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Orlicz sequence spaces of a nonabsolute type

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The well-known l_p and Orlicz spaces are of absolute type in the sense that given a sequence $x=\{x_k\}$ the norm of $|x|=\{|x_k|\}$ is equal to that of x. However, there are sequence spaces which are of non-absolute type, for example, the space H of all sequences $x=\{x_k\}$ such that the series $\sum x_k$ is conditionally convergent with norm given by

$$||x||=\sup\left\{\left|\sum_{k=1}^{n} x_k\right|;\ n\geqq 1\right\}.$$

Similarly, if we consider the convergence field of many a summability method as a normed linear space, we again obtain a sequence space of nonabsolute type. In this note we shall give some examples of such sequence spaces, find their associate spaces in the sense of Köthe, and prove a necessary and sufficient condition for matrix transformations mapping from the spaces into a given space.

Let $A=(a_{nk})$ be a lower semi-matrix with non-zero diagonal, i.e. $a_{nk}=0$ for $k>n$ and $a_{nk}\neq 0$ for $k=n$. Then the inverse of A always exists and is again a lower semi-matrix with non-zero diagonal ([1] p. 22). For convenience, we shall denote in what follows by B the inverse of A and by B' the transpose of B. Therefore when $s=Ax$, we have $x=Bs$. Further we shall assume throughout that B is a band matrix, i.e., $b_{nk}=0$ for $n>k+p$ and some fixed p.

Next let φ be an Orlicz function, i.e. φ is continuous and convex on $[0, \infty)$ with $\varphi(0)=0$, $\varphi(x)>0$ whenever $x>0$, and $\varphi(|x|)=\varphi(x)$. Then Orlicz sequence space l_φ is the linear hull of all sequences $x=\{x_k\}$ such that

$$\sum_{k=1}^{\infty}\varphi(x_k)<\infty .$$

The complementary function ψ of φ is defined as follows:

$$\psi(y)=\sup\{|xy|-\varphi(x);\ x>0\} .$$

For simplicity, we shall assume that ψ is also an Orlicz function.

Now we define X_φ to be the space of all sequences $x=\{x_k\}$ such that $Ax\in l_\varphi$ where A is given as above. Given a sequence space X, the associate space of X, denoted by X', is defined to be the space of all sequences $y=\{y_k\}$ such that whenever $x=\{x_k\}\in X$ we have

$$\sum_{k=1}^{\infty} x_k y_k < \infty .$$

It is known that the associate space of l_φ is l_ψ where ψ is the conplementary function of φ. Also, the associate space of H (the space of all convergent series) is the space consisting of all sequences $y=\{y_k\}$ such that

$$\sum_{k=1}^{\infty} |y_k - y_{k+1}| < \infty .$$

The following theorem describes the assoicate space of X_φ.

THEOREM 1. *Let A be a matrix given as above and $B=(b_{nk})$ the inverse of A satisfying*

(i) $\sup\left\{\sum_{n=k}^{k+p} |b_{nk}|;\ n \geqq 1\right\} < \infty$;

(ii) $(-1)^k b_{nk} > 0$ *for* $n=1, 2, \cdots$ *and* $k=n-p, \cdots, n$;

(iii) $|b_{nn}| \geqq \alpha > 0$ *for some* α *and all* n.

Then the associate space X'_φ of X_φ is the space of all bounded sequences $y=\{y_k\}$ such that $B'y \in l_\psi$ where B' is the transpose of B and ψ is the complementary function of φ.

Proof. We write $s=Ax$ and $t=By$. We recall that B is a band matrix. Therefore $x=Bs$ and

$$\begin{aligned}\sum_{n=1}^{m} x_n y_n &= \sum_{n=1}^{m}\left(\sum_{k=n-p}^{n} b_{nk} s_k\right) y_n \\ &= \sum_{k=1}^{m} s_k \sum_{n=k}^{k+p} b_{nk} y_n - \sum_{k=m-p+1}^{m} s_k \sum_{n=m+1}^{k+p} b_{nk} y_n \\ &= \sum_{k=1}^{m} s_k t_k - \sum_{k=m-p+1}^{m} s_k \sum_{n=m+1}^{k+p} b_{nk} y_n\end{aligned}$$

Suppose that $x \in X_\varphi$ and y is a bounded sequence such that $t=B'y \in l_\psi$. Since $s \in l_\varphi$, s is a null sequence. In view of condition (i), the last term in the above equation tends to 0. Then there exist $\alpha>0$ and $\beta>0$ such that

$$\begin{aligned}\left|\sum_{n=1}^{\infty} x_n y_n\right| &= \left|\sum_{k=1}^{\infty} s_k t_k\right| \\ &\leqq (\alpha\beta)^{-1}\left[\sum_{k=1}^{\infty} \varphi(\alpha s_k) + \sum_{k=1}^{\infty} \psi(\beta t_k)\right] < \infty .\end{aligned}$$

Hence $y \in X'_\varphi$.

Conversely, suppose that $y \in X'_\varphi$, i.e.,

$$\sum_{n=1}^{\infty} x_n y_n < \infty \quad \text{for every } x \in X_\varphi .$$

Then $x_n y_n \to 0$ as $n \to \infty$, or

$$y_n \sum_{k=n-p}^{n} b_{nk} s_k \to 0 \quad \text{as} \quad n \to \infty .$$

We may replace $\{s_k\}$ above by $\{(-1)^k|s_k|\}$ which still belongs to l_φ. In view of conditions (ii) and (iii), it follows that $y_n s_n \to 0$ as $n \to \infty$. We claim that y is bounded. If not, choose a subsequence $\{y_{n(i)}\}$ of $y=\{y_k\}$ such that

$$\sum_{i=1}^{\infty} \varphi(1/y_{n(i)}) < \infty .$$

Put $s_{n(i)}=1/y_{n(i)}$ and 0 elsewhere. Thus $s=\{s_n\} \in l_\varphi$ and with $x=Bs \in X_\varphi$ and yet $s_n y_n$ does not tend to 0 as $n \to \infty$. In view of condition (i) again, we have

$$\sum_{n=1}^{\infty} x_n y_n = \sum_{k=1}^{\infty} s_k t_k < \infty .$$

where $t=B'y$. Hence $B'y \in l_\psi$ and the proof is complete.

We note that conditions (ii) and (iii) were not required in the first half of the proof. As an example, we may take A to be σ^r where r is a positive integer, $\sigma^r x = \sigma(\sigma^{r-1}x)$, and

$$(\sigma x)_n = \sum_{k=1}^{n} x_k .$$

Then the inverse of A is Δ^r where $\Delta^r x = \Delta(\Delta^{r-1}x)$ and

$$(\Delta x)_n = x_n - x_{n-1} .$$

We note that conditions (i), (ii) and (iii) in Theorem 1 are satisfied. If we define $H(r, p)$ to be the space of all sequences $x=\{x_k\}$ such that $\sigma^r x \in l_p$ for $1 \leqq p \leqq \infty$, then the following is a corollary of Theorem 1.

THEOREM 2. *The associate space of $H(r, p)$ for $1<p<\infty$ is the set of all bounded sequences $y=\{y_k\}$ such that $\Delta^r y \in l_q$ where $1/p+1/q=1$.*

In fact, it is not difficult to verify that Theorem 2 also holds for $p=1$ and $p=\infty$. It is interesting to note that sequence spaces of nonabsolute type do not necessarily include the class of all finite sequences and that the so-called associate norm of a given norm defined in the usual way is not necessarily a norm. For example, the sequence $(1, 0, 0, \cdots)$ does not belong to $H(1, 2)$ and given

$$\|y\|' = \sup\left\{\left|\sum_{n=1}^{\infty} x_n y_n\right| : \|x\| \leqq 1\right\}$$

where $x \in H(1, 2)$ and

$$\|x\| = \left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n} x_k\right|^2\right)^{1/2} ,$$

if we take $y_k=1$ for all k, then $\|y\|'=0$ and yet $y \neq 0$. However, if a sequence space X does include the class of all finite sequences, then it is a standard procedure ([4] p. 473) to show that $y \in X'$ if and only if $\|y\|' < \infty$ and that $\|y\|'$ is a bona fide norm.

A norm in Orlicz sequence space may be defined as follows:

$$||x||_\varphi=\inf\left\{\varepsilon>0;\ \sum_{k=1}^{\infty}\varphi(x_k/\varepsilon)\leqq 1\right\}.$$

The following is easy.

THEOREM 3. *The spaces X_φ and X'_φ are complete under respectively the following norms*

$$||x||_{X(\varphi)}=||Ax||_\varphi\ ,$$

$$||y||_{X'(\varphi)}=\sum_{n=1}^{p}|y_n|+||B'y||_\psi\ ,$$

where ψ is the complementary function of φ.

Naturally, we may define

$$||y||'_{X(\varphi)}=\sup\left\{\left|\sum_{n=1}^{\infty}x_n y_n\right|;\ ||x||_{X(\varphi)}\leqq 1\right\}.$$

However we have only $||y||'_{X(\varphi)}\leqq||y||_{X'(\varphi)}$. In fact, $||y||'_{X(\varphi)}$ is equivalent to $||B'y||_\psi$.

Let $C=(c_{nk})$ be a lower semi-matrix with non-zero diagonal. We define Y to be the space of all sequences $y=\{y_k\}$ such that

$$||y||_Y=\sup\left\{\left|\sum_{k=1}^{n}c_{nk}y_k\right|;\ n\geqq 1\right\}<\infty\ .$$

It is easy to show that Y is a Banach space.

THEOREM 4. *In order that $T=(t_{nk})$ be a matrix transformation from X_φ into Y, it is necessary and sufficient that*

$$\sup\left\{\left|\left|\left\{\sum_{k=1}^{n}c_{nk}t_{ki}\right\}_{i\geqq 1}\right|\right|'_{X(\varphi)};\ n\geqq 1\right\}<\infty\ .$$

Proof. Suppose T maps X_φ into Y. Then T in continuous ([5] p. 29) and for some $N>0$

$$||Tx||_Y\leqq N||x||_{X(\varphi)}\ .$$

It follows that whenever $||x||_{X(\varphi)}\leqq 1$ we have

$$\left|\sum_{i=1}^{\infty}\left(\sum_{k=1}^{n}c_{nk}t_{ki}\right)x_i\right|\leqq N\ .$$

Hence

$$\left|\left|\left\{\sum_{k=1}^{n}c_{nk}t_{ki}\right\}_{i\geqq 1}\right|\right|'_{X(\varphi)}\leqq N\ ,$$

and the condition holds.

Conversely, if the condition holds, then

$$\left|\sum_{k=1}^{n}c_{nk}\sum_{i=1}^{\infty}t_{ki}x_i\right|=\left|\sum_{i=1}^{\infty}\left(\sum_{k=1}^{n}c_{nk}t_{ki}\right)x_i\right|$$

$$\leqq\left|\left|\left\{\sum_{k=1}^{n}c_{nk}t_{ki}\right\}_{i\geqq 1}\right|\right|'_{X(\varphi)}||x||_{X(\varphi)}\ .$$

Consequently, $Tx \in Y$ for $x \in X_\varphi$.

We remark that

$$\sup\left\{\left\|\left\{\sum_{k=1}^{n} c_{nk}t_{ki}\right\}_{i\geqq 1}\right\|_{X'(\varphi)};\ n\geqq 1\right\}<\infty$$

is sufficient but not necessary for T mapping X_φ into Y. For example, let X_φ be the space $H(1, 2)$, and Y the space l_∞ of all bounded sequences, i.e., C being an identity matrix. Now put $T=(t_{nk})$ with $t_{nk}=n$ for all k. Then

$$\sup\{\|\{t_{nk}\}_{k\geqq 1}\|'_{H(1,2)};\ n\geqq 1\}=\sup\left\{\left(\sum_{k=1}^{\infty}|t_{nk}-t_{n,k+1}|^2\right)^{1/2};\ n\geqq 1\right\}=0\ ,$$

$$\sup\{\|\{t_{nk}\}_{k\geqq 1}\|_{H'(1,2)};\ n\geqq 1\}=\sup\{|t_{n1}|;\ n\geqq 1\}=\infty\ .$$

It is easy to verify that T maps an element in $H(1, 2)$ into the zero element in l_∞.

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