\mathrm{def}}\{\langle\varphi_2(\mathrm{seg}(a, i)*\hat{0}), (a)_i\dot{-}1\rangle: \mathrm{Rule}(\mathrm{seg}(a, i))=\exists^0\mathrm{E} \text{ and } 2\leq i<\mathrm{lth}(a)\dot{-}1\} .$$

$b\in^*\mathfrak{G}(a)$ and $f\in^*\mathfrak{G}(a)$ mean the followings respectively:

$$\forall x\in\mathfrak{G}(a)((b)_{(x)_0}=(x)_1) \text{ and } \forall x\in\mathfrak{G}(a)((x)_0<\mathrm{lth}(b)) ,$$
$$\forall x\in\mathfrak{G}(a)(f((x)_0)=(x)_1) .$$

Note that in the above definitions $\langle(x)_0, (x)_1\rangle=x$. We show the following proposition (P) by the induction over T:

$$(P) \qquad \forall x\in T(\forall y\in^*\mathfrak{G}(x)Q(y)) .$$

We can deduce $\forall xQ(x)$ from (P), since $\hat{0}_2\in T$ and $G(\hat{0}_2)=\varnothing$. Hence our proof is completed when (P) is proved. It is sufficent to prove that $\forall x\in T(\forall i(x*\hat{i}\in T\rightarrow E(x*\hat{i}))\rightarrow E(x))$, where $E(x)$ means $\forall y\in^*\mathfrak{G}(x)Q(y)$. We prove this proposition by cases.

Case 1. $\mathrm{Rule}(a)=\exists^0\mathrm{I}$. Then by Lemma 4 and the definition of T

$$\mathrm{Asp}(\Sigma_{a*\hat{0}})\subseteq\{\mathfrak{F}(F)\}\cup\{\mathfrak{A}(F, \bar{n}_1, \bar{n}_2): \langle n_1, n_2\rangle\in\mathfrak{G}(a)\}$$

and

$$\mathrm{Con}(a*\hat{0})=A(F, \varphi_1(a)) .$$

If $f\in^*\mathfrak{G}(a)$ and $B\in\mathrm{Asp}(a*\hat{0})$, then $\mathrm{Sat}^{(n)}(f; B)$. Hence by the partial reflection principles if $f\in^*\mathfrak{G}(a)$, then $\mathrm{Sat}^{(n)}(f; \mathrm{Con}(a*\hat{0}))$. We now see if $f\in^*\mathfrak{G}(a)$, then $A(f, \varphi_1(a))$. Assume $b\in^*\mathfrak{G}(a)$. We show $Q(b)$ from this.

Subcase 1. $\varphi_1(a)\leq\mathrm{lth}(b)$. Assume $b\in^*\mathfrak{G}(a)$. Then $A([b], \varphi_1(a))$, since $[b]\in^*\mathfrak{G}(a)$. By H1 and the assumption of Subcase 1 we see $A([b], \mathrm{lth}(b))$. By H2, $Q(b)$ is true.

Subcase 2. $\varphi_1(a)>\mathrm{lth}(b)$. Similarly to Subcase 1 we can see that if $b\in^*\mathfrak{G}(a)$, then $\forall x(\mathrm{lth}(x)=\varphi_1(a)-\mathrm{lth}(b)\rightarrow Q(b*x))$. By an induction on $\varphi_1(a)-\mathrm{lth}(b)$ with use of H3, we can conclude $Q(b)$.

Case 2. $\mathrm{Rule}(a)=\curlywedge$-rule. Then by the same argument as Case 1 if $f\in^*\mathfrak{G}(a)$, then $\mathrm{Sat}^{(n)}(f; \curlywedge)$. Hence there is no f such that $f\in^*\mathfrak{G}(a)$. If $b\in^*\mathfrak{G}(a)$, then $[b]\in^*\mathfrak{G}(a)$. Hence there is no b such that $b\in^*\mathfrak{G}(a)$. Namely $E(a)$ is vacuously true. So the desired proposition is valid.

Case 3. $\mathrm{Rule}(a)=\exists^0\mathrm{E}$. Then by Lemma 4 and the definition of

T we see the major premiss Con $(a*\hat{0})$ is the formula $\exists y\mathfrak{A}(F, \overline{\varphi_2(a*\hat{0})}, y)$, Con $(a)=\exists nA(F, n)$, $\mathrm{Asp}(a*\langle i+1\rangle)\subseteq\{\mathfrak{F}(F), \mathfrak{A}(F, \overline{\varphi_2(a*\hat{0})}, \bar{i})\}\cup\{\mathfrak{A}(F, \bar{n}_1, \bar{n}_2)$: $\langle n_1, n_2\rangle\in\mathfrak{S}(a)\}$ and $a*\langle i+1\rangle\in T$. Since $\forall i(a*\langle i+1\rangle\in T)$, it is sufficent to prove $\forall b\in^*\mathfrak{S}(a)Q(b)$ from $\forall i\forall b\in^*\mathfrak{S}(a*\langle i+1\rangle)Q(b)$. We prove this by two subcases.

Subcase 1. $\exists x\in\mathfrak{S}(a)((x)_0=\varphi_2(a*\hat{0}))$. Assume $x\in\mathfrak{S}(a)$ and $(x)_0=\varphi_2(a*\hat{0})$. Then $\mathfrak{S}(a*\langle(x)_1+1\rangle)=\mathfrak{S}(a)$. Since $\forall i\forall b\in^*\mathfrak{S}(a*\langle i+1\rangle)Q(b)$, we conclude $\forall b\in^*\mathfrak{S}(a)Q(b)$.

Subcase 2. $\forall x\in\mathfrak{S}(a)((x)_0\neq\varphi_2(a*\hat{0}))$. Assume $b\in^*\mathfrak{S}(a)$.

Subcase 2a. $\mathrm{lth}(b)>\varphi_2(a*\hat{0})$. Then $b\in^*\mathfrak{S}(a*\langle(b)_{\varphi_2(a*\hat{0})}+1\rangle)$. Since $\forall i\forall b\in^*\mathfrak{S}\in(b*\langle i+1\rangle)Q(b)$, we see $\forall b\in^*\mathfrak{S}(a)Q(b)$.

Subcase 2b. $\mathrm{lth}(b)\leq\varphi_2(a*\hat{0})$. Then every c whose length is $\varphi_2(a*\hat{0})+1-\mathrm{lth}(b)$, satisfies $b*c\in^*\mathfrak{S}(a*\langle(b*c)_{\varphi_2(a*\hat{0})}+1\rangle)$. By the assumption $\forall i\forall x\in^*\mathfrak{S}(a*\langle i+1\rangle)Q(x)$, we see $Q(b*c)$ for all c whose length is $\varphi_2(a*\hat{0})+1-\mathrm{lth}(b)$. By an induction on $\varphi_2(a*\hat{0})+1-\mathrm{lth}(b)$ with uses of H3, we can conclude $Q(b)$.

The possible cases are only above three. Hence we now complete the proof.

3.6. Corollary 1 (Transfinite Induction Rule). *Let* ρ *be a term of type* (0, 0), $\mathrm{WF}(\rho)$ *and* $\mathrm{I}(\rho)$ *be the formulae*

$$\forall f\exists n\neg(f(n)\rho f(n+1))$$

and

$$\forall Q^{(0)}[\forall x\{\forall y(x\rho y\rightarrow Qy)\rightarrow Qx\}\rightarrow\forall xQx] ,$$

respectively. If $\mathrm{S}\vdash\mathrm{WF}(\rho)$, *then* $\mathrm{S}\vdash\mathrm{I}(\rho)$.

Proof. We follow the proof of Theorem 5A of [7]. Let $A(f, n)$ be the following formula:

$$\neg(\forall m<n\dot{-}1(fm\rho f(m+1))) .$$

The $\mathrm{S}\vdash\mathrm{WF}(\rho)$ implies $\mathrm{S}\vdash\forall f\exists nA(f, n)$. By Theorem 4 we see $\mathrm{S}\vdash\forall Q^{(0)}[\{\mathrm{H1}\ \&\ \mathrm{H2}\ \&\ \mathrm{H3}\}\rightarrow\forall xQx]$. Let Qx be the formula $(x=\hat{0}_0\ \&\ \forall yXy)\vee(x\neq\hat{0}_0\ \&\ (Bx\rightarrow\forall y((x)_{\mathrm{lth}(x)\dot{-}1}\rho y\rightarrow Xy)))$, where X is a free variable of type (0) and Bx is the formula $\forall m<\mathrm{lth}(x)\dot{-}1((x)_m\rho(x)_{m+1})$. It is easy to verify $\mathrm{S}\vdash\{\mathrm{H1}\ \&\ \mathrm{H2}\}$. Hence $\mathrm{S}\vdash[\forall x\{(\forall yQ(x*\hat{y}))\rightarrow Qx\}\rightarrow\forall xQx]$. This implies $\mathrm{S}\vdash\mathrm{I}(\rho)$.

3.7. Lemma 5. *If* $\mathrm{S}\vdash\forall f\exists X^{(0,0)}A(f, X)$, *then there is a natural number* n *such that* $\mathrm{S}\vdash\exists h\forall f[!h(f)\ \&\ h(f)$ *is an index of a term which belongs to* $\mathrm{T}^{(\bar{n})}$ *and has type* (0, 0) & $A(f, \lambda xy\mathrm{Sat}^{(n)}(f; \{h(f)\}(\bar{x}, \bar{y})))]$, *where* $\{h(f)\}$ *means the term whose index is* $h(f)$.

Proof. Assume $\mathrm{Pv}^n(\forall f\exists XA(f, X))$. Then $\mathrm{S}\vdash\mathrm{NPv}^{\bar{n}}(\forall f\exists XA(f, X))$.

The following considerations are formalizable in S.

Since $\mathrm{NPv}^{n}(\forall f \exists X A(f, X))$, we find a derivation Σ so that $\mathrm{NPf}^{n}(\Sigma)$, $\mathrm{Asp}(\Sigma)=\varnothing$ and $\mathrm{Con}(\Sigma)=\forall f \exists X A(f, X)$. Rule$(a)$, Con$(a)$, $\cdots$ are defined as in 3.5. We can define a partial recursive function χ so that

$$\chi(b, x) \simeq \begin{cases} \varphi_3(b)+1 & \text{if Rule}(b)=\exists^0\mathrm{I} \\ \chi(b*\langle (x)_{\varphi_2(\tau*\hat{0})}+1\rangle, x) & \text{if Rule}(b)=\exists^0\mathrm{E} \ \& \ \mathrm{lth}(x)>\varphi_2(b*\hat{0}) \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to verify χ is totally defined by the well-foundness of Σ. Let h be the function $\chi(\langle 0, 0\rangle, x)$. Now we show h is the desired as function.

We define a primitive recursive functional (in the sence of [10]) ψ as follows:

$$\psi(0, f)=\langle 0, 0\rangle$$

$$\psi(n+1, f)=\begin{cases} \psi(n, f)*\langle f(\varphi_2(\psi(n, f)*\hat{0}))+1\rangle & \text{if Rule}(\psi(n, f))=\exists^0\mathrm{E} \\ 0 & \text{otherwise}. \end{cases}$$

By the well-foundness of Σ, we see $\forall f \exists n(\psi(n, f)=0)$. Let a be the number $\psi(\mu y(\psi(y, f)=0)\dot{-}1, f)$. Rule$(a)$ is $\exists^{(0,0)}\mathrm{I}$ or $\curlywedge$-rule, and $\mathrm{Asp}(a*\hat{0})\subseteq\{\mathfrak{F}(F)\}\cup\{\mathfrak{A}(F, \bar{n}, \overline{fn}): n$ is a natural number$\}$. Hence if $B\in\mathrm{Asp}(a*\hat{0})$, then $\mathrm{Sat}^{(n)}(f; B)$. Hence by the partial reflection principle we see $\mathrm{Sat}^{(n)}(f; \mathrm{Con}(a*\hat{0}))$. Hence Rule$(a)$ is $\exists^{(0,0)}\mathrm{I}$-rule and $\mathrm{Con}(a*\hat{0})=A(F, t^{(0,0)})$. By the definition of h, $h(f)$ is an index of t. Hence we see $\mathrm{Sat}^{(n)}(f, \lambda xy\,\mathrm{Sat}^{(n)}(f; \{t\}(\bar{x}, \bar{y})); A(F, G))$. By 4.5.7 of [17] we conclude $A(f, \lambda xy\,\mathrm{Sat}^{(n)}(f; \{h(f)\}(x, y)))$.

3.8. THEOREM 5 (Continuity Rule). *If* $\mathrm{S}\vdash\forall f \exists g A(f, g)$, *then* $\mathrm{S}\vdash\exists h\{h$ *is primitive recursive* & $\forall f(!(h|f)$ & $A(f, (h|f)))\}$.

Proof. If $\mathrm{S}\vdash\forall f \exists g A(f, n)$, then by Lemma 5 and EP we see there is a formula $B(F, G)$ such that $\mathrm{S}\vdash\{\forall f \exists ! g B(f, g)$ & $\forall fg(B(f, g)\rightarrow A(f, g))\}$. By Theorem 2 there is a natural number n such that $\mathrm{S}\vdash\mathrm{NPv}^{\bar{n}}(\forall x \forall f \exists y \Gamma_B(x, f, y))$, where $\Gamma_B(x, F, y)$ means $\exists G(\mathfrak{F}(G)$ & $B(F, G)$ & $G(x, y))$. The following proof is formalizable in S.

Assume $\mathrm{NPf}^{n}(\Sigma)$, $\mathrm{Asp}(\Sigma)=\varnothing$ and $\mathrm{Con}(\Sigma)=\forall x \forall f \exists y \Gamma_B(x, f, y)$. The functional χ is defined as in the proof of Lemma 5 except that φ_3 is replaced by φ_1. Let h' be the function $\chi(\langle (x)_0, 0, 0\rangle, \mathrm{tl}(x))$. Then by a similar proof to Lemma 5 we see $\forall f(!(h'|f)$ & $\forall x \Gamma_B(x, f, (h'|f)(x)))$. Namely $\forall f(!(h'|f)$ & $\forall x \exists g(B(f, g)$ & $g(x)=(h'|f)(x)))$. Since $\forall f \exists ! g B(f, g)$, we see $\forall fg(B(f, g)\Leftrightarrow g=(h'|f))$. Hence we see $\forall f(!(h'|f)$ & $A(f, (h'|f)))$, since $\forall fg(B(f, g)\rightarrow A(f, g))$. Since h' is a recursive function, there is a number e such that $h'(x)=U(\mu y T(e, x, y))$. Let $R(x, y)$ be $T(e, x, y)$. We define a primitive recursive function h so that

$$h(x)=\begin{cases} U(\mu y<\mathrm{lth}(x)(\exists z\leqq x(R(z, y)\ \&\ Uy\neq 0))) & \\ & \text{if}\quad \exists y<\mathrm{lth}(x)\exists z\leqq x(R(z, y)\ \&\ Uy\neq 0) \\ 0 & \text{otherwise}. \end{cases}$$

Since $(h'(x)>0\ \&\ x\leqq y)\rightarrow h'(x)=h'(y)$, we see $\forall f(!(h|f)\ \&\ A(f, (h|f)))$.

3.9. THEOREM 6 (Fan Rule). *If* $\mathrm{S}\vdash\forall f\exists nA(f, n)$, *then* $\mathrm{S}\vdash\exists f[f$ *is primitive recursive* & $\forall g\{!f(g)\ \&\ \forall h\leq g\exists n\forall k(\bar{h}(f(g))=\bar{k}(f(g))\rightarrow A(k, n))\}]$.

Proof. By Theorem 5 it is sufficent to prove that if $\mathrm{S}\vdash\forall f\exists nA(f, n)$, then $\mathrm{S}\vdash\forall g\exists m\{\forall f\leq g\exists n\forall k(\bar{g}(m)=\bar{k}(m)\rightarrow A(k, n))\}$. Without loss of generality we may assume $\forall f\exists nA(f, n)$ satisfies the condition of Lemma 4. Let $\mathrm{NPv}^{\bar{n}}(\forall f\exists nA(f, n))$ be provable in S. The following proof is formalizable in S.

Assume $\mathrm{NPf}^{n}(\Sigma)$, $\mathrm{Asp}(\Sigma)=\varnothing$ and $\mathrm{Con}(\Sigma)=\forall f\exists nA(f, n)$. Rule (a), $\cdots$ are the same as in the proof of Theorem 4. Note that the assumption is the same as in the proof of Theorem 4. So we use the same definitions as in the proof of Theorem 4.

For each function g we assign a tree T_g as follows:

$$T_g=\{a\colon a\in T\ \&\ \forall i(2\leq i<\mathrm{lth}(a)\dot{-}1\rightarrow(a)_i\dot{-}1\leq g(\varphi_2(\mathrm{seg}(a, i)*\hat{0})))\}\ .$$

Let $P(a, m)$ be the following predicate:

$$\forall y(y\in T_g\ \&\ a\leqq y\ .\rightarrow.\ \forall x\in\mathfrak{S}(y)((x)_0\leq m))\ .$$

We prove $\forall x\in T_g\exists mP(x, m)$ by the induction of T_g. Assume $a\in T_g$ and $\forall x(a*\hat{x}\in T_g\rightarrow\exists mP(a*\hat{x}, m))$. We must show $P(a, m)$ from this.

Case 1. Rule $(a)\in\{\wedge\text{-rule}, \exists^0\mathrm{I}\}$. If $a\leqq b$ and $b\in T_g$, then b is a itself. Hence we may take $\max\{(x)_0\colon x\in\mathfrak{S}(a)\}$ as m.

Case 2. Rule $(a)=\exists^0\mathrm{E}$. If $b\in T_g$ and $a\leqq b$, then $(a)_{\mathrm{lth}(a)}\leq g(\varphi_2(a*\hat{0}))$. From the induction hypothesis for each i which satisfies $i\leq g(\varphi_2(a*\hat{0}))$ we can find m_i so that $P(a*\hat{i}, m_i)$. We take $\max\{m_i\colon i\leq g(\varphi_2(a*\hat{0}))\}$ as m.

We define a functional ψ by the same definition as in the case Lemma 5. Then $\forall f\exists n(\psi(n, f)=0)$. Let $\delta(f)$ be the following number:

$$\psi(\mu y(\psi(y, f)=0)\dot{-}1, f)\ .$$

Then $\mathrm{Asp}(\delta(f))\subseteq\{\mathfrak{F}(F)\}\cup\{\mathfrak{A}(F, \bar{n}_1, \bar{n}_2)\colon\langle n_1, n_2\rangle\in\mathfrak{S}(\delta(f))\ \&\ f(n_1)=n_2\}$ and $\mathrm{Con}(\delta(f))\in\{A(F, \bar{n})\colon n$ is a natural number$\}$.

Fix a function g. Let m be a number which satisfies $P(\hat{0}_2, m)$. Since $\hat{0}_2$ is the root of the tree T_g, we see $\forall a\in T_g\forall x\in\mathfrak{S}(a)((x)_0\leq m)$. Assume $f\leq g$. Then $\delta(f)\in T_g$. Let $\mathrm{Con}(\delta(f))$ be the formula $A(F, \bar{n}_0)$. Assume $\bar{f}(m)=\bar{k}(m)$. If B is an element of $\mathrm{Asp}(\delta(f))$, then $\mathrm{Sat}^{(n)}(k; B)$. Hence by the partial reflection principle and 4.5.7 of [17] we see $A(k, n_0)$. Namely we have shown that $\forall f\leq g\exists n(\bar{h}(m)=\bar{k}(m)\rightarrow A(k, n))$.

3.10. COROLLARY 2. (i) *If* $\mathrm{S}\vdash\{A$ *is a function from* $[0,1]$ *to* $\boldsymbol{R}\}$, *then* $\mathrm{S}\vdash\{A$ *is uniformly continuous on* $[0,1]\}$. (ii) *If* $\mathrm{S}\vdash\{A$ *is a function from* $\boldsymbol{R}$ *to* $\boldsymbol{R}\}$, *then* $\mathrm{S}\vdash\{A$ *is continuous on* $\boldsymbol{R}\}$.

Proof. (i) By a formalization of Theorem 1 of 3.4.3 of [9] we can see that there is a formula S such that $\mathrm{S}\vdash\{S$ is a finitary spread and S consides with the interval $[0,1]\}$. In this formalization we must show that for every real number there is a canonical number-generater which equals to the real number. In [9] this is proved by the use of the axiom of choice. However, we need not the axiom of choice, because in our definition real numbers have their modulus of convergence. Note that a fan is represented by a function (see [18]). There is a formula Γ such that $\mathrm{S}\vdash\{\forall f\exists!g\Gamma(f,g)\ \&\ \forall fg(\Gamma(f,g)\to g\in S)\ \&\ \forall f(f\in S)\to\Gamma(f,f)\}$, where $g\in S$ means g is an element of the spread S (see 2.1 of [18]). By a formalization of the first half of the proof of Theorem 1 of 3.4.5 of [9] we see $\mathrm{S}\vdash\forall x\forall f\in S\exists nB(x,f,n)$, where B is the formula $\exists y\in\boldsymbol{R}\{A(r(f),y)$ & there is a canonical number-generater $\{\eta_n 2^{-n}\}$ which equals to y and $\eta_x=n\}$ and $r(f)$ denotes the real number which is represented in S by f. Thence $\mathrm{S}\vdash\forall f\exists n\exists g(\Gamma(\lambda xf(x+1),g)\ \&\ B(f(0),g,n))$. By the fan rule we see $\mathrm{S}\vdash\forall f\exists m\forall g\leq f\exists n\forall h\{\bar{g}(m)=\bar{h}(m)\to\exists k(\Gamma(\lambda xh(x+1),k)\ \&\ B(h(0),k,n))\}$. It is easy to see $\mathrm{S}\vdash\forall x\exists f\forall g\in S(\langle x\rangle*g\leq f))$, where $\langle x\rangle*g$ is the function which satisfies $(\langle x\rangle*g)(0)=x$ and $(\langle x\rangle*g)(n+1)=g(n)$. Hence we can see $\mathrm{S}\vdash\forall x\exists m\forall f\in S\exists n\forall g\in S(\bar{f}(m)=\bar{g}(m)\to B(x,g,n))$. Hence by the same way as the second half of Theorem 1 of 3.4.5 of [9] we see $\mathrm{S}\vdash\{A$ is uniformly continuous on $[0,1]\}$.

(ii) Assume $\mathrm{S}\vdash\{A$ is a function $\boldsymbol{R}$ to $\boldsymbol{R}\}$. Then $\mathrm{S}\vdash\forall x^0\{A(p(f,x),g)$ represents a function from $[0,1]$ to $\boldsymbol{R}\}$, where $p(f,x)$ means the addition of the real number f and the rational number which is represented by x. Similarly to (i) we see $\mathrm{S}\vdash\forall x\{A(p(f,x),g)$ is uniformly continuous on $[0,1]\}$. Since $\mathrm{S}\vdash\forall f\in\boldsymbol{R}\exists x\exists g\in[1/3,2/3]\ (f=p(g,x))$, we see $\mathrm{S}\vdash\{A$ is continuous on $\boldsymbol{R}\}$.

3.11. LEMMA 6. *If* $\mathrm{S}\vdash\forall f\neg\neg\exists n\exists zT(\bar{e},fn,f(Sn),z)$, *then*

$$\mathrm{S}\vdash\forall f\exists n\exists zT(\bar{e},fn,f(Sn),z)\,,$$

where T *is the constant corresponding to the Kleene's* T*-predicate.*

Proof. Note that $\forall f\neg\neg\exists nzT(\bar{e},fn,f(Sn),z)$ is an abbrebiation. Its original form is equivalent to the following form:

$$\forall F(\mathfrak{F}(F)\to(((\exists xy_1y_2z\mathfrak{T}(F,x,y_1,y_2,z))\to\curlywedge)\to\curlywedge))\,,$$

where $\mathfrak{T}(F,x,y_1,y_2,z)$ denotes $(F(x,y_1)\ \&\ F(Sx,y_2))\ \&\ T(\bar{e},y_1,y_2,z)$. For simplicity we abbreviate the above form to $\mathfrak{S}1$. Assume $\mathrm{Pv}^n(\mathfrak{S}1)$. By Theorem 2 we see $\mathrm{S}\vdash\mathrm{NPv}^n(\mathfrak{S}1)$. The following proof can be formalized in S.

Assume $\mathrm{NPf}^n(\Sigma)$, $\mathrm{Asp}(\Sigma)=\varnothing$ and $\mathrm{Con}(\Sigma)=\mathfrak{S}1$. Let $\mathfrak{S}2$ denote the form $((\exists x y_1 y_2 z \mathfrak{T}(F, x, y_1, y_2, z))\rightarrow\curlywedge)\rightarrow\curlywedge$. Σ has the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{F}(F)\\ \Sigma_{\hat{0}_2}\\ \mathfrak{S}2\end{matrix}}{\mathfrak{F}(F)\rightarrow\mathfrak{S}2}}{\forall F(\mathfrak{F}(F)\rightarrow\mathfrak{S}2)}\,.$$

Let Π be the $\Sigma_{\hat{0}_2}$. Rule (a), Con (a), $\cdots$ mean Rule (Π_a), Con (Π_a), $\cdots$ respectively. Let $\mathfrak{S}3$ be $(\exists x y_1 y_2 z \mathfrak{T}(F, x, y_1, y_2, z))\rightarrow\curlywedge$.

Fix a function f. We define sets of formulae as follows:

$B=\{\mathfrak{F}(F)\}\cup\{\mathfrak{A}(F, \bar{a}, \bar{b}): fa=b\}\cup\{\mathfrak{S}3\}$,

$D_1=\{A$: A is a numerical closed subformulae of $\exists x y_1 y_2 z \mathfrak{T}(F, x, y_1, y_2, z)$ and A is not quantifier free$\}$,

$D_2=\{$false closed prime formulae$\}\cup\{A$: A is $F(\bar{a}, \bar{b})$ or $F(\bar{a}, \bar{b})\&F(S\bar{a}, \bar{c})$, and A is false if $\lambda xyF(x, y)$ is interpreted by $\lambda xy(fx=y)\}$,

$D_3=\{(F(\bar{a}, \bar{b})\&F(S\bar{a}, \bar{c}))\&T(\bar{e}, \bar{b}, \bar{c}, \bar{z})$: a, b, c, z are natural numbers$\}$.

$D=\bigcup_{i=1}^{3} D_i$.

We define function g as follows:

$$g0=\langle\ \rangle$$

$$g(n+1)=\begin{cases} gn*\langle f(\varphi_2(gn*\hat{0}))+1\rangle & \text{if } (1.1)\\ gn*\hat{0} & \text{if } (1.2)\\ gn*\hat{0} & \text{if } (2)\&(3.1)\\ gn*\hat{1} & \text{if } (2)\&(3.3)\\ gn*\langle\varphi_4(gn, f)\rangle & \text{if } (2)\&(3.2)\&(4)\\ gn & \text{otherwise}\,, \end{cases}$$

where (1.1)-(4) indicate the following conditions:
(1.1) Rule $(gn)=\exists^0$E, (1.2) Rule $(n)=\rightarrow$I, (2) Asp $(gn)\ni\mathfrak{S}3$,
(3.1) Rule $(gn)\in\{\curlywedge$-rule, $\exists^0$I$\}$, (3.2) Rule $(gn)\in\{$atomic rules, &I$\}$,
(3.3) Rule $(gn)=\rightarrow$E, (4) Con $(gn)\in D_2\cup D_3$, and if Con $(gn)\in D_3$ and $\lambda xyF(x, y)$ is interpreted by $\lambda xy(fx=y)$ then Con (gn) is false.

By induction on n we can easily see that if n is greater then zero, then $gn\in\Pi$, Asp $(gn)\subseteq B$ and Con $(gn)\in D$, e.g., if Rule $(gn)=\rightarrow$E, then Π_{gn} is one of the following forms

$$\dfrac{\dfrac{\dfrac{\mathfrak{A}(F, \bar{a}, \bar{b})}{\forall z(F(\bar{a}, z)\rightarrow\bar{b}=z)}}{F(\bar{a}, \bar{c})\rightarrow\bar{b}=\bar{c}} \quad \begin{matrix}\Pi_{gn\cdot\hat{1}}\\ F(\bar{a}, \bar{c})\end{matrix}}{\bar{b}=\bar{c}}\,, \qquad \dfrac{\mathfrak{S}3 \quad \begin{matrix}\Pi_{gn*\hat{2}}\\ \exists x y_1 y_2 z \mathfrak{T}(F, x, y_1, , y_2, z)\end{matrix}}{\curlywedge}\,,$$

by the induction hypothesis we see $g(n+1)$, i.e. $\Pi_{gn*\hat{1}}$, satisfies the condition. By the well-foundness of Π, we see there is a natural number n such that $gn=g(n+1)$. Set $n_0=\mu n(gn=g(n+1))$. Note that $n_0>0$.

We now show $\exists nzT(e, fn, f(n+1), z)$. Assume gn_0 does not satisfy (2). Then by $\mathrm{Asp}(gn_0)\subseteq B$ we see all of the open assumptions of Π_{gn_0} are true. Hence $\mathrm{Con}(gn_0)$ is true. Hence $\exists nzT(e, fn, f(n+1), z)$. Assume gn_0 satisfies (2). Then by the definition of g and n_0, we see $\mathrm{Con}(gn_0)\in D_3$ and $\mathrm{Con}(gn_0)$ is true provided $\lambda xyF(x, y)$ is interpreted by $\lambda xy(fx=y)$. Hence we see $\exists nzT(e, fn, f(n+1), z)$.

3.12. THEOREM 7. *If* $\mathrm{S}\vdash\forall xy(x\rho y\vee\neg x\rho y)$ *and* $\mathrm{S}^c\vdash\mathrm{WF}(\rho)$, *then* $\mathrm{S}\vdash\mathrm{I}(\rho)$, *where* S^c *is the system obtained from* S *by adjoining the axiom* $\forall X(\neg\neg X\ .\!\rightarrow.\ X)$, *and* $\mathrm{WF}(\rho)$ *and* $\mathrm{I}(\rho)$ *have the same meaning as in* 3.6.

Proof. Note that $(x\rho y\vee\neg x\rho y)$ is the formula $\exists z(z=0\rightarrow x\rho y\ .\&.\ z\neq 0\rightarrow\neg x\rho y)$. By the Church's rule and the assumption of the theorem, we see $\mathrm{S}\vdash\exists e\forall xy\{\exists zT(e, x, y, z)\&\exists z(T(e, x, y, z)\&(U(z)=0\rightarrow\neg x\rho y)\&(U(z)\neq 0\rightarrow x\rho y))\}$. By the formalized version of Theorem IV of §57 of [10], we see $\mathrm{S}\vdash\exists e\forall xy(\exists zT(e, x, y, z)\Leftrightarrow\neg x\rho y)$. By EP there is a natural number e such that $\mathrm{S}\vdash\forall xy(\exists zT(\bar{e}, x, y, z)\Leftrightarrow\neg x\rho y)$. By the assumption of the theorem, we see $\mathrm{S}^c\vdash\forall f\exists nzT(\bar{e}, fn, f(n+1), z)$. By the translation of 1.10.2 of [17], we see $\mathrm{S}\vdash\forall f\neg\neg\exists nzT(\bar{e}, fn, f(n+1), z)$.[4] By the above lemma and the transfinite induction rule we see $\mathrm{S}\vdash\mathrm{I}(\rho)$.

3.13. COROLLARY 3 (Markov's Rule of Type 1). *If* $\mathrm{S}\vdash\forall fg(A(f, g)\vee\neg A(f, g))$ *and* $\mathrm{S}^c\vdash\forall f\exists gA(f, g)$, *then* $\mathrm{S}\vdash\forall f\exists gA(f, g)$, *where* S^c *has the same meaning as in Theorem* 7.

Proof. In this proof j, j_1, j_2 are the functions of (B) of 1.3.9 of [17]. Since $\mathrm{S}\vdash\forall fg\exists n\{(n=0\rightarrow A(f, g))\&(n\neq 0\rightarrow\neg A(f, g))\}$, we see

$$\mathrm{S}\vdash\exists h\forall fg\{!h(\lambda x.j(fx, gx))\&(h(\lambda x.j(fx, gx))=0\Leftrightarrow A(f, g))\}$$

by the Continuity Rule. By EP there is a formla $h(x^0, y^0)$ such that $\mathrm{S}\vdash[\mathfrak{F}(h)\&\forall fg\{!h(\lambda x.j(fx, gx))\&(h(\lambda x.j(fx, gx))=0\Leftrightarrow A(f, g))\}]$. Let Pa be the formula $\exists x<\mathrm{lth}(a)\{\mathrm{lth}(a)>\mathrm{lth}(x)\ \&\ \forall i<\mathrm{lth}(x)((a)_i=j_1((x)_i))\ \&\ h(x)=1\ \&\ \forall y\prec x(h(y)=0)\}$. Since $\mathrm{S}^c\vdash\forall f\exists gA(f, g)$, we see $\mathrm{S}^c\vdash\forall f\exists nP(\bar{f}(n))$. Let $x\rho y$ be the formula $(x\prec y\ \&\ \neg\exists z\prec yPz)$. Then we see

$$\mathrm{S}^c\vdash\forall f\exists n\neg(fn\rho f(n+1))$$

(cf. Theorem 5C of [7]). Since $\mathrm{S}\vdash\forall xy(x\rho y\vee\neg x\rho y)$, we deduce $\mathrm{S}\vdash\forall f\exists n\neg(fn\rho f(n+1))$ by the above theorem. Since

[4] It is an easy exercise to verify the translation of $\forall f\exists nzT(\bar{e}, fn, f(n+1), z)$ is equivalent to $\forall f\neg\neg\exists nzT(\bar{e}, fn, f(n+1), z)$ in S.

$$\mathrm{S} \vdash \forall f \exists n[\neg \bar{f} n \rho \bar{f}(n+1) .\longrightarrow. \exists z \prec \bar{f}(n+1) \exists x < \mathrm{lth}(z)\{\mathrm{lth}(z) > \mathrm{lth}(x) \\ \& \forall i < \mathrm{lth}(x)(fi = j_1((x)_i) \& h(x) = 1 \& \forall y \prec x(h(y) = 0)\}] ,$$

we see $\mathrm{S} \vdash \forall f \exists x\{\forall i < \mathrm{lth}(x)(j_1((x)_i) = fi) \& h(x) = 1 \& \forall y \prec x(h(y) = 0)\}$. This implies $\mathrm{S} \vdash \forall f \exists x A(f, [x])$. Finally we conclude $\mathrm{S} \vdash \forall f \exists g A(f, g)$.

Appendix 1

Let S^ω be the higher order intuitionistic arithmetic with the axioms of extensionality. Our derived rules hold also for S^ω. The proofs of the present paper can be applied to S^ω without modifications except treatments on the class Φ and the axioms of extensionality.

In the proof of Theorem 3 we used the property of Φ that if $A \in \Phi$, then $\mathrm{S} \vdash A \in \Phi$. This is evident if the all of *s.p.p.* of A are finite. The finiteness of *s.p.p.*s is trivially proved for S, however, it is not so for S^ω. However, the finiteness of *s.p.p.*s can be proved by the method of 2.4 of [15].[5)]

If the axioms of extentionality are assumptions of the normal proofs, then the analyses of the proofs do not work as in §3. We eliminate the axioms of extensionality by the usual relativization (see [5]). Note that the relativation of the second order variables are not necessary, since we have the rule of equality. If a proof of S^ω is relativized, the additional assumptions, which have the form $\mathrm{EXT}(X^\tau)$ (see [5]), may occur. However, $\mathrm{EXT}(X^\tau)$ has no *s.p.p.* which is the form $\exists x^\sigma Ax$ or $X\mathbf{t}$ (X is a bound variable). Hence the method of the section 3 holds good for the relativized proof figures (cf. [6]). Since a formula is equivalent to its relativized form under the axioms of extensionality, we can see the theorems of §3 hold for S^ω.

Let $\mathrm{TI}(e)$ be the formula

$$\forall xy \exists z T(e, x, y, z) \& \mathrm{TI}(\lambda xy(U(\mu z T(e, x, y, z)) = 0)) .$$

Then it is easy to see $\mathrm{S} + \mathrm{TI}(e) \vdash \exists \Sigma(\mathrm{IPf}^{\bar{1}}(\Sigma) \& \mathrm{Asp}(\Sigma) = \varnothing \& \mathrm{Con}(\Sigma) = \mathrm{TI}(e))$. Hence if $\mathrm{S} + \mathrm{TI}(e) \vdash A$, then $\exists n(\mathrm{S} + \mathrm{TI}(e) \vdash \mathrm{NPv}^n(\mathrm{TI}(e)))$ by the footnote 2. Hence we can extend the results of the present paper to $\mathrm{S} + \mathrm{TI}(e)$, $\mathrm{S}^\omega + \mathrm{TI}(e)$.

Remark. $\mathrm{TI}(e)$ may be a false proposition.

Appendix 2.

In this appendix we present a sketch of a proof of the provability of Theorem 5 in **HA**. It is not so difficult to prove the other derived

5) Schütte used the König's lemma, however, the use is not essential. Actually the finiteness of the subexpressions can be proved in **HA** provided a formula is called regular, if and only if all of all of the subexpressions of the formula are finite.

rules of the present paper in **HA** by similar ways, though some of them need more refined treatments.[6]

It is not so difficult to see that Theorem 2 is provable in **HA**. Hence we can see

(i) $\mathbf{HA}\vdash\forall n\{\mathrm{S}\vdash\forall A(\mathrm{Pv}^{\bar{n}}(A)\rightarrow\mathrm{NPv}^{\bar{n}}(A))\}$,

(ii) $\mathbf{HA}\vdash\forall n\{\mathrm{S}\vdash\forall A(\mathrm{Pv}^{\bar{n}}(\exists xAx)\rightarrow\exists t\,\mathrm{NPv}^{\bar{n}}(At))\}$.

By the formalized version of 4.5.8 of [17] and somewhat refined proofs of Lemma 5 and Theorem 5, we can see

(iii) $\mathbf{HA}\vdash\forall nA[\mathrm{S}\vdash\{\mathrm{NPv}^{\bar{n}}(\forall f\exists X^{(0,0)}A(f,X))\rightarrow\exists h\forall f(!h(f)\ \&\ h(f)$ is an index of a term belonging to $\mathrm{T}^{(\bar{n})}$ and has type $(0,0)$

$$\&A(f,\lambda xy\,\mathrm{Sat}^{(n)}(f;\{h(f)\}(\bar{x},\bar{y}))))\}]\ ,$$

(iv) $\mathbf{HA}\vdash\forall n\forall A[\mathrm{S}\vdash\forall B\{\mathrm{NPv}^{\bar{n}}(\forall x\forall f\exists y\Gamma_B(x,f,y))\&\mathrm{Sat}^{(n)}(\varnothing;\forall f\exists!gB(f,g))$ $\&\ B\subseteq A)\ .\!\rightarrow\!.\ \exists h(h$ is primitive recursive $\&\ \forall f(!(h|f)\ \&\ \mathrm{Sat}^{(n)}(f,(h|f);$ $A(F,G))))\}]$,

where $B\subseteq A$ denotes $\forall fg(B(f,g)\rightarrow A(f,g))$.

We now prove Theorem 5 in **HA** by the following steps.

1. $\mathbf{HA}\vdash\{(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow(\exists n(\mathrm{S}\vdash\mathrm{Pv}^{\bar{n}}(\forall f\exists gA(f,g))))\}$.
2. By (i) and (iii), we see

$$\mathbf{HA}\vdash\forall A[(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow\exists n\{\mathrm{S}\vdash\exists h\forall f(!h(f)\ \&\ (h(f)$$

is an index of term belonging to $T^{(\bar{n})}$ and has type $(0,0))$

$$\&\ A(f,\lambda xy\,\mathrm{Sat}^{(n)}(f;\{h(f)\}(\bar{x},\bar{y})))\ \&\ F(\lambda xy\,\mathrm{Sat}^{(n)}(f;\{h(f)\}(x,y))))\}]\ .$$

3. $B^n_h(f,g)$ is the formula $(!h(f)\ \&\ \forall xy(g(x)=y\Leftrightarrow\mathrm{Sat}^{(n)}(f;\{h(f)\}(x,y)))$. $\mathbf{HA}\vdash\forall A\{(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow\exists n(\mathrm{S}\vdash\exists h(\forall f\exists!gB^n_h(f,g)\ \&\ B^n_h\subseteq A))\}$.
4. $\mathbf{HA}\vdash\forall A[(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow\exists n\{\mathrm{S}\vdash\mathrm{Pv}^{\bar{n}}(\exists h(\forall x\forall f\exists y\Gamma_{B^n_h}(x,f,y)\ \&$ $\forall f\exists!gB^n_h(f,g)\ \&\ B^n_h\subseteq A))\}]$.
5. By (ii) we see

$$\mathbf{HA}\vdash\forall A[(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow\exists n\{\mathrm{S}\vdash\exists B(\mathrm{NPv}^{\bar{n}}(\forall x\forall f\exists y\Gamma_B(x,f,y)\ \&\ \forall f\exists!gB(f,g)\ \&\ B\subseteq A))\}]\ .$$

6. By formalizing 4.5.8 and 4.5.9 of [17] we see

$$\mathbf{HA}\vdash\forall A[(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow\exists n\{\mathrm{S}\vdash\exists B(\mathrm{NPv}^{\bar{n}}(\forall x\forall f\exists y\Gamma_B(x,f,y))\ \&\ \mathrm{Sat}^{(n)}(\varnothing;\forall f\exists!gB(f,g)\ \&\ B\subseteq A))\}]\ .$$

7. By (iv) and the formalized partial reflection principles we see $\mathbf{HA}\vdash\forall A[(\mathrm{S}\vdash\forall f\exists gA(f,g))\rightarrow\mathrm{S}\vdash\exists h\{h$ is primitive recursive $\&\ \forall f(!(h|f)\ \&$ $A(f,(h|f)))\}]$.

This is the desired conclusion.

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[6] For example we use the locally formalized EP for S, i.e. $\mathbf{HA}\vdash\forall n\{\mathrm{S}\vdash\forall A(\mathrm{Pv}^{\bar{n}}(\exists xAx)\rightarrow\exists t\,\mathrm{Pv}^{\bar{n}}(At))\}$. This can be proved by formalizing Lemma 2 in **HA**.

Appendix 3.

The purpose of this appendix to correct a slight error in the proof of [17] for the partial reflection principles. The point to be corrected is the proof of 4.5.8. If we acceede the method in [17], we obtain only a weak form of 4.5.8, e.g. $\forall n \forall A(\mathrm{S}\vdash \mathrm{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \Leftrightarrow \forall x\, \mathrm{Sat}^{(n)}(X, \ulcorner A\bar{x} \urcorner))$. However, we need the strong form, e.g.

$$\forall n(\mathrm{S}\vdash \forall A \in \mathrm{Fm}^{(n)}(\mathrm{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \Leftrightarrow \forall x\, \mathrm{Sat}^{(n)}(X, \ulcorner A\bar{x} \urcorner)))\ .$$

In order to prove the strong form we introduce the concept of formations of a formula A ($A \in \mathrm{Fm}^{(n)}$). *A formation of* A ($A \in \mathrm{Fm}^{(n)}$) is a sequence $\langle\langle a_0, b_0, c_0, t_0\rangle, \cdots, \langle a_m, b_m, c_m, t_m\rangle\rangle$ such that $t_m = A$, $t_i \in \mathrm{T}^{(n)}$ and if $a_{i+1} \leq i$, then t_{i+1} is obtained by the substitution of $t_{a_{i+1}}$ for the variables whose indexes are equal to b_{i+1} in the term that belongs to Fm^n and has the index c_{i+1}. Namely a formation represents the way of interated substitutions by which a formula A is constructed from the elements of T^n. We use f, g, $\cdots$ as variables for formations. Naturally we can define a relation $\mathrm{Sat}_{\mathrm{f}}^{(n)}(X, \ulcorner A \urcorner)$ which means the results of the interated substitutions of Sat^n according to the way of the substitutions represented by f. Then we can prove the following form of 4.5.8:

LEMMA A. *In* S *we can prove*

(i) $\forall \mathrm{f} \forall AB \in \mathrm{Fm}^{(n)} \exists \mathrm{gh} \forall X(\mathrm{Sat}_{\mathrm{f}}^{(n)}(X, \ulcorner A \circ B \urcorner) \Leftrightarrow \mathrm{Sat}_{\mathrm{g}}^{(n)}(X, \ulcorner A \urcorner) \circ \mathrm{Sat}_{\mathrm{h}}^{(n)}(X, \ulcorner B \urcorner))$ *for* $\circ \equiv \rightarrow, \&, \vee$.

(ii) $\forall \mathrm{f} \forall A \in \mathrm{Fm}^{(n)} \exists \mathrm{g} \forall X(\mathrm{Sat}_{\mathrm{f}}^{(n)}(X, \ulcorner Q v_i A(v_i) \urcorner) \Leftrightarrow (Q v_i)\, \mathrm{Sat}_{\mathrm{g}}^{(n)}(X, \ulcorner A(\bar{v}_i) \urcorner))$ *for* $Q \equiv \forall_1, \exists_1$.

(iii) *Similar to* (ii) *for the second order quantifiers.*

By Lemma A we can prove the following lemma:

LEMMA B. *In* S *we can prove* $\forall X \forall A \forall \mathrm{fg}$ (if f, g are formations of A, then $\mathrm{Sat}_{\mathrm{f}}^{(n)}(X, \ulcorner A \urcorner) \Leftrightarrow \mathrm{Sat}_{\mathrm{g}}^{(n)}(X, \ulcorner B \urcorner)$).

Since $\mathrm{Sat}^{(n)}(X, \ulcorner \mathrm{A} \urcorner)$ is equivalent to $\exists \mathrm{f}\, \mathrm{Sat}_{\mathrm{f}}^{(n)}(X, \ulcorner A \urcorner)$ in S, we obtain the strong form of 4.5.8 by Lemma A and Lemma B.

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Added in proof (August 5, 1977). The auther wishes to correct the errors of his paper [8].

(1) p. 110, footnote 1. "the Bar Induction Rule" is changed to "the Rule of Bar Recursion of type 0 and 1".

(2) p. 110. In the definitions of the symbols, "$!f(g) \equiv_{\mathrm{def}} \exists y(f(g) \simeq x)$" is changed to "$!f(g) \equiv_{\mathrm{def}} \exists y(f(g) \simeq y)$".

(3) p. 111. In the second to last line, "(cf. [4])" is changed to "(cf. [3])".