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On the space of entire functions of (p, q) order ρ

by

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1. Introduction. Let Γ denote the space of all functions entire in the complex plane endowed with the topology τ of uniform convergence on compact subsets of the complex plane. A number of topologies equivalent to τ have been defined on $C^{\dagger}$ by Iyer [5], Arsove [1] and others. Various properties of space Γ such as dual, linear transformations, proper bases etc. have also been investigated by these authors [2; 6].

Different subclasses of C have also been considered and topologies analogous to those considered by Iyer and Arsove have been defined. Thus Krishnamurthy [8] considered the subclasses $C(\rho)$ and $C(\rho, d)$ of C where $C(\rho)$ consists of all entire functions whose order does not exceed ρ while $C(\rho, d)$ contains all entire functions of order not exceeding ρ and type not exceeding d if of order ρ. Ekblaw [4] considered the subspaces of entire functions consisting of functions of bounded index and of periodic entire functions, whereas Laird [9] considered space of entire mean periodic functions.

Recently, a new classification scheme for entire functions has been proposed in [7] by introducing concepts of index pair and (p, q) order. Thus for $\alpha \in C$ and $p \geqq 2$, $q \geqq 1$ set

$$(1.1) \qquad \rho(p, q) \equiv \rho = \limsup_{r\to\infty} \frac{\log^{[p]} M(r; \alpha)}{\log^{[q]} r},$$

where

$$(1.2) \qquad M(r; \alpha) = \max_{|z|=r} |\alpha(z)| \quad \text{and for } m = 1, 2, \cdots$$

$$(1.3) \qquad \log^{[m]} x = \log(\log^{[m-1]} x), \quad \log^{[0]} x = x.$$

The entire function $\alpha(z)$ is said to have *index pair* (p, q), $p \geqq 2$, $q \geqq 1$, if $b < \rho(p, q) < \infty$ and $\rho(p-1, q-1)$ is not a non-zero finite number, where

$$(1.4) \qquad b = 0 \text{ if } p > q \text{ and } b = 1 \text{ if } p = q.$$

If $f(z)$ has the index pair (p, q) then $\rho = \rho(p, q)$ is called its (p, q) order (For details regarding index pair and (p, q) order, see [7]).

If $\alpha(z)$ is represented by the power series $\sum_0^\infty a_n z^n (|z| < \infty)$, the

† C denotes the class of all entire functions.

coefficient characterisation for $\rho(p, q)$ has also been obtained [7]. In fact, it has been shown that

$$(1.5)\qquad \rho(p, q)=P(L(p, q))$$

where,

$$(1.6)\qquad L(p, q)=\limsup_{n\to\infty}\frac{\log^{[p-1]} n}{\log^{[q-1]}(\log|a_n|^{-1/n})}$$

$$\begin{aligned} P(L(p, q)) &= L(p, q) && \text{if } p>q \\ &= 1+L(p, q) && \text{if } p=q=2 \\ &= \max(1, L(p, q)) && \text{if } 3\leqq p=q<\infty . \end{aligned}$$

Let $C_{(p,q)}(\rho)$ denote the class of all entire functions whose index pair does not exceed (p, q) and whose (p, q) order does not exceed ρ if of index pair (p, q). It is easily seen that $C_{(p,q)}(\rho)$ is a linear space over the complex field E with the usual addition and scalar multiplication. Further, any element $\alpha(z)=\sum_0^\infty a_n z^n \in C_{(p,q)}(\rho)$ is characterised by the equation

$$(1.7)\qquad |a_n|^{1/n}\exp^{[q-1]}(\log^{[p-2]} n)^{1/\rho+\delta-A}\to 0 \quad\text{as}\quad n\to\infty \quad \text{for every } \delta>0 ,$$

where

$$(1.8)\qquad \begin{aligned} A &= 1 \quad \text{for} \quad (p, q)=(2, 2) \\ &= 0 \quad \text{otherwise} . \end{aligned}$$

Define

$$(1.9)\qquad ||\alpha; \rho+\delta||=\sum_{n=0}^{\infty}|a_n|\exp(n\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta-A})$$

where for $m=0, 1, 2, \cdots$ $\exp^{[m]} x=\exp(\exp^{[m-1]} x)$, $\exp^{[-m]} x=\log^{[m]} x$,

$$(1.10)\qquad \begin{aligned} \lambda_n &= N_0 \quad \text{for} \quad 0\leqq n\leqq N_0 \\ &= n \quad \text{for} \quad n>N_0 \end{aligned}$$

and

$$N_0=[\exp^{[p-3]} 1]+1 .$$

Clearly, for every $\delta>0$ and $\alpha\in C_{(p,q)}(\rho)$, (1.9) defines a norm. Denote the corresponding normed space by $\Gamma_{(p,q)}(\rho, \delta)$ and let $\Gamma_{(p,q)}(\rho)$ be the weakest topology which is stronger than each $\Gamma_{(p,q)}(\rho, \delta)$. Obviously, $\Gamma_{(p,q)}(\rho)$ is generated by the family $\{\Gamma_{(p,q)}(\rho, \delta), \delta>0\}$. Further, it can be easily verified that $\Gamma_{(p,q)}(\rho)$ is an F-space [3] under the induced metric

$$(1.11)\qquad d(\alpha, \beta)=|||\alpha-\beta|||=\sum_{p=1}^{\infty}\frac{1}{2^p}\frac{||\alpha-\beta; \rho+1/p||}{1+||\alpha-\beta; \rho+1/p||} .$$

In this paper, we study different properties of the space $\Gamma_{(p,q)}(\rho)$

such as its dual, linear transformations of $\Gamma_{(p,q)}(\rho)$ into itself, classical interpretation of convergence in $\Gamma_{(p,q)}(\rho)$ etc. In Theorems 4 and 5 we give necessary and sufficient conditions for a basis in $\Gamma_{(p,q)}(\rho)$ to be a proper basis.

2. Preliminary Lemmas. In this section we state a few lemmas which are either well known or can be easily proved on the lines adopted in [8].

LEMMA 1. *Let Γ_0 be a subspace of $\Gamma_{(p,q)}(\rho)$. Let f be a linear functional defined and continuous on Γ_0 in the topology of $\Gamma_{(p,q)}(\rho)$. Then f is continuous on Γ_0 regarded as a subspace of $\Gamma_{(p,q)}(\rho, \delta)$ for some $\delta>0$.*

LEMMA 2. *The set $\Gamma^*_{(p,q)}(\rho)$ of continuous linear functionals on $\Gamma_{(p,q)}(\rho)$ is the union of the sets $\Gamma^*_{(p,q)}(\rho, \delta)$ for all $\delta>0$.*

LEMMA 3 [3, Theorem 5, p. 58]. *If a linear space V is an F-space under each of two metrics, and if one of the corresponding topologies contains the other, the two topologies are equal.*

LEMMA 4. *If $|||\alpha|||\geqq k$ $(0<k<2)$, then $||\alpha; \rho+\delta||\geqq k/(2-k)$ for some $\delta=\delta_0$ where $0<\delta_0\leqq 1$, and therefore for all values of $\delta\leqq\delta_0$.*

Remark. A consequence of this Lemma is that if a series converges in $\Gamma_{(p,q)}(\rho, \delta)$ for each $\delta>0$, then it converges in $\Gamma_{(p,q)}(\rho)$.

LEMMA 5. *Let T be a linear transformation of $\Gamma_{(p,q)}(\rho)$ into itself. In order that T be continuous, it is necessary and sufficient that, for each $\delta_2>0$, there exists some $\delta_1>0$ such that T is continuous linear transformation from $\Gamma_{(p,q)}(\rho, \delta_1)$ into $\Gamma_{(p,q)}(\rho, \delta_2)$.*

3. Dual Space $\Gamma^*_{(p,q)}(\rho)$. In this section we obtain the general form of continuous linear functionals on $\Gamma_{(p,q)}(\rho)$. Thus we have

THEOREM 1. (a) *A functional $f\in\Gamma^*_{(p,q)}(\rho, \delta)$ is of the form*

$$f(\alpha)=\sum_{n=0}^{\infty} c_n a_n,\quad \alpha(z)=\sum_{0}^{\infty} a_n z^n\in\Gamma_{(p,q)}(\rho, \delta)$$

where

$$|c_n|\exp\{-n\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta-A}\} \tag{3.1}$$

is bounded and conversely.

(b) *Every functional $f\in\Gamma^*_{(p,q)}(\rho)$ is of the form $f(\alpha)=\sum_0^\infty c_n a_n$, $\alpha(z)=\sum_0^\infty a_n z^n\in\Gamma_{(p,q)}(\rho)$ where*

$$(\log^{[q-1]}|c_n|^{1/n})(\log^{[p-2]}\lambda_n)^{-1/\rho-A}\to 0 \quad as \quad n\to\infty \tag{3.2}$$

and conversely.

Proof. (a) Suppose $f(z^n)=c_n$. Assume $f\in\Gamma^*_{(p,q)}(\rho, \delta)$, then

$$f(\alpha)=\lim_{n\to\infty} f\left(\sum_0^n a_k z^k\right)=\lim_{n\to\infty}\sum_0^n c_k a_k=\sum_0^\infty c_n a_n .$$

Further, by definition of continuity of f, given $\delta>0$ there exists $K(\delta)$ such that

$$|f(\alpha)|\leqq K||\alpha;\ \rho+\delta|| .$$

Putting $\alpha=z^n$, we get

$$|c_n|\leqq K||z^n,\ \rho+\delta||=K\exp\{n\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta-A}\}$$

which yields (3.1).

Conversely, let $f(\alpha)=\sum_0^\infty c_n a_n$, where $\{c_n\}$ satisfies (3.1). Obviously f is linear. To show f is continuous, consider

$$\begin{aligned}|f(\alpha)|&=\left|\sum_0^\infty c_n a_n\right|\leqq\sum_0^\infty |c_n||a_n|\\&=K_1\sum_0^\infty |a_n|\exp\{n\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta-A}\}\\&=K_1||\alpha;\ \rho+\delta||\end{aligned}$$

so that $f\in\Gamma^*_{(p,q)}(\rho,\ \delta)$.
The result in 1(b) immediately follows in view of the above result and Lemma 2.

4. **Convergence in $\Gamma_{(p,q)}(\rho)$.** The main result of this section is the classical interpretation of convergence in $\Gamma_{(p,q)}(\rho)$. The following observations about convergence in $\Gamma_{(p,q)}(\rho)$ can be easily noted.

(i) The topology induced on $C_{(p,q)}(\rho)$ by Γ is weaker then $\Gamma_{(p,q)}(\rho)$.

(ii) Convergence in $\Gamma_{(p,q)}(\rho)$ is equivalent to convergence in $\Gamma_{(p,q)}(\rho,\ \delta)$ for every $\delta>0$.

(iii) The sequence of partial sums of $\alpha(z)=\sum_0^\infty a_n z^n$ converges to α in $\Gamma_{(p,q)}(\rho)$.

Next we have

THEOREM 2. *Let $\{\alpha_n\}$ be a sequence of elements of $\Gamma_{(p,q)}(\rho)$. The statement $\alpha_n\to\alpha$ in $\Gamma_{(p,q)}(\rho)$ is equivalent to the statement that for every $\delta>0$, the sequence $\alpha_n(z)$ of entire functions converges to $\alpha(z)$ uniformly over compact subsets of D_a: $\{z: |z|>a\}$ relative to the function*

$$\exp\left(+\int_a^{|z|}\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right),$$

where $a=\max\{1,\ \exp^{[q-2]}1\}$.

Proof. Define

$$||\alpha,\ \rho+\delta||_1=\max_{z\in D_a}\left\{\exp\left(-\int_a^{|z|}\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right)\right\}|\alpha(z)| . \tag{4.1}$$

Clearly, for every $\delta>0$ and $\alpha\in C_{(p,q)}(\rho)$, (4.1) defines a norm. Denote the corresponding normed space by $\Gamma'_{(p,q)}(\rho, \delta)$. Let $\Gamma'_{(p,q)}(\rho)$ be the topology generated by the family $\{\Gamma'_{(p,q)}(\rho, \delta); \delta>0\}$. Now we shall show that $\Gamma'_{(p,q)}(\rho)$ is an F-space under the metric

$$\text{(4.1a)} \qquad d'(\alpha, \beta)\equiv|||\alpha-\beta|||_1=\sum_{p=1}^{\infty}\frac{1}{2^p}\frac{||\alpha-\beta; \rho+1/p||_1}{1+||\alpha-\beta; \rho+1/p||_1} .$$

Clearly d' is an invariant metric and for $\lambda \in E$, $\alpha \in C_{(p,q)}(\rho)$ the mapping $(\lambda, \alpha)\rightarrow\lambda\alpha$ is simultaneously continuous. Thus, for $\Gamma'_{(p,q)}(\rho)$ to be an F-space, it is sufficient to show that it is complete. Consider $\alpha_{p'}(z)=\sum_{n=0}^{\infty} a_{p',n}z^n$ as Cauchy sequence in $\Gamma'_{(p,q)}(\rho)$. Then it is Cauchy in $\Gamma'_{(p,q)}(\rho, \delta)$ for every $\delta>0$. Thus, given $\varepsilon>0$ there exists $p_0(\varepsilon)$ such that for $p', q'\geqq p_0(\varepsilon)$, we have

$$\text{(4.2)} \quad \max_{z\in D_a}\left\{\exp\left(-\int_a^{|z|}\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right)\right\}|\alpha_{p'}(z)-\alpha_{q'}(z)|\leqq\varepsilon .$$

Let S be a compact subset of D_a then $r_1=\sup_{z\in S}|z|>a$.

$$\left\{\exp\left(-\int_a^{r_1}\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right)\right\}|\alpha_{p'}(z)-\alpha_{q'}(z)|\leqq\varepsilon \text{ for every } z\in S .$$

Then, for every $z\in S$ and $p', q'\geqq p_0$, we have

$$|\alpha_{p'}(z)-\alpha_{q'}(z)|\leqq\varepsilon\exp\left(\int_a^{r_1}\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right) .$$

Hence $\{\alpha_{p'}(z)\}$ converges uniformly over compact subsets of D_a to a regular function $\alpha(z)$. Now consider a circle $\Lambda\equiv\{z: |z|=r_0>a\}$. Since $\{\alpha_{p'}(z)\}$ is a sequence of entire functions converging uniformly on Λ, it also converges uniformly inside Λ to a function $\beta(z)$ regular inside Λ. But $\alpha(z)$ and $\beta(z)$ coincide in the region $\equiv\{z: a<|z|\leqq r_0\}$. Thus $\alpha(z)$ can be analytically continued over the whole complex plane. Hence $\alpha(z)$ is an entire function which belongs to $C_{(p,q)}(\rho)$. From (4.2), it follows that $\alpha_{p'}\rightarrow\alpha$ in $\Gamma'_{(p,q)}(\rho, \delta)$ for every $\delta>0$. Hence $\alpha_{p'}\rightarrow\alpha$ in $\Gamma'_{(p,q)}(\rho)$. Thus $\Gamma'_{(p,q)}(\rho)$ is a complete linear metric space, metrizable with invariant metric d'. So it is an F-space.

We now show that the topology $\Gamma'_{(p,q)}(\rho)$ is comparable with $\Gamma_{(p,q)}(\rho)$. For this, consider

$$\begin{aligned}
&\max_{\substack{r\\ a<r<\infty}}|a_n|r^n\exp\left(-\int_a^r\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right)\\
&\quad=|a_n|\max_r\exp\left(-\int_a^r\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt+n\log r\right)\\
&\quad=|a_n|\exp\left(-\int_a^{\exp^{[q-1]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta}}\frac{\exp^{[p-2]}(\log^{[q-1]}t)^{\rho+\delta}}{t}dt\right.\\
&\qquad\qquad\left.+n\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta}\right)\\
&\quad<|a_n|\exp(+n\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\delta-\Delta}) .
\end{aligned}$$

Hence two topologies are comparable. Thus Lemma 3 gives that metrics (4.1a) and (1.11) are equal. This completes the proof.

THEOREM 3. *A necessary and sufficient condition that there exists a continuous linear transformation T from $\Gamma_{(p,q)}(\rho)$ into itself with $T(z^n)=\alpha_n$, $n=0, 1, 2, \cdots$, is that, for each $\delta>0$,*

$$\underset{n\to\infty}{\text{Lt Sup}} \frac{\log^{[q]} ||\alpha_n; \rho+\delta||^{1/n}}{\log^{[p-1]} \lambda_n} < \frac{1}{\rho-A} . \tag{4.5}$$

Proof. Suppose T is a continuous linear transformation from $\Gamma_{(p,q)}(\rho)$ into itself. Then, by Lemma 5, for every $\delta>0$, there exists a $\delta_1(\delta)>0$ such that T is continuous linear transformation from $\Gamma_{(p,q)}(\rho, \delta_1)$ into $\Gamma_{(p,q)}(\rho, \delta)$. Hence there exists a constant $M(\delta)$ such that

$$||T(z^n); \rho+\delta|| \leqq M||z^n; \rho+\delta_1||$$

i.e.,

$$||\alpha_n; \rho+\delta|| \leqq M \exp\{n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\delta_1-A}\}$$

which gives (4.5).

Conversely, let $\alpha(z)=\sum_0^\infty a_n z^n \in \Gamma_{(p,q)}(\rho)$, so that

$$|a_n|^{1/n} \exp^{[q-1]} (\log^{[p-2]} \lambda_n)^{1/\rho+\delta-A} \to 0 \quad \text{as } n\to\infty \text{ for every } \delta>0.$$

So, given $\eta>0$, there exists $n_0(\eta)>N_0$, such that

$$|a_n| \leqq \exp\{-n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\eta-A}\} \quad \text{for} \quad n\geqq n_0(\eta) . \tag{4.6}$$

Take $\eta'>\eta$. From (4.5) there exists $n_0'(\eta')>N_0$ such that for $n\geqq n_0'$

$$||\alpha_n; \rho+\delta|| \leqq \exp\{n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\eta'-A}\} . \tag{4.7}$$

Let $n'\geqq \max(n_0, n_0')$, then (4.6) and (4.7) lead to

$$\begin{aligned}|a_n|^{1/n}||\alpha_n; \rho+\delta||^{1/n} \leqq \exp\{&\exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\eta'-A} \\ &-\exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\eta-A}\} \to 0 \quad \text{as} \quad n\to\infty .\end{aligned}$$

So the series $\sum |a_n| ||\alpha_n; \rho+\delta||$ is convergent for every $\delta>0$. Thus, Lemma 4 gives that $\sum_{n=0}^\infty a_n\alpha_n(z)$ converges to an element in $\Gamma_{(p,q)}(\rho)$. Define $T(\alpha)=\sum_0^\infty a_n\alpha_n$, for $\alpha\in\Gamma_{(p,q)}(\rho)$. Then $T(z^n)=\alpha_n$. Given $\delta>0$, $\delta'>0$, we have, from (4.5), for $n\geqq n_0''=n_0''(\delta, \delta_1)$,

$$||\alpha_n; \rho+\delta|| \leqq \exp\{n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\delta_1-A}\}$$

or

$$||\alpha_n; \rho+\delta|| \leqq K_1 \exp\{n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\delta_1-A}\}$$

for all n. Thus

$$\begin{aligned}||T(\alpha); \rho+\delta|| &\leqq \sum |a_n| ||\alpha_n; \rho+\delta|| \\ &= K_1 \sum_0^\infty |a_n| \exp\{n \exp^{[q-2]} (\log^{[p-2]} \lambda_n)^{1/\rho+\delta_1-A}\} \\ &= K_1||\alpha; \rho+\delta_1||\end{aligned}$$

which shows that T is continuous linear transformation from $\Gamma_{(p,q)}(\rho, \delta_1)$ into $\Gamma_{(p,q)}(\rho, \delta)$ for every $\delta>0$. Therefore by Lemma 2, T is continuous linear transformation from $\Gamma_{(p,q)}(\rho)$ into itself.

5. **Proper Bases in $\Gamma_{(p,q)}(\rho)$ and their Characterisation.** We now obtain a characterisation for proper basis in $\Gamma_{(p,q)}(\rho)$. As is well known, a basis in $\Gamma_0 \subset \Gamma_{(p,q)}(\rho)$ is a linearly independent set spanning the closed subspace Γ_0 whereas a proper basis is a basis which has in addition the property:

for all sequences $\{c_n\}$ of complex numbers,

(5.1) $$\sum c_n \alpha_n \text{ converges in } \Gamma_{(p,q)}(\rho), \text{ if and only if } \sum c_n e_n \text{ converges in } \Gamma_{(p,q)}(\rho),$$

where $e_n = z^n$ for every $n=1, 2, \cdots$.

Now, it can be easily seen that

(5.2) $$\sum c_n e_n \text{ converges in } \Gamma_{(p,q)}(\rho), \text{ if and only if } \lim_{n\to\infty} |c_n|^{1/n} \exp^{[q-1]} (\log^{[p-2]} \lambda_n)^{1/\rho+\delta-A} = 0 \quad \text{for every } \delta>0.$$

We first have,

THEOREM 4. *The following three conditions are equivalent:*

(A) $$\limsup_{n\to\infty} \frac{\log^{[q]} \|\alpha_n; \rho+\delta\|^{1/n}}{\log^{[p-1]} \lambda_n} < \frac{1}{\rho-A} \quad \textit{for every } \delta>0.$$

(B) *For all the sequences $\{c_n\}$ of complex numbers, $\sum c_n e_n$ converges in $\Gamma_{(p,q)}(\rho) \Rightarrow \sum c_n \alpha_n$ converges in $\Gamma_{(p,q)}(\rho)$.*

(C) *For all sequences $\{c_n\}$ of complex numbers*

$$\sum c_n e_n \textit{ couverges in } \Gamma_{(p,q)}(\rho) \Rightarrow c_n \alpha_n \to 0 \textit{ in } \Gamma_{(p,q)}(\rho).$$

Proof. Clearly (B)$\Rightarrow$(C). (A)$\Rightarrow$(B) has been proved in sufficient part of Theorem 3. So we have only to show (C)$\Rightarrow$(A). Assume (C) holds and (A) does not. So, given $\delta'>0$, there exists a sequence $\{n_k\}$ of positive integers such that

(5.3) $$\frac{\log^{[q]} \|\alpha_n; \rho+\delta'\|^{1/n}}{\log^{[p-1]} \lambda_n} > \frac{1}{\rho+\frac{1}{n}-A} \quad \text{for} \quad n=n_k .$$

Define

(5.4) $$c_n = \begin{cases} \|\alpha_n; \rho+\delta'\|^{-1} & \text{when} \quad n=n_k \\ 0 & \text{when} \quad n \neq n_k \end{cases} .$$

(5.3) and (5.4) give that $\{c_n\}$ satisfies (1.7). So $\sum c_n e_n$ converges in $\Gamma_{(p,q)}(\rho)$. Thus (C) gives $c_n \alpha_n \to 0$ in $\Gamma_{(p,q)}(\rho)$. But for all $n=n_k$,

$$\|c_n \alpha_n; \rho+\delta'\| = |c_n| \|\alpha_n; \rho+\delta'\| = 1 ,$$

gives the contradiction to $c_n\alpha_n \to 0$ in $\Gamma_{(p,q)}(\rho)$. This proves (C)$\Rightarrow$(A).

THEOREM 5. *The following three conditions are equivalent:*

(α) $\underset{\delta\to 0}{\mathrm{Lt}}\left\{\liminf_{n\to\infty}\frac{\log^{[q]}\|\alpha_n;\rho+\delta\|^{1/n}}{\log^{[p-1]}\lambda_n}\right\}\geqq\frac{1}{\rho-A}$.

(β) *For all sequences* $\{c_n\}$ *of complex numbers*, $\sum c_n\alpha_n$ *converges in* $\Gamma_{(p,q)}(\rho)\Rightarrow\sum c_n e_n$ *coverges in* $\Gamma_{(p,q)}(\rho)$.

(γ) *For all sequences* $\{c_n\}$ *of complex numbers*, $c_n\alpha_n\to 0$ *in* $\Gamma_{(p,q)}(\rho)\Rightarrow\sum c_n e_n$ *converges in* $\Gamma_{(p,q)}(\rho)$.

Proof. Obviously (γ)$\Rightarrow$(β). To prove (β)$\Rightarrow$(α), suppose (β) holds and (α) does not. Then

$$\underset{\delta\to 0}{\mathrm{Lt}}\left\{\underset{n\to\infty}{\mathrm{Lt\,inf}}\frac{\log^{[q]}\|\alpha_n;\rho+\delta\|^{1/n}}{\log^{[p-1]}\lambda_n}\right\}<\frac{1}{\rho-A}\tag{5.5}$$

so that

$$\liminf_{n\to\infty}\frac{\log^{[q]}\|\alpha_n;\rho+\delta\|^{1/n}}{\log^{[p-1]}\lambda_n}<\frac{1}{\rho-A}\quad\text{for each}\quad\delta>0\,.\tag{5.6}$$

Fix $\eta>0$. In view of (5.5) and (5.6), we can find, for each $r>0$, a positive integer n_r such that, for all r, $n_{r+1}>n_r$, and

$$\frac{\log^{[q]}\left\|\alpha_{n_r};\rho+\dfrac{1}{r}\right\|^{1/n_r}}{\log^{[p-1]}\lambda_{n_r}}<\frac{1}{\rho+\eta-A}\,.\tag{5.7}$$

Let $0<\eta_1<\eta$ and define a sequence $\{c_n\}$ by

$$c_n=\begin{cases}(\exp^{[q-1]}(\log^{[p-2]}\lambda_n)^{1/\rho+\eta_1-A})^{-n} & \text{for}\quad n=n_r\\ 0 & \text{otherwise.}\end{cases}\tag{5.8}$$

Then, for any $\delta>0$,

$$\sum_{n=1}^{\infty}|c_n|\,\|\alpha_n;\rho+\delta\|=\sum_{r=1}^{\infty}|c_{n_r}|\,\|\alpha_{n_r};\rho+\delta\|$$

which is dominated by $\sum|c_{n_r}|\,\|\alpha_{n_r};\rho+1/r\|$ after omitting finitely many terms for which $1/r>\delta$(given). The convergence of this series follows from (5.7) and (5.8). Thus, the sequence $\{c_n\}$ has the property that $\sum c_n\alpha_n$ converges in $\Gamma_{(p,q)}(\rho,\delta)$ for each $\delta>0$ and therefore in $\Gamma_{(p,q)}(\rho)$. Because of (β), $\sum c_n e_n$ converges in $\Gamma_{(p,q)}(\rho)$ so that (5.2) is satisfied. But by (5.8),

$$|c_n|^{1/n}\exp^{[q-1]}(\log^{[p-2]}\lambda_n)^{1/\rho+\eta_1-A}=1\not\to 0\,,$$

which contradicts (5.2). Thus (β)$\Rightarrow$(α).

To show (α)$\Rightarrow$(γ). Assume that (α) holds but (γ) does not. So there exists a sequence $\{c'_n\}$ for which

$$c'_n\alpha_n\to 0\quad\text{in}\quad\Gamma_{(p,q)}(\rho)\tag{5.9}$$

but

$$\limsup_{n\to\infty} \frac{\log^{[p-1]} \lambda_n}{\log^{[q-1]}\left(\frac{1}{n} \log |c_n'|^{-1}\right)} > \rho - A .$$

Thus, given $\lambda>0$, there exists a sequence (n_k) of positive integers such that

(5.10) $$|c_n'|^{1/n} \geqq \exp\{-\exp^{[q-2]}(\log^{[p-2]}\lambda_n)^{1/\rho+\lambda-A}\}$$
for $n=n_k$, $k=1, 2, 3, \cdots$.

Choose a positive number η such that $\lambda>3\eta/2$. From (α), there exists a $\delta=\delta(\eta)$ such that

$$\liminf_{n\to\infty} \frac{\log^{[q]} ||\alpha_n; \rho+\delta||^{1/n}}{\log^{[p-1]} \lambda_n} \geqq \frac{1}{\rho+\eta-A}$$

so that

(5.11) $$||\alpha_n; \rho+\delta||^{1/n} \geqq \exp^{[q-1]}(\log^{[p-2]}\lambda_n)^{1/\rho+3\eta/2-A} \quad \text{for} \quad n \geqq n_0(\eta) .$$

Thus $\max_n ||c_n'\alpha_n; \rho+\delta|| \geqq \max_{n_k} |c_{n_k}'|\,||\alpha_{n_k}; \rho+\delta||>1$ from (5.10) and (5.11). It follows that for this δ, $c_n'\alpha_n \not\to 0$ in $\Gamma_{(p,q)}(\rho, \delta)$. So $c_n'\alpha_n \not\to 0$ in $\Gamma_{(p,q)}(\rho)$ which is a contradiction to (5.9). This proves $(\alpha) \Rightarrow (\gamma)$. Hence the theorem.

Combining Theorems 4 and 5, we get

THEOREM 6. *A basis $\{\alpha_n\}$ in a closed subspace Γ_0 of $\Gamma_{(p,q)}(\rho)$ is proper if and only if conditions (A) and (α) hold.*

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