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A new class of orthogonal expansions for the H-function of several complex variables

by

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For the H-function of several complex variables, which was introduced and studied systematically in a series of recent papers by H. M. Srivastava and R. Panda (*cf.* [5], [6] and [7]), the present authors derive a new class of expansions in series of Gegenbauer (or ultraspherical) polynomials. The main results (1.12) and (1.13) below, as well as their variations (1.17) and (1.18), do not seem to follow easily from the general expansions for the multivariable H-function in series of hypergeometric polynomials [6, p. 132, Eq. (3.1); p. 137, Eq. (4.6)]; indeed, on specializing the various parameters involved, these new expansion formulas will yield a number of known results including, for example, the Fourier sine and cosine series for the multivariable H-function (contained in Theorem 3 on page 179 of the earlier paper [7, Part II]).

1. Introduction and the main results

Throughout the present paper we shall make use of the various notations explained fairly fully in the earlier works [6] and [7], and let

$$(1.1)\qquad H^{0,\lambda:\ (\mu',\nu');\ \cdots;\ (\mu^{(r)},\nu^{(r)})}_{A,C:\ [B',D'];\ \cdots;\ [B^{(r)},D^{(r)}]}\begin{pmatrix} z_1\\ \vdots\\ z_r\end{pmatrix}$$

$$\equiv H^{0,\lambda:\ (\mu',\nu');\ \cdots;\ (\mu^{(r)},\nu^{(r)})}_{A,C:\ [B',D'];\ \cdots;\ [B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:\\ [(c):\psi',\cdots,\psi^{(r)}]:\end{matrix}\ \begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right)$$

denote the H-function of r complex variables $z_1,\cdots,z_r$ (see also [5], p. 271, Eq. (4.1) *et seq.*). Suppose, as usual, that the parameters

$$(1.2)\qquad \begin{cases} a_j,\ j=1,\cdots,A;\ b_j^{(i)},\ j=1,\cdots,B^{(i)};\\ c_j,\ j=1,\cdots,C;\ d_j^{(i)},\ j=1,\cdots,D^{(i)};\ \forall i\in\{1,\cdots,r\}\,,\end{cases}$$

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are complex numbers, and the associated coefficients

$$(1.3)\qquad \begin{cases} \theta_j^{(i)},\ j=1,\cdots,A;\ \phi_j^{(i)},\ j=1,\cdots,B^{(i)}; \\ \psi_j^{(i)},\ j=1,\cdots,C;\ \delta_j^{(i)},\ j=1,\cdots,D^{(i)};\ \forall i\in\{1,\cdots,r\}, \end{cases}$$

are positive real numbers such that

$$(1.4)\qquad \lambda_i\equiv\sum_{j=1}^{A}\theta_j^{(i)}+\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}-\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}<0$$

and

$$(1.5)\qquad \begin{aligned} \Lambda_i\equiv\sum_{j=1}^{\lambda}\theta_j^{(i)}-\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} \\ -\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0, \\ \forall i\in\{1,\cdots,r\}, \end{aligned}$$

where the integers λ, $\mu^{(i)}$, $\nu^{(i)}$, A, $B^{(i)}$, C and $D^{(i)}$ are constrained by the inequalities $0\leqq\lambda\leqq A$, $0\leqq\mu^{(i)}\leqq D^{(i)}$, $C\geqq 0$, and $0\leqq\nu^{(i)}\leqq B^{(i)}$, $\forall i\in\{1,\cdots,r\}$.

Then it is known that the multiple Mellin-Barnes contour integral (*cf.*, *e.g.*, [6], p. 130, Eq. (1.3)) representing the H-function (1.1) converges absolutely when

$$(1.6)\qquad |\arg(z_i)|<\frac{1}{2}\Lambda_i\pi, \qquad \forall i\in\{1,\cdots,r\},$$

the points $z_i=0$, $i=1,\cdots,r$, and various exceptional parameter values, being tacitly excluded. Furthermore, we recall here the known asymptotic expansions [6, p. 131, Eq. (1.9)] in the following convenient form:

$$(1.7)\qquad H\begin{matrix} 0,\lambda:\ (\mu',\nu');\cdots;\ (\mu^{(r)},\nu^{(r)}) \\ A,C:[B',D'];\cdots;[B^{(r)},D^{(r)}] \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}$$

$$=\begin{cases} O(|z_1|^{\alpha_1}\cdots|z_r|^{\alpha_r}), & \max\{|z_1|,\cdots,|z_r|\}\to 0, \\ O(|z_1|^{\beta_1}\cdots|z_r|^{\beta_r}), & \lambda\equiv 0, \quad \min\{|z_1|,\cdots,|z_r|\}\to\infty, \end{cases}$$

where, with $i=1,\cdots,r$,

$$(1.8)\qquad \begin{cases} \alpha_i=\min\{\mathrm{Re}\,(d_j^{(i)})/\delta_j^{(i)}\}, & j=1,\cdots,\mu^{(i)}, \\ \beta_i=\max\{\mathrm{Re}\,(b_j^{(i)}-1)/\phi_j^{(i)}\}, & j=1,\cdots,\nu^{(i)}, \end{cases}$$

provided that each of the inequalities in (1.4), (1.5) and (1.6) holds.

Expansions for the multivariable H-function (1.1) in series of the Gegenbauer (or ultraspherical) polynomials [8, p. 81, Eq. (4.7.6)]

$$(1.9)\qquad P_n^{(\nu)}(x)=\binom{n+2\nu-1}{n}\,{}_2F_1\left[\begin{matrix}-n,\ 2\nu+n; \\ \nu+1/2;\end{matrix}\ \frac{1-x}{2}\right], \qquad n\geqq 0,$$

can, of course, be deduced as obvious special cases of certain classes of expansions {given, for example, by Theorem 1 (p. 133) and Theorem 3 (p. 137) of the recent work [6]} in series of generalized hypergeometric polynomials of the type

$$\Pi_n^{(\gamma)}(x) = {}_{A+2}F_B\left[\begin{matrix} -n, \gamma+n, a_1, \cdots, a_A; \\ b_1, \cdots, b_B; \end{matrix} x\right], \qquad n \geqq 0 . \tag{1.10}$$

In the present paper, however, we aim at giving a class of ultraspherical series expansions for the multivariable H-function (1.1), which do not seem to follow easily from the aforementioned general classes of known expansions. Indeed, we first state our main results contained in the following

THEOREM. *Let each of the inequalities in* (1.4), (1.5) *and* (1.6) *hold. Also let* $\sigma_i > 0$, $\forall i \in \{1, \cdots, r\}$, *and suppose that*

$$F_n[z_1, \cdots, z_r] = H\begin{matrix} 0, \lambda+3: (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)}) \\ A+3, C+3: [B', D']; \cdots; [B^{(r)}, D^{(r)}] \end{matrix} \tag{1.11}$$
$$\left(\begin{matrix} [\mu-\nu: \sigma_1, \cdots, \sigma_r], [1-\nu-l/2: \sigma_1, \cdots, \sigma_r], [-\nu-(l-1)/2: \sigma_1, \cdots, \sigma_r], \\ [(c): \psi', \cdots, \psi^{(r)}], [1-\nu: \sigma_1, \cdots, \sigma_r], [-\nu-n-l: \sigma_1, \cdots, \sigma_r], \end{matrix}\right.$$
$$\left.\begin{matrix} [(a): \theta', \cdots, \theta^{(r)}]: [(b'): \phi']; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [\mu-\nu+n: \sigma_1, \cdots, \sigma_r]: [(d'): \delta']; \cdots; [(d^{(r)}): \delta^{(r)}]; \end{matrix} z_1, \cdots, z_r\right),$$
$$n \geqq 0 .$$

Then

$$\sum_{n=0}^{\infty} (-1)^n \binom{2n+l}{l} \frac{(2n)!}{n!} \frac{(\mu+2n+l)\Gamma(\mu+n+l)}{\Gamma(2\mu+2n+l)} F_n[z_1, \cdots, z_r] P_{2n+l}^{(\mu)}(\cos\theta) \tag{1.12}$$
$$= \frac{\sqrt{\pi}\,(\csc\theta)^{2(\mu-\nu)}}{2^{2\mu-1}\Gamma(\mu)} \sum_{k=0}^{l} \frac{(\cos\theta-1)^k}{k!(l-k)!} H\begin{matrix} 0, \lambda+2: (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)}) \\ A+2, C+2: [B', D']; \cdots; [B^{(r)}, D^{(r)}] \end{matrix}$$
$$\left(\begin{matrix} [1-\nu-(k+l)/2: \sigma_1, \cdots, \sigma_r], [-\nu-(k+l-1)/2: \sigma_1, \cdots, \sigma_r], [(a): \theta', \cdots, \theta^{(r)}]: \\ [(c): \psi', \cdots, \psi^{(r)}], [1-\nu: \sigma_1, \cdots, \sigma_r], [1/2-\nu-k: \sigma_1, \cdots, \sigma_r]: \end{matrix}\right.$$
$$\left.\begin{matrix} [(b'): \phi']; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [(d'): \delta']; \cdots; [(d^{(r)}): \delta^{(r)}]; \end{matrix} z_1(\sin\theta)^{2\sigma_1}, \cdots, z_r(\sin\theta)^{2\sigma_r}\right),$$

or equivalently

$$\sum_{n=0}^{\infty} (-1)^n \binom{2n+l}{l} \frac{(2n)!}{n!} \frac{(\mu+2n+l)\Gamma(\mu+n+l)}{\Gamma(2\mu+2n+l)} F_n[z_1, \cdots, z_r] P_{2n+l}^{(\mu)}(\cos\theta) \tag{1.13}$$
$$= \frac{\sqrt{\pi}\,(\csc\theta)^{2(\mu-\nu)}}{2^{2\mu+l-1}\Gamma(\mu)} \sum_{k=0}^{l} \frac{\cos(l-2k)\theta}{k!(l-k)!} H\begin{matrix} 0, \lambda+2: (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)}) \\ A+2, C+2: [B', D']; \cdots; [B^{(r)}, D^{(r)}] \end{matrix}$$
$$\left(\begin{matrix} [1-\nu-k: \sigma_1, \cdots, \sigma_r], [1-\nu+k-l: \sigma_1, \cdots, \sigma_r], [(a): \theta', \cdots, \theta^{(r)}]: \\ [(c): \psi', \cdots, \psi^{(r)}], [1-\nu: \sigma_1, \cdots, \sigma_r], [1-\nu: \sigma_1, \cdots, \sigma_r]: \end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix} z_1(\sin\theta)^{2\sigma_1},\cdots,z_r(\sin\theta)^{2\sigma_r}\right),$$

provided that $0<\theta<\pi$, *and*

$$\text{(1.14)}\qquad \operatorname{Re}(\mu-2\nu)<1+2\sum_{i=1}^{r}\sigma_i\alpha_i ,$$

$\alpha_1,\cdots,\alpha_r$ *being given by* (1.8).

Remark 1. A number of interesting variations of the orthogonal expansions (1.12) and (1.13) can be given in a rather straightforward manner. For example, if we define $G_n[z_1,\cdots,z_r]$ by

$$\text{(1.15)}\qquad G_n[z_1,\cdots,z_r]=H^{0,\lambda+1:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}_{A+3,C+3:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}$$
$$\left(\begin{matrix}[1-\mu+\nu-n:\sigma_1,\cdots,\sigma_r],[(a):\theta',\cdots,\theta^{(r)}],[\nu:\sigma_1,\cdots,\sigma_r],\\ [(c):\psi',\cdots,\psi^{(r)}],[1-\mu+\nu:\sigma_1,\cdots,\sigma_r],[\nu+l/2:\sigma_1,\cdots,\sigma_r],\end{matrix}\right.$$
$$\left.\begin{matrix}[\nu+n+l+1:\sigma_1,\cdots,\sigma_r]:[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [\nu+(l+1)/2:\sigma_1,\cdots,\sigma_r]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix} z_1,\cdots,z_r\right),\qquad n\geqq 0,$$

and assume that, for *some* $\gamma_1,\cdots,\gamma_r$,

$$\text{(1.16)}\qquad H^{0,\lambda:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}_{A,C:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}\begin{pmatrix}z_1\\ \vdots\\ z_r\end{pmatrix}$$
$$=O(|z_1|^{\gamma_1}\cdots|z_r|^{\gamma_r}),\qquad \lambda\neq 0,\qquad \min\{|z_1|,\cdots,|z_r|\}\to\infty ,$$

which evidently would complement the second part of the asymptotic expansions in (1.7), we shall readily obtain the expansion formula

$$\text{(1.17)}\qquad \sum_{n=0}^{\infty}\binom{2n+l}{l}\frac{(2n)!}{n!}\frac{(\mu+2n+l)\Gamma(\mu+n+l)}{\Gamma(2\mu+2n+l)}G_n[z_1,\cdots,z_r]P^{(\mu)}_{2n+l}(\cos\theta)$$
$$=\frac{\sqrt{\pi}\,(\csc\theta)^{2(\mu-\nu)}}{2^{2\mu-1}\Gamma(\mu)}\sum_{k=0}^{l}\frac{(1-\cos\theta)^k}{k!(l-k)!}H^{0,\lambda:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}_{A+2,C+2:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}$$
$$\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}],[\nu:\sigma_1,\cdots,\sigma_r],[\nu+k+1/2:\sigma_1,\cdots,\sigma_r]:\\ [(c):\psi',\cdots,\psi^{(r)}],[\nu+(k+l)/2:\sigma_1,\cdots,\sigma_r],[\nu+(k+l+1)/2:\sigma_1,\cdots,\sigma_r]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix} z_1(\csc\theta)^{2\sigma_1},\cdots,z_r(\csc\theta)^{2\sigma_r}\right),$$

or equivalently

$$\text{(1.18)}\qquad \sum_{n=0}^{\infty}\binom{2n+l}{l}\frac{(2n)!}{n!}\frac{(\mu+2n+l)\Gamma(\mu+n+l)}{\Gamma(2\mu+2n+l)}G_n[z_1,\cdots,z_r]P^{(\mu)}_{2n+l}(\cos\theta)$$
$$=\frac{\sqrt{\pi}\,(\csc\theta)^{2(\mu-\nu)}}{2^{2\mu+l-1}\Gamma(\mu)}\sum_{k=0}^{l}\frac{\cos(l-2k)\theta}{k!(l-k)!}H^{0,\lambda+1:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}_{A+3,C+3:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{array}{c} [\nu:\sigma_1,\cdots,\sigma_r],\ [(a):\theta',\cdots,\theta^{(r)}],\ [\nu:\sigma_1,\cdots,\sigma_r],\ [\nu+1/2:\sigma_1,\cdots,\sigma_r]: \\ [(c):\psi',\cdots,\psi^{(r)}],\ [\nu+k:\sigma_1,\cdots,\sigma_r],\ [\nu-k+l:\sigma_1,\cdots,\sigma_r],\ [\nu+1/2:\sigma_1,\cdots,\sigma_r]: \end{array}\right.$$
$$\left.\begin{array}{c} [(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}]; \end{array} z_1(\csc\theta)^{2\sigma_1},\cdots,z_r(\csc\theta)^{2\sigma_r}\right),$$

provided that $0<\theta<\pi$, $\sigma_i>0$, $\lambda_i<0$, $\Lambda_i>4\sigma_i$, $|\arg(z_i)|<\pi(\Lambda_i-4\sigma_i)/2$, $\forall i\in\{1,\cdots,r\}$, and

$$\text{Re}(\mu-2\nu)<1-2\sum_{i=1}^{r}\sigma_i\gamma_i\,, \tag{1.19}$$

where λ_i, Λ_i and γ_i are given by (1.4), (1.5) and (1.16), respectively.

Remark 2. The confluent cases of the orthogonal expansions (1.12), (1.13), (1.17) and (1.18), when $\sigma_j\downarrow 0$, $\forall j\neq i$, will obviously lead to expansion formulas in which the parameters corresponding *only* to the H-function variable z_i, $1\leqq i\leqq r$, are affected, and we omit the details.

2. Proof of the theorem

In order to establish the expansion formula (1.12) or (1.13) as a *formal* identity, we begin by replacing $F_n[z_1,\cdots,z_r]$ on their left-hand sides by its Mellin-Barnes contour integral derivable from the definition of the multivariable H-function involved (*cf.*, *e.g.*, [6], p. 130, Eq. (1.3)), change the order of integration and summation (which can easily be justified when the integral and the series are absolutely convergent), and simplify the resulting summand by applying Legendre's duplication formula:

$$\sqrt{\pi}\,\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma(z+1/2)\,. \tag{2.1}$$

Denoting the first member of the expansion formula (1.12) or (1.13) by $\Omega(\theta)$, we thus find that

$$\begin{aligned}\Omega(\theta)=&\frac{1}{(2\pi\omega)^r}\int_{\mathscr{L}_1}\cdots\int_{\mathscr{L}_r}\Phi_1(\zeta_1)\cdots\Phi_r(\zeta_r)\Psi(\zeta_1,\cdots,\zeta_r)z_1^{\zeta_1}\cdots z_r^{\zeta_r}\\ &\times\frac{\sqrt{\pi}}{2^{2\nu+2\Delta+l-1}}\left[\frac{\Gamma(1-\mu+\nu+\Delta)\Gamma(2\nu+2\Delta+l)}{\Gamma(\nu+\Delta)}\sum_{n=0}^{\infty}(-1)^n\binom{2n+l}{l}\frac{(2n)!}{n!}\right.\\ &\times\frac{(\mu+2n+l)\Gamma(\mu+n+l)}{\Gamma(2\mu+2n+l)\Gamma(\nu+\Delta+n+l+1)\Gamma(1-\mu+\nu+\Delta-n)}\\ &\left.\times P_{2n+l}^{(\mu)}(\cos\theta)\right]d\zeta_1\cdots d\zeta_r\,,\end{aligned} \tag{2.2}$$

where $\omega=\sqrt{-1}$, $\Phi_i(\zeta_i)$, $i=1,\cdots,r$, and $\Psi(\zeta_1,\cdots,\zeta_r)$ are defined by Equations (1.4) and (1.5) on page 130 in the Srivastava-Panda paper [6], and, for convenience,

$$\Delta=\sigma_1\zeta_1+\cdots+\sigma_r\zeta_r\,. \tag{2.3}$$

Now the series on the right-hand side of (2.2) can be summed by appealing to Askey's formula [2, p. 1192, Eq. (6)]

$$(2.4)\qquad P_l^{(\nu)}(\cos\theta)=(2\sin\theta)^{2(\mu-\nu)}\frac{\Gamma(\mu)\Gamma(1-\mu+\nu)\Gamma(2\nu+l)}{\Gamma(\nu)}$$
$$\times\sum_{n=0}^{\infty}(-1)^n\binom{2n+l}{l}\frac{(2n)!}{n!}\frac{(\mu+2n+l)\Gamma(\mu+n+l)}{\Gamma(2\mu+2n+l)\Gamma(\nu+n+l+1)\Gamma(1-\mu+\nu-n)}$$
$$\times P_{2n+l}^{(\mu)}(\cos\theta)\ ,$$

where, for convergence, $0<\theta<\pi$ and $\mathrm{Re}\,(\mu-2\nu)<1$, and the resulting ultraspherical polynomial $P_l^{(\nu+\Delta)}(\cos\theta)$ can be expressed as a finite sum by using *either* the definition (1.9) *or* the known result (*cf.* [8], p. 95, Eq. (4.9.19); see also [1], p. 776, Eq. (22.3.12));

$$(2.5)\qquad P_n^{(\nu)}(\cos\theta)=\sum_{k=0}^{n}\binom{\nu+k-1}{k}\binom{\nu+n-k-1}{n-k}\cos(n-2k)\theta\ ,$$
$$(\nu\neq 0,\ -1,\ -2,\ \cdots)\ .$$

On inverting the order of the (finite) summation and integration in either case, if we interpret the multiple Mellin-Barnes contour integral thus obtained as an H-function, we shall arrive at the identity (1.12) or (1.13) as the case may be.

So far we have only shown that the expansion formula (1.12) or (1.13) is a formal identity. We now proceed to demonstrate that (1.12) and (1.13) are indeed valid under the hypotheses of the theorem.

First of all we notice that, since $0<\theta<\pi$ and $\sigma_i>0$, $\forall i\in\{1,\cdots,r\}$, the inequalities in (1.4), (1.5) and (1.6) are sufficient to insure that the H functions occurring on both sides of (1.12) and (1.13) are well defined. On the other hand, the precise condition of convergence of the infinite series in (1.12) or (1.13) may be determined by considering the asymptotic behaviours of the ultraspherical polynomial $P_{2n+l}^{(\mu)}(\cos\theta)$ and the function $F_n[z_1,\cdots,z_r]$ defined by (1.11) for large n and fixed l, θ, $|\mu|$, $|\nu|$, and $|z_1|,\cdots,|z_r|$. The asymptotic estimate of the former is well known, and it is easily verified from (1.7), (1.11), and the familiar result

$$(2.6)\qquad \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)}=n^{\alpha-\beta}[1+O(n^{-1})]\ ,\qquad n\to\infty\ ,$$

that, for fixed $|z_1|,\cdots,|z_r|$,

$$(2.7)\qquad F_n[z_1,\cdots,z_r]\sim n^{-2(\sigma_1\alpha_1+\cdots+\sigma_r\alpha_r)}\ ,\qquad n\to\infty\ ,$$

where, as before, $\alpha_1,\cdots,\alpha_r$ are given by (1.8).

These considerations will readily yield the hypothesis (1.14), and the proof of our Theorem is evidently completed.

We remark in passing that the above method of derivation of the orthogonal expansions (1.12) and (1.13) would apply *mutatis mutandis* to their variations (1.17) and (1.18), respectively. Indeed, Askey's formula (2.4) is used in its alternative (but essentially equivalent) form:

$$(2.8)\qquad P_l^{(\nu)}(\cos\theta)=(2\sin\theta)^{2(\mu-\nu)}\frac{\Gamma(\mu)\Gamma(2\nu+l)}{\Gamma(\mu-\nu)\Gamma(\nu)}$$
$$\times\sum_{n=0}^{\infty}\binom{2n+l}{l}\frac{(2n)!}{n!}\frac{(\mu+2n+l)\Gamma(\mu+n+l)\Gamma(\mu-\nu+n)}{\Gamma(2\mu+2n+l)\Gamma(\nu+n+l+1)}P_{2n+l}^{(\mu)}(\cos\theta)\,,$$

where, as before, $0<\theta<\pi$ and $\mathrm{Re}\,(\mu-2\nu)<1$, and the hypothesis (1.19) stems, among other things already indicated, from the readily verifiable fact that, for fixed $|z_1|,\cdots,|z_r|$,

$$(2.9)\qquad G_n[z_1,\cdots,z_r]\sim n^{2(\sigma_1\gamma_1+\cdots+\sigma_r\gamma_r)}\,,\qquad n\to\infty\,,$$

where $G_n[z_1,\cdots,z_r]$ is defined by (1.15), and $\gamma_1,\cdots,\gamma_r$ are given by (1.16). We omit the details involved.

Remark 3. In the asymptotic estimate (2.9), and hence also in the convergence condition (1.19), γ_i can be replaced by the β_i defined by (1.8), $i=1,\cdots,r$, provided that $\lambda\equiv 0$ in *each* of the orthogonal expansions (1.17) and (1.18).

3. Some interesting deductions

The expansion formulas (1.12), (1.13), (1.17) and (1.18) are quite general in character. Indeed, these and their variations (indicated, for example, by Remark 2) can be suitably applied to derive various classes of (known or new) orthogonal expansions in series of ultraspherical polynomials (or their such special cases as Legendre polynomials $P_n(x)$, the Tchebycheff polynomials $T_n(x)$ and $U_n(x)$, the trigonometric polynomials $\sin nx$ and $\cos nx$, and so on) for a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of the E, F, G or H functions of one or more variables. We do not find it worthwhile to give the details of the analysis involved in deriving *all* of these special or confluent cases of our results. For convenience of the interested reader, however, we record here the following well-known relationships which would enable one to deduce the aforementioned expansion formulas (*cf.* [1], pp. 777-779):

$$(3.1)\qquad P_n(x)=P_n^{(1/2)}(x)\,,\qquad T_n(x)=\frac{1}{2}nP_n^{(0)}(x)\,,\qquad U_n(x)=P_n^{(1)}(x)\,;$$

$$(3.2)\qquad P_n^{(1)}(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}\,,\qquad 0<\theta<\pi\,;$$

and

(3.3) $$P_n^{(0)}(\cos\theta)=\frac{2}{n}\cos n\theta\,,\qquad n=1,2,3,\cdots;\ P_0^{(0)}(\cos\theta)=1\,,$$

where, by definition,

(3.4) $$P(x)_n^{(0)}=\lim_{\mu\to 0}\,\{P_n^{(\mu)}(x)/\mu\}\,,$$

or equivalently,

(3.5) $$\lim_{\mu\to 0}\,\{\Gamma(\mu)P_n^{(\mu)}(\cos\theta)\}=\frac{2}{n}\cos n\theta\,,\qquad n=1,2,3,\cdots.$$

For $l=0$, the series on the right-hand side of *each* of our expansion formulas (1.12), (1.13), (1.17) and (1.18) would obviously reduce to its first term given by $k=0$, and we shall be led immediately to the following simpler expansions:

(3.6) $$\frac{\sqrt{\pi}\,(\csc\theta)^{2(\mu-\nu)}}{2^{2\mu-1}\Gamma(\mu)}H\begin{matrix}0,\lambda\colon\ (\mu',\nu');\cdots;\ (\mu^{(r)},\nu^{(r)})\\ A,C\colon[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}\begin{pmatrix}z_1(\sin\theta)^{2\sigma_1}\\ \vdots\\ z_r(\sin\theta)^{2\sigma_r}\end{pmatrix}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{n!}\,\frac{(\mu+2n)\Gamma(\mu+n)}{\Gamma(2\mu+2n)}F_n^*[z_1,\cdots,z_r]P_{2n}^{(\mu)}(\cos\theta)$$

and

(3.7) $$\frac{\sqrt{\pi}\,(\csc\theta)^{2(\mu-\nu)}}{2^{2\mu-1}\Gamma(\mu)}H\begin{matrix}0,\lambda\colon\ (\mu',\nu');\cdots;\ (\mu^{(r)},\nu^{(r)})\\ A+2,C+2\colon[B',D'];\cdots;[B^{(r)},D^{(r)}]\end{matrix}$$
$$\begin{pmatrix}[(a)\colon\theta',\cdots,\theta^{(r)}],[\nu\colon\sigma_1,\cdots,\sigma_r],[\nu+1/2\colon\sigma_1,\cdots,\sigma_r]\colon\\ [(c)\colon\psi',\cdots,\psi^{(r)}],[\nu\colon\sigma_1,\cdots,\sigma_r],[\nu+1/2\colon\sigma_1,\cdots,\sigma_r]\colon\end{pmatrix.$$
$$\left.\begin{matrix}[(b')\colon\phi'];\cdots;[(b^{(r)})\colon\phi^{(r)}];\\ [(d')\colon\delta'];\cdots;[(d^{(r)})\colon\delta^{(r)}];\end{matrix}\ z_1(\csc\theta)^{2\sigma_1},\cdots,z_r(\csc\theta)^{2\sigma_r}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(2n)!}{n!}\,\frac{(\mu+2n)\Gamma(\mu+n)}{\Gamma(2\mu+2n)}G_n^*[z_1,\cdots,z_r]P_{2n}^{(\mu)}(\cos\theta)\,,$$

where, for convenience,

(3.8) $$\begin{cases}F_n^*[z_1,\cdots,z_r]=F_n[z_1,\cdots,z_r]|_{l=0}\\ G_n^*[z_1,\cdots,z_r]=G_n[z_1,\cdots,z_r]|_{l=0}\end{cases}$$

$F_n[z_1,\cdots,z_r]$ and $G_n[z_1,\cdots,z_r]$ being defined by (1.11) and (1.15), respectively.

Formulas (3.6) and (3.7) are valid under the same hypotheses as those of their parent formulas (1.12) and (1.17), or (1.13) and (1.18), respectively. {See also Remark 3 of the preceding section.} As a matter of fact, such expansions as (3.6) and (3.7) can also be obtained *directly* by applying the familiar orthogonality property of the ultraspherical polynomials in conjunction with Lemma 1 on page 171 of the earlier work [7, Part II], *without* using Askey's result (2.4) or (2.8).

Two further consequences of (3.6) and (3.7) are worthy of mention here. For example, if we multiply both sides of (3.6) by $\Gamma(\mu)$, set $\nu=\rho/2$, replace θ by $\theta/2$, and proceed to the limit as $\mu\to 0$ using (3.3) and (3.5), we shall obtain the elegant result

$$(3.9)\qquad \left(\sin\frac{\theta}{2}\right)^{\rho} H^{0,\,\lambda:\ (\mu',\,\nu');\,\cdots;\,(\mu^{(r)},\,\nu^{(r)})}_{A,\,C:\,[B',\,D'];\,\cdots;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1\left(\sin\frac{\theta}{2}\right)^{2\sigma_1}\\ \vdots \\ z_r\left(\sin\frac{\theta}{2}\right)^{2\sigma_r}\end{pmatrix}$$

$$=\frac{1}{\sqrt{\pi}}H^{0,\,\lambda+1:\ (\mu',\,\nu');\,\cdots;\,(\mu^{(r)},\,\nu^{(r)})}_{A+1,\,C+1:\,[B',\,D'];\,\cdots;\,[B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(1-\rho)/2:\sigma_1,\cdots,\sigma_r],\\ [(c):\psi',\cdots,\psi^{(r)}],\end{matrix}\ \begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [-\rho/2:\sigma_1,\cdots,\sigma_r]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right)$$

$$+\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n\gamma_n^{(\rho)}[z_1,\cdots,z_r]\cos n\theta\,,\qquad 0<\theta<2\pi\,,$$

where

$$(3.10)\qquad \gamma_n^{(\rho)}[z_1,\cdots,z_r]=H^{0,\,\lambda+2:\ (\mu',\,\nu');\,\cdots;\,(\mu^{(r)},\,\nu^{(r)})}_{A+2,\,C+2:\,[B',\,D'];\,\cdots;\,[B^{(r)},\,D^{(r)}]}$$

$$\left(\begin{matrix}[-\rho/2:\sigma_1,\cdots,\sigma_r],\ [(1-\rho)/2:\sigma_1,\cdots,\sigma_r],\quad [(a):\theta',\cdots,\theta^{(r)}]:\\ [(c):\psi',\cdots,\psi^{(r)}],\ [-n-\rho/2:\sigma_1,\cdots,\sigma_r],\ [n-\rho/2:\sigma_1,\cdots,\sigma_r]:\end{matrix}\ \begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right),\qquad n\geqq 1\,,$$

and, for convergence of the infinite series in (3.9),

$$(3.11)\qquad \mathrm{Re}\,(\rho)>-1-2\sum_{i=1}^{r}\sigma_i\alpha_i\,,$$

$\alpha_1,\cdots,\alpha_r$ being defined by (1.8).

On the other hand, if in (3.6) we set $\mu=1$ and $\nu=1+\sigma/2$, and appeal to the known relationship (3.2), we shall get the interesting formula

$$(3.12)\qquad (\sin\theta)^{\sigma}H^{0,\,\lambda:\ (\mu',\,\nu');\,\cdots;\,(\mu^{(r)},\,\nu^{(r)})}_{A,\,C:\,[B',\,D'];\,\cdots;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1(\sin\theta)^{2\sigma_1}\\ \vdots\\ z_r(\sin\theta)^{2\sigma_r}\end{pmatrix}$$

$$=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\Gamma_n^{(\sigma)}[z_1,\cdots,z_r]\sin(2n+1)\theta\,,\qquad 0<\theta<\pi\,,$$

where, for convergence,

$$\mathrm{Re}\,(\sigma) > -2\left(1+\sum_{i=1}^{r}\sigma_i\alpha_i\right), \tag{3.13}$$

and

$$\Gamma_n^{(\sigma)}[z_1, \cdots, z_r] = H^{0,\,\lambda+2:\ (\mu',\,\nu');\ \cdots;\ (\mu^{(r)},\,\nu^{(r)})}_{A+2,\,C+2:\,[B',\,D'];\ \cdots;\ [B^{(r)},\,D^{(r)}]} \tag{3.14}$$

$$\left(\begin{matrix} [-\sigma/2: \sigma_1, \cdots, \sigma_r], [(1-\sigma)/2: \sigma_1, \cdots, \sigma_r], [(a): \theta', \cdots, \theta^{(r)}]: \\ [(c): \psi', \cdots, \psi^{(r)}], [-n-(\sigma+1)/2: \sigma_1, \cdots, \sigma_r], [n-(\sigma-1)/2: \sigma_1, \cdots, \sigma_r]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'): \phi']; \cdots; [(b^{(r)}): \phi^{(r)}]; \\ [(d'): \delta']; \cdots; [(d^{(r)}): \delta^{(r)}]; \end{matrix}\ z_1, \cdots, z_r\right), \qquad n \geqq 0 .$$

Similar consequences of the orthogonal expansion (3.7) can be derived in a similar manner, and we leave the details as an exercise to the interested reader.

The trigonometrical Fourier series (3.9) and (3.12) for multivariable H functions were given earlier by Srivastava and Panda [7, Part II, p. 179, Theorem 3], who indeed considered the general problem of orthogonal expansions for the H-function (1.1). {See also [6], p. 137, Theorem 3; p. 138, Theorem 4; p. 140, Eq. (5.2).} Other known special cases* of our main results in this paper include the expansion formula [3, p. 527, Eq. (2.1)], which follows readily from (1.17) when $r=2$ and $\nu=1$, and its *alternative* (but essentially *equivalent*) form [4, p. A40, Eq. (2.1)], whose *corrected* version (without any *redundant* parameters) is substantially the same as our formula (1.18) with, of course, $r=2$ and $\nu=1$.

Finally, we should like to remark that, with each of the coefficients listed in (1.3) equated to 1, and with $\sigma_i=N$, $\forall i \in \{1, \cdots, r\}$, where N is an arbitrary positive integer, our main expansion formulas as well as their various special or confluent cases discussed in this paper can be reduced to the corresponding results for the relatively more familiar G-function of several complex variables.

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* Incidentally, relevant references to the literature containing several *further* special cases of the trigonometrical Fourier series (3.9) and (3.12) are indeed given in the Srivastava-Panda paper [7, Part II, pp. 194–197]. {See also [6], pp. 144–145.}

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