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A formulation of cancellation theory and its application to modules

by

Makoto ISHIBASHI*

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§ 0. Introduction

As is well known, E. Noether recognized the importance of the concept of representation module and restated the classical representation theory by G. Frobenius and W. Burnside module-theoretically.

Usually we consider representations over a field, and many fertile theorems can be obtained from this direction. On the contrary, the so-called integral representation isc complicated by the fact that the useful theorems no longer hold. It is well-known that the Krull-Schmidt theorem does not hold for finitely generated modules over a Dedekind ring. By Berman-Gudivok [8]**, the cancellation theorem need not be true for $Z[G]$-modules, where G is a cyclic group of order p^2 (p: an odd prime). Furthermore, by results of R. Swan [7], the cancellation theorem is not necessarily true for the category of projective $Z[G]$-lattices, where G is a generalized quaternion group of order 32. In this aspect of the cancellation theorem, H. Jacobinski [1] obtained the extremely remarkable results.

The Section 1 of this note consists of the presentation of notations and definitions concerning abelian categories with decomposition theory. We need the supplementary hypothesis on abelian category for the later discussion. In the Section 2, we shall give an example of the Grothendieck group associated with a decomposition theory in categorical formulation. The Proposition 2.3, which is essentially related to cancellation, will be proved without difficulty. The Section 3, the origin of this note, is the main part of our cancellation theory. In this section, we introduce some new concepts and investigate the cancellation theorem for certain categories. The concepts of Section 3 are applied in the Section 4 to modules which are finitely generated torsion-free over a ring of algebraic integers.

Our main Theorem 4.1 gives a certain characterization of linkable pairs using the class number of algebraic number field.

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** Numbers in brackets refer to the references at the end of this note.

Throughout this note, in order to generalize the propositions as possible as we can, we resort to categorical notations. We use the terminology "*property*" instead of "*proposition*", when it is a immediate consequence of the preceeding definitions. Therefore, the proofs of properties are omitted.

§ 1. Preliminaries of notations

It is clear that if the Krull-Schmidt theorem holds, then so does the cancellation theorem. Of course, the Krull-Schmidt theorem for modules over a ring asserts under suitable conditions the existence and uniqueness of indecomposable factors up to operator isomorphisms. However we say the Krull-Schmidt theorem holds with respect to a given module M, when this module M can be uniquely decomposable into indecomposable submodules up to operator isomorphisms. In a category $\mathfrak{C}$, if the Krull-Schnidt theorem holds for every object in $\mathfrak{C}$, then we say that the Krull-Schmidt theorem holds for $\mathfrak{C}$.

On the other hand, we say that $M\in \mathrm{OB}(\mathfrak{C})$ is cancellative in a category $\mathfrak{C}$, if every relation $M\oplus X\cong N\oplus X$ with X, $N\in\mathrm{OB}(\mathfrak{C})$ implies $M\cong N$. Furthermor, if every object in $\mathfrak{C}$ is cancellative, we say that the cancellation theorem holds for $\mathfrak{C}$. One of our purposes is to investigate the conditions under which the cancellation theorem holds for a given category. For the sake of this, we introduce some terminologies and use the sheaf-theoretical ideas.

We recall briefly the definitions and elementary properties of abelian categories, but for full details we refer to A. Grothendieck [6].

Definition 1.1. A decomposition of an object $M=M_1\oplus\cdots\oplus M_n$ is called a *Remak decomposition* of M, if each M_r is indecomposable and non-zero.

Definition 1.2. An abelian category $\mathfrak{C}$ is called AB3)-*finite*, if there exists in $\mathfrak{C}$ the direct sum of any finite number of objects in $\mathfrak{C}$.

Definition 1.3. A *bi-chain* of a category $\mathfrak{C}$ is a sequence of triples $\{A_n, i_n, p_n\}$ with the following properties.

(i) $A_n\in\mathrm{OB}(\mathfrak{C})$ $(n\geqq 0)$;

(ii) $i_n\colon A_n\to A_{n-1}$ is a monomorphism $(n\geqq 1)$;

(iii) $p_n\colon A_{n-1}\to A_n$ is an epimorphism $(n\geqq 1)$.

Definition 1.4. A bi-chain $\{A_n, i_n, p_n\}$ is said to *terminate* if there exists an integer m such that, for all $n\geqq m$, $p_n i_n=\mathrm{id}_{A_n}$ and $i_n p_n=\mathrm{id}_{A_{n-1}}$.

Definition 1.5. The *bi-chain condition* holds in $\mathfrak{C}$ if every bi-chain of $\mathfrak{C}$ terminates.

There is the following well-known fact, which is found in N. Popescu [9].

PROPOSITION 1.6. *If the bi-chain condition holds in an abelian category $\mathfrak{C}$, then any object of $\mathfrak{C}$ admits a unique Remak decomposition.*

Therefore, in Section 3, we are going to discuss the cancellation theorem for a special type of abelian categories which does not necessarily satisfy the bi-chain condition.

§ 2. An example of the Grothendieck group

From now on, unless otherwise specified, the symbol $\mathfrak{C}$ denotes AB3)-finite category and any object of $\mathfrak{C}$ admits a Remak decomposition.

Let $\mathscr{I}(\mathfrak{C})$ be the subcategory of $\mathfrak{C}$ whose objects are all the indecomposable objects in $\mathfrak{C}$. And let $\mathscr{I}[\mathfrak{C}]$ be the isomorphism classes in $\mathscr{I}(\mathfrak{C})$. We can define the canonical identical correspondence

$$\varpi : \mathscr{S}_{\mathfrak{C}} \to \mathfrak{C} ,$$

where $\mathscr{S}_{\mathfrak{C}}$ denotes the free monoid whose generators are objects in $\mathscr{I}[\mathfrak{C}]$.

Choose $\mathfrak{U}=\{\mathfrak{C}_\lambda\}_{\lambda\in\Lambda}$ (a family of subcategories of $\mathfrak{C}$) satisfying the next three conditions.

(i) each $\mathfrak{C}_\lambda$ has at most finite isomorphism classes;

(ii) every object of $\mathfrak{C}$ is an object of $\mathfrak{C}_\lambda$ for some $\lambda\in\Lambda$;

(iii) $\mathfrak{C}_\lambda$, $\mathfrak{C}_\mu\in\mathfrak{U}$ implies $\mathfrak{C}_\lambda+\mathfrak{C}_\mu\in\mathfrak{U}$, where we define the sum of two subcategories to be the subcategory whose objects are $M_\lambda\oplus M_\mu$ with $M_\lambda\in \mathrm{OB}(\mathfrak{C}_\lambda)$ and $M_\mu\in\mathrm{OB}(\mathfrak{C}_\mu)$.

Remark. For instance, let $\mathfrak{C}$ be the full set of lattices over an order in a semi-simple algebra over an algebraic number field which is the quotient field of a Dedekind domain. Then the family of genera of $\mathfrak{C}$ preserves our three conditions (See H. Jacobinski [1]).

Now, for two correspondences

$$\sigma_\lambda,\ \tau_\lambda : \mathfrak{C}_\lambda \to \mathscr{S}_{\mathfrak{C}} ,$$

we define the *equality* $\sigma_\lambda=\tau_\lambda$ if it holds $M_{\sigma_i}=M_{\tau_i}$ after a suitable reordering of the suffixes $1\leqq i\leqq r$ for all $M\in\mathrm{OB}(\mathfrak{C}_\lambda)$, where r depends on M,

$$\sigma_\lambda(M)=M_{\sigma_1}\oplus\cdots\oplus M_{\sigma_r} ,$$
$$\tau_\lambda(M)=M_{\tau_1}\oplus\cdots\oplus M_{\tau_r} .$$

Let $\Gamma(\mathfrak{C}_\lambda, \mathscr{S}_{\mathfrak{C}})$ denote the family of all the correspondences σ_λ: $\mathfrak{C}_\lambda\to\mathscr{S}_{\mathfrak{C}}$ such that $\varpi\sigma_\lambda(M)\cong M$ for all $M\in\mathrm{OB}(\mathfrak{C}_\lambda)$.

Definition 2.1. For $\sigma_\lambda\in\Gamma(\mathfrak{C}_\lambda, \mathscr{S}_{\mathfrak{C}})$ and $\sigma_\mu\in\Gamma(\mathfrak{C}_\mu, \mathscr{S}_{\mathfrak{C}})$, we define *addition* $\sigma_\lambda+\sigma_\mu$ by

$$(\sigma_\lambda+\sigma_\mu)(M_\lambda\oplus M_\mu)=\sigma_\lambda(M_\lambda)\oplus\sigma_\mu(M_\mu)$$

with each $M_\lambda \in \mathrm{OB}(\mathfrak{C}_\lambda)$ and $M_\mu \in \mathrm{OB}(\mathfrak{C}_\mu)$.

Then this addition is well-defined, and $\sigma_\lambda + \sigma_\mu \in \Gamma(\mathfrak{C}_\lambda + \mathfrak{C}_\mu, \mathscr{S}_\mathfrak{C})$. By the condition (iii) on $\mathfrak{U} = \{\mathfrak{C}_\lambda\}_{\lambda \in \Lambda}$, we can make $\Gamma(\mathfrak{U}, \mathscr{S}_\mathfrak{C}) = \{\Gamma(\mathfrak{C}_\lambda, \mathscr{S}_\mathfrak{C})\}_{\mathfrak{C}_\lambda \in \mathfrak{U}}$ to be a monoid. Hence we can define a kind of Grothendieck groups.

Definition 2.2. Let $G(\Gamma(\mathfrak{U}, \mathscr{S}_\mathfrak{C}))$ be a commutative group generated by symbols $[\sigma_\lambda]$ ($\sigma_\lambda \in \Gamma(\mathfrak{C}, \mathscr{S}_\mathfrak{C})$, $\mathfrak{C}_\lambda \in \mathfrak{U}$, $\lambda \in \Lambda$) with relations $[\sigma_\lambda] + [\sigma_\mu] = [\sigma_\nu]$, where $\mathfrak{C}_\lambda + \mathfrak{C}_\mu = \mathfrak{C}_\nu \in \mathfrak{U}$, $\lambda, \mu, \nu \in \Lambda$.

PROPOSITION 2.3. *The canonical map*

$$\gamma : \Gamma(\mathfrak{U}, \mathscr{S}_\mathfrak{C}) \to G(\Gamma(\mathfrak{U}, \mathscr{S}_\mathfrak{C})) \text{ is injective.}$$

Proof. It is sufficient to show that the cancellation theorem holds in $\Gamma(\mathfrak{U}, \mathscr{S}_\mathfrak{C})$. (For the reason, see S. Lang [4], p. 43–44.) Suppose that $\sigma_\lambda + \sigma_\mu = \tau_\lambda + \sigma_\mu$, i.e., $\sigma_\lambda(M_\lambda) \oplus \sigma_\mu(M_\mu) = \tau_\lambda(M_\lambda) \oplus \sigma_\mu(M_\mu)$ for all $M_\lambda \in \mathrm{OB}(\mathfrak{C}_\lambda)$ and all $M_\mu \in \mathrm{OB}(\mathfrak{C}_\mu)$.

We can put

$$\sigma_\lambda(M_\lambda) = M_{\sigma_\lambda 1} \oplus \cdots \oplus M_{\sigma_\lambda r},$$
$$\sigma_\mu(M_\mu) = M_{\sigma_\lambda 1} \oplus \cdots \oplus M_{\sigma_\mu s},$$
$$\tau_\lambda(M_\lambda) = M_{\tau_\lambda 1} \oplus \cdots \oplus M_{\tau_\lambda r},$$

where $M_{\sigma_\lambda i}, M_{\sigma_\mu j}, M_{\tau_\lambda k} \in \mathscr{I}[\mathfrak{C}]$ ($1 \leqq i \leqq r$, $1 \leqq j \leqq s$, $1 \leqq k \leqq r$).

Then there exists a bijection

$$\varphi : \mathfrak{M}_\sigma = \{M_{\sigma_\lambda 1}, \cdots, M_{\sigma_\lambda r}, M_{\sigma_\mu 1}, \cdots, M_{\sigma_\mu s}\}$$
$$\to \mathfrak{M}_\tau = \{M_{\tau_\lambda 1}, \cdots, M_{\tau_\lambda r}, M_{\sigma_\mu 1}, \cdots, M_{\sigma_\mu s}\}$$

such that $M \cong \varphi(M)$ for all $M \in \mathfrak{M}_\sigma$.

If there exists an indecomposable M_μ in $\mathrm{OB}(\mathfrak{C}_\mu)$, then $\sigma_u(M_\mu) = M_{\sigma_\mu 1} \in \mathscr{I}[\mathfrak{C}]$, and $\mathfrak{M}_\sigma = \{M_{\sigma_\lambda 1}, \cdots, M_{\sigma_\lambda r}, M_{\sigma_\mu 1}\}$, $\mathfrak{M}_\tau = \{M_{\tau_\lambda 1}, \cdots, M_{\tau_\lambda r}, M_{\sigma_\mu 1}\}$ (i.e., $s=1$). Let $\mathfrak{M}_{\sigma_\mu} = \{M_1, \cdots, M_k\} = \{M \in M_\sigma | M \cong M_{\sigma_\mu 1}\} = \{M \in M_\tau | M \cong M_{\sigma_\mu 1}\}$ and $\mathfrak{M}'_\sigma = \mathfrak{M}_\sigma \setminus \{M_{\sigma_\mu 1}\}$, $\mathfrak{M}'_\tau = \mathfrak{M}_\tau \setminus \{M_{\sigma_\mu 1}\}$. In both $\mathfrak{M}'_\sigma$ and $\mathfrak{M}'_\tau$, there exist $(k-1)$'s indecomposable M_i which are isomorphic to $M_{\sigma_\mu 1}$. On the other hand, we have a bijection

$$\varphi' : \mathfrak{M}'_\sigma \setminus \mathfrak{M}_{\sigma_\mu} \to \mathfrak{M}'_\tau \setminus \mathfrak{M}_{\sigma_\mu},$$

which is a restriction map of $\varphi : \mathfrak{M}_\sigma \xrightarrow{\sim} \mathfrak{M}_\tau$. Therefore we have $M_{\sigma_\lambda 1} \oplus \cdots \oplus M_{\sigma_\lambda r} = M_{\tau_\lambda 1} \oplus \cdots \oplus M_{\tau_\lambda r}$ for all $M_\lambda \in \mathrm{OB}(\mathfrak{C}_\lambda)$.

In general, we cannot deduce that there exists an indecomposable M_μ in $\mathrm{OB}(\mathfrak{C}_\mu)$. But the same discussion is available. For $s > 1$, without losing the generality, we can assume that

$$M_{\sigma_\mu 1} \cong M_{\sigma_\mu 2} \cong \cdots\cdots\cdots\cdots \cong M_{\sigma_\mu i_1}$$
$$\not\cong M_{\sigma_\mu, i_1+1} \cong \cdots\cdots\cdots\cdots \cong M_{\sigma_\mu, i_1+i_2}$$
$$\not\cong M_{\sigma_\mu, i_1+i_2+1} \cong \cdots\cdots\cdots\cdots \cong M_{\sigma_\mu, i_1+i_2+i_3}$$
$$\vdots$$
$$, \quad \not\cong M_{\sigma_\mu, s-i_t-1} \cong \cdots\cdots\cdots\cdots \cong M_{\sigma_\mu s}.$$

Apply the former discussion for each $M_{\sigma_\mu j}$ ($j=1, i_1+1, i_1+i_2+1, \cdots, s-i_t-1$). Then we get a bijection

$$\varphi': \mathfrak{M}'_\sigma \to \mathfrak{M}'_\tau ,$$

where $\mathfrak{M}'_\sigma = \mathfrak{M}_\sigma \setminus \{M_{\sigma_\mu 1}, \cdots, M_{\sigma_\mu s}\}$, $\mathfrak{M}'_\tau = \mathfrak{M}_\tau \setminus \{M_{\sigma_\mu 1}, \cdots, M_{\sigma_\mu s}\}$.

Hence $M_{\sigma_\lambda 1} \oplus \cdots \oplus M_{\sigma_\lambda r} = M_{\tau_\lambda 1} \oplus \cdots \oplus M_{\tau_\lambda r}$ for all $M_\lambda \in \mathrm{OB}(\mathfrak{C}_\lambda)$, i.e., $\sigma_\lambda = \tau_\lambda$. This proves the proposition.

§ 3. The cancellation theorem

We keep here the notations explained in the preceding sections. For any fixed $M \in \mathrm{OB}(\mathfrak{C})$, let $\varpi^{-1}(M)$ denote all the decompositions $D(\in \mathscr{S}_{\mathfrak{C}})$ of M into indecomposable factors. We shall give some elementary properties of the following concepts.

Definition 3.1. Let $D_i, D_j \in \varpi^{-1}(M)$ be decomposition $M_{i1} \oplus \cdots \oplus M_{ir}$, $M_{j1} \oplus \cdots \oplus M_{js}$ respectively, where each $M_{il}, M_{jm} \in \mathscr{I}[\mathfrak{C}]$ ($1 \leqq l \leqq r$, $1 \leqq m \leqq s$). A pair $[D_i, D_j]$ will be called a *redundant pair*, if

$$M_{i1} \oplus \cdots \oplus M_{ir_1} \cong M_{j1} \oplus \cdots \oplus M_{js_1}$$
$$M_{i,r_1+1} \oplus \cdots \oplus M_{ir} \cong M_{j,s_1+1} \oplus \cdots \oplus M_{js}$$

for some r_1 ($1 \leqq r_1 < r$) and some s_1 ($1 \leqq s_1 < s$) after a suitable reordering of the suffixes. Otherwise, $[D_i, D_j]$ will be called a *irredundant pair*.

Definition 3.2. Let $\mathscr{F}(M)$ be all the irredundant pairs $[D_i, D_j]$ about M. The family of all $\mathscr{F}(M)$ ($M \in \mathrm{OB}(\mathfrak{C}) \setminus \mathscr{I}(\mathfrak{C})$) will be called the *relational data* of the decomposition theory of $\mathfrak{C}$, and we denote it by $\mathscr{F}(\mathfrak{C})$. By the *decomposition data* of $\mathfrak{C}$, we understand the pair $\{\mathscr{I}[\mathfrak{C}], \mathscr{F}(\mathfrak{C})\}$.

PROPERTY 3.3. *If the Krull-Schmidt theorem holds for* $\mathfrak{C}$, *then* $\mathscr{F}(M) = \varnothing$ *for all* $M \in \mathrm{OB}(\mathfrak{C}) \setminus \mathscr{I}(\mathfrak{C})$.

Definition 3.4. Let $M_i, N_j \in \mathscr{I}(\mathfrak{C})$ ($1 \leqq i \leqq l$, $1 \leqq j \leqq m$). We say that the decompositions $M_1 \oplus \cdots \oplus M_l$ and $N_1 \oplus \cdots \oplus N_m$ are *relatively prime*, if and only if $M_i \not\cong N_j$ for every i ($1 \leqq i \leqq l$) and every j ($1 \leqq j \leqq m$). Then we write $(M_1 \oplus \cdots \oplus M_l, N_1 \oplus \cdots \oplus N_m) = 1$.

PROPERTY 3.5. *Suppose that the cancellation theorem holds for* $\mathfrak{C}$. *If* $\mathscr{F}(M) \neq \varnothing$, *then for every pair* $[D_i, D_j] \in \mathscr{F}(M)$, *we get* $(D_i, D_j) = 1$.

Put $\mathscr{F}_0(\mathfrak{C}) = \{[D_i, D_j] \in \mathscr{F}(\mathfrak{C}) | (D_i, D_j) = 1\}$, then we can restate the above property 3.5 as the following statement.

PROPERTY 3.6. *If the cancellation theorem holds for* $\mathfrak{C}$, *then* $\mathscr{F}_0(\mathfrak{C}) = \mathscr{F}(\mathfrak{C})$.

Definition 3.7. Write $n(M) = \min\{r_j \in N | M_1 \oplus \cdots \oplus M_{r_j} \in \varpi^{-1}(M)\}$, and we call it the *minimal indecomposable number* of $M \in \mathrm{OB}(\mathfrak{C})$.

PROPOSITION 3.8. *Suppose that* $\mathscr{F}_0(\mathfrak{C})=\mathscr{F}(\mathfrak{C})$. *Then* $M\oplus X\cong N\oplus X$ *implies* $M\cong N$.

Proof. In order to prove our proposition, we have only to prove the following statement;

(*) If $X\in\mathscr{I}(\mathfrak{C})$ and $M\oplus X\cong N\oplus X$, then $M\cong N$.

Admitting this, we get $M\oplus X'\cong N\oplus X'$ for some $X'\in \mathrm{OB}(\mathfrak{C})$, where $X\cong X'\oplus X_0$ and $X_0\in\mathscr{I}[\mathfrak{C}]$. To show that $M\cong N$, we note that there exists a Remak decomposition of X, by the assumption of $\mathfrak{C}$ in the beginning of §2. Thus our propostion is valid by the induction on $n(X)$.

It remains to prove (*). We shall use the induction on $r=n(M)$. If $r=1$, then (*) is obvious. Thus we assume that $r=n(M)>1$ and $s=n(N)>1$. Let $M\cong M_1\oplus\cdots\oplus M_r$ and $N\cong N_1\oplus\cdots\oplus N_s$, where $M_1, \cdots, M_r, N_1, \cdots, N_s\in\mathscr{I}[\mathfrak{C}]$. Then $[M_1\oplus\cdots\oplus M_r\oplus X, N_1\oplus\cdots\oplus N_s\oplus X]$ must be a redundant pair, because of the assumption $\mathscr{F}_0(\mathfrak{C})=\mathscr{F}(\mathfrak{C})$. Therefore we have the following two cases:

(a)
$$M_1\oplus\cdots\oplus M_{r_1}\cong N_1\oplus\cdots\oplus N_{s_1}$$
$$M_{r_1+1}\oplus\cdots\oplus M_r\oplus X\cong N_{s_1+1}\oplus\cdots\oplus N_s\oplus X$$

for some $r_1(1\leqq r_1<r)$ and for some $s_1(1\leqq s_1<s)$ after a suitable renumbering of the suffixes.

(b)
$$M_1\oplus\cdots\oplus M_{r_1}\cong N_{s_1+1}\oplus\cdots\oplus N_s\oplus X$$
$$M_{r_1}\oplus\cdots\oplus M_r\oplus X\cong N_1\oplus\cdots\oplus N_{s_1}$$

for some $r_1(1\leqq r_1<r)$ and for some $s_1(1\leqq s_1<s)$ after a suitable renumbering of the suffixes.

In case (a), we get $M_{r_1+1}\oplus\cdots\oplus M_r\cong N_{s_1+1}\oplus\cdots\oplus N_s$ by the inductive hypothesis on $n(M_{r_1+1}\oplus\cdots\oplus M_r)\leqq r-r_1<r$. It follows immediately that $M\cong N$. In case (b), we deduce that

$$\begin{aligned} M&\cong(N_{s_1+1}\oplus\cdots\oplus N_s\oplus X)\oplus M_{r_1+1}\oplus\cdots\oplus M_r \\ &\cong N_{s_1+1}\oplus\cdots\oplus N_s\oplus(M_{r_1+1}\oplus\cdots\oplus M_r\oplus X) \\ &\cong N\,. \end{aligned}$$

This completes the proof of (*). Hence our proposition has been proved.

Definition 3.9. Let $D_i, D_j\in\varpi^{-1}(M)$ be two decompositions of M. A pair $[D_i, D_j]$ will be called *linkable* if and only if there exist $D_1, \cdots, D_k\in\varpi^{-1}(M)$ such that $[D_i, D_1]$, $[D_1, D_2]$, $[D_2, D_3]$, $\cdots$, $[D_{k-1}, D_k]$, $[D_k, D_j]$ are redundant pairs.

Definition 3.10. Let $\mathscr{H}(M)$ be all the non-linkable pairs $[D_i, D_j]$ of $\mathscr{F}(M)$. The family of all $\mathscr{H}(M)$ $(M\in\mathrm{OB}(\mathfrak{C})\setminus\mathscr{I}(\mathfrak{C}))$, which is trivially contained in $\mathscr{F}(\mathfrak{C})$, will be denoted by $\mathscr{H}(\mathfrak{C})$.

PROPERTY 3.11. *If $[D_i, D_j]$ is a redundant pair, then it is of course linkable, i.e. non-linkable ⇒ irredundant.*

§ 4. Application to algebraic number fields

We shall apply the preceding concepts to the following case.

Let K be an algebraic number field, and let $\mathfrak{C}$ be the full set of the fractional ideals in K. The motivation of our discussion in this section is the classical Steinitz-Chevalley-Kaplansky [10] theory of modules over a Dedekind domain. The following is our main result in this note.

THEOREM 4.1. *Let h_K denote the class number of K. Then the following conditions on h_K and $\mathscr{F}(\mathfrak{C})$, $\mathscr{H}(\mathfrak{C})$ are equivalent:*

(i) $h_K \leqq 2$.

(ii) $\mathscr{F}(\mathfrak{C}) = \mathscr{H}(\mathfrak{C})$ *(i.e. every linkable pair is always redundant).*

Proof. Step 1. (i) ⇒ (ii).

First recall the classical theory. Let $D_\mu = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ and $D_\nu = \mathfrak{O}_K^{(1)} \oplus \cdots \oplus \mathfrak{O}_K^{(r-1)} \oplus \mathfrak{A}_1 \cdots \mathfrak{A}_r$, where each $\mathfrak{A}_i$ is a fractional ideal of K $(1 \leqq i \leqq r)$ and $\mathfrak{O}_K^{(1)} = \cdots = \mathfrak{O}_K^{(r-1)} = \mathfrak{O}_K$ denote the ring of all algebraic integers in K. By the well-known Steinitz's result of modules over a Dedakind domain, we can assert that $D_u \cong D_\nu$, and that $[D_\mu, D_\nu]$ is linkable if $r \geqq 3$. In general, for $n(M) = r = 2$, it is trivially verified that the linkable pair about M is necessarily redundant.

Now we proceed the proof of Step 1. Let $[D_\mu, D_\nu]$ be a linkable pair, where $D_\mu = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$, $D_\nu = \mathfrak{A}'_1 \oplus \cdots \oplus \mathfrak{A}'_r$, for $r \geqq 3$. If $h_K = 1$, then the statement (ii) is obvious by the definitions in Section 3. Therefore, let $h_K = 2$ and let $\mathfrak{B}$ be a fixed non-principal ideal of K. Without loss of generality, we may assume that

$$\mathfrak{A}_1 \cong \mathfrak{B}, \cdots, \mathfrak{A}_{s_1} \cong \mathfrak{B}, \mathfrak{A}_{s_1+1} \cong \mathfrak{O}_K, \cdots, \mathfrak{A}_r \cong \mathfrak{O}_K ,$$
$$\mathfrak{A}'_1 \cong \mathfrak{B}, \cdots, \mathfrak{A}'_{s_2} \cong \mathfrak{B}, \mathfrak{A}'_{s_2+1} \cong \mathfrak{O}_K, \cdots, \mathfrak{A}'_r \cong \mathfrak{O}_K ,$$
$$s_1 \geqq s_2 .$$

By the classical theory, $\mathfrak{A}_1 \cdots \mathfrak{A}_{s_1}$ and $\mathfrak{A}'_1 \cdots \mathfrak{A}'_{s_2}$ are contained in the same ideal class. Hence $s_1 \equiv s_2 \pmod{2}$. In case $s_1 = s_2$, we easily deduce that $[D_\mu, D_\nu]$ is redundant. If $s_1 > s_2 > 0$, then $\mathfrak{A}'_1 \oplus \cdots \oplus \mathfrak{A}'_{s_2} \cong \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_{s_2}$ and $\mathfrak{A}'_{s_2+1} \oplus \cdots \oplus \mathfrak{A}'_r \cong \mathfrak{A}_{s_2+1} \oplus \cdots \oplus \mathfrak{A}_r (0 < s_2 < r)$, therefore $[D_\mu, D_\nu]$ is redundant. (We note that $s_1 - s_2$ is even and that $\mathfrak{B} \oplus \mathfrak{B} \cong \mathfrak{O}_K \oplus \mathfrak{B}\mathfrak{B} \cong \mathfrak{O}_K \oplus \mathfrak{O}_K$.) If $s_1 > s_2 = 0$, then $\mathfrak{A}'_1 \oplus \mathfrak{A}'_2 \cong \mathfrak{A}_1 \oplus \mathfrak{A}_2$ and $\mathfrak{A}'_3 \oplus \cdots \oplus \mathfrak{A}'_r \cong \mathfrak{A}_3 \oplus \cdots \oplus \mathfrak{A}_r$, since s_1 is even and $s_1 \geqq 2$, $r > 2$. Thus $[D_\mu, D_\nu]$ is always redundant. This complets the Step 1 of our proof.

Step 2. (ii) ⇒ (i).

We have only to prove that $h_K \geqq 3$ implies $\mathscr{F}(\mathfrak{C}) \supsetneqq \mathscr{H}(\mathfrak{C})$ (i.e. there exists a linkable pair which is not redundant).

Let C_K be the ideal class group of K, and C_m be a cyclic group of order $m \in N$. We classify the following three cases;

(a) $C_K \cong C_3$.

(b) $C_K \cong C_2 \times C_2$.

(c) $C_K \cong C_4$ or $|C_K| \geqq 5$.

In case (a), let $D_1 = \mathfrak{A} \oplus \mathfrak{A} \oplus \mathfrak{A}$, $D_2 = \mathfrak{O}_K \oplus \mathfrak{O}_K \oplus \mathfrak{O}_K$, where $\mathfrak{A} \in \mathfrak{C}$, $\langle [\mathfrak{A}] \rangle = C_K$.
In case (b), let $D_1 = \mathfrak{A} \oplus \mathfrak{B} \oplus \mathfrak{A}\mathfrak{B}$, $D_2 = \mathfrak{O}_K \oplus \mathfrak{O}_K \oplus \mathfrak{O}_K$, where $\mathfrak{A}, \mathfrak{B} \in \mathfrak{C}$, $\langle [\mathfrak{A}] \rangle \times \langle [\mathfrak{B}] \rangle = C_K$.
In case (c), let $D_1 = \mathfrak{A} \oplus \mathfrak{A} \oplus \mathfrak{A}$, $D_2 = \mathfrak{O}_K \oplus \mathfrak{O}_K \oplus \mathfrak{A}^3$, where $\mathfrak{A} \in \mathfrak{C}$, $\langle [\mathfrak{A}] \rangle = C_K \cong C_4$.
If $|C_K| \geqq 5$, we decompose as $C_K = \langle [\mathfrak{A}_1] \rangle \times \langle [\mathfrak{A}_2] \rangle \times \cdots \times \langle [\mathfrak{A}_r] \rangle \cong C_{p_1^{e_1}} \times C_{p_2^{e_2}} \times C_{p_3^{e_3}} \times \cdots \times C_{p_r^{e_r}}$ (since C_K is a finite abelian group), and $p_1^{e_1} + p_2^{e_2} + \cdots + p_r^{e_r} - r - 1 \geqq 2$, where $|C_K| = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ ($p_1, \cdots, p_r$ are not necessarily distinct primes) and $\mathfrak{A}_1, \cdots, \mathfrak{A}_r \in \mathfrak{C}$. In this case, let

$$D_1 = \mathfrak{A}_1^{(1)} \oplus \cdots \oplus \mathfrak{A}_1^{(p_1^{e_1}-1)} \oplus \mathfrak{A}_2^{(1)} \oplus \cdots \oplus \mathfrak{A}_2^{(p_2^{e_2}-1)} \oplus \cdots \oplus \mathfrak{A}_r^{(1)} \oplus \cdots \oplus \mathfrak{A}_r^{(p_{s_r}^{e_r}-1)},$$
$$D_2 = \mathfrak{O}_K^{(1)} \oplus \mathfrak{O}_K^{(2)} \oplus \cdots \oplus \mathfrak{O}_K^{(p_1^{e_1} + \cdots + p_r^{e_r} - r - 1)} \oplus \mathfrak{A}_1^{p_1^{e_1}-1} \mathfrak{A}_2^{p_2^{e_2}-1} \cdots \mathfrak{A}_r^{p_r^{e_r}-1},$$

where for each $i (1 \leqq i \leqq r)$, $\mathfrak{A}_i^{(j)} = \mathfrak{A}_i^{(k)} = \mathfrak{A}_i$ $(1 \leqq j, k \leqq p_i^{e_i} - 1)$ and $\mathfrak{O}_K^{(1)} = \mathfrak{O}_K^{(2)} = \cdots = \mathfrak{O}_K^{(p_1^{e_1} + \cdots + p_r^{e_r} - r - 1)} = \mathfrak{O}_K$.

In any case, $[D_1, D_2]$ is linkable by the consequence of the Steinitz's theorem, however $[D_1, D_2]$ is never redundant. Thus the Step 2 is proved. This completes the proof of our theorem.

References

[1] JACOBINSKI, H.; Genera and decomposition of lattices over orders, *Acta. Math.* **121** (1968) p. 1-29.

[2] JACOBINSKI, H.; Lattices over orders, *Proc. of Symp. in pure Math. of Amer. Math Soc.* **XXI**, Representation theory of finite groups and related topics, (1971) p. 85-88.

[3] REINER, I.; A survey of integral representation theory, *Bull. Amer. Math. Soc.* **76** (1970) p. 159-227.

[4] LANG, S.; *Algebra*, Addison-Wesley, (1965).

[5] ATIYAH, M.; On the Krull-Schmidt theorem with application to sheaves, *Bull. Soc. Math. Erance* **84** (1956) p. 307-317.

[6] GROTHENDIECK, A.; Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957) p. 119-221.

[7] SWAN, R. G.; Projective modules over group rings and maximal orders, *Annals of Math.* **76** (1962) p. 55-61.

[8] BERMAN, S. D. and GUDIVOK, P. M.; Integral representations of finite groups, *Soviet Math. Dokl.* **3** (1962) p. 1176-1174.

[9] POPESCU, N.; *Abelian categories with applications to rings and modules*, Academic Press (1973).

[10] KAPLANSKY I.; Modules over a Dedekind rings and valuation rings, *Trans. Amer. Math. Soc.* **72** (1952) p. 327-340.

Department of Mathematics,
St. Paul's University
Ikebukuro, Tokyo, Japan