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Abelian p-groups of not limit length

by

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In this note we consider additively written abelian p-groups. p-groups A of lengths $\lambda+n$ (where λ is a limit ordinal and n a positive integer) are best understood if they are considered as elongations of p^n-bounded groups $p^\lambda A$ by p-groups $A/p^\lambda A$ of limit lengths (we use the term "elongation" in the sense of Nunke [8]). Our principal aim is to point out that these groups can also be viewed from the opposite angle, viz. as quotients of p-groups T of lengths λ modulo p^n-bounded subgroups U (see Theorem 1). In several important cases, T can be chosen from the same class of p-groups to which $A/p^\lambda A$ belongs. Moreover, we show that it is always possible to find $A\cong T/U$ such that U is dense in $T[p^n]$ in the generalized p-adic topology; here $T[p^n]=\{a\in T\,|\,p^n a=0\}$.

As an application of our results, we prove a theorem on totally injective p-groups of lengths $\omega+n$. We show that they are uniquely determined, up to isomorphism, by their p^n-socles regarded as abelian groups furnished with the height function as valuation (Theorem 5). This result generalizes a theorem by Richman [9].

We follow the notations and terminology of [3]. In particular, $p^\sigma A$ is defined for all ordinals σ inductively by $p^{\sigma+1}A=\{pa\,|\,a\in p^\sigma A\}$ and $p^\rho A=\bigcap_{\sigma<\rho} p^\sigma A$ for limit ordinals ρ. The smallest λ with $p^{\lambda+1}A=p^\lambda A$ is the length of A; $p^\lambda A=0$ whenever A is reduced.

1. Let λ be a limit ordinal, and $\mathscr{C}_\lambda$ a non-empty class of reduced p-groups. It will be called a *p^λ-class* if it satisfies

P1. $X\in\mathscr{C}_\lambda$ and $Y\cong X$ imply $Y\in\mathscr{C}_\lambda$;

P2. $p^\lambda X=0$ for all $X\in\mathscr{C}_\lambda$;

P3. $X\oplus Y\in\mathscr{C}_\lambda$ if and only if $X, Y\in\mathscr{C}_\lambda$;

P4. $X\in\mathscr{C}_\lambda$ if and only if $pX\in\mathscr{C}_\lambda$.

We call $\mathscr{C}_\lambda$ a *full p^λ-class* if, in addition, the following holds:

P5. if $p^\lambda X=0$ and if $X/P\in\mathscr{C}_\lambda$ for some $P\leqq X[p]$, then $X\in\mathscr{C}_\lambda$.

Examples for such classes are abundant.

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E1. The class of all reduced p-groups of lengths$\leqq\lambda$ is a full p^{λ}-class.

E2. The class of direct sums of cyclic p-groups is a p^{ω}-class. It is not a full p^{ω}-class, since a proper $p^{\omega+1}$-projective separable p-group is not in this class, though it contains a subsocle P satisfying the hypotheses of P5 (see [5]).

E3. The class of torsion-complete p-groups is a full p^{ω}-class. Note that if X, P are as in P5, then from the exact sequence

$$0\longrightarrow \mathrm{Ext}\,(\boldsymbol{Z}(p^{\infty}),\,P)\overset{\alpha}{\longrightarrow}\mathrm{Ext}\,(\boldsymbol{Z}(p^{\infty}),\,X)\longrightarrow\mathrm{Ext}\,(\boldsymbol{Z}(p^{\infty}),\,X/P)\longrightarrow 0$$

we obtain $p^{\omega}\,\mathrm{Ext}\,(\boldsymbol{Z}(p^{\infty}),\,X)\leqq\mathrm{Im}\,\alpha$ (since X/P torsion-complete means $p^{\omega}\,\mathrm{Ext}\,(\boldsymbol{Z}(p^{\infty}),\,X/P)$ vanishes [3, 68.5]). In this inclusion, the smaller group is torsion-free (because of $p^{\omega}X=0$ and [3, 56.3]), while the second group is p-bounded. Thus the first group vanishes, and X is torsion-complete.

E4. The class $\mathscr{D}_{\lambda}$ of p^{λ}-high injective (reduced) p-groups (Megibben [7] defined them as p-groups of lengths$\leqq\lambda$ which are direct summands of every p-group containing them as p^{λ}-high subgroups; for an alternative characterization, see Dubois [2]) is a full p^{λ}-class. This follows easily from the fact that a p-group A is p^{λ}-high injective if and only if $p^{\lambda}\,\mathrm{Ext}\,(\boldsymbol{Z}(p^{\infty}),\,A)=0$.

E5. The smallest (full) p^{ω}-class is readily seen to coincide with the class of all bounded p-groups. In fact, by P3, 0 is contained in every class, and by P4 the same holds for all bounded p-groups.

Let $\mathscr{C}_{\lambda}$ be a p^{λ}-class and n a positive integer. We define a new class to consist of "elongations" of p^{n}-bounded groups by groups in $\mathscr{C}_{\lambda}$, i.e.

$$\mathscr{C}_{\lambda}(n)=\{A\mid p^{\lambda+n}A=0 \text{ and } A/p^{\lambda}A\in\mathscr{C}_{\lambda}\}\,.$$

E6. In view of Proposition 1 in [7], a p-group A of length$\leqq\lambda+n$ is $p^{\lambda+n}$-high injective exactly if $A/p^{\lambda}A$ is p^{λ}-high injective. Hence $\mathscr{D}_{\lambda}(n)$ is the class of $p^{\lambda+n}$-high injectives.

Our next objective is to show that for full p^{λ}-classes $\mathscr{C}_{\lambda}$, the class $\mathscr{C}_{\lambda}(n)$ is the same as the class of all quotients of groups in $\mathscr{C}_{\lambda}$ modulo p^{n}-bounded subgrops. We phrase our theorem in a slightly more general form, so as to include the so-called Switch Lemma [1] as a special case.

THEOREM 1. *For a reduced p-group A, integer n and a p^{λ}-class $\mathscr{C}_{\lambda}$, the following conditions are equivalent;*

(a) *there is a subgroup $P\leqq A[p^{n}]$ such that $A/P\in\mathscr{C}_{\lambda}$;*

(b) *there exist a group $T\in\mathscr{C}_{\lambda}$ and a subgroup U of $T[p^{n}]$ such that $T/U\cong A$.*

Both are consequences of the following condition:

(c) $A \in \mathscr{C}_\lambda(n)$.

All three conditions are equivalent for a full p^λ-class $\mathscr{C}_\lambda$.

(a)$\Rightarrow$(b). There is no difficulty in constructing an exact sequence $0 \to X[p^n] \to X \xrightarrow{\pi} A/P \to 0$ where π is multiplication by p^n, i.e. $\pi(x)=p^n x$ ($x \in X$). By induction on n, P4 implies $X \in \mathscr{C}_\lambda$. Starting with the bottom row and the last column, we can get a commutative diagram

$$\begin{array}{ccccccc} & & & & X[p^n] & = & X[p^n] \\ & & & & \downarrow & & \downarrow \\ E_1\colon & P & \rightarrowtail & & T & \twoheadrightarrow & X \\ & \| & & & \downarrow & & \downarrow{\scriptstyle \pi} \\ E_2\colon & P & \rightarrowtail & & A & \twoheadrightarrow & A/P \end{array}$$

The map $\operatorname{Ext}(A/P, P) \xrightarrow{\pi^*} \operatorname{Ext}(X, P)$ induced by π is likewise a multiplication by p^n [3, 52.1]. From $E_1 = \pi^*(E_2) = p^n E_2$ and $p^n \operatorname{Ext}(A/P, P)=0$ we infer that E_1 is splitting. Thus $P, X \in \mathscr{C}_\lambda$ implies $T \in \mathscr{C}_\lambda$, proving (b). ($U$ is the image of $X[p^n]$ in T.)

(b)$\Rightarrow$(a). In view of P4, $T \in \mathscr{C}_\lambda$ implies $T/T[p^n] \in \mathscr{C}_\lambda$. The choice $T = T[p^n]/U$ proves the assertion.

(c)$\Rightarrow$(a) is clear, since $p^\lambda A \leqq A[p^n]$ whenever $p^{\lambda+n}A=0$.

Assuming P5, (a)$\Rightarrow$(c). $A/P \in \mathscr{C}_\lambda$ implies $p^\lambda(A/P)=0$, thus $p^\lambda A \leqq P$, and $P/p^\lambda A \leqq (A/p^\lambda A)[p^n]$. By induction on n, P5 leads to (c).

Consequently, the p-groups of lengths $\lambda+n$ are quotients of p-groups of lengths λ mod p^n-bounded subgroups.

2. In order to improve on theorem 1, we want to show that U can be chosen in some special way. We consider a topology which is essentially a generalized p-adic topology.

Given the limit ordinal λ, consider the linear topology of A where $\{p^\sigma A\}_{\sigma<\lambda}$ is a base of neighborhoods of 0. We follow Megibben [7] in referring to this topology as the λ-*topology* of A.

If T is a p-group of limit length λ, the λ-topology of T induces a topology on $T[p^n]$. When we say a subgroup of $T[p^n]$ is *closed* or *dense*, we always refer to this induced topology.

The following lemma is easily verified.

LEMMA 1. *If T is a p-group and $U \leqq T[p^n]$, then for every $a \in T$,*

$$h_{T/U}(a+U) \leqq \sup_{u \in U}\,[h_T(a+u)+1]+n-1$$

where the index indicates the group where the heights are computed.

First, assume $U \leqq T[p]$. We induct on $\rho = h_{T/U}(a+U)$. If $\rho=0$, there is nothing to prove. Let $\rho\geqq 1$, and observe that by the definition of height and induction hypothesis,

$$\rho=\sup_{pb\in a+U}[h_{T/U}(b+U)+1]\leqq\sup_{pb\in a+U}\{\sup_{u\in U}[h_T(b+u)+1]+1\}\ .$$

Here $h_T(b+u)+1\leqq h_T(pb+pu)=h_T(pb)$ whence the assertion follows at once for $n=1$. Next assume that $n\geqq 2$ and that lemma has been established for $n-1$. Set $U'=U\cap T[p^{n-1}]$; then U/U' is contained in the socle of T/U'. By induction hypothesis,

$$h_{T/U'}(a+U')=\sup_{u'\in U'}[h_T(a+u')+1]+(n-2)$$

for each $a\in T$. Thus, by what has been proved we infer that for every $a\in T$,

$$\begin{aligned}h_{T/U}(a+U)&=h_{(T/U')/(U/U')}(a+u)\leqq\sup_{u\in U}[h_{T/U'}(a+u+U')+1]\\&\leqq\sup_{u\in U}\{\sup_{u'\in U'}[h_T(a+u+u')+1]+n-1\}\\&\leqq\sup_{u\in U}[h_T(a+u)+1]+n-1\ .\end{aligned}$$

We shall require the following simple lemma.

LEMMA 2. *Let T be a p-group of limit length λ, and U a subgroup of $T[p^n]$ for some integer n. Then T/U is at most of length $\lambda+n$. If U is closed, then T/U has length exactly λ.*

The first assertion follows at once from Theorem 1. Evidently, T/U has length$\geqq\lambda$. Let $a+U$ (for some $a\in T$) be of height λ in T/U. In view of the preceding lemma, there exists an increasing chain of ordinals $\{\sigma_i\}_{i\in I}$ with $\sup\sigma_i=\lambda$, together with elements $u_i\in U$ $(i\in I)$, such that $h_T(a-u_i)=\sigma_i$ (where h_T denotes height in T). Since

$$p^\lambda(T/T[p^n])\cong p^\lambda(p^nT)=p^\lambda T=0\ ,$$

from $T/T[p^n]\cong(T/U)/(T[p^n]/U)$ we deduce $p^\lambda(T/U)\leqq T[p^n]/U$ whence $a\in T[p^n]$ follows. Thus a is the limit in $T[p^n]$ of elements in U. If U is closed, then $a\in U$, and consequently, $p^\lambda(T/U)=0$.

3. Let T be a p-group whose length is a limit ordinal λ, and let U be a subgroup of $T[p^n]$. We are mainly interested in the case when U is dense in $T[p^n]$. For the sake of brevity, we then say that T/U is a *dense representation* of the group $A=T/U$.

From Lemma 2 we know that the length of A is then at most $\lambda+n$. A more precise information is given by

LEMMA 3. *If T and U are as above, and if $A=T/U$ is a dense representation of A, then $p^\lambda A=T[p^n]/U$.*

From $p^\lambda(T/T[p^n])\cong p^\lambda(p^nT)=p^\lambda T=0$ we obtain $p^\lambda A\leqq T[p^n]/U$. On the other hand, by the density of U, for each ordinal $\sigma<\lambda$, every $t\in T[p^n]$ can be written in the form $t=t_\sigma+u_\sigma$ with $t_\sigma\in p^\sigma T[p^n]$, $u_\sigma\in U$. Hence $t+U=t_\sigma+U\in(p^\sigma T[p^n]+U)/A\leqq p^\sigma A$, and the assertion follows.

The following lemma is of technical character.

LEMMA 4. *If A has a dense representation T/U with p^n-bounded U, then it has also one where T does not contain any p^n-bounded summand$\neq 0$.*

In fact, $A=T/U$ is any dense representation with $U\leqq T[p^n]$, then select a maximal p^n-bounded direct summand V of T that is contained in U. Thus $T=V\oplus T'$ and $U=V\oplus(T'\cap U)$, so that $A\cong T'/(T'\cap U)$ will be a dense representation of A. Suppose that $T'=\langle a\rangle\oplus T''$ where $\langle a\rangle$ is of order $p^m (m\leqq n)$. By the density of $T'\cap U$ in $T'[p^n]$, we can write $a=u+b$ with $u\in T'\cap U$ and $b\in(pT')[p^n]$. Manifestly, neither u nor b vanishes, and $h_T(p^r a)=h_T(p^r u)$ for all $r\geqq 0$. But then $\langle u\rangle$ is likewise a summand of T', a contradiction to the choice of V. Hence T' does not have p^n-bounded summands$\neq 0$.

Next we establish one of our main results: the existence of dense representations.

THEOREM 2. *Let $\mathscr{C}_\lambda$ be a p^λ-class of p-groups and $A\in\mathscr{C}_\lambda(n)$. Then A has a dense representation T^*/U^* with $U^*\leqq T^*[p^n]$.*

Here $T^*\in\mathscr{C}_\lambda$ can be chosen whenever $\mathscr{C}_\lambda$ is a full p^λ-class.

By Theorem 1, $A\cong T/U$ for some $T\in\mathscr{C}_\lambda$ and subgroup U of $T[p^n]$. Let T' be a p-group satisfying $p^nT'=T$; then by P4, also $T'\in\mathscr{C}_\lambda$. Let $\bar{U}$ be the closure of U, and let $U'=p^{-n}U\leqq T'$. Then clearly, $\bar{U}\leqq T'[p^n]\leqq U'$, and we can define

$$T^*=T'/\bar{U} \quad\text{and}\quad U^*=U'/\bar{U}\,.$$

The λ-topology of T coincides with the topology induced on T by the λ-topology of T', so $\bar{U}$ is closed in T', too. From Lemma 2 we infer that T^* is of length λ. If $\mathscr{C}_\lambda$ is a full p^λ-class, then T^* has a p^n-bounded subgroup $T'[p^n]/\bar{U}$ modulo which the quotient is $T'/T'[p^n]\cong T$, thus $T^*\in\mathscr{C}_\lambda$. In view of $T^*/U^*\cong T'/U'\cong T/U\cong A$, only the density of U^* in $T^*[p^n]$ remains to be verified, i.e. that

$$T^*[p^n]=U^*+p^\sigma T^*[p^n] \quad\text{for all}\quad \sigma<\lambda\,.$$

Since $p^\sigma(T'/\bar{U})[p^n]\geqq((p^\sigma T'+\bar{U})/\bar{U})[p^n]$, it suffices to establish the inclusion

$$p^{-n}\bar{U}\leqq p^{-n}U+[(p^\sigma T'+\bar{U})\cap p^{-n}\bar{U}]\,.$$

The last group is equal to $(p^{-n}U+p^\sigma T')\cap p^{-n}\bar{U}$, so the proof can be completed by showing that $p^{-n}\bar{U}\leqq p^{-n}U+p^\sigma T'$ for all $\sigma<\lambda$. If $t\in p^{-n}\bar{U}$, then $p^nt\in\bar{U}$, so there are $u\in U$ and $s\in p^{n+\sigma}T=p^\sigma T'$ such that $p^nt=$

$u+p^ns$. Hence $t-s\in p^{-n}U$ and $t=(t-s)+s\in p^{-n}U+p^{\sigma}T'$ completes the proof.

Consequently, *every p-group A of length* $\lambda+n$ (λ *a limit ordinal*) *can be written as* $A=T/U$ *where* T *is of length* λ *and* U *is a* p^n-*bounded dense subgroup of* T.

4. As an application of our theorem on the existence of dense representations, we show how isomorphisms of p-groups of lengths $\lambda+n$ can be related to isomorphisms of p-groups of lengths λ.

THEOREM 3. *Two p-groups, A and A', of lengths* $\lambda+n$ *are isomorphic if and only if there is an isomorphism*

$$\phi\colon A/p^{\lambda}A\longrightarrow A'/p^{\lambda}A'$$

such that $\phi(A[p^n]/p^{\lambda}A)=A'[p^n]/p^{\lambda}A'$.

Necessity is clear. To verify sufficiency, we appeal to Theorem 2 and start with dense representations $A=T/U$ and $A'=T'/U'$. By Lemma 4, without loss of generality, we may assume that neither T nor T' contains p^n-bounded summands. From Lemma 3 we obtain

$$A/p^{\lambda}A=(T/U)/(T[p^n]/U)\cong T/T[p^n]\cong p^nT\ .$$

Under the obvious isomorphisms, $A[p^n]/p^{\lambda}A$ maps upon $U\cap p^nT$. Arguing analogously for A', we can establish from the given ϕ an isomorphism $\psi\colon p^nT\longrightarrow p^nT'$ such that $\psi(U\cap p^nT)=U'\cap p^nT'$. Hence

$$\begin{aligned}p^nA&\cong(p^nT+U)/U\cong p^nT/(U\cap p^nT)\cong p^nT'/(U'\cap p^nT')\\&\cong(p^nT'+U')/U'\cong p^nA'\ .\end{aligned}$$

From [10] it follows that apart from p^n-bounded direct summands, A and A' are isomorphic. It suffices to note that ϕ induces an isomorphism between the maximal p^n-bounded summands of A and A' in order to conclude that $A\cong A'$, in fact.

In the special case when $n=1$, Theorem 3 reduces to Lemma 2 of [9].

The next result states that in certain cases the p^n-socles of groups in $\mathscr{C}_{\lambda}(n)$ determine the groups up to isomorphism. Here we use the term "*isometry*" to mean height-preserving isomorphism (cf. e.g. [4]).

THEOREM 4. *Let* $\mathscr{C}_{\lambda}$ *be a full* p^{λ}-*class and* $n>0$ *an integer such that if* $T, T'\in\mathscr{C}_{\lambda}$ *and if* U, U' *are isometric dense subgroups of* $T[p^n]$ *and* $T'[p^n]$, *respectively, then there is an isomorphism* $T\longrightarrow T'$ *carrying* U *onto* U'.

Then two groups, A and A', in $\mathscr{C}_{\lambda}(n)$ *are isomorphic if and only if there is an isometry*

$$\phi\colon A[p^n]\longrightarrow A'[p^n]\ .$$

Assume ϕ is an isometry for $A, A' \in \mathscr{C}_\lambda(n)$. Let $T, T' \in \mathscr{C}_\lambda$, and let $T/U = A$, $T'/U' = A'$ be dense representations where neither T nor T' contains p^n-bounded summands. From Lemma 3 we infer $A/p^\lambda A \cong T/T[p^n] \cong p^n T$ and $A'/p^\lambda A' \cong p^n T'$ where both $p^n T$ and $p^n T'$ belong to $\mathscr{C}_\lambda$. Manifestly, the map $A[p^n]/p^\lambda A \to A'[p^n]/p^\lambda A'$ induced by ϕ is an isometry between dense subgroups of the p^n-socles of $A/p^\lambda A$ and $A'/p^\lambda A'$. From hypothesis we conclude that there is an isomorphism $A/p^\lambda A \to A'/p^\lambda A'$ mapping $A[p^n]/p^\lambda A$ onto $A'[p^n]/p^\lambda A'$. A simple appeal to Theorem 3 completes the proof.

5. We now specialize $\mathscr{C}_\lambda = \mathscr{D}_\omega$, the class of torsion-complete p-groups which is in view of E3, a full p^ω-class. In order to show that the class $\mathscr{D}_\omega$ satisfies the hypotheses of Theorem 4 for every n, note that a dense subgroup U of $T[p^n] (T \in \mathscr{D}_\omega)$ necessarily contains $B[p^n]$ for some basic subgroup B of T. Therefore, every isometry between dense subgroups $\psi\colon U \to U' \leqq T'[p^n] (T' \in \mathscr{D}_\omega)$ induces an isomorphism $B \to B'$ between basic subgroups of T and T'. By torsion-completeness, this extends uniquely to an isomorphism $\phi\colon T \to T'$. It is easy to check that $\phi | U = \psi$, since they are identical on a dense subgroup $U \cap B$ of U. Hence, by virtue of Theorem 4, we obtain:

Theorem 5. *Two groups in $\mathscr{D}_\omega(n)$ are isomorphic if and only if there is an isometry between their p^n-socles.*

From Theorem 1 we know that the class $\mathscr{D}_\omega(n)$ is exactly the class of p-groups A such that $A/p^\omega A$ is torsion-complete and $p^\omega A$ is p^n-bounded. These groups are the totally injective p-groups of lengths $\leqq \omega + n$ (see [6]). For $n=1$, these groups were discussed by Richman [9]; he proved that they are uniquely determined, up to isomorphism, by their socles as valued vector spaces. Our Theorem 5 is a generalization of Richman's result for arbitrary n.

We wish to give an alternative proof of Theorem 5. This is a direct approach based on immediate generalizations of Richman's ideas. For the proof we need the following lemma.

Lemma 5. *Let $A_i (i=1, 2)$ be p-groups, B_i pure and dense subgroups of A_i, and P_i subgroups of $A_i[p^n]$. Suppose there exist isomorphisms $\beta\colon B_1 \to B_2$ and $\gamma\colon A_1/P_1 \to A_2/P_2$ with the following properties:*

(i) $\gamma(A_1[p^n]/P_1) = A_2[p^n]/P_2$,

(ii) *the diagram*

$$\begin{array}{ccc} B_1 & \xrightarrow{\beta} & B_2 \\ \downarrow & & \downarrow \\ A_1/P_1 & \xrightarrow{\gamma} & A_2/P_2 \end{array}$$

commutes where the vertical maps are the injections followed by the canonical maps.

Then there exists an isomorphism $\alpha: A_1 \to A_2$ *such that* $\alpha | B_1 = \beta$ *and the following square commutes:*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow & & \downarrow \\ A_1/P_1 & \xrightarrow{\gamma} & A_2/P_2 \end{array}$$

(*the vertical maps are the canonical maps*).

First we define α on $p^n A_1$ as follows. Given $a_1 \in p^n A_1$, choose $c_1 \in A_1$ so as to satisfy $p^n c_1 = a_1$. If $\gamma: c_1 + P_1 \mapsto c_2 + P_2 (c_2 \in A_2)$, then we let $\alpha: a_1 \mapsto a_2 = p^n c_2$. If we select $c_1 + x_1 (x_1 \in A[p^n])$ rather than c_1, then because of (i), $\gamma(x_1 + P_1) \in A_2[p^n]/P_1$, so a_2 will be independent of the selections of c_1 and c_2. That α is a homomorphism on $p^n A_1$ is quite clear. Now, if $a_1 \in B_1 \cap p^n A = p^n B_1$, then to this a_1 we can choose $c_1 \in B_1$ and $c_2 \in B_2$, thus $\alpha(a_1) = \beta(a_1)$ which shows that the homomorphisms $\beta: B_1 \to B_2$ and $\alpha: p^n A_1 \to p^n A_2$ agree on the intersection of their domains. Hence they can be uniquely extended to a homomorphism $\alpha: B_1 + p^n A = A_1 \to A_2$. In the same way, an inverse $A_2 \to A_1$ can be constructed showing that α is an isomorphism. The commutativity of the square above follows at once from (ii) and the definitions.

We can now verify the sufficiency statement of Theorem 5. Suppose A_1 and A_2 are in $\mathscr{D}_\omega(n)$ and $\phi: A_1[p^n] \to A_2[p^n]$ is an isometry.

We set $P_i = p^\omega A_i (i = 1, 2)$ and notice that $P_i \leqq A_i[p^n]$. Pick a basic subgroup B_1 of A_1, $B_1 = \bigoplus_{i \in I} \langle b_{1i} \rangle$ where, say, b_{1i} is of order p^{n_i}. For $n_i \leqq n$, set $b_{2i} = \phi b_{1i}$, and for $n_i > n$, let $b_{2i} \in A_2$ satisfy $p^{n_i - n} b_{2i} = \phi(p^{n_i - n} b_{1i})$; since ϕ is height-preserving, such b_{2i}'s exist. As in p-groups, direct sums, purity and density can be recognized in the socles, it is clear that $B_2 = \bigoplus_{i \in I} \langle b_{2i} \rangle$ will be a basic subgroup of A_2. The map $\phi | B_1[p^n]$ can be extended via $b_{1i} \mapsto b_{2i}$ to an isomorphism $\beta: B_1 \to B_2$. From $B_i \cap P_i = 0$ $(i = 1, 2)$ we infer that β induces an isomorphism $\beta_0: (B_1 + P_1)/P_1 \to (B_2 + P_2)/P_2$ between basic subgroups of A_1/P_1 and A_2/P_2. The latter groups are torsion-complete, thus β_0 can be extended to an isomorphism $\gamma: A_1/P_1 \to A_2/P_2$.

The isomorphism ϕ induces an isomorphism $A_1[p^n]/P_1 \to A_2[p^n]/P_2$ which must act as γ. In fact, given $a_1 \in A_1[p^n]$, for every $m > 0$ there is a $b_m \in B_1$ such that $h(a_1 - b_m) \geqq m$; it is easy to check that $b_m \in B_1[p^n]$ can be assumed without loss of generality. Application of ϕ yields $h(\phi a_1 - \phi b_m) \geqq m$. Hence $\phi b_m (m = 1, 2, \cdots)$ is a bounded Cauchy sequence in A_2, and $\phi b_m + P_2 (m = 1, 2, \cdots)$ is one in A_2/P_2; clearly, $\phi a_1 + P_2$ is its limit. But ϕ and β are identical on $B_1[p^n]$, and $\gamma(a_1 + P_1)$ is precisely the limit of $\beta b_m + P_2 (m=1, 2, \cdots)$.

Thus we have constructed isomorphisms β and γ, satisfying the hypotheses of Lemma 5. This completes the second proof of Theorem 5.

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